A SIMPLE SEPARABLE C*-ALGEBRA WHICH IS NOT SINGLY GENERATED

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Abstract. It is shown that there is a simple unital separable AH algebra which is not singly generated.

1. Introduction

In [5], among other questions, Kadison asked whether every von Neumann algebra acting on a separable Hilbert space is singly generated (see [2] for a survey of Kadison's problem list). Many von Neumann algebras are known to be singly generated, and this generator problem was reduced to the II_1 factors ([15]), but it is still open in general. For C*-algebras, the same question has been asked: whether every simple separable C*-algebra is singly generated (without simplicity, C*-algebras such as $\mathrm{C}(S^2)$ provide examples which are not singly generated; see [8]). Stable C*-algebras, UHF-absorbing C*-algebras, $\mathcal{Z}$-absorbing C*-algebras, as well as Villadsen's algebras of the first type, are known to be singly generated ([13], [10], [7], [11] [6]). Clearly, a positive answer for this question of C*-algebras will imply a positive answer to the von Neumann algebra question. In this note, using Villadsen's second type construction ([14]), we show that there are simple separable C*-algebras which are not singly generated (Theorem 3.1).

Theorem 1.1. *There are simple unital separable AH algebras A such that $\mathrm{C}^*(1, g) \neq A$ for all $g \in A$.*

To prove the theorem, let us suppose the AH algebra (constructed in Section 2) were generated by a single element. Then, in the base space of a building block of A, we consider a distinguished set on which the approximation of the generator has a diagonal block which is cut by the trivial rank-one projection. Then, restricted to this 1×1 diagonal block, one can show that the projection map from this distinguished set into a given manifold can be approximated by a map factoring through a compact subset of $\mathbb{C}$ (the spectrum of the 1×1 block of the approximating element of the generator). To see that this leads to to a contradiction, we use the Thom-Porteous formula of intersection theory (which was discovered by A. Toms ([12]) to have beautiful applications to the study of AH algebras) to show that the projection map above actually is nonzero on the homology group at some degree at least 2, but such a homology group of any compact planar set vanishes.

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Our example does not have stable rank one or real rank zero (Proposition 3.3). A natural question then is whether every simple (or possibly even not simple in the real rank zero case) separable C*-algebra of stable rank one or real rank zero is singly generated. Note that a positive answer in either cases will still imply a positive answer to Kadison's generator question.

AI is not used during the research of this work.

2. Villadsen's construction of the second type

Following Villadsen's construction of the second type, let us construct a simple unital AH algebra which will be shown in the next section not to be singly generated.

2.1. **Complex projective spaces.** Recall that $\mathbb{C}\mathrm{P}^n$, n-dimensional complex projective space, is a compact orientable differentiable manifold of dimension $2n$. The homology group $\mathrm{H}_{2i}(\mathbb{C}\mathrm{P}^n)$ is $\mathbb{Z}$ for all $0 \leq i \leq n$, and all other homology groups are $\{0\}$. Denote by $x_n \in \mathrm{H}_{2n}(\mathbb{C}\mathrm{P}^n)$ the fundamental class of $\mathbb{C}\mathrm{P}^n$.

The cohomology group $\mathrm{H}^{2i}(\mathbb{C}\mathrm{P}^n)$ is $\mathbb{Z}$ for $0 \leq i \leq n$, and all other cohomology groups are $\{0\}$. Set

$$\gamma_n = c_1(\xi_n) \in \mathrm{H}^2(\mathbb{C}\mathrm{P}^n),$$

where ξ_n is the canonical line bundle of $\mathbb{C}\mathrm{P}^n$ and $c_1(\cdot)$ is the first Chern class. Then γ_n is a generator of $\mathrm{H}^2(\mathbb{C}\mathrm{P}^n)$, and its cup power $\gamma_n^i \in \mathrm{H}^{2i}(\mathbb{C}\mathrm{P}^n)$ is a generator of $\mathrm{H}^{2i}(\mathbb{C}\mathrm{P}^n)$. In fact, γ_n is the generator of the cohomology ring of $\mathbb{C}\mathrm{P}^n$. By Poincaré duality, for each $i = 0, ..., n$, the element

$$x_n^{(n-i)} := \left\langle \gamma_n^i, x_n \right\rangle \in \mathrm{H}_{2(n-i)}(\mathbb{C}\mathrm{P}^n)$$

is a generator of $\mathrm{H}_{2(n-i)}(\mathbb{C}\mathrm{P}^n)$, where $\langle \cdot, \cdot \rangle$ denotes the cap product. (See, for instance, [3].)

