SINGLE GENERATION OF TRANSFORMATION GROUP C^*-ALGEBRAS

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Abstract. We prove that $C(X) \rtimes \Gamma$ is singly generated whenever Γ is a finitely generated discrete amenable group and its action on the nonempty compact metrizable space X is free, minimal, and has the uniform Rokhlin property and Cuntz comparison of open sets. In particular, every free minimal $\mathbb{Z}^d$-crossed product with metrizable underlying space is singly generated.

1. Introduction

In [7], Kadison asked whether every von Neumann algebra acting on a separable Hilbert space is singly generated (see also in [6]). This problem was reduced to the II_1 factor case([18]). Recently, OpenAI [14] resolves the remaining case. In the C^*-algebra setting, one can ask whether each simple separable C^*-algebra singly generated? Simplicity is necessary, otherwise $C(S^2)$ is not singly generated. Many classes of C^*-algebras are proved to be singly generated, such as stable C^*-algebras, UHF-absorbing C^*-algebras, $\mathcal{Z}$-absorbing C^*-algebras, the first type Villadsen's algebras, $C_r^*(\mathbb{F}_2)$, and $C_r^*(\mathbb{F}_n)$ for each $n \in \{3, 4, \ldots\} \cup \{\infty\}$([13, 16, 15, 9, 3, 4]). On the other hand, we construct a simple separable unital AH algebra which is not singly generated([5]). Thus simplicity and separability alone do not suffice.

For transformation group algebras, the uniform Rokhlin property (URP) and Cuntz comparison of open sets (COS) provide large matrix models and control their complements. It is showed that these two properties imply stable rank one for a free minimal action of a countable amenable group [8, Theorem 7.8]. The purpose of this paper is to prove a single-generation theorem under the same dynamical hypotheses when the acting group is finitely generated. In fact, we get the following result.

Theorem 1.1. *Let Γ be a finitely generated discrete amenable group, and let (X, Γ) be a free and minimal dynamical system, where X is a nonempty compact metrizable space. If (X, Γ) has the URP and COS, then $C(X) \rtimes \Gamma$ is singly generated.*

Since any free and minimal $\mathbb{Z}^d$ action has URP and COS([10]), we have the following result.

Corollary 1.2. *Let $d \geq 1$, and let $(X, \mathbb{Z}^d)$ be a free and minimal dynamical system on a nonempty compact metrizable space. Then $C(X) \rtimes \mathbb{Z}^d$ is singly generated.*

The proof has two stages. In the first stage, a finite family of self-adjoint Fourier sums is recovered from two self-adjoint elements (Proposition 4.1). The interior of each Rokhlin tower provides a matrix algebra. Its complement has uniformly small measure, so COS and stable rank one move the remainder into a small collection of tower rows. The available off-diagonal positions accommodate both the Fourier coefficients and every corner of the transported remainder. Three positions are left unused. Applied to the real and imaginary parts of finitely many group generators, this construction recovers all implementing unitaries.

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The second stage recovers a prescribed diagonal function using a pair in any nonzero diagonal hereditary subalgebra, relative to these known unitaries (Proposition 5.2). The central reconstruction identity is the following. If a finite family $(V_\gamma)_{\gamma\in F}$ has $S=\sum_\gamma V_\gamma^* V_\gamma$ invertible, then

$$a = S^{-1}\left(\sum_{\gamma,\nu\in F} V_\gamma^*(V_\gamma a V_\nu^*)V_\nu\right)S^{-1}. \tag{1.1}$$

Minimality supplies such a frame with $V_\gamma=\chi^{1/2}u_\gamma^*$ for any nonzero $\chi\in C(X)_+$. We encode the inner corners in (1.1); the outer frame operators are already known. This recovers the actual diagonal function without recovering a tower-base projection. Disjoint spectral intervals then combine countably many local pairs, and the three unused positions store the resulting three hereditary elements.

The matrix encoding follows the principle of Olsen and Zame [13]. Here the matrix units are multipliers, the tower algebras need not be full corners, and the complement must be recovered in norm. We address these points by scalar cutoffs, separate coding of all remainder corners, and the finite-frame identity. Uniform smallness in measure is used for comparison and is never converted into smallness in norm.

The finite generation assumption is used to recover all implementing unitaries in the first construction. Countably many local pairs can then be placed in disjoint spectral intervals and combined by norm convergent series. The method does not assume strict comparison of all positive elements or $\mathcal{Z}$-stability. For comparison, under URP and COS, zero mean dimension already gives $\mathcal{Z}$-stability by theorem [11], and hence single generation by [15]. The argument here does not impose zero mean dimension. The theorem asserts existence of a generator; it does not assert density of generators or a bound on generator rank.

Section 2 fixes the tower and comparison conventions. Section 3 gives the finite-frame identity and the off-diagonal filtering lemmas. Sections 4 and 5 establish the two recovery constructions. Section 6 combines them and proves the main theorem. Section 7 discusses the role of finite generation and the scope of the method.

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2. Notation and preliminaries

2.1. Dynamical systems and Rokhlin towers.

Definition 2.1. Consider a topological dynamical system (X,Γ), where X is a separable compact Hausdorff space, and Γ is a discrete group which acts on X from the right. The dynamical system (X,Γ) is said to be minimal if

$$Y\gamma = Y, \quad \gamma\in\Gamma,$$

for some closed $Y\subseteq X$ implies $Y=\emptyset$ or $Y=X$; it is said to be free if

$$x\gamma = x$$

for some $x\in X$ and $\gamma\in\Gamma$ implies $\gamma=e$.

Definition 2.2. A Borel measure μ on X is invariant under the action σ if for any Borel set $E\subseteq X$, one has

$$\mu(E)=\mu(E\gamma), \quad \gamma\in\Gamma.$$

Denote by $\mathcal{M}_1(X,\Gamma)$ the compact space of invariant Borel probability measures on X. It is a Choquet simplex under the weak* topology.

Definition 2.3. Let Γ be a countable discrete group. Let $K \subseteq \Gamma$ be a finite set and let $\varepsilon > 0$. Then a finite set $E \subseteq \Gamma$ is said to be (K, ε)-invariant if

$$\frac{|EK \Delta E|}{|E|} < \varepsilon.$$

The group Γ is amenable if there exists a sequence (Γ_n) of finite subsets of Γ such that for any (K, ε), there is N such that Γ_n is (K, ε)-invariant for any $n > N$. The sequence (Γ_n) is called a Følner sequence.

The K-interior of a finite set $E \subseteq \Gamma$ is defined as

$$\mathrm{int}_K(E) = \{\gamma \in E : \gamma K \subseteq E\}.$$

Consider a topological dynamical system (X, Γ). Then the group Γ acts (from the left) on the C^*-algebra $C(X)$ by

$$\alpha_\gamma(f) = f \circ \gamma.$$

The (full) crossed product C^*-algebra $A = C(X) \rtimes \Gamma$ is defined to be the universal C^*-algebra generated by $C(X)$ and unitaries u_γ, $\gamma \in \Gamma$, with respect to the relations

$$u_\gamma f u_\gamma^* = f \circ \gamma, \quad u_{\gamma_1} u_{\gamma_2}^* = u_{\gamma_1 \gamma_2^{-1}}, \quad u_e = 1_A, \quad \gamma, \gamma_1, \gamma_2 \in \Gamma.$$

The C^*-algebra A is nuclear if Γ is amenable (see, for instance, Corollary 7.18 of [17]). If, moreover, (X, Γ) is minimal, then A is simple. (Theorem 5.16 of [2] and Théorème 5.15 of [19]), i.e. A has no non-trivial closed two-sided ideals.

Definition 2.4. A topological dynamical system (X, Γ) is said to have the uniform Rokhlin property (URP) if for any finite set F, and any $\varepsilon > 0$, there are open Rokhlin towers $(B_1, F_1, \ldots, (B_S, F_S)$ such that $F_1, F_2, \ldots, F_S$ are (F, ε)-invariant, the sets

$$B_s\gamma, \quad \gamma \in F_s, s = 1, 2, \ldots, S$$

are mutually disjoint, and

$$\mu(X \setminus \bigsqcup_{s=1}^{S} \bigsqcup_{\gamma \in F_s} B_s\gamma) < \varepsilon, \quad \mu \in \mathcal{M}_1(X, \Gamma).$$

Definition 2.5. [12] A topological dynamical system (X, Γ) is said to have (λ, m)-Cuntz comparison of open sets, where $\lambda \in (0, +\infty)$ and $m \in \mathbb{N}$, if, for any open sets $E, F \subseteq X$ with

$$\mu(E) \leq \lambda \mu(F), \quad \mu \in \mathcal{M}_1(X, \Gamma),$$

one has

$$\varphi_E \precsim \varphi_F^{\oplus m}.$$

A topological dynamical system is said to have Cuntz-comparison of open sets (COS) if it has (λ, m)-Cuntz-comparison of open sets for some λ and m.

