ON THE GENERATORS OF SEPARABLE UNITAL C*-ALGEBRAS

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Abstract. Let A be a separable unital C*-algebra with nonempty tracial state space and strict comparison by ordinary tracial states. If $\mathcal{Z}$ embeds unitally into A, then A is singly generated. In particular, every simple separable unital C*-algebra A with $\mathrm{T}(A) \neq \varnothing$ and strict comparison is singly generated.

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1. Introduction

In [24], among other questions, Kadison asked whether every von Neumann algebra acting on a separable Hilbert space is singly generated (see [17] for a survey of Kadison's problem list). Many von Neumann algebras are known to be singly generated, and the problem was reduced to the II_1 factor case [47]. The recent preprint [32] resolves that remaining case.

The analogous question for C*-algebras asks which separable C*-algebras are singly generated. Commutative examples such as $\mathrm{C}(S^2)$ show that separability alone is insufficient; see [29]. Among separable C*-algebras, stable algebras and unital UHF-absorbing, $\mathcal{Z}$-absorbing, and approximately divisible algebras are known to be singly generated [45, 31, 27, 43]. An explicit single generator for the reduced group algebra $\mathrm{C}^*_r(\mathbb{F}_2)$ was constructed in [13]. Single generation has also been established for Villadsen's algebras of the first type

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[25], while simple unital AH algebras which are not singly generated were constructed in [12]. Beyond simple C*-algebras, Hayes [23] constructed a W*-bundle over a compact metrizable space whose fibers are II_1 factors with separable predual, but no finite family of continuous sections generates every fiber as a von Neumann algebra.

Related results were recently obtained by Bouwen, Castillejos, and Curda [7]. They prove that simple separable tracially AF algebras have generator rank at most one, and obtain single-generation criteria using comparison and diffuse Cantor-spectrum subalgebras.

Here we prove single generation for separable unital algebras with nonempty trace space, stabilized strict comparison by ordinary tracial states, and a unital embedding of the Jiang–Su algebra $\mathcal{Z}$. This embedding supplies the matrix cones and hereditary coefficient algebras used in the proof. There is no nuclearity assumption or restriction on the ideal structure or the extreme trace boundary.

We specify the comparison convention, since different formulations need not agree for nonsimple algebras.

Definition 1.1. Let A be unital with $\mathrm{T}(A) \neq \varnothing$. We say that A has *strict comparison by tracial states* if, for $a, b \in (A \otimes \mathcal{K})_+$,

$$\mathrm{d}_\tau(a) < \mathrm{d}_\tau(b) \quad (\tau \in \mathrm{T}(A)) \quad \Longrightarrow \quad a \precsim b.$$

All traces here are ordinary traces, extended using the unnormalized matrix trace, and $\mathrm{d}_\tau(a) = \lim_m (\tau \otimes \mathrm{Tr})(a^{1/m})$. The relation $a \precsim b$ is Cuntz subequivalence. We distinguish this condition from comparison by the full cone of lower semicontinuous traces in Section 2.

For $X = \mathrm{T}(A)$ put

$$\|a\|_{2,X} = \sup_{\tau \in X} \tau(a^*a)^{1/2}.$$

We write $\mathcal{M}_A$ for the bounded uniform tracial completion. Its construction first identifies elements at zero 2-seminorm and then completes norm-bounded balls. The canonical map $A \to \mathcal{M}_A$ is trace preserving and may have a kernel. Generation throughout the paper is norm generation in the original C*-algebra, without automatically adjoining a unit.

Theorem 1.2. *Let A be a separable unital* C*-*algebra with* $\mathrm{T}(A) \neq \varnothing$ *and strict comparison by tracial states. If $\mathcal{Z}$ embeds unitally into A, then A is singly generated.*

The embedding hypothesis can be verified from intrinsic Cuntz-semigroup regularity. Recall that A is *pure* if $\mathrm{Cu}(A)$ is almost unperforated and almost divisible. A unital pure algebra of stable rank one admits a unital $\mathcal{Z}$-embedding by [4, Proposition 7.6.5]. When its lower semicontinuous 2-quasitraces are traces, comparison by tracial states follows from comparison on the full trace cone; the quotient argument establishing this implication is included below.