2.2. **The product space.** Let M be a compact orientable differentiable manifold such that $d = \dim(M) \geq 2$. For each $n = 0, 1, 2, ...$, define

$$X_n = M \times \underbrace{\mathbb{C}\mathrm{P}^2 \times \cdots \times \mathbb{C}\mathrm{P}^{2^i} \times \cdots \times \mathbb{C}\mathrm{P}^{2^n}}_{n}. \tag{2.1}$$

Note that

$$\dim(X_n) = d + 2(2 + \cdots + 2^n) = d + 2(2^{n+1} - 2). \tag{2.2}$$

The product space X_n, $n = 0, 1, 2, ...$, is still a compact orientable manifold, and by the Künneth theorem, the fundamental class of X_n is given by

$$[X_n] := e \otimes x_2 \otimes \cdots \otimes x_{2^i} \otimes \cdots \otimes x_{2^n},$$

where $e \in \mathrm{H}_d(M)$ is the fundamental class of M.

2.3. **The construction of A.** Let X and Y be compact metrizable spaces. Let $\lambda_1, ..., \lambda_n : Y \to X$ be continuous and let $q_1, ..., q_n \in \mathrm{C}(Y) \otimes \mathcal{K}$ be mutually orthogonal projections, where $\mathcal{K}$ is the algebra of compact operators on a separable Hilbert space. Then the generalized diagonal map $\mathrm{C}(X) \to \mathrm{C}(Y) \otimes \mathcal{K}$ associated with the spectral data (λ_i, q_i), $i = 1, ..., n$, is defined by

$$f \mapsto (x \mapsto \sum_{i=1}^{n} (f \circ \lambda_i)(x) q_i(x)).$$

Let us denote this map just by $(\lambda_i, q_i)_{1 \leq i \leq n} = (\lambda_i, q_i)$.

The composition of the two maps associated with spectral data,

$$(\alpha_i, q_i^{(1)}) : \mathrm{C}(X) \to \mathrm{C}(Y) \otimes \mathcal{K},\ i = 1, ..., n,$$

and

$$(\beta_j, q_j^{(2)}) : \mathrm{C}(Y) \to \mathrm{C}(Z) \otimes \mathcal{K} \quad j = 1, ..., m,$$

is the map associated with the spectral data

$$(\alpha_i \circ \beta_j, \beta_j^*(q_i^{(1)}) \otimes q_j^{(2)}), \quad i = 1, ..., n,\ j = 1, ..., m,$$

where $p \otimes q$ denotes the tensor product of p and q considered as vector bundles.

Let us construct the C*-algebra $A = \varinjlim A_n$: The base space of A_n, $n = 0, 1, ...$, is given by (2.1), i.e.,

$$X_0 = M, \quad X_n = X_{n-1} \times \mathbb{C}\mathrm{P}^{2^n}, \quad n = 1, 2, ...$$

Choose $y_n \in X_n$ in such a way that for every $n = 0, 1, 2, ...$, the sequence of points

$$y_n, \pi_{n+1}^{(1)}(y_{n+1}), ..., (\pi_{n+1}^{(1)} \circ \cdots \circ \pi_{n+k}^{(1)})(y_{n+k}), ...$$

is dense in X_n, where

$$\pi_{n+i}^{(1)} : X_{n+i} = X_{n+i-1} \times \mathbb{C}\mathrm{P}^{2^n} \to X_{n+i-1}, \quad i = 0, 1, 2, ...,$$

is the projection to the first factor.

Consider the inductive system

$$\mathrm{C}(X_0) \to p_1(\mathrm{C}(X_1) \otimes \mathcal{K})p_1 \to \cdots \to A = \varinjlim A_n,$$

where the map $A_{n-1} \to A_n$, $n = 1, 2, ...$, is induced by the spectral data

$$(\pi_n^{(1)}, \theta) \quad \text{and} \quad (y_n, (\pi_n^{(2)})^*(\xi_{2^n})),$$

where θ is the (trivial) constant line bundle over X_n, and $\pi_n^{(2)} : X_n \to \mathbb{C}\mathrm{P}^{2^n}$ is the standard projection (and recall that ξ_{2^n} is the canonical line bundle over $\mathbb{C}\mathrm{P}^{2^n}$). Then A is a simple unital separable AH algebra.