For $a \in A_+$, put $\mathrm{Her}(a) = \overline{aAa}$ and let p_a be its support projection in A^{**}. For a positive continuous function, write $\mathrm{osupp}(a) = \{x : a(x) > 0\}$; $\mathrm{supp}(a)$ denotes closed support. If $W \subseteq X$ is open, choose $\varphi_W \in C(X)_+$ with $\mathrm{osupp}(\varphi_W) = W$, and write $\mathrm{Her}(W) = \mathrm{Her}(\varphi_W)$. Such a function exists because X is metrizable. Functions with the same open support are Cuntz equivalent already in $C(X)$, and determine the same hereditary subalgebra in A.

Indeed, for $a, b \in C(X)_+$ with $\mathrm{osupp}(a) \subseteq \mathrm{osupp}(b)$, continuous quotients with a cutoff near $b = 0$ give $a \precsim b$ in $C(X)$.

Uniform smallness of the measure of an open support is used below only to obtain comparison. The remainder itself is recovered exactly or by a norm limit; no norm estimate is inferred from a measure estimate.

Throughout Sections 2–5, Γ is countably infinite, and the action is free and minimal with the URP and COS. The finite-group case will be treated separately in the proof of Theorem 1.1.

An open Rokhlin tower (U, E) has mutually disjoint levels Ua, $a \in E$. In a family of towers, all levels are required to be mutually disjoint. By replacing U with Ua_0 and E with $a_0^{-1}E$, we may assume $e \in E$.

For a right Følner sequence (Φ_n) and a closed set $K \subseteq X$, write

$$\operatorname{ocap}(K) = \limsup_{n\to\infty} \sup_{x\in X} \frac{1}{|\Phi_n|} \sum_{\gamma\in\Phi_n} 1_K(x\gamma).$$

On closed sets this agrees with orbit capacity in [8] and satisfies

$$\operatorname{ocap}(K) = \max_{\mu\in\mathcal{M}_1(X,\Gamma)} \mu(K) \quad [8, \text{Remark 2.20}].$$

2.2. **Preparation of towers.**

Lemma 2.6. *Given a finite buffer L, $\varepsilon > 0$, and $N \in \mathbb{N}$, the URP towers may be chosen with $|E_s| \geq N$, mutually disjoint closed cores $B_s a \subset U_s a$, and*

$$\mu\left(\bigsqcup_s \bigsqcup_{a\in E_s} B_s a\right) > 1 - \varepsilon, \qquad |E_s \setminus \operatorname{int}_L(E_s)| < \varepsilon |E_s|.$$

There are functions $g_s, \xi_s, f_s \in C_0(U_s)_+$, bounded by 1, strictly positive g_s on U_s, and equal to 1 near B_s, with

$$\operatorname{supp}\xi_s \Subset \operatorname{int}\{g_s = 1\}, \qquad \operatorname{supp} f_s \Subset \operatorname{int}\{\xi_s = 1\}.$$

Proof. Enlarge the prescribed buffer to a finite set L' containing e and more than $2N$ elements, and choose the URP error $\delta < 1$ sufficiently small. For any nonempty shape E and any $a \in E$,

$$|L'| = |aL'| \leq |EL'| < (1+\delta)|E|,$$

so $|E| \geq N$. Also

$$|E \setminus \operatorname{int}_L(E)| \leq |L|\,|EL \setminus E| \leq |L|\,|EL' \setminus E|.$$

Taking δ smaller gives the asserted interior estimate. To obtain cores uniformly in the invariant measures, approximate the indicator functions of the finitely many open bases increasingly by continuous contractions of compact support. The corresponding integrals are continuous on the compact space of invariant measures. Their sets of values greater than a fixed number below 1 form an increasing open cover, so one approximation works for every measure. A level set of that approximation gives compact cores with the required measure estimate, after slightly relaxing the error. Discard towers whose chosen cores are empty. The remaining cores still have the asserted measure estimate, and their translated levels are disjoint. Urysohn functions give the nested plateaux; adding a small strictly positive function off the plateau makes g_s strictly positive on U_s without changing its value near B_s. □

2.3. **Tower algebras and exact comparison.** The tower algebra associated with (U_s, E_s) is the image of

$$\iota_s : M_{|E_s|}(C_0(U_s)) \longrightarrow A, \qquad [c_{ab}] \longmapsto \sum_{a,b\in E_s} u_a^* c_{ab} u_b. \tag{2.1}$$

It is an injective $*$-homomorphism. Disjointness of the levels gives multiplicativity, and its restriction to each diagonal corner is faithful. Distinct tower algebras are orthogonal.

We use multiplier matrix units e^s_{ab} for this algebra, represented in A^{**} by

$$e^s_{ab} = u_a^* p_{U_s} u_b, \qquad e^s_{ab} e^s_{cd} = \delta_{bc} e^s_{ad},$$

where p_{U_s} is the characteristic projection of U_s. For $c \in C_0(U_s)$, the notation ce^s_{ab} means the actual element $u_a^* c u_b \in A$. A scalar base function acts as the corresponding central multiplier of the whole matrix block, not just as a function on its identity level. In particular, every use of a multiplier below is accompanied by a coefficient vanishing at the boundary of its base.

Lemma 2.7. *Let $C = M_n(C_0(U))$ be a tower algebra, with its multiplier algebra represented in A^{**}. A bounded continuous matrix coefficient $b \in M(C)$ satisfies $cb, bc \in C \subseteq A$ for every scalar $c \in C_0(U)$. If $cb = p_\chi(cb)p_\chi$, then $cb \in \mathrm{Her}(\chi)$.*

More generally, let C_s, C_t be two tower algebras, let T_s, T_t be positive scalar elements in them, and let v_s, v_t be bounded multiplier columns. If $d \in A$ satisfies $T_s d = d = dT_t$, then

$$v_s^* d v_t = (T_s v_s)^* d (T_t v_t) \in A.$$

The same observation applies to translating this coefficient to other columns. It requires no multiplier matrix unit between different towers.

Proof. A bounded continuous matrix function multiplied by a function in $C_0(U)$ again vanishes at infinity. The hereditary assertion follows from $\mathrm{Her}(\chi) = p_\chi A^{**} p_\chi \cap A$. For the last formula, $T_s v_s$ and $T_t v_t$ are actual elements of A, and scalar multiplier functions commute with the columns. □

For clarity, $p_{s,a}$ will always mean the multiplier projection e^s_{aa} of a tower row. Such a projection may be used to describe support, but its membership in a generated subalgebra will never be assumed. We call $t_s e^s_{ab}$ a *soft matrix unit*: it is an actual element of A, although e^s_{ab} need not be.

Lemma 2.8. *The algebra A has stable rank one. If $a, b \in A_+$ and $a \precsim b$, then there is $v \in A$ with $v^*v = a$ and $vv^* \in \mathrm{Her}(b)$. If $a, b \in \mathrm{Her}(\chi)$, then $v \in \mathrm{Her}(\chi)$.*

Proof. Stable rank one is [8, Theorem 7.8]. We deduce the required implementation from the Hilbert-module description of the Cuntz semigroup of Coward, Elliott, and Ivanescu [1, Theorem 3 and Appendix 6]. Put

$$E_a = \overline{aA}, \qquad E_b = \overline{bA}, \qquad \langle x, y \rangle = x^* y.$$

These are countably generated Hilbert A-modules, and $a \precsim b$ means $[E_a] \leq [E_b]$ in the Hilbert-module picture. Since A has stable rank one, the cited theorem says that this order is realized by an inner-product preserving A-linear embedding $U : E_a \to E_b$. The element $a^{1/2}$ belongs to E_a by continuous functional calculus. Hence $v = U(a^{1/2}) \in E_b \subseteq A$ satisfies

$$v^*v = \langle U(a^{1/2}), U(a^{1/2}) \rangle = \langle a^{1/2}, a^{1/2} \rangle = a.$$

Choose $x_n \in A$ with $bx_n \to v$. Then $bx_n x_n^* b \to vv^*$, so $vv^* \in \overline{bAb} = \mathrm{Her}(b)$. No adjointability of the module embedding is needed. For the last assertion, both v^*v and vv^* belong to $\mathrm{Her}(\chi)$. An approximate unit (e_n) of $\mathrm{Her}(\chi)$ satisfies $e_n v \to v$ and $v e_n \to v$, since

$$\|(1 - e_n)v\|^2 = \|(1 - e_n)vv^*(1 - e_n)\| \longrightarrow 0$$

and the analogous identity uses v^*v on the other side. Hence $e_n v e_n \to v$. □

Lemma 2.9. *Let t be a strictly positive contraction in a scalar tower algebra. If a continuous bounded multiplier coefficient a satisfies $\|a(x)\| \leq Ct(x)$, then $\theta_\varepsilon(t)a \to a$ in norm, where θ_ε is a smooth function taking values in $[0, 1]$ and $\theta_\varepsilon = 0$ on $[0, \varepsilon]$ and 1 on $[2\varepsilon, 1]$. Thus expressions involving inverse powers of t can be recovered with continuous functions which vanish near zero, provided the final coefficient is $O(t)$. If $c \in \mathrm{Her}(r)$ and r and rcr are recovered, then c is recovered by*

$$k_\varepsilon(r)(rcr)k_\varepsilon(r) = \theta_\varepsilon(r)c\theta_\varepsilon(r) \longrightarrow c, \qquad k_\varepsilon(t) = \theta_\varepsilon(t)/t.$$

Proof. The first error is at most $2C\varepsilon$. The second convergence is the approximate-unit property of $\mathrm{Her}(r)$. □

3. Finite frames and spectral recovery

Lemma 3.1. *Let A be unital and let $D \subseteq A$ be a unital C^*-subalgebra. Suppose $V_1, \dots, V_M \in D$ and $S = \sum_j V_j^* V_j$ is invertible in A. If $a \in A$ and $V_j a V_k^* \in D$ for every j, k, then $a \in D$, with*

$$a = S^{-1} \left(\sum_{j,k} V_j^* (V_j a V_k^*) V_k \right) S^{-1}.$$

For a minimal action and $0 \neq \chi \in C(X)_+$, a finite family $V_\gamma = \chi^{1/2} u_\gamma^*$ always has this property.