Corollary 1.3 (Corollary 9.2)**.** *Let A be a separable unital pure* C*-*algebra of stable rank one. Suppose that every lower semicontinuous* 2*-quasitrace on A is an ordinary trace. Then A is singly generated.*

In particular, the conclusion holds for separable unital nowhere scattered algebras of stable rank one with strict comparison by the full ordinary trace cone; see Corollary 9.4.

Here nowhere scattered means that no nonzero ideal-quotient is elementary. Pureness in this case follows from [5, Theorem 5.12].

The simple tracial comparison theorem is a further consequence.

Corollary 1.4 (Corollary 9.3)**.** *Let A be a simple separable unital* C*-*algebra with* $\mathrm{T}(A) \neq \varnothing$ *and strict comparison by tracial states. Then A is singly generated.*

In the infinite-dimensional simple case, the embedding hypothesis of Theorem 1.2 follows from comparison. Indeed, Lemma 2.2 passes to comparison by the full ordinary trace cone. Thiel's theorem [42, Corollary B] then gives pureness, and Lin's theorem [28, Corollary 4.5] yields stable rank one. Proposition 2.4 supplies the required unital $\mathcal{Z}$-embedding. The finite-dimensional case is elementary.

For nonsimple algebras, the unital $\mathcal{Z}$-embedding in Theorem 1.2 supplies noncommutative matrix structure uniformly over the trace space. Strict comparison alone does not imply single generation in this generality. For example, if X is the geometric realization of the nonplanar finite graph K_5, then $\mathrm{Cu}(\mathrm{C}(X)) \cong \mathrm{Lsc}(X, \overline{\mathbb{N}})$ by [3, Theorem 3.4]. Thus $\mathrm{C}(X)$ has strict comparison by tracial states, but it is not singly generated: a generator would give a continuous embedding $X \to \mathbb{C}$.

The proof begins with a completely positive contractive order-zero map $\phi : \mathrm{M}_n \to \mathcal{Z} \subseteq A$ such that $\tau(\phi(1_n))$ is arbitrarily close to one, uniformly over $\mathrm{T}(A)$. A cutdown of one diagonal entry defines a full hereditary sub-C*-algebra D. The defining positive element has constant positive rank across the trace space, so the normalized trace space of D is affinely homeomorphic to $\mathrm{T}(A)$. Its strictly positive Cuntz class is almost divisible, and D inherits strict comparison. Proposition 1.4 and Theorem 1.3 of [16] therefore give real rank zero of the uniform tracial completion of D, together with realization and comparison of projections by their continuous affine trace profiles.

These regularity properties allow us to construct a unital simple AF sub-C*-algebra in the completion, with the same trace simplex as D and a uniform tracial averaging property. Its tracial closures provide the auxiliary inclusions for our adaptation of the relative-generation method of [32]. For an irreducible inclusion of finite factors, that method produces a unitary which generates the ambient factor together with the subfactor. Here the estimates must hold uniformly over the entire trace simplex and ultimately yield norm generation in D.

Comparison first captures the AF generators in the uniform 2-closure of a finitely generated sub-C*-algebra of D. Finite-word estimates and spectral clipping then give unitary perturbations small in operator norm. A Baire argument that allows the trace to vary gives relative generation simultaneously over the trace space. A further comparison argument stores the remaining errors in actual elements of D, upgrading tracial density to norm generation and giving seven selfadjoint generators for D.

The matrix cone provided by ϕ then compresses these seven generators to two selfadjoint generators of A. To recover the hereditary bimodules in this final step, each connecting element is chosen so that its initial square is full in the initial hereditary algebra. This fullness condition makes the compression work without simplicity.

Real rank zero gives a density theorem for the intrinsic class. For a C*-algebra A, write

$$\mathrm{Gen}(A) = \{g \in A : \mathrm{C}^*(g) = A\}.$$

We use Thiel's generator rank gr [40].

Theorem 1.5 (Theorem 10.4)**.** *Let A be a separable unital pure* C*-*algebra of real rank zero and stable rank one. Suppose that every lower semicontinuous 2-quasitrace on A is an ordinary trace. Then* $\mathrm{Gen}(A)$ *is a norm-dense G_δ subset of A. Equivalently,* $\mathrm{gr}(A) \leq 1$.

The density argument uses finite spectral refinements and single generation of all nonzero projection corners. In a separable real-rank-zero algebra, elements that generate an entire rectangular corner as a bimodule are dense in that corner. These connecting elements recover every off-diagonal block and allow the argument to accommodate arbitrary ideals. The simple tracial comparison result follows as a corollary.