Recall the notation

$$X_n = M \times \mathbb{C}\mathrm{P}^2 \times \cdots \times \mathbb{C}\mathrm{P}^{2^i} \times \cdots \times \mathbb{C}\mathrm{P}^{2^n}, \quad n = 1, 2, ... \tag{2.3}$$

The vector bundle (over X_n) associated to the projection p_n above is

$$\zeta_n := \theta \oplus \eta_2 \oplus \cdots \oplus \underbrace{(\eta_{2^i} \oplus \cdots \oplus \eta_{2^i})}_{2^{i-1}} \oplus \cdots \oplus \underbrace{(\eta_{2^n} \oplus \cdots \oplus \eta_{2^n})}_{2^{n-1}}, \tag{2.4}$$

where $\eta_{2^i} = (\pi_n^{(i)})^*(\xi_{2^i})$, and $\pi_n^{(i)} : X_n \to \mathbb{C}\mathrm{P}^{2^i}$, $i = 1, ..., n$, is the standard projection map associated with (2.3).

Note that

$$c_{2^n-1}(\zeta_n \ominus \theta) = (\pi_n^{(1)})^*(\gamma_2) \cup \cdots \cup ((\pi_n^{(i)})^*(\gamma_{2^i}))^{2^{i-1}} \cup \cdots \cup ((\pi_n^{(n)})^*(\gamma_{2^n}))^{2^{n-1}}, \tag{2.5}$$

where $c_i(\cdot)$ denotes the i-th Chern class.

Also note that

$$\mathrm{rank}(p_n) = 2^n, \quad n = 0, 1, 2, ...,$$

and the map $\phi_{0,n} : A_0 \to A_n$ is given by

$$\begin{aligned} f \quad \mapsto \quad & (f \circ \pi_n^M) \oplus (f(y_1)\eta_2) \oplus \cdots \oplus (\underbrace{f(y_{i-1})\eta_{2^i} \oplus \cdots \oplus f(y_{i-1})\eta_{2^i}}_{2^{i-1}}) \\ & \oplus \cdots \oplus (\underbrace{f(y_{n-1})\eta_{2^n} \oplus \cdots \oplus f(y_{n-1})\eta_{2^n}}_{2^{n-1}}), \end{aligned} \tag{2.6}$$

where $\pi_n^M : X_n \to M$ is the standard projection map associated with (2.3).

3. Generators of A

3.1. **Generators of A.** Let us show that the C*-algebra A constructed in the previous section is not singly generated:

Theorem 3.1. *For every $g \in A$, one has $\mathrm{C}^*(1, g) \neq A$. In fact, there is a finite set $\mathcal{E} \subseteq A$ such that for every $g \in A$, the sub-C*-algebra $\mathrm{C}^*(1, g)$ does not contain $\mathcal{E}$.*

Proof. First, regard

$$M \subseteq \mathbb{C}^{\tilde{d}}$$

as a submanifold for some $\tilde{d} \in \mathbb{N}$ (where $\mathbb{C}$ is only regarded as the manifold $\mathbb{R}^2$), and consider the coordinate maps

$$e_i : M \to \mathbb{C}, \ i = 1, ..., \tilde{d},$$

which we shall regard as elements of $A_0 = \mathrm{C}(M)$.

Let us show that for every $g \in A$, the sub-C*-algebra $\mathrm{C}^*(g)$ does not contain

$$\mathcal{E} := \{e_1, ..., e_{\tilde{d}}\}.$$

In particular, the C*-algebra A is not singly generated.

First, note that there is $\varepsilon > 0$ such that

(1) there is a retraction from the $\varepsilon\tilde{d}$-neighbourhood of $M \subseteq \mathbb{C}^{\tilde{d}}$ to M, and
(2) if $f, h : Z \to M$ are continuous maps such that $\|f(x) - h(x)\| < \varepsilon$ for all $x \in Z$, where Z is a compact space, then $[f]_* = [h]_*$ on $\mathrm{H}_*(Z)$.

Now, assume that the set $\mathcal{E}$ were contained in a singly generated sub-C*-algebra. Then, there would be (noncommutative) polynomials $P_1, ..., P_{\tilde{d}}$ such that for sufficiently large n, there is $g \in A_n = p_n(\mathrm{C}(X_n) \otimes \mathcal{K})p_n$ such that

$$\phi_{0,n}(e_i) \approx_\varepsilon P_i(g), \quad i = 1, 2, ..., \tilde{d}. \tag{3.1}$$

Write

$$g = (q_1 + \cdots + q_{2^n})g(q_1 + \cdots + q_{2^n}) = \sum_{i,j=1}^{2^n} q_i g q_j,$$

where $q_1, ..., q_{2^n}$ are the partition of unity in A_n given by the rank-one projections of the line bundles of (2.4).