Proof. Spectral permanence gives $S^{-1} \in D$, and the middle sum equals SaS. For the last assertion, translates of $\{\chi > 0\}$ cover X by minimality. A finite subcover, enlarged to a symmetric finite set F containing the identity, gives $S = \sum_{\gamma \in F} \alpha_\gamma(\chi) > 0$ everywhere. Compactness then gives $S \geq \kappa 1$ for some $\kappa > 0$. □

Lemma 3.2. *Let $H = H^*$ in a unital C^*-algebra and x be any element. For $F \in C_c^\infty(\mathbb{R}^2)$ define*

$$\Phi_F^H(x) = \frac{1}{(2\pi)^2} \iint \widehat{F}(s,t) e^{isH} x e^{itH} \, ds \, dt, \quad \widehat{F}(s,t) = \iint F(a,b) e^{-i(sa+tb)} \, da \, db.$$

Then $\Phi_F^H(x) \in C^*(1, H, x)$. If, in a tower block, $H = t \operatorname{diag}(\lambda_i)$ and $x = a e_{ij}$, then

$$\Phi_F^H(x) = F(\lambda_i t, \lambda_j t) x.$$

If $Hx = \lambda x$ and $x = x p_J(H)$ for a spectral interval J, then $\Phi_F^H(x) = x F(\lambda, H) p_J(H)$.

Proof. The Fourier transform is integrable, so the displayed integral is a norm Bochner integral of elements of $C^*(1, H, x)$. The two formulas follow from Fourier inversion, first in the scalar tower algebra and then in the continuous functional calculus of the right copy of H. □

Lemma 3.3. (1) *A finite family of labels $\lambda_1, \dots, \lambda_N$ in $(2, 2.1)$ can be chosen so that all ratios λ_i / λ_j, $i \neq j$, are distinct.*

(2) *Let $C = M_N(C_0(U))$ be a tower algebra, let $t \in C_0(U)$ be a strictly positive contraction, and suppose that $H = H^* \in A$ acts on this block as $t \operatorname{diag}(\lambda_i)$. Let*

$$x = \sum_{(i,j) \in J} a_{ij} e_{ij}, \qquad J \subseteq \{(i,j) : i \neq j\},$$

with distinct ordered positions, scalar coefficients $a_{ij} \in C_0(U)$, and $|a_{ij}| \leq C_{ij} t$. Then each $a_{ij} e_{ij}$ belongs to $C^*(1, H, x)$. The same conclusion holds for finitely many orthogonal tower blocks if the labels are chosen together.

The statement in (2) concerns off-diagonal positions only.

Proof. For (1), choose

$$\lambda_i = 2 \exp(c 2^i), \qquad c = \frac{\log(21/20)}{2^{N+1}}, \qquad 1 \leq i \leq N.$$

These labels lie in $(2, 2.1)$. The nonzero differences $2^i - 2^j$ are distinct for distinct ordered pairs: for a positive difference, its power of 2 determines the smaller index, and its odd part determines the larger index. Negative differences are treated by reversing the pair. This proves the ratio assertion.

For (2), the rays $R_{ij} = \{(\lambda_i \tau, \lambda_j \tau) : 0 \leq \tau \leq 1\}$ meet only at the origin. Fix $\varepsilon > 0$. A smooth compactly supported function on $\mathbb{R}^2$ can be chosen equal to $\theta_\varepsilon(\tau)$ on the selected ray and zero on all other rays, where θ_ε is the cutoff of Lemma 2.9. One obtains it by a smooth extension along the ray and a narrow tubular cutoff on its portion away from zero. Lemma 3.2 gives $\theta_\varepsilon(t) a_{ij} e_{ij} \in C^*(1, H, x)$. Its distance from $a_{ij} e_{ij}$ is at most $2 C_{ij} \varepsilon$. The finite-block assertion uses the same construction with the globally distinct ratios. □

Lemma 3.4. *Let $(e_{ij})_{i,j=1}^{N}$, $N \geq 2$, be multiplier matrix units of a matrix algebra over $C_0(U)$, and let $t \in C_0(U)$ be a strictly positive contraction. If a C^*-subalgebra $D \subseteq A$ contains te_{ij} for the edges of a connected graph on the rows, then it contains every te_{ij} and every $f(t)e_{ij}$ with f continuous and $f(0) = 0$.*

Proof. For an edge $x = te_{ij}$, the positive square roots of x^*x and xx^* are te_{jj} and te_{ii}. Products along a path give $t^k e_{ij}$. Multiplying by the continuous function $\theta_\varepsilon(t)t^{1-k}$ of the known diagonal te_{ii}, extended by zero near zero, gives $\theta_\varepsilon(t)te_{ij} \to te_{ij}$ in norm. Continuous functional calculus recovers $f(t)e_{ii}$. Multiplication by $k_\varepsilon(te_{ii})(te_{ij})$, where $k_\varepsilon(\tau) = \theta_\varepsilon(\tau)/\tau$, then gives $\theta_\varepsilon(t)f(t)e_{ij} \to f(t)e_{ij}$. The last convergence uses continuity at zero, rather than a boundedness assumption on $f(t)/t$. □

Remark 3.5. We use smooth functions with integrable Fourier transforms, not an unjustified continuous two-variable functional calculus on A. No uniform bound on the filters as $\varepsilon \downarrow 0$ is needed: at each stage all unwanted entries are annihilated exactly and the remaining scalar cutoff error tends to zero in norm. Fixed point entries are assigned positions off every occupied ray, including the reverse rays. Entries on a vertical or horizontal axis are recovered later by row cutoffs. Diagonal entries all lie on the ray of ratio 1, so the off-diagonal separation lemma makes no assertion about separating them. In the constructions below, every diagonal remainder corner is first moved to an unused off-diagonal position.

4. Recovery of a finite Fourier family

Proposition 4.1. *Let Γ be a countably infinite amenable group acting freely and minimally on the compact metrizable space X, and suppose the action has URP and COS. For every finite family $a_1, \ldots, a_q$ of self-adjoint finite Fourier sums in A, there are $H_0 \in C(X)_+$ and $K_0 \in A_{\mathrm{sa}}$ such that $D_0 = C^*(1, H_0, K_0)$ contains all the a_i. There is a nonempty open set $W \subset X$ such that, for every $P_1, P_2, P_3 \in \mathrm{Her}(W)_{\mathrm{sa}}$, one can replace K_0 by a self-adjoint $K \in A$ satisfying*

$$D_0 \subset C^*(1, H_0, K), \qquad P_j \in C^*(1, H_0, K) \quad (j = 1, 2, 3).$$

The three additional entries occupy distinct previously unused off-diagonal matrix positions of one tower block.

Proof. We give the construction and then verify recovery in its required order. All matrix units below are multiplier matrix units; every expression asserted to belong to A has a coefficient in the appropriate $C_0(U_s)$. *(1) Compression and comparison.* Enlarge a common Fourier support to a finite symmetric set L containing the identity, and write $\ell = |L|$. We may suppose $q \geq 1$. Let λ, m be the COS constants. Fix

$$0 < \rho \leq \frac{1}{100\sqrt{q+1}}, \qquad 0 < \delta < \min\left\{\frac{1}{100}, \frac{\rho}{100}, \frac{\lambda\rho}{100m}\right\}, \qquad 0 < \eta < \frac{1}{100}.$$

Choose the towers of Lemma 2.6 with buffer L, error δ, and shapes of size

$$n_s = |E_s| \geq N, \qquad N \geq 100(q\ell + q + 10) + \frac{2m}{\rho}.$$

Put $I_s = \mathrm{int}_L(E_s)$ and $d_s = |I_s|$. In their tower algebra $B = \bigoplus_s M_{n_s}(C_0(U_s))$, set

$$h = \sum_s \sum_{a \in I_s} g_s e^s_{aa}, \qquad r = 1 - h, \qquad c_i = r^{1/2} a_i r^{1/2}, \qquad b_i = a_i - c_i.$$

The diagonal functions in this formula are extended by zero off their tower levels. Since $h = 1$ on every interior closed core,

$$\mu(\{r > 0\}) < 2\delta \quad \text{for every invariant probability measure } \mu. \tag{4.1}$$

Indeed, discarding boundary cores loses at most $\delta \sum_s n_s \mu(B_s) \leq \delta$ in addition to the core error.