Corollary 1.6 (Corollary 10.5)**.** *Let A be a simple separable unital* C*-*algebra with* $\mathrm{T}(A) \neq \varnothing$*, real rank zero, and strict comparison by tracial states. Then* $\mathrm{Gen}(A)$ *is a norm-dense G_δ subset of A.*

Thiel proved the nonsimple density conclusion for separable $\mathcal{Z}$-stable algebras of real rank zero [41, Theorem 5.3]. Our intrinsic theorem uses hereditary Cuntz-semigroup regularity and the single-generation theorem above.

The matrix-cone compression also controls topological free entropy dimension. Its scalar spectral labels and sparse connecting blocks give an operator-norm microstate covering estimate. Increasing the matrix size while keeping the seven-generator coefficient bound fixed makes the upper bound approach one. We use

$$\delta_{\mathrm{top}}(g) = \delta_{\mathrm{top}}(\mathrm{Re}\, g, \mathrm{Im}\, g), \qquad \chi_{\mathrm{top}}(g) = \chi_{\mathrm{top}}(\mathrm{Re}\, g, \mathrm{Im}\, g)$$

as in [46]. Recall that a separable unital C*-algebra is MF if it embeds unitally into a quotient $\prod_j \mathrm{M}_{q_j} / \bigoplus_j \mathrm{M}_{q_j}$. Such an algebra has a tracial state.

Theorem 1.7 (Theorem 11.5)**.** *Let A be a separable unital* MF C*-*algebra with strict comparison by tracial states and a unital embedding of $\mathcal{Z}$. Then*

$$\inf_{\substack{g \in A \\ \mathrm{C}^*(g) = A}} \delta_{\mathrm{top}}(g) = 1.$$

For every $\eta > 0$, there is an invertible generator $g = uh$, with $u \in \mathcal{U}(A)$ and $h = |g| \geq 1_A$, such that

$$1 \leq \delta_{\mathrm{top}}(g) < 1 + \eta, \qquad \chi_{\mathrm{top}}(g) = -\infty.$$

The infimum statement does not assert that a generator of dimension exactly one exists.

Note that, for $n \in \{2, 3, \ldots, \infty\}$, the reduced free group C*-algebras $\mathrm{C}^*_r(\mathbb{F}_n)$ are simple [33] and MF [21, Corollary 8.4]. For finite n, strict comparison follows from [1, Theorem A], using Robert's selflessness [35] and hence the work of Dykema and Rørdam [9]; for $n = \infty$, see [34, Proposition 6.3.2]. Thus Corollary 1.4 and Theorem 1.7 apply.

Applying Corollary 1.4 to a suitable weakly dense separable sub-C*-algebra of a II_1 factor also recovers the finite-factor conclusion of [32]; see Remark 9.5.

Single generation is substantially weaker than strict comparison even among simple separable unital algebras with nonempty trace space. The first-type Villadsen algebras in [25] include examples without strict comparison [44, Section 8]. Our transformation-group

theorem [14] also applies beyond strict comparison, including to the crossed products of Giol and Kerr [18, Theorem 2.2].

Section 2 records comparison, $\mathcal{Z}$-embeddings, and the regularity of the coefficient tracial completions. Section 3 constructs the AF algebra, and Section 4 establishes uniform finite-word estimates. Section 5 proves comparison storage and finite capture. Section 6 proves relative generation by a finite perturbation and a Baire argument; the perturbation is adapted from [32]. Section 7 obtains a uniform finite generator bound, and Section 8 gives the matrix-cone compression. Section 9 proves Theorem 1.2 and its intrinsic and simple corollaries. Section 10 proves Theorem 1.5 and Corollary 1.6. Section 11 proves the entropy estimates.

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2. Comparison, Jiang–Su embeddings, and tracial completions

We extend tracial states to matrix algebras using the unnormalized matrix trace. Unless explicitly stated otherwise, strict comparison means the stabilized comparison by ordinary tracial states in Definition 1.1. The following distinction is relevant when simplicity is not assumed.