Regard

$$(q_1 g q_i)^* = q_i g^* q_1 \quad \text{and} \quad q_i g q_1, \quad i = 2, ..., 2^n,$$

as morphisms from the trivial line bundle θ to the line bundle corresponding to q_i, and then consider the map

$$\sigma := (q_2 g^* q_1, ..., q_{2^n} g^* q_1, q_2 g q_1, ..., q_{2^n} g q_1)$$

which is a morphism from the trivial line bundle

$$E := \theta$$

to the vector bundle

$$F := (\zeta_n \ominus \theta) \oplus (\zeta_n \ominus \theta).$$

(Recall ζ_n is given by (2.4).) Note that

$$\mathrm{rank}(F) = 2 \cdot \mathrm{rank}(\zeta_i \ominus \theta) = 2(2^n - 1),$$

and by (2.5),

$$c_{2(2^n-1)}(F) = (c_{2^n-1}(\zeta_n \ominus \theta))^2 = (\pi_n^{(1)})^*(\gamma_2^2) \cup \cdots \cup (\pi_n^{(i)})^*(\gamma_{2^i}^{2^i}) \cup \cdots \cup (\pi_n^{(n)})^*(\gamma_{2^n}^{2^n}) \neq 0. \tag{3.2}$$

Since E is the trivial line bundle, the morphism σ also can be identified as a continuous section of F:

$$\sigma : X_n \ni x \mapsto \sigma_x(1_{\mathbb{C}}) \in F_x \subseteq F.$$

Since $c_{2(2^n-1)}(F) \neq 0$, the section σ has non-empty intersection with the zero section $0_F \subseteq F$. By the Whitney approximation theorem and the transversality theorem ([4]), the section σ can be perturbed within arbitrarily small tolerance to a smooth section such that σ is transverse to the zero section $0_F \subseteq F$. Thus, one may assume that the set

$$Z(g) := \{x \in X_n : \sigma(x) = 0\} = \sigma^{-1}(0_F)$$

is a differentiable (orientable) submanifold of X_n with codimension $2 \cdot \mathrm{rank}(F) = 2(2^{n+1} - 2)$.

Over the set $Z(g)$. Rewrite the set $Z(g)$ as

$$Z(g) = \{x \in X_n : q_1 g q_i(x) = 0,\ q_i g q_1(x) = 0,\ i = 2, ..., 2^n\}. \tag{3.3}$$

In other words, over the set $Z(g)$, the element g has the diagonal form $q_1 g q_1 + q_1^\perp g q_1^\perp$.

For each function e_i, $i = 1, ..., \tilde{d}$, note that, by (3.1),

$$\begin{aligned} q_1 \phi_{0,n}(e_i) q_1 &\approx_\varepsilon q_1 P_i((q_1 + q_1^\perp) g (q_1 + q_1^\perp)) q_1 \\ &= P_i(q_1 g q_1) + q_1 (\sum q_1 \cdots q_1^\perp \cdots q_1) q_1. \end{aligned}$$

Restricting to $Z(g)$, one has

$$(q_1 \phi_{0,n}(e_i) q_1)|_{Z(g)} \approx_\varepsilon P_i((q_1 g q_1)|_{Z(g)}), \quad i = 1, ..., \tilde{d}. \tag{3.4}$$

Note that, by (2.6),
$$q_1\phi_{0,n}(f)q_1 = f\circ\pi_n^M,\quad f\in \mathrm{C}(M),$$
and hence
$$(e_i\circ\pi_n^M)(x)\approx_\varepsilon P_i((q_1gq_1)(x)),\quad x\in Z(g),\ i=1,...,\tilde{d}, \tag{3.5}$$
where, since q_1 is the (trivial) constant rank-one projection, the q_1gq_1 is regarded naturally as a continuous function on X_n.