We have $b_i \in B$ exactly. For a Fourier term $f_\gamma u_\gamma$, the coefficient of its difference from the compression by $r^{1/2}$ is

$$f_\gamma(1 - r^{1/2}\alpha_\gamma(r^{1/2})).$$

Whenever this coefficient is nonzero, one endpoint is an L-interior level. Symmetry of L places the other endpoint in the same tower: if the range row is a, its source row is $a\gamma$; conversely, a source row b has range row $b\gamma^{-1}$. In base coordinates the scalar multiplier is 0, g_s, or $1 - \sqrt{1 - g_s}$. Since $1 - \sqrt{1 - g_s} \le g_s$, all three are bounded by g_s. Consequently each block of each b_i has at most ℓn_s possibly nonzero matrix coefficients, and every such coefficient β satisfies $|\beta| \le C_i g_s$ for a fixed finite constant C_i.

In each I_s reserve $k_s = m\lfloor \rho n_s/m \rfloor$ rows and divide them into m groups of equal size k_s/m. The groups in different blocks are combined using the same group index. Let V be the Cartan function obtained by putting f_s in the first group of each block, and let R put f_s in all reserved rows. For each invariant μ,

$$\mu(\{V > 0\}) \ge \frac{\rho}{2m} \sum_s n_s \mu(B_s) > \frac{\rho}{2m}(1 - \delta).$$

Our choices imply $2\delta < \lambda\rho(1 - \delta)/(2m)$. Thus COS and (4.1) give $r \precsim V^{\oplus m}$. The m row groups are identical copies of the scalar base functions, so $V^{\oplus m}$ is Cuntz equivalent to R. More explicitly, the row identifications, multiplied by $f_s^{1/2}$, form a rectangular matrix over A whose source square is $V^{\oplus m}$ and whose range square is R. Thus the stabilized comparison has a single-element target $R \in A$. No bare multiplier is asserted to belong to A. Lemma 2.8 therefore provides $v \in A$ with

$$v^*v = r, \qquad vv^* \in \mathrm{Her}(R). \tag{4.2}$$

Put $d_i = vc_iv^* \in \mathrm{Her}(R)$.

Use the nested cutoffs of Lemma 2.6 and put

$$t_s = \eta g_s + (1 - \eta)\xi_s.$$

This function is strictly positive on U_s, belongs to $C_0(U_s)$, and equals 1 on $\mathrm{supp}\, f_s$. Fix a tower s_* with nonempty core, and require that the largest high label occur in this tower. Choose globally distinct high labels $\lambda_{s,a} \in [2, 2.1]$, $a \in I_s$, with all nontrivial ordered ratios distinct, as in Lemma 3.3. Give each boundary row a distinct positive label $\nu_{s,b} < \eta$, and set

$$H_0 = \sum_s t_s \left(\sum_{a \in I_s} \lambda_{s,a} e^s_{aa} + \sum_{b \in E_s \setminus I_s} \nu_{s,b} e^s_{bb} \right). \tag{4.3}$$

On the support of r, every high row has $g_s < 1$, hence $\xi_s = 0$, and $t_s = \eta g_s$. Thus

$$(H_0 - 2.1\eta)_+ r^{1/2} = 0, \qquad p_r \le 1_{[0,2.1\eta]}(H_0). \tag{4.4}$$

On a reserved row supporting vv^*, however, $t_s = 1$ and H_0 has the fixed corresponding high label.

(2) Assignment of the matrix coefficients. We now assign the entries of K_0. An unordered high–high position means a pair of distinct high rows; both orientations of that position are occupied together. First choose, in each block, a chain through all its high rows, and put $t_s e^s_{a,a'}$ on its successive edges. For each actual scalar coefficient β of each b_i, assign a new high–high position in the same block and put $t_s \beta e^s_{p,q}$ there. Assign one further position to $t_s g_s e^s_{p,q}$.

For the entries of d_i, write $Q_{s,a} = \xi_s e^s_{aa}$ on each reserved row. Since $\xi_s = 1$ on $\mathrm{supp}\, f_s$, the actual corners

$$d_i^{s,a;t,b} = Q_{s,a} d_i Q_{t,b}$$

sum to d_i over all ordered reserved row pairs. For an unordered pair of different reserved rows, encode one corner, obtaining its adjoint automatically. Encode each diagonal corner separately in

an off-diagonal position. Choose a new position (p,q) between the same two tower blocks (within that block for a diagonal corner), and put

$$T_i^{s,a;t,b} = (t_s e_{p,a}^s)\, d_i^{s,a;t,b}\, (t_t e_{b,q}^t) \tag{4.5}$$

there. These are actual elements of A. Their endpoint functions equal 1 on the support of the original corner, so

$$H_0 T_i^{s,a;t,b} = \lambda_{s,p} T_i^{s,a;t,b}, \qquad T_i^{s,a;t,b} H_0 = \lambda_{t,q} T_i^{s,a;t,b}.$$

Only the two endpoint labels specify a position between different blocks; no multiplier matrix unit between those blocks is used.

Here is an explicit check that these assignments are possible. Within block s they require at most

$$(d_s - 1) + q\ell n_s + 1 + \frac{qk_s(k_s+1)}{2} + 3 \le (q\ell+1)n_s + \frac{q\rho^2 n_s^2 + q\rho n_s}{2} + 3 \tag{4.6}$$

unordered positions, including the three positions we leave vacant. The available number is $d_s(d_s-1)/2$, with $d_s > (1-\delta)n_s$. Since $q\rho^2 \le 10^{-4}$, $q\rho \le q/100$, and $n_s \ge 100(q\ell + q + 10)$, the right side of (4.6) is less than $0.011 n_s^2$, whereas $d_s(d_s-1)/2 > 0.48 n_s^2$. Between different blocks the number required is $qk_sk_t \le 10^{-4} n_s n_t$, whereas there are $d_s d_t > 0.98 n_s n_t$ available positions. Thus every assignment can be made injectively. Fix three vacant unordered positions in block s_*.

For every boundary row $b \in E_s \backslash I_s$, choose a distinct high target row $j(b) \in I_s$ outside the reserved rows, and put $t_s e_{j(b),b}^s$ in the packet. This is possible because $|E_s \backslash I_s| < \delta n_s$ and $d_s - k_s > (1-\delta-\rho)n_s$. Let S be the sum of all the chain entries, scalar entries, corner entries, and boundary entries just described, using one orientation for each assigned unordered position. Define

$$K_0 = S + S^* + v + v^* \in A_{\mathrm{sa}}.$$

(3) Spectral separation and recovery. We verify recovery from $D_0 = C^*(1, H_0, K_0)$. For a high label pair i,j, write $R_{ij} = \{(\lambda_i\tau, \lambda_j\tau) : 0 \le \tau \le 1\}$. Here and in Table 1, a joint spectral type means that the formulas of Lemma 3.2 annihilate the indicated entry whenever the filter vanishes on the displayed set.

TABLE 1. Spectral types in the finite construction. Each adjoint has the coordinates reversed.

Entry	Joint type for H_0	Recovery
Chain and scalar data	R_{ij}, with $i \ne j$	Smooth ray filter; origin cutoff
Moved remainder corner	$\{(\lambda_i, \lambda_j)\}$, $i \ne j$	Smooth point filter
Boundary link	$\{(\lambda_j\tau, \nu_b\tau) : 0 \le \tau \le 1\}$	High row cutoff after subtraction
Reserved row of v	$\{\lambda_i\} \times [0, 2.1\eta]$	Reserved row and high spectral cutoff
Hereditary insertion	$\{(\lambda_{p_j}, \lambda_{q_j})\}$	Smooth point filter in a vacant position

The first two types have coordinate ratios in $[2/2.1, 2.1/2]$. Distinct assigned ordered positions have distinct ratios, including the reverse orientations. Boundary rays have ratios greater than $2/\eta$, and their adjoints have ratios less than $\eta/2$. For a component of v, its range is on a reserved plateau row and its source is in $1_{[0,2.1\eta]}(H_0)$, by (4.4). Thus the mixed rectangles in the table are disjoint from every high–high ray and point. Indeed, a high–high ray with first coordinate at least 2 has second coordinate at least $4/2.1 > 2.1\eta$.

Fix a scalar or chain entry and $\varepsilon > 0$. Its ray segment away from the origin is separated by a positive distance from every other type. Choose the smooth joint filter equal to $\theta_\varepsilon(\tau)$ on that ray and zero on every unwanted type. The resulting element is the selected entry times $\theta_\varepsilon(t_s)$. Since

its coefficient is $O(t_s)$, Lemma 2.9 gives convergence to the entry in norm. For a moved corner, its single point has positive distance from all other types; a filter equal to 1 at that point recovers it exactly. Consequently all high–high entries, and their finite self-adjoint sum K_{hh}, belong to D_0. Boundary links and comparison links may overlap each other; they are recovered subsequently, in the order below.