Convention 2.1. Write $\mathrm{T}_{\mathrm{lsc}}(A)$ for the cone of all lower semicontinuous $[0,\infty]$-valued traces on A_+, and $\mathrm{QT}_{\mathrm{lsc}}(A)$ for the corresponding cone of 2-quasitraces. Densely finite weights are not the only weights included in these cones. A 2-quasitrace is a homogeneous positive function that is additive on commuting positive elements, satisfies $q(x^*x) = q(xx^*)$, and has a compatible such extension to $\mathrm{M}_2(A)$; additivity on all positive elements makes it a trace. All these weights are extended to $A \otimes \mathcal{K}$.

We say that A has *strict comparison by the full ordinary trace cone* if, for $a, b \in (A \otimes \mathcal{K})_+$, the existence of $0 < \gamma < 1$ with

$$\mathrm{d}_\rho(a) \le \gamma\, \mathrm{d}_\rho(b) \qquad (\rho \in \mathrm{T}_{\mathrm{lsc}}(A))$$

implies $a \precsim b$. This is the full-cone convention of [30, Definition 3.1]. It differs from the pointwise strict comparison by normalized tracial states used in our main theorem. We keep the two conventions separate.

Recall that a Cu-semigroup S is *almost unperforated* if $(k+1)x \le ky$ implies $x \le y$, for $x, y \in S$ and $k \ge 1$. It is *almost divisible* if, for $x' \ll x$ and $n \ge 1$, there is $z \in S$ such that

$$nz \le x, \qquad x' \le (n+1)z.$$

An element x is called almost divisible if this condition holds for that element. A C*-algebra is *pure* if its Cuntz semigroup is almost unperforated and almost divisible. Almost unperforation is equivalent to strict comparison by the full cone of lower semicontinuous 2-quasitraces, with the uniform-ratio convention above; see [30, Section 3, before Definition 3.1]. A unital algebra has *stable rank one* if its invertible elements are norm dense.

Lemma 2.2. *Let A be a simple unital C^*-algebra with $\mathrm{T}(A) \neq \varnothing$ and strict comparison by ordinary tracial states. Then A has strict comparison by the full ordinary trace cone, and every lower semicontinuous 2-quasitrace on A is an ordinary trace.*

Proof. Suppose that $a, b \in (A \otimes \mathcal{K})_+$ and $0 < \gamma < 1$ satisfy

$$\mathrm{d}_\rho(a) \leq \gamma \mathrm{d}_\rho(b) \qquad (\rho \in \mathrm{T}_{\mathrm{lsc}}(A)).$$

If $b = 0$, faithfulness of a tracial state gives $a = 0$. Otherwise every $\tau \in \mathrm{T}(A)$ is faithful, so $\mathrm{d}_\tau(b) > 0$. Fix $\varepsilon > 0$. Finite-matrix localization gives $c \in \mathrm{M}_m(A)_+$ with $[(a - \varepsilon)_+] \leq [c]$. Thus $\mathrm{d}_\tau((a - \varepsilon)_+) \leq m < \infty$. If $\mathrm{d}_\tau(b) < \infty$, then

$$\mathrm{d}_\tau((a - \varepsilon)_+) \leq \mathrm{d}_\tau(a) \leq \gamma \mathrm{d}_\tau(b) < \mathrm{d}_\tau(b).$$

If $\mathrm{d}_\tau(b) = \infty$, finiteness of the left side gives the same strict inequality. Comparison by tracial states gives $(a - \varepsilon)_+ \precsim b$. Letting $\varepsilon \downarrow 0$ proves $a \precsim b$, as required by Convention 2.1. Ng–Robert's theorem [30, Theorem 3.6(i)] now gives the assertion about quasitraces. □

Lemma 2.3. *Let A be a nonzero unital pure C^*-algebra of stable rank one. Suppose that every lower semicontinuous 2-quasitrace on A is a trace. Then $\mathrm{T}(A) \neq \varnothing$, and A has strict comparison by ordinary tracial states in the sense of Definition 1.1.*

Proof. Every nonzero unital quotient of A has stable rank one and is therefore stably finite. By [22, Theorem 2.4], it admits a unital homomorphism into a finite AW^*-factor. Pulling back the canonical normalized quasitrace gives a normalized lower semicontinuous 2-quasitrace by [6, Proposition II.1.6 and Corollary II.1.10]. Its pullback to A is a trace by hypothesis, so the quotient has a tracial state. Taking the quotient to be A first shows that $\mathrm{T}(A)$ is nonempty.