Consider the maps
$$\Pi := (e_1\circ\pi_n^M, e_2\circ\pi_n^M, ..., e_{\tilde{d}}\circ\pi_n^M) : Z(g)\to M\subseteq \mathbb{C}^{\tilde{d}}$$
and
$$\Psi := (P_1(q_1gq_1), P_2(q_1gq_1), ..., P_{\tilde{d}}(q_1gq_1)) : Z(g)\to \mathrm{sp}(q_1gq_1)\to\mathbb{C}^{\tilde{d}}.$$
By (3.5),
$$\Pi(x)\approx_{\varepsilon\tilde{d}}\Psi(x),\quad x\in Z(g). \tag{3.6}$$

Note that the map Π is just the projection to M,
$$Z(g)\xrightarrow{\iota} X_n = M\times Y_n\xrightarrow{\pi_n^M} M\subseteq\mathbb{C}^{\tilde{d}}, \tag{3.7}$$
where $Y_n := \mathbb{C}\mathrm{P}^2\times\cdots\times\mathbb{C}\mathrm{P}^{2^i}\times\cdots\times\mathbb{C}\mathrm{P}^{2^n}$, and also note that the map Ψ factors through the planar set $\mathrm{sp}(q_1gq_1)\subseteq\mathbb{C}$.

By the choice of ε, we may assume that the image of Ψ is actually inside M, and then, by (3.6) and the choice of ε again, one has
$$[\Pi]_* = [\Psi]_* : \mathrm{H}_*(Z(g))\to\mathrm{H}_*(M).$$

The fundamental class of $Z(g)$ and the Thom-Porteous formula. Recall that, by the choice of g, the set $Z(g)$ is a submanifold of X_n with codimension $2(2^{n+1}-2)$, and the image of the fundamental class of $[Z(g)]$ in $\mathrm{H}_m(X_{n+1})$, where $m=\dim(X_n)-2(2^{n+1}-2)=d=\dim(M)$, is independent of σ, and is given by the (special case of the) Thom-Porteous formula,
$$\iota_*([Z(g)]) = \left\langle c_{2(2^n-1)}(F-E), [X_n]\right\rangle, \tag{3.8}$$
where
$$c(F-E) = c(F)/c(E).$$

Let us calculate the right hand side of (3.8): By (3.2) and note that E is trivial, we have
$$\begin{aligned}
& \left\langle c_{2(2^n-1)}(F-E)), [X_n]\right\rangle \\
= & \left\langle ((\pi_n^{(1)})^*(\gamma_2))^2\cup\cdots\cup((\pi_n^{(i)})^*(\gamma_{2^i}))^{2^i}\cup\cdots\cup((\pi_n^{(n)})^*(\gamma_{2^n}))^{2^n}, e\otimes x_2\otimes\cdots\otimes x_{2^n}\right\rangle \\
= & \ e\otimes\left\langle\gamma_2^2, x_2\right\rangle\otimes\cdots\otimes\left\langle\gamma_{2^i}^{2^i}, x_{2^i}\right\rangle\otimes\cdots\otimes\left\langle\gamma_{2^n}^{2^n}, x_{2^n}\right\rangle \\
= & \ e\otimes 1\otimes\cdots\otimes 1\in\mathrm{H}_d(X_n)\setminus\{0\}.
\end{aligned}$$
This implies that
$$(\Pi)_d([Z(g)]) = (\pi_n^M)_d(\iota([Z(g)])) = (\pi_n^M)_d(e\otimes 1\otimes\cdots\otimes 1) = e\neq 0.$$

But the map Π is homotopic to the map Ψ which factors through a planar set, and thus

$$(\Pi)_d([Z(g)]) = (\Psi(Z(g)))_d = 0,$$

which is a contradiction.

Thus, the set $\mathcal{E}$ is not contained in any singly generated sub-C*-algebra, as desired. $\square$

3.2. **Stable rank and real rank of A.** Let us show that both the stable rank of A and the real rank of A are at least 2.

Let $f : M \to N$ be a continuous map, where M and N are orientable manifolds with the same dimension d. Fix orientations of M and N. Let $y \in N$ be a regular point. Then the degree of f at y, denoted by $\deg(f, y)$, is defined by

$$\deg(f, y) = \sum_{x \in f^{-1}(y)} w_x,$$

where $w_x = \pm 1$ depending on whether $\mathrm{d}f(x) : \mathrm{T}_M(x) \to \mathrm{T}_N(y)$ preserves the orientation or reverses it. If N is connected, then $\deg(f, y)$ is independent of the choice of y, and it is denoted by $\deg(f)$, and it satisfies $f_*(\mu) = \deg(f)\nu$ (note that $\mathrm{H}_d(N) \cong \mathbb{Z}$ in this case). (See, for instance, [4].)