The chain entries give every $t_s e^s_{ab}$ for $a, b \in I_s$, by Lemma 3.4. The coded entry $t_s g_s e^s_{pq}$ gives $g_s e^s_{aa}$ for every high row a: multiply on the two sides by the known soft matrix units to obtain $t_s^3 g_s e^s_{aa}$, and use continuous inverse cutoffs of $t_s e^s_{aa}$. The error tends to zero because $g_s \le t_s/\eta$. Thus

$$Q_{s,a} = \frac{t_s e^s_{aa} - \eta g_s e^s_{aa}}{1-\eta} \in D_0 \quad (a \in I_s).$$

This argument uses only high rows and precedes recovery of v.

Choose a scalar continuous function χ equal to zero on $[0, 2.1\eta]$ and to one on $[2, 2.1]$. On a reserved row the cutoff $Q_{s,a}$ equals the row projection on the range support of v. Its product with every boundary entry vanishes because boundary targets lie outside the reserved rows. Also $\chi(H_0)v^* = 0$, by (4.4). Consequently

$$Q_{s,a}\chi(H_0)(K_0 - K_{\mathrm{hh}}) = p_{s,a}v \in D_0, \qquad v = \sum_{s,a \text{ reserved}} p_{s,a}v \in D_0. \tag{4.7}$$

The projection notation on the right merely identifies the recovered element; the expression on the left consists of actual known elements.

Subtract $v + v^*$ as well. The remaining packet consists just of the boundary links and their adjoints. Multiplication on the left by $t_s e^s_{j(b),j(b)}$ isolates $t_s^2 e^s_{j(b),b}$, because the high targets are distinct. Inverse cutoffs of that known diagonal recover $t_s e^s_{j(b),b}$. Its modulus gives $t_s e^s_{bb}$. The high chain and these boundary links form a connected graph through every row of the block, so products and inverse cutoffs give

$$t_s e^s_{ab} \in D_0 \quad (a, b \in E_s). \tag{4.8}$$

We can now decode every original coefficient of b_i. If $t_s \beta e^s_{pq}$ was its coded entry and its original position was (a, b), multiplication by the two known soft matrix units gives $t_s^3 \beta e^s_{ab}$. Inverse cutoffs on the left diagonal recover βe^s_{ab} in norm, since $\beta = O(g_s) = O(t_s)$. Summing gives $b_i \in D_0$. Similarly, multiplying (4.5) by $t_s e^s_{a,p}$ and $t_t e^t_{q,b}$ recovers $d_i^{s,a;t,b}$ exactly: the endpoint functions equal 1 on its support. Include the adjoints for the opposite corners and sum to obtain $d_i \in D_0$. Finally $r = v^* v \in D_0$ and

$$v^* d_i v = r c_i r \in D_0.$$

To complete the norm recovery, use the actual functions $k_\varepsilon(\tau) = \theta_\varepsilon(\tau)/\tau$, extended by zero near zero. Then

$$k_\varepsilon(r)(v^* d_i v)k_\varepsilon(r) = \theta_\varepsilon(r) c_i \theta_\varepsilon(r) \longrightarrow c_i \quad \text{in norm.} \tag{4.9}$$

This follows from $c_i = r^{1/2} a_i r^{1/2} \in \mathrm{Her}(r)$ and the approximate-unit property of $\theta_\varepsilon(r)$. Thus $a_i = b_i + c_i \in D_0$. The measure estimate (4.1) was used only to obtain v; the recovery in (4.9) is a norm argument. *(4) Three additional hereditary elements.* It remains to verify the three additional positions. Let w be the row in block s_* with the largest high label. Choose a nonempty open set W whose closure lies in the w-level of $\mathrm{int}\{\xi_{s_*} = 1\}$. In particular, $t_{s_*} = 1$ on $\overline{W}$. Give each of the three vacant positions an orientation (p_j, q_j), $j = 1, 2, 3$, and define the known actual elements

$$A_j = t_{s_*} e^{s_*}_{p_j,w}, \qquad B_j = t_{s_*} e^{s_*}_{w,q_j}.$$

For any prescribed $P_j \in \mathrm{Her}(W)_{\mathrm{sa}}$, put

$$E_j = A_j P_j B_j, \qquad K = K_0 + \sum_{j=1}^{3} (E_j + E_j^*).$$

The support plateau gives

$$H_0 E_j = \lambda_{s_*,p_j} E_j, \qquad E_j H_0 = \lambda_{s_*,q_j} E_j.$$

The point $(\lambda_{s_*,p_j}, \lambda_{s_*,q_j})$ lies on no old occupied high–high ray or point, since its unordered position was vacant and all ordered ratios are distinct. It also lies off every boundary ray and every mixed spectral set of $v + v^*$. It differs from the three reverse points and the other new points. A smooth joint filter, equal to one near this point and zero on all the other spectral types, recovers E_j exactly from H_0, K. Subtracting all six entries recovers K_0, hence D_0, and

$$A_j^* E_j B_j^* = (t_{s_*}^2 e_{ww}^{s_*}) P_j (t_{s_*}^2 e_{ww}^{s_*}) = P_j.$$

This proves the proposition. □

5. Local recovery of diagonal functions

The group unitaries are assumed to be available in this section. This assumption is used twice: in a finite frame, and when recovered base functions are moved back to their original tower levels. The argument does not require a hereditary subalgebra to inherit URP.

Lemma 5.1. *Let $F \subseteq \Gamma$ be finite and symmetric. In a finite family of towers (U_s, E_s) choose centers $c_{s,l}$ with $c_{s,l}F^2 \subseteq E_s$, and orthogonal multiplier projections $q_{s,l}$ whose matrix rows are supported in $c_{s,l}F$. Let g_s, ξ_s be the nested tower functions, put*

$$h = \bigoplus_s g_s 1_{|E_s|}, \quad r = 1 - h, \quad t_s = \eta g_s + (1-\eta)\xi_s, \quad H = \sum_s t_s \sum_l \lambda_{s,l} q_{s,l},$$

where $0 < \eta < 1$ and $0 < \lambda_{s,l} \le b$. If $e_\gamma \in C(X)_+$ vanishes wherever $\alpha_\gamma^{-1}(r)$ vanishes, then, for $\gamma \in F$,

$$(H - b\eta)_+ e_\gamma^{1/2} = 0, \qquad p_{e_\gamma} \le 1_{[0,b\eta]}(H). \tag{5.1}$$

Proof. Compute in the original tower matrix coordinates. At $y \in U_s$ with $g_s(y) < 1$, the nesting of the plateaux implies $\xi_s(y) = 0$. Hence $t_s(y) \le \eta$ and $0 \le H(y) \le b\eta 1$; the positive part in (5.1) is zero on the whole fibre. If $g_s(y) = 1$, a selected row is $y c_{s,l}\delta$ with $\delta \in F$. For $\gamma \in F$,

$$c_{s,l}\delta\gamma^{-1} \in c_{s,l}F^2 \subseteq E_s, \qquad r(y c_{s,l}\delta\gamma^{-1}) = 0.$$

Thus e_γ vanishes on every row supporting any $q_{s,l}$. In this fibre, $q_{s,l} e_\gamma^{1/2} = 0$, and consequently $(H - b\eta)_+ e_\gamma^{1/2} = 0$. Outside the tower levels, $H = 0$. The products just computed are actual elements of the tower algebras: multiplication of their bounded matrix coefficients by a function vanishing with t_s is covered by Lemma 2.7. Faithfulness of the tower embeddings therefore proves the first identity in A.

In A^{**}, the closure of the range of $e_\gamma^{1/2}$ lies in the kernel of $(H - b\eta)_+$. Since $H \ge 0$, the projection onto this kernel is $1_{[0,b\eta]}(H)$. This proves the second assertion. The argument establishes annihilation, not only a bound on a compression of H. □

Proposition 5.2. *Let $0 \ne \chi \in C(X)_+$ and let $f \in C(X)_{\mathrm{sa}}$. There are $H \in \mathrm{Her}(\chi)_+$ and $K \in \mathrm{Her}(\chi)_{\mathrm{sa}}$ such that*

$$f \in C^*(1, \chi, \{u_\gamma : \gamma \in \Gamma\}, H, K).$$

Proof. Write

$$D_0 = C^*(1, \chi, \{u_\gamma : \gamma \in \Gamma\}).$$

Replacing χ by a scalar multiple, assume $\|\chi\| \leq 1$. Minimality and compactness give a finite symmetric set $F \subseteq \Gamma$, containing e, such that

$$V_\gamma = \chi^{1/2} u_\gamma^*, \qquad S = \sum_{\gamma \in F} V_\gamma^* V_\gamma = \sum_{\gamma \in F} \alpha_\gamma(\chi) \geq \kappa 1 \quad (\kappa > 0). \tag{5.2}$$

In particular, $V_\gamma, S^{-1} \in D_0$. Put $M = |F|$ and $C_F = 2|FF^{-1}|$. The functions

$$a_\gamma = \alpha_\gamma(\chi) S^{-1} \quad (\gamma \in F) \tag{5.3}$$

belong to $D_0 \cap C(X)$, are positive, and sum to 1.