Suppose that $a, b \in (A \otimes \mathcal{K})_+$ satisfy

$$\mathrm{d}_\tau(a) < \mathrm{d}_\tau(b) \qquad (\tau \in \mathrm{T}(A)).$$

The element b must be full. Otherwise its ideal is $I \otimes \mathcal{K}$ for a proper ideal I of A, and a tracial state on the nonzero unital quotient A/I pulls back to a tracial state of A annihilating b. This contradicts the displayed strict inequality.

Fix $\varepsilon > 0$. Finite-matrix localization gives $c \in \mathrm{M}_m(A)_+$ and $\eta > 0$ such that

$$[(a - \varepsilon)_+] \leq [(c - \eta)_+], \qquad [c] \leq [a].$$

For completeness, put $p_m = 1_A \otimes \sum_{j=1}^m e_{jj}$ and use $a^{1/2} p_m a^{1/2} \to a$ in norm. Cutdown comparison and the Cuntz equivalence of the corresponding cutdowns of vv^* and v^*v, with $v = a^{1/2} p_m$, give these assertions with $c = p_m a p_m$ for sufficiently large m. Choose a continuous function f on $[0, \|c\|]$ with $0 \leq f \leq 1$, $f(0) = 0$, and $f = 1$ on $[\eta, \|c\|]$. The finite continuous affine function $F(\tau) = \tau(f(c))$ satisfies

$$\mathrm{d}_\tau((a - \varepsilon)_+) \leq F(\tau) \leq \mathrm{d}_\tau(c) \leq \mathrm{d}_\tau(a) < \mathrm{d}_\tau(b).$$

Lower semicontinuity and compactness of $\mathrm{T}(A)$ yield a number $\delta > 0$ with $F(\tau) + \delta \leq \mathrm{d}_\tau(b)$ for every τ. If $M = \max F > 0$, take $\gamma = M/(M + \delta)$; if $M = 0$, take $\gamma = 1/2$. Then $0 < \gamma < 1$ and

$$\mathrm{d}_\tau((a - \varepsilon)_+) \leq \gamma \, \mathrm{d}_\tau(b) \qquad (\tau \in \mathrm{T}(A)). \tag{2.1}$$

Every lower semicontinuous trace ρ with $0 < \rho(1_A) < \infty$ is a scalar multiple of a tracial state, so (2.1) holds for it as well. A weight with $\rho(1_A) = 0$ is zero. If $\rho(1_A) = \infty$, fullness of b and compactness of $[1_A]$ give $[1_A] \le k[b]$ for some k, hence $\mathrm{d}_\rho(b) = \infty$. The same inequality therefore holds for every $\rho \in \mathrm{T}_{\mathrm{lsc}}(A)$. Pureness, the hypothesis on quasitraces, and full-cone comparison now give $(a-\varepsilon)_+ \precsim b$. Letting $\varepsilon \downarrow 0$ proves $a \precsim b$. □

Proposition 2.4. *Let A be a nonzero unital C*-algebra of stable rank one. If $\mathrm{Cu}(A)$ is almost unperforated and $[1_A]$ is almost divisible, then $\mathcal{Z}$ embeds unitally in A. In particular this holds for every nonzero unital pure algebra of stable rank one.*

Proof. By [4, Paragraph 7.6.4], almost unperforation is the strict comparison hypothesis used in [4, Proposition 7.6.5]. The implication (2)⇒(4) of that proposition gives a unital homomorphism $\mathcal{Z} \to A$. It is injective because $\mathcal{Z}$ is simple. The criterion does not require simplicity of A or identification of its Cuntz semigroup with a space of affine functions. □

Proposition 2.5. *Let A be a unital C*-algebra with $\mathrm{T}(A) \neq \varnothing$, containing a fixed unital copy of $\mathcal{Z}$. For every $n \ge 1$ and $\eta > 0$ there is a c.p.c. order zero map $\phi : \mathrm{M}_n \to A$ with range in this copy and*

$$\tau(\phi(1_n)) > 1 - \eta \qquad (\tau \in \mathrm{T}(A)).$$

Write $t = \phi(1_n)$. If $0 < \delta < \tau_{\mathcal{Z}}(t)$, then $d = (\phi(e_{11}) - \delta)_+$ is full in A and

$$\mathrm{d}_\tau(d) = c_\delta > 0 \quad (\tau \in \mathrm{T}(A)), \qquad n c_\delta \ge \tau_{\mathcal{Z}}(t) - \delta.$$

The class $[d]$ is almost divisible in $\mathrm{Cu}(\mathrm{Her}_A(d))$.