Lemma 3.2. *Let M and N be differentiable manifolds with the same dimension, and assume that N is connected. Let $L \subseteq N$ be a connected differentiable submanifold, and let $f : M \to N$ be a differentiable map which transverses to L. Assume that* $\deg(\pi) = 1$. *Then* $\deg(f|_{f^{-1}(L)}) \neq 0$.

Proof. Since f transverses to L, the preimage $f^{-1}(L)$ is a submanifold of M with same dimension as L. Fix an orientation of $f^{-1}(L)$, and then fix an orientation of M which is compatible with $f^{-1}(L)$. Also fix an orientations of L and N which are compatible.

Pick a regular value $y \in L \subseteq N$ for $f : f^{-1}(L) \to L$. Then $f^{-1}(y)$ is a zero-dimensional submanifold of M, and hence

$$f^{-1}(y) = \{x_1, x_2, ..., x_n\}.$$

Consider the tangent spaces

$$\mathrm{T}_M(x_i), \quad i = 1, ..., n,$$

and the tangent maps

$$\mathrm{d}f(x_i) : \mathrm{T}_M(x_i) \to \mathrm{T}_N(y).$$

Then

$$\deg(f) = \sum_{i=1}^{n} w_i,$$

where $w_i = \pm 1$, $i = 1, ..., n$, depending on whether the tangent map $\mathrm{d}f : \mathrm{T}_M(x_i) \to \mathrm{T}_N(y)$ preserves the orientation or reverses it. Since $\deg(f) = 1$, necessarily n is odd.

On the other hand

$$\deg(f|_{f^{-1}(L)}) = \sum_{i=1}^{n} w'_i,$$

where $w'_i = \pm 1$, $i = 1, ..., n$, depends on whether the tangent map $\mathrm{d}(f|_{f^{-1}(L)}) : \mathrm{T}_{f^{-1}(L)}(x_i) \to \mathrm{T}_L(y)$ preserves the orientation or reverses the orientation. Since n is odd, the sum $\sum_{i=1}^{n} w'_i$ must be nonzero, as desired. ☐

Proposition 3.3. *With A the C*-algebra considered above, the following bounds on the stable rank and real rank hold:*

$$2 \leq \mathrm{tsr}(A) \leq \left\{ \begin{array}{ll} 3, & d \leq 4, \\ 4, & d > 4; \end{array} \right.$$

and

$$2 \leq \mathrm{rr}(A) \leq \left\{ \begin{array}{ll} 2, & d = 2, \\ 3, & d > 2. \end{array} \right.$$

(Recall that $d = \dim(M) \geq 2$.)

Proof. Stable Rank. Let us work on the stable rank first.

By [9],

$$\mathrm{tsr}(A_n) = \lceil \lfloor 2^{n+1} - 2 + d/2 \rfloor / 2^n \rceil + 1,$$

and therefore

$$\mathrm{tsr}(A) \leq \left\{ \begin{array}{ll} 3, & d \leq 4, \\ 4, & d > 4. \end{array} \right.$$

Let us show that $\mathrm{tsr}(A) \geq 2$. It is enough to find an element of A which is not in the closure of the invertible elements.

On the manifold M, choose a two-dimensional orientable connected submanifold S (if $\dim(M) = 2$, just choose S to be a connected component of M; otherwise, S may be chosen to be a two-dimensional sphere). On S, choose $S^+ \subseteq S$ which is homeomorphic to the unit disk, and fix a homeomorphism

$$a : S^+ \to \{z \in \mathbb{C} : |z| \leq 1\}.$$

Extend a to a continuous function from M to the unit disk, and still denote it by a. Regard a as an element of $\mathrm{C}(M) = A_1$.

Suppose there were $g \in A_n$ for a sufficiently large n that g is invertible and

$$\|\phi_{0,n}(a) - g\| < 1. \tag{3.9}$$

Write

$$A_n = (q_1 + \cdots + q_{2^n})(\mathrm{C}(X_n) \otimes \mathcal{K})(q_1 + \cdots + q_{2^n}),$$

and consider the set $Z(g) \subseteq X_n = M \times Y_n$ as below:

$$Z(g) = \{x \in X_n : q_1 g q_i(x) = q_1 g^* q_i(x) = 0, \ i = 2, ..., 2^n\}.$$

With the same argument as in the proof of Theorem generator-thm, by the transversality theorem, the element g can be chosen further that $Z(g) \subseteq X_n$ is a differentiable (orientable) submanifold of X_n with the same dimension as M. Note that, restricted to $Z(g)$, the element g has the diagonal form $q_1 g q_1 + q_1^{\perp} g q_1^{\perp}$. Using the Thom-Porteous formula, the same computation as in the proof of Theorem 3.1 shows that