We first specify the numerical room needed later. Fix $0 < \eta < 10^{-3}$ and $0 < \rho < 10^{-2}$. If (λ, m) are the COS constants, choose

$$0 < \varepsilon < \min\left\{\frac{1}{4}, \frac{\lambda\rho}{8C_F M m}\right\}. \tag{5.4}$$

The tower heights will be taken large enough to satisfy all the finite inequalities below. This is possible in Lemma 2.6, since the constants F, M, C_F, ρ, m have already been fixed. For example, it suffices to require $N_s \geq C_F R_*$, where

$$R_* = \max\left\{\frac{2Mm}{\rho},\, 4(M+1),\, 16(C_F + M + 2)\right\}.$$

Apply Lemma 2.6 with buffer F^2, error ε, and the prescribed minimum height. Since $\varepsilon < 1/4$, its interior estimate gives

$$|\operatorname{int}_{F^2}(E_s)| \geq \tfrac{1}{2}|E_s|.$$

Use its mutually disjoint towers (U_s, E_s), closed cores B_s, and nested functions g_s, ξ_s, f_s. Write $N_s = |E_s|$. Let $C_s \cong M_{N_s}(C_0(U_s))$ be the original tower algebra, with multiplier matrix units e_{ab}^s, $a, b \in E_s$. In this identification a point $y \in U_s$ represents all the physical levels ya, $a \in E_s$. Define

$$h = \bigoplus_s g_s 1_{N_s}, \qquad r = 1 - h, \qquad t_s = \eta g_s + (1-\eta)\xi_s, \qquad T_s = t_s 1_{N_s} \in C_s. \tag{5.5}$$

The direct-sum notation denotes the inclusions into A, and h is a diagonal function on X. The function g_s is strictly positive on U_s, and

$$t_s \geq \eta g_s > 0 \text{ on } U_s, \qquad t_s \leq \eta \text{ where } g_s < 1. \tag{5.6}$$

Moreover, $t_s = 1$ on an open neighborhood of $\operatorname{supp} f_s$, containing B_s. The open support $E = \{r > 0\}$ satisfies

$$\mu(E) < \varepsilon \qquad (\mu \in \mathcal{M}_1(X, \Gamma)), \tag{5.7}$$

because $h = 1$ on all the closed core levels.

The positive functions and the remainder functions to be transported are

$$e_\gamma = \alpha_\gamma^{-1}(a_\gamma r) = \chi\, \frac{\alpha_\gamma^{-1}(r)}{\alpha_\gamma^{-1}(S)}, \qquad c_\gamma^{(0)} = e_\gamma \alpha_\gamma^{-1}(f) \quad (\gamma \in F). \tag{5.8}$$

Thus $e_\gamma, c_\gamma^{(0)} \in \operatorname{Her}(\chi)$, $c_\gamma^{(0)} \in \operatorname{Her}(e_\gamma)$, and

$$f = hf + \sum_{\gamma \in F} \alpha_\gamma(c_\gamma^{(0)}). \tag{5.9}$$

By invariance, the open support of each e_γ has measure less than ε for every invariant measure.

(1) Return columns inside $\operatorname{Her}(\chi)$. Choose a maximal family of mutually disjoint sets $c_{s,l}F \subseteq E_s$ with $c_{s,l} \in \operatorname{int}_{F^2}(E_s)$, and denote its cardinality by R_s. If $cF \cap c'F \neq \varnothing$, then $c' \in cFF^{-1}$. Hence

each chosen tile excludes at most $|FF^{-1}|$ centers. Maximality and the preceding interior estimate give

$$R_s \geq \frac{N_s}{2|FF^{-1}|} = \frac{N_s}{C_F}. \tag{5.10}$$

Index these tiles by $0 \leq l < R_s$, and take $c_{s,0}$ as the reference center. Put

$$p_s = e^s_{c_{s,0},c_{s,0}}.$$

This is an open projection in A^{**}; it is not assumed to be in the generated algebra. The following are bounded multiplier columns of the *original* algebra C_s:

$$v_{s,l}(y) = \sum_{\delta\in F} \frac{\chi(yc_{s,l}\delta)^{1/2}}{S(yc_{s,l})^{1/2}}\, e^s_{c_{s,l}\delta,c_{s,0}} \qquad (y \in U_s). \tag{5.11}$$

The denominator is at least $\kappa^{1/2}$. Disjointness of the tiles and the normalization imply

$$v^*_{s,l}v_{s,k} = \delta_{lk}p_s, \qquad q_{s,l} = v_{s,l}v^*_{s,l}. \tag{5.12}$$

Every $q_{s,l}$ has its range in the support projection of χ. Indeed, its component at a level where χ vanishes is zero. Consequently

$$T_s v_{s,l}v^*_{s,k} \in A \cap \mathrm{Her}(\chi).$$

The factor $t_s \in C_0(U_s)$ guarantees membership in A. The operator T_s is central in C_s and commutes with all its multiplier columns and scalar coefficient functions.

Choose, simultaneously for all pairs (s,l), labels $\lambda_{s,l} \in [2, 2.1]$ as in Lemma 3.3, and set

$$H = \sum_s T_s \sum_{l=0}^{R_s-1} \lambda_{s,l}q_{s,l} \in \mathrm{Her}(\chi)_+. \tag{5.13}$$

Apply Lemma 5.1 with $b = 2.1$ to the functions e_γ in (5.8). Their open supports lie in that of $\alpha_\gamma^{-1}(r)$, and the return centers satisfy the required F^2 buffer. We obtain the exact identity

$$(H - 2.1\eta)_+ e_\gamma^{1/2} = 0, \qquad p_{e_\gamma} \leq 1_{[0,2.1\eta]}(H) \quad (\gamma \in F). \tag{5.14}$$

This is the support statement needed for the comparison links below.

(2) Comparison and the reserved columns. Let

$$k_s = \left\lfloor \frac{\rho R_s}{Mm} \right\rfloor.$$

By increasing the minimum tower height, assume $k_s \geq \rho R_s/(2Mm)$. Reserve, for each $\gamma \in F$, m mutually disjoint sets $J_{s,\gamma,1}, \ldots, J_{s,\gamma,m}$ of k_s column indices. All these sets, for all γ, are disjoint. Their union has at most ρR_s elements. Write

$$T_{s,\gamma} = \bigcup_{j=1}^m J_{s,\gamma,j}.$$

Avoid the reference column and, in addition, leave M different nonreserved column indices $l_{s,\gamma}$ for the frame links below. This is possible for sufficiently large R_s.

Choose an open set Ω_s containing B_s, with compact closure in the open plateau where $t_s = 1$, and a positive function $\beta_s \in C_0(U_s)$ with open support Ω_s. For a fixed γ, select one original center level from each of k_s distinct return tiles, and let $b_\gamma \in C(X)_+$ be the sum of the copies of β_s on these levels, over all s. Its open support O_γ is a *Cartan* open set. By invariance and (5.10),

$$\mu(O_\gamma) \geq \sum_s k_s\mu(B_s) \geq \frac{\rho}{2C_FMm} \sum_s N_s\mu(B_s) > \frac{\rho(1-\varepsilon)}{2C_FMm}. \tag{5.15}$$

Equations (5.4) and (5.7) give the strict COS inequality. Hence

$$e_\gamma \precsim b_\gamma^{\oplus m}.$$

For each of these m copies, match its selected original center rows with the column indices in $J_{s,\gamma,j}$. If a is a selected center row and l its matched column, the multiplier $v_{s,l}e^s_{c_{s,0},a}$ carries that row to $q_{s,l}$. Multiplication by $\beta_s^{1/2}$ makes this an element of A. The relations in (5.12) show that these matches give Cuntz equivalence of the matrix direct sum with

$$B_\gamma = \sum_s \sum_{l\in T_{s,\gamma}} \beta_s q_{s,l} \in \mathrm{Her}(\chi)_+. \tag{5.16}$$

Thus $e_\gamma \precsim B_\gamma$ in A. Lemma 2.8 gives $w_\gamma \in \mathrm{Her}(\chi)$ with

$$w_\gamma^* w_\gamma = e_\gamma, \qquad w_\gamma w_\gamma^* \in \mathrm{Her}(B_\gamma). \tag{5.17}$$

If $e_\gamma = 0$, take $w_\gamma = 0$ and omit its unused data. The range of w_γ is supported on its own reserved indices $T_{s,\gamma}$ and on $t_s = 1$. On each of those rows the left H-value is the constant $\lambda_{s,l} \in [2, 2.1]$, whereas the right H-spectrum is contained in $[0, 2.1\eta]$ by (5.14).