Proof. There is a c.p.c. order zero map $\psi : \mathrm{M}_n \to \mathcal{Z}$ with $\tau_{\mathcal{Z}}(\psi(1_n)) > 1 - \eta$; see [36, opening paragraph of Section 6]. Compose with the fixed embedding. Every tracial state of A restricts to the unique tracial state on this copy of $\mathcal{Z}$, which gives the first assertion. The supporting-homomorphism description of order zero maps [48, Theorem 2.3 and Corollary 3.2] gives

$$\mathrm{d}_\tau((\phi(e_{11}) - \delta)_+) = \frac{1}{n}\mathrm{d}_\tau((t-\delta)_+).$$

The ranks are independent of τ, since both cutdowns belong to $\mathcal{Z}$. If μ is the spectral distribution of t for its trace, then

$$\tau_{\mathcal{Z}}(t) = \int_0^1 s \, \mathrm{d}\mu(s) \le \delta + \mu((\delta, 1]) = \delta + n c_\delta.$$

Thus $d \neq 0$. Simplicity of $\mathcal{Z}$ makes d full in $\mathcal{Z}$, hence full in A because the embedding is unital.

The description $\mathrm{Cu}(\mathcal{Z}) = \mathbb{N}_0 \sqcup (0, \infty]$ [4, Paragraph 7.3.2], with compact integers distinguished from soft integers, shows that for every $x \in \mathrm{Cu}(\mathcal{Z})$ and $k \ge 1$ there is a class y with $ky \le x \le (k+1)y$. For a soft $x = r$, use the soft class r/k (and ∞ when $r = \infty$). For a nonzero compact integer r, choose a soft number s with $r/(k+1) < s < r/k$. The mixed order rules give the same inequalities; the case $x = 0$ is immediate. Apply this to $x = [d]$ and pass the inequalities to $\mathrm{Cu}(A)$. The full hereditary inclusion $\mathrm{Her}_A(d) \subseteq A$ induces an isomorphism on Cuntz semigroups, so the asserted almost divisibility follows. □

We next recall the form of regularity of a uniform tracial completion that we use. A *tracially complete C*-algebra* is a pair $(\mathcal{M}, X)$ in which $\mathcal{M}$ is unital, $X \subseteq \mathrm{T}(\mathcal{M})$ is nonempty,

compact, and convex, the uniform 2-seminorm

$$\|x\|_{2,X} = \sup_{\tau\in X} \tau(x^*x)^{1/2}$$

is a norm, and the norm-closed unit ball of $\mathcal{M}$ is complete for this norm. The pair is *factorial* if X is a face of $\mathrm{T}(\mathcal{M})$: whenever $t\rho + (1-t)\sigma \in X$, with $0 < t < 1$ and $\rho, \sigma \in \mathrm{T}(\mathcal{M})$, both ρ and σ belong to X. Equivalently, $\pi_\tau(\mathcal{M})''$ is a factor for every extreme point τ of X. The pair is of type II_1 if $\pi_\tau(\mathcal{M})''$ is a type II_1 von Neumann algebra for every $\tau \in X$.

For a factorial pair of type II_1, we use the term *regular* in the sense of [16, Theorem 1.3]. The equivalent formulation needed here consists of real rank zero of $\mathcal{M}$ and the following two projection properties:

(i) for projections $p, q \in \mathcal{M}$, one has $p \precsim_{\mathrm{MvN}} q$ if and only if $\tau(p) \le \tau(q)$ for all $\tau \in X$;
(ii) for every continuous affine $f : X \to [0,1]$, there is a projection $p \in \mathcal{M}$ with $\tau(p) = f(\tau)$ for every $\tau \in X$.

Here $p \precsim_{\mathrm{MvN}} q$ means that some partial isometry $v \in \mathcal{M}$ satisfies $v^*v = p$ and $vv^* \le q$.

If B is a C*-algebra with compact nonempty normalized trace space, its *uniform tracial completion* $\overline{B}^{\mathrm{T}(B)}$ is the quotient of the algebra of norm-bounded, $\|\cdot\|_{2,\mathrm{T}(B)}$-Cauchy sequences by the ideal of $\|\cdot\|_{2,\mathrm{T}(B)}$-null sequences. Every $\tau \in \mathrm{T}(B)$ extends to this completion by taking ordinary limits along such sequences. We denote the resulting compact set of traces by X_B. The canonical map from B to its completion need not be injective: its kernel is the common trace kernel. Tracial calculations with elements of B use their canonical images; this identification does not assert that their original norms are preserved.