$$(\pi_n^M)_*[Z(g)] = [M],$$

and therefore, restricted to the connected component of M which contains S, one has

$$\deg(\pi_n^M|_{Z(g)}) = 1.$$

Let us consider the cut-down $q_1 g q_1$. Since q_1 is the rank-one trivial projection, it is a function on X_n. To simply the notation, let us still denote it by g. Then, restricted to $Z(g)$, by (3.9), one has

$$\left\| a \circ \pi_n^M|_{Z(g)} - g \right\| < 1.$$

By the transversality theorem, there is a map $\tilde{\pi} : Z(g) \to M$ which is arbitrarily close to π_n^M and transverse to $S \subseteq M$. Then, $\tilde{\pi}$ can be chosen such that

$$\|a \circ \tilde{\pi} - g\| < 1, \tag{3.10}$$

$$\deg(\tilde{\pi}) = \deg(\pi_n^M|_{Z(g)}) = 1,$$

and the set

$$\Sigma := \tilde{\pi}^{-1}(S)$$

is a 2-dimensional orientable submanifold of $Z(g)$.

Define

$$\Sigma^1 := \tilde{\pi}^{-1}(S^1), \quad \Sigma^+ := \tilde{\pi}^{-1}(S^+), \quad \Sigma^- := \tilde{\pi}^{-1}(S^-),$$

where

$$S^- = S \setminus \mathrm{int}(S^+) \quad \text{and} \quad S^1 = S^+ \cap S^-.$$

Then, applying the Mayer-Vietoris sequence (and its naturality), one has the maps between exact sequences

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \mathrm{H}_2(\Sigma) & \xrightarrow{\partial_\Sigma} & \mathrm{H}_1(\Sigma^1) & \xrightarrow{(\iota^+,\iota^-)} & \mathrm{H}_1(\Sigma^+) \oplus \mathrm{H}_1(\Sigma^-) & \longrightarrow & \cdots, \\
 & & \downarrow{\scriptstyle (\tilde{\pi}|_\Sigma)_2} & & \downarrow{\scriptstyle (\tilde{\pi}|_{\Sigma^1})_1} & & \downarrow & & \\
\cdots & \longrightarrow & \mathrm{H}_2(S) & \xrightarrow{\partial_S} & \mathrm{H}_1(S^1) & \xrightarrow{(\iota^+,\iota^-)} & \mathrm{H}_1(S^+) \oplus \mathrm{H}_1(S^-) & \longrightarrow & \cdots
\end{array}$$

where $\iota^\pm$ are the embeddings of the corresponding spaces.

By Lemma 3.2,

$$\deg(\tilde{\pi}|_\Sigma) \neq 0,$$

and hence the map $(\tilde{\pi}|_\Sigma)_2$ is nonzero. Since the boundary map ∂_S is an isomorphism, the element

$$\sigma := \partial_\Sigma(\mu) \in \mathrm{H}_1(\Sigma^1)$$

is not zero, where $\mu \in \mathrm{H}_2(\Sigma)$ is the fundamental class. Note that, by the exactness,

$$\iota^+(\sigma) = 0 \in \mathrm{H}_1(\Sigma^+). \tag{3.11}$$

Note that the restriction of a to S^+ is a homeomorphism to the disc which sends S^1 to the unit circle in $\mathbb{C}$, and also note that

$$\mathrm{H}_1(\mathbb{C} \setminus \{0\}) \ni (a \circ \tilde{\pi}|_{\Sigma^1})_1(\sigma) \neq 0.$$

Consider the map

$$g : Z(g) \to \mathbb{C} \setminus \{0\}.$$

By (3.10), its restriction to Σ^1 is homotopic to $(a \circ \tilde{\pi})|_{\Sigma^1}$ (as a map to $\mathbb{C} \setminus \{0\}$), and therefore

$$(g|_{\Sigma^1})_1(\sigma) = (a \circ \tilde{\pi}|_{\Sigma^1})_1(\sigma) \neq 0.$$

On the other hand, the map $g|_{\Sigma^1}$ factors through

$$\Sigma^1 \xrightarrow{\iota^+} \Sigma^+ \xrightarrow{g} \mathbb{C} \setminus \{0\},$$

and hence, by (3.11),

$$(g|_{\Sigma^1})_1(\sigma) = (g)_1(\iota^+(\sigma)) = 0,$$

which is a contradiction. Therefore, the element a cannot be approximated by invertible elements within distance 1.