(3) The coefficients to be recovered. In every column algebra include the adjacent chain $T_s v_{s,l} v_{s,l+1}^*$, $0 \le l < R_s - 1$. For every original level $a \in E_s$, the scalar base function

$$F_{s,a}(y) = g_s(y) f(ya) \tag{5.18}$$

satisfies $|F_{s,a}| \le \eta^{-1}\|f\| t_s$. Assign an unused off-diagonal position (i, j) to the actual entry

$$F_{s,a} v_{s,i} v_{s,j}^* = T_s(F_{s,a}/t_s) v_{s,i} v_{s,j}^*.$$

The quotient is continuous and bounded on U_s, and the entry belongs to A since $F_{s,a} \in C_0(U_s)$. Also assign positions to the M entries $T_s c'_\gamma v_{s,i} v_{s,j}^*$, using the scalar functions defined below in (5.20).

The other high/high data are the corners of

$$A_\gamma = w_\gamma c_\gamma^{(0)} w_\gamma^*. \tag{5.19}$$

For each pair of its reserved range rows $(s, p), (t, q)$, move the corner to an unused off-diagonal pair of coding rows. More explicitly, if

$$c = v_{s,p}^* A_\gamma v_{t,q},$$

place $v_{s,i} c v_{t,j}^*$ in the chosen position $((s, i), (t, j))$. The coefficient c is supported on $t_s = t_t = 1$. In fact

$$c = (T_s v_{s,p})^* A_\gamma (T_t v_{t,q}) \in A.$$

Its translated corner is also in A, because

$$v_{s,i} c v_{t,j}^* = (T_s v_{s,i}) c (T_t v_{t,j})^*.$$

All four multiplied columns are actual elements of their tower algebras. For $s = t$ use a distinct unused unordered pair $\{i, j\}$, with one orientation; even a diagonal corner is moved to such an off-diagonal pair. Encode only original pairs $p \le q$ when $s = t$: restore an off-diagonal source corner together with its adjoint, and restore a diagonal source corner once. For $s < t$ use distinct pairs between the two blocks and let adjoints encode the reverse corners. The operators A_γ are self-adjoint, since f is real-valued.

Here is explicit room for these assignments. Put $d_{s,\gamma} = |T_{s,\gamma}| = mk_s \le \rho R_s/M$. Within one block, it suffices to assign at most

$$(R_s - 1) + N_s + M + \sum_{\gamma\in F} d_{s,\gamma}^2 \le (R_s - 1) + C_F R_s + M + \frac{\rho^2}{M} R_s^2$$

unordered pairs, each followed by its adjoint in K. This is less than $R_s(R_s - 1)/2$ with the stated choice of R_*. Indeed, $R_s \ge 16(C_F + M + 2) \ge 80$ bounds the linear terms by $R_s^2/16$, while

$\rho^2/M < 10^{-4}$. The required number is therefore less than $0.063R_s^2$, and the available number is greater than $0.49R_s^2$. Between two blocks the number of positions is at most

$$\sum_{\gamma\in F} d_{s,\gamma}d_{t,\gamma} \le \frac{\rho^2}{M}R_sR_t < R_sR_t.$$

These inequalities explain why one may require all heights to be sufficiently large before choosing the towers. They also avoid counting a diagonal corner as a separate self-adjoint generator.

(4) The frame links and the second self-adjoint element. Fix s temporarily and omit its subscript. In the reference tile set

$$w'_\gamma = V_\gamma p, \qquad c'_\gamma(y) = \frac{\chi(yc_0\gamma)}{S(yc_0)^{1/2}}, \qquad v_0^* w'_\gamma = c'_\gamma p, \qquad z_\gamma = w'_\gamma - v_0 c'_\gamma. \tag{5.20}$$

Here c'_γ is the bounded scalar base function for which a position was reserved above. We regard it as a central multiplier of the original tower block; $c'_\gamma p$ is its corner on the reference source row. The tower relations give

$$v_0^* z_\gamma = 0, \qquad q_l z_\gamma = 0 \quad \text{for every } l, \qquad Hz_\gamma = 0.$$

For $l \neq 0$, the second identity uses disjointness of the reference tile and the other tiles. Introduce the additional links

$$x_{s,\gamma} = T_s z_\gamma v_{s,l_{s,\gamma}}^*. \tag{5.21}$$

They belong to $A \cap \mathrm{Her}(\chi)$. Indeed, they are bounded multiplier coefficients of the original tower block multiplied by $t_s \in C_0(U_s)$, and both their endpoints lie in the support of χ. They have left H-value 0 and right joint values $\lambda_{s,l_{s,\gamma}} t_s$.

Let $\mathcal{P}$ be the finite list of all the high/high entries just assigned, including the chain, the scalar data, and the moved corners. Define the self-adjoint element

$$K = \sum_{x\in\mathcal{P}} (x + x^*) + \sum_{s,\gamma\in F} (x_{s,\gamma} + x_{s,\gamma}^*) + \sum_{\gamma\in F} (w_\gamma + w_\gamma^*) \in \mathrm{Her}(\chi). \tag{5.22}$$

For the rest of the proof put $D = C^*(D_0, H, K)$.

(5) Recovery of the columns and the remainder. Table 2 records all terms of K. Here $R_{ij} = \{(\lambda_i\tau, \lambda_j\tau) : 0 \le \tau \le 1\}$, with the tower index included in a label index.

TABLE 2. Spectral types in the local construction. The adjoint of each entry has the reversed type.

Entry	Joint type for H	Recovery
Chain and scalar data	R_{ij}, $i \neq j$, coefficient $O(t_s)$	Smooth ray filter; origin cutoff
Moved corner of A_γ	$\{(\lambda_i, \lambda_j)\}$, $i \neq j$	Smooth point filter
Component of w_γ	$\{\lambda_i\} \times [0, 2.1\eta]$	Reserved column and high cutoff
Frame defect $x_{s,\gamma}$	$\{0\} \times [0, \lambda_{s,l_{s,\gamma}}]$	Right column cutoff after subtraction

Every scalar high/high entry has joint spectral support on its assigned ray

$$\{(\lambda_{s,i}t, \lambda_{s,j}t) : 0 \le t \le 1\}.$$

Its coefficient is $O(t)$, including the scalar functions $F_{s,a}$ and $t_s c'_\gamma$. A moved remainder corner in one block has its support at the endpoint of its assigned ray; one between blocks has the single joint point $(\lambda_{s,i}, \lambda_{t,j})$. The generic ratios make all these assigned types distinct, including their adjoints. By contrast, the additional links have joint types on the axes, and the comparison links

have high/low or low/high joint types. A comparison link has a left value in $[2, 2.1]$ and a right value in $[0, 2.1\eta]$, or the reverse values for its adjoint. Such a type cannot meet a high/high ray: a point on that ray has the ratio of its coordinates in $[2/2.1, 2.1/2]$. The axis types meet the high/high rays only at the origin. This proves the required separation.

Lemma 3.2, using smooth functions which avoid a shrinking neighborhood of the origin, therefore recovers every member of $\mathcal{P}$. The omitted origin part is $O(t)$ and tends to zero in norm. Lemma 3.4 then recovers

$$E^s_{ij} = T_s v_{s,i} v^*_{s,j}, \qquad T_s q_{s,i} \quad (0 \le i, j < R_s). \tag{5.23}$$

For example, a scalar entry $T_s \phi v_{s,i} v^*_{s,j}$ is transported to the reference diagonal by

$$E^s_{0i}\, k_\delta(T_s q_{s,i})(T_s \phi v_{s,i} v^*_{s,j}) k_\delta(T_s q_{s,j})\, E^s_{j0} \longrightarrow T_s \phi q_{s,0}, \qquad k_\delta(t) = \theta_\delta(t)/t.$$

The product equals $\theta_\delta(t_s)^2 T_s \phi q_{s,0}$; the convergence follows from its $O(t_s)$ coefficient. Thus $F_{s,a} q_{s,0}$ and $T_s c'_\gamma q_{s,0}$ are in D. Similarly, the moved remainder corners are transported back to their reserved rows using (5.23). Here no inverse limit is needed, because these corners have $t_s = t_t = 1$. Summing the restored corners, using each diagonal corner once and adding adjoints only for the off-diagonal source corners, gives $A_\gamma \in D$.