Proposition 2.6. *Let B be a separable C*-algebra such that $\mathrm{T}(B)$ is nonempty and compact. Suppose that B has strict comparison by ordinary tracial states, the class of a strictly positive element is almost divisible, and B has no nonzero finite-dimensional representations. Then*

$$(\mathcal{M}_B, X_B) := (\overline{B}^{\mathrm{T}(B)}, X_B)$$

is a regular factorial tracially complete C-algebra of type II_1. Moreover, $\mathrm{T}(\mathcal{M}_B) = X_B$.*

Proof. The completion with its designated traces is factorial and tracially complete by [16, Section 2.4.2]. Bounded uniform 2-density identifies the corresponding tracial GNS von Neumann algebras before and after completion. The finite GNS von Neumann algebras of B have no type I summand: a nonzero type I summand, disintegrated into finite-dimensional factor representations, would give a nonzero finite-dimensional representation of the separable algebra B. Thus the pair is of type II_1. Proposition 1.4 of [16] applies to the stated comparison and almost-divisibility hypotheses and gives regularity. The equality of the full trace space with the designated traces follows from [16, Theorem 1.3]. Neither simplicity nor nuclearity is required here. □

Lemma 2.7 (Constant-rank hereditary trace spaces)**.** *Let A be unital with $\mathrm{T}(A) \neq \varnothing$, and let $d \in A_+$ be full and satisfy $\mathrm{d}_\tau(d) = c$ for every $\tau \in \mathrm{T}(A)$, where $c > 0$ is constant. Set $D = \mathrm{Her}_A(d)$. Then*

$$\mathrm{T}(A) \longrightarrow \mathrm{T}(D), \qquad \tau \longmapsto c^{-1}\tau|_D$$

is an affine homeomorphism. In particular $\mathrm{T}(D)$ *is compact. If* $\mathcal{M}_A$ *is the bounded uniform tracial completion of* A*, the support of the image of* d *is a projection* $p \in \mathcal{M}_A$*, and the normalized completion of* D *identifies with* $p\mathcal{M}_A p$*.*

Proof. Fullness of d gives a finite family $\xi_i = x_i d^{1/2} \in A$ such that $\sum_i \xi_i \xi_i^* = 1$. To justify this form, choose a finite sum $\sum_i z_i d w_i$ within $1/2$ of 1. Writing it as XY, with $X = (z_i d^{1/2})_i$ and $Y = (d^{1/2} w_i)_i^{\mathsf{T}}$, shows that $XX^* = \sum_i z_i d z_i^*$ is invertible: use $(XY)(XY)^* \le \|Y\|^2 XX^*$. Left normalization by $(XX^*)^{-1/2}$ gives the required ξ_i.

For a bounded trace σ on D, define

$$\tau_\sigma(a) = \sum_i \sigma(\xi_i^* a \xi_i), \qquad a \in A_+.$$

Every summand is in D, so this defines a bounded positive functional. Inserting the finite frame on both sides of $x \in A$ yields

$$\begin{aligned}\tau_\sigma(x^*x) &= \sum_{i,j} \sigma((\xi_j^* x \xi_i)^* (\xi_j^* x \xi_i)) \\ &= \sum_{i,j} \sigma((\xi_j^* x \xi_i)(\xi_j^* x \xi_i)^*) = \tau_\sigma(xx^*).\end{aligned}$$

Thus τ_σ is a trace. For $a \in D_+$, the elements $a^{1/2}\xi_i = a^{1/2} x_i d^{1/2}$ lie in D, and cyclicity gives $\tau_\sigma(a) = \sigma(a)$. Any trace extending σ must have the displayed formula, by $\sum_i \xi_i \xi_i^* = 1$, so the extension is unique.

For a tracial state τ on A, the norm of $\tau|_D$ is $\mathrm{d}_\tau(d) = c$. Conversely, if $\sigma \in \mathrm{T}(D)$ and $\tau_\sigma = \lambda\tau$ for $\tau \in \mathrm{T}(A)$, then $1 = \|\sigma\| = \lambda c$, so $\lambda = 1