Real Rank. Let us now work on the real rank of A. The argument is almost the same as the argument for stable rank.

By [1] (and argument of [9]; see proof of Theorem 10 of [14]),

$$\mathrm{rr}(A_n) = \lceil \frac{d + 2^{n+2} - 4}{2^{n+1} - 1} \rceil = \lceil 2 + \frac{d-2}{2^{n+1}-1} \rceil,$$

and therefore

$$\mathrm{rr}(A) \leq \begin{cases} 2, & d = 2, \\ 3, & d > 2. \end{cases}$$

Let us show that $\mathrm{rr}(A) \geq 2$. As the case of stable rank, choose a two-dimensional (orientable) submanifold S, and choose $S^+ \subseteq S$ which is homeomorphic to the unit disk. Fix a homeomorphism a from S^+ to the unit disk, and extend it to M. Then consider the real-valued functions

$$\Re(a),\ \Im(a) \in \mathrm{C}(M) = A_1.$$

Let us show that there do not exist self-adjoint elements $b_1, b_2 \in A$ such that

$$\left\|(b_1^2 + b_2^2) - (\phi_{0,\infty}(\Re(a))^2 + \phi_{0,\infty}(\Im(a))^2)\right\| < 1 \tag{3.12}$$

and $b_1^2 + b_2^2$ is invertible.

Assume such self-adjoint elements b_1 and b_2 exist. Without loss of generality, one may assume that

$$b_1, b_2 \in A_n = (q_1 + \cdots + q_{2^n})(\mathrm{C}(X_n) \otimes \mathcal{K})(q_1 + \cdots + q_{2^n})$$

for some n, and then, consider the set $Z(b_1, b_2) \subseteq X_n = M \times Y_n$ as below:

$$Z(b_1, b_1) = \{x \in X_n : q_1 b_1 q_i(x) = q_1 b_2 q_i(x) = 0,\ i = 2, ..., 2^n\},$$

and one may assume further that $Z(b_1, b_2) \subseteq X_n$ is a differentiable (orientable) submanifold of X_n with the same dimension as M. Note that, restricted to $Z(b_1, b_2)$, both b_1 and b_2 have the diagonal form $q_1 \cdot q_1 + q_1^\perp \cdot q_1^\perp$. Using the Thom-Porteous formula, the same computation as Theorem 3.1 shows that

$$\deg(\pi_n^M|_{Z(b_1,b_2)}) = 1.$$

Let us consider the cut-downs $q_1b_1q_1$ and $q_1b_2q_1$. Since q_1 is the rank-one trivial projection, both are functions on X_n. To simply the notation, let us still denote them by b_1 and b_2. Then, restricted to $Z(b_1, b_2)$, by (3.12), one has

$$\left\|(b_1^2 + b_2^2) - ((\Re(a) \circ \pi_n^M|_{Z(b_1,b_2)})^2 + (\Im(a) \circ \pi_n^M|_{Z(b_1,b_2)})^2)\right\| < 1 \tag{3.13}$$

By the transversality theorem, there is a map $\tilde{\pi} : Z(b_1, b_2) \to M$ which is arbitrarily close to π_n^M and transverses to $S \subseteq M$. Then, $\tilde{\pi}$ can be chosen such that

$$\left\|(b_1^2 + b_2^2) - ((\Re(a) \circ \tilde{\pi})^2 + (\Im(a) \circ \tilde{\pi})^2)\right\| < 1 \tag{3.14}$$

$$\deg(\tilde{\pi}) = \deg(\pi_n^M|_{Z(b_1,b_2)}) = 1,$$

and the set

$$\Sigma := \tilde{\pi}^{-1}(S)$$

is a 2-dimensional submanifold of $Z(b_1, b_2)$. Then the same argument as in the stable rank case applyed to the function

$$g := b_1 + ib_2 : Z(b_1, b_2) \to \mathbb{C}$$

leads to a contradiction with the vanishing of the homology class $g|_{\Sigma_1}(\sigma)$. This shows that such b_1 and b_2 do not exist. Therefore the real rank of A is at least 2. □

Remark 3.4. Is every simple separable C*-algebra of stable rank one singly generated? (Without simplicity, the C*-algebra C(X), where X is a one-dimensional non-planar compact metrizable space, provides an example of a separable C*-algebra of stable rank one but is not singly generated; see Proposition 2 of [8].) Is every separable C*-algebra (not necessarily simple) of real rank zero singly generated?

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