Subtract the recovered high/high terms from K. Choose a continuous function G equal to 0 on $[0, 2.1\eta]$ and equal to 1 on $[2, 2.1]$. For the reserved range of a fixed γ, set

$$P_{\gamma,\delta} = \sum_s \sum_{l \in T_{s,\gamma}} \theta_\delta(T_s q_{s,l}) \in D.$$

For $\delta < 1/2$, this acts as the identity on the range of w_γ, since that range has $t_s = 1$. Other comparison links have disjoint reserved ranges. The adjoints of all comparison links have low left H-spectrum. The $x_{s,\gamma}$ have left value 0, and their adjoints have high ranges on the indices $l_{s,\gamma}$, all outside the reserved sets. It follows that

$$P_{\gamma,\delta} G(H) \left(K - \sum_{x \in \mathcal{P}} (x + x^*) \right) = w_\gamma.$$

Hence *each* $w_\gamma \in D$. The relations in (5.17) and (5.19) give

$$e_\gamma \in D, \qquad w^*_\gamma A_\gamma w_\gamma = e_\gamma c^{(0)}_\gamma e_\gamma \in D.$$

Division by soft cutoffs of e_γ, as in Lemma 2.9, recovers

$$c^{(0)}_\gamma \in D. \tag{5.24}$$

After subtracting all comparison links, the remaining expression is $\sum_{s,\gamma}(x_{s,\gamma} + x^*_{s,\gamma})$. Right multiplication by $\theta_\delta(T_s q_{s,l_{s,\gamma}})$ selects $x_{s,\gamma}\theta_\delta(t_s)$: the other links have different right column indices, and the adjoints have right supports orthogonal to every column. This converges in norm to $x_{s,\gamma}$, because $x_{s,\gamma}$ has an $O(t_s)$ coefficient. Therefore all these links belong to D.

(6) Recovery of the diagonal function. Again fix one block and suppress s. From its recovered link and soft matrix units,

$$x_\gamma k_\delta(T q_{l_\gamma}) E_{l_\gamma,0} = \theta_\delta(t) T z_\gamma v^*_0 \longrightarrow T z_\gamma v^*_0.$$

Adding the recovered scalar term $T c'_\gamma q_0$ gives

$$L_\gamma = T w'_\gamma v^*_0 \in D. \tag{5.25}$$

Notice that p has not been recovered. Instead, the recovered L_γ records its image under each known frame operator.

Let $\phi = F_{s,a}$ be one of the scalar functions in (5.18). Define the actual diagonal function $A_\phi = \phi p \in C(X)$ on the reference level $U_s c_{s,0}$, extended by zero. It belongs to $C(X)$ because $\phi \in C_0(U_s)$. The corresponding column corner $Y_\phi = \phi q_0$ has already been recovered. Put

$$b_\delta(t) = \theta_\delta(t)/t^2,$$

extended by zero near zero. Functional calculus on the known element Tq_0, followed by the column relations, gives

$$L_\gamma b_\delta(Tq_0) Y_\phi L_\nu^* = \theta_\delta(t) V_\gamma A_\phi V_\nu^* \longrightarrow V_\gamma A_\phi V_\nu^*. \tag{5.26}$$

Here $\theta_\delta(t)$ is the scalar base cutoff on the indicated tower entries. The frame buffer ensures that all their endpoints remain in the same original block. Since $|\phi| \le \eta^{-1}\|f\| t$, the error in (5.26) is at most $2\delta\eta^{-1}\|f\|\|V_\gamma\|\|V_\nu\|$. Thus each $V_\gamma A_\phi V_\nu^*$ belongs to D.

Apply Lemma 3.1 to the known frame $(V_\gamma)_{\gamma\in F}$ and the just recovered inner corners. It gives

$$A_\phi = S^{-1}\left(\sum_{\gamma,\nu\in F} V_\gamma^*(V_\gamma A_\phi V_\nu^*)V_\nu\right) S^{-1} \in D. \tag{5.27}$$

The middle sum equals $SA_\phi S$. Every outer operator in this identity belongs to D_0. In particular, no inverse of a compressed frame and no membership assertion for p is needed.

For the chosen $\phi = F_{s,a}$, the diagonal function on the original level $U_s a$ is

$$u_{a^{-1}c_{s,0}} A_\phi u_{a^{-1}c_{s,0}}^*.$$

Its value at ya is $A_\phi(yc_{s,0}) = g_s(y)f(ya)$, by the right-action convention. Its support is contained in $U_s a$. All these unitaries belong to D_0, so summing over the levels gives $hf \in D$. Finally, (5.24), the group unitaries, and (5.9) give $f \in D$. □

Remark 5.3. The hypothesis that all implementing group unitaries are already available cannot simply be removed from Proposition 5.2. The finite frame depends on χ, and the final translates of its recovered base functions use the full group. Neither operation is inferred from weak tracial approximation alone.

6. Countable assembly and the proof of the main theorem

We first place local constructions in orthogonal diagonal hereditary subalgebras. This separation is spectral, so both elements of each local pair can subsequently be extracted by the same cutoff.

Lemma 6.1. *Every nonempty open subset W of X contains the compact support of a positive function z for which there are pairwise disjoint open intervals $I_n \subset (0,\infty)$ meeting its range. There are nonzero $\chi_n = \psi_n(z)$ and continuous functions ϑ_n supported in I_n such that*

$$\operatorname{supp}\psi_n \Subset I_n, \qquad \vartheta_n\psi_n = \psi_n, \qquad \vartheta_n\psi_m = 0 \quad (m \ne n).$$

If $a_m \in \operatorname{Her}(\chi_m)$, then $\vartheta_n(z)a_m = a_m\vartheta_n(z) = \delta_{nm}a_m$.

Proof. The space has no isolated points: otherwise the orbit of an isolated point would have invariant closed complement, and minimality would make X a compact discrete orbit, contradicting freeness of the infinite group. Choose a smaller open set with closure in W, a sequence of distinct points x_n converging to a point there, and disjoint small neighborhoods of the x_n. Choose bump functions b_n of compact support in those neighborhoods, with $b_n(x_n) = 1$. Then $z = \sum_n 4^{-n}b_n$ converges uniformly, has compact support in W, and takes each value 4^{-n}. Take disjoint intervals around these values and choose the indicated functions. The last assertion holds first on $\chi_m A \chi_m$ and then on its norm closure. □

Proof of Theorem 1.1. If Γ is finite, each orbit is closed. Minimality and freeness therefore imply that X is one orbit with $|X| = |\Gamma|$, and $A \cong M_{|\Gamma|}(\mathbb{C})$. A diagonal self-adjoint matrix with distinct positive entries, together with the self-adjoint sum of the adjacent matrix units and their adjoints, generates this matrix algebra.

Suppose henceforth that Γ is infinite. Let $s_1, \dots, s_k$ generate Γ. Apply Proposition 4.1 to the real and imaginary parts of the u_{s_j}. Obtain a pair H_0, K_0, three unused positions, and an open plateau W. Then $D_0 = C^*(1, H_0, K_0)$ contains every u_γ.

Choose z, χ_n, ϑ_n as in Lemma 6.1, and let (f_n) be norm dense in $C(X)_{\mathrm{sa}}$. By Proposition 5.2, for each n there are $H_n \in \mathrm{Her}(\chi_n)_+$ and $K_n \in \mathrm{Her}(\chi_n)_{\mathrm{sa}}$ such that

$$f_n \in C^*(1, \{u_\gamma : \gamma \in \Gamma\}, \chi_n, H_n, K_n).$$

Replace H_n and K_n, respectively, by $H_n/(1 + \|H_n\|)$ and $K_n/(1 + \|K_n\|)$. Both norms are then at most one, and multiplication by the original scalar constants recovers the unscaled pair. The series

$$P = \sum_{n=1}^{\infty} 2^{-n} H_n, \qquad Q = \sum_{n=1}^{\infty} 2^{-n} K_n$$

converge in norm and belong to $\mathrm{Her}(W)$. Encode the three self-adjoint elements z, P, Q in the unused positions of Proposition 4.1. For the resulting $K = K^*$, $D = C^*(1, H_0, K)$ contains D_0, z, P, Q. Moreover,

$$2^n \vartheta_n(z) P \vartheta_n(z) = H_n, \qquad 2^n \vartheta_n(z) Q \vartheta_n(z) = K_n.$$

Thus D contains every f_n, and therefore $C(X)$ and all implementing unitaries. Hence $D = A$.

Set $z_0 = 1 + H_0 + iK$. Since $\operatorname{Re} z_0 = 1 + H_0$ has spectrum in $[1, 1 + \|H_0\|]$, choose a continuous scalar function vanishing at zero and equal to one on this interval. Its functional calculus shows that $1 \in C^*(z_0)$. Consequently $H_0, K \in C^*(z_0)$, and $C^*(z_0) = A$. □

Remark 6.2. Finite generation is used only in the first paragraph of the infinite-group argument: recovery of the finitely many u_{s_j} gives every u_γ. The finite Fourier proposition itself is valid for countably infinite amenable groups. The local proposition is also a countable-group statement, but it is relative to the implementing unitaries. Since the finite frame depends on the prescribed local support, these two propositions do not by themselves establish the theorem for arbitrary countable groups. A countable-group extension would require a construction which recovers the unitaries needed by each successive local frame while preserving all previously recovered data. No such extension is claimed here.

Remark 6.3. The proof uses the diagonal functions and the explicit matrix coordinates of the Rokhlin towers. Weak tracial approximation alone does not provide the finite-frame recovery formula (5.27). The result proved here is therefore a theorem for the stated dynamical hypotheses, rather than a statement about all weakly tracially homogeneous C^*-algebras.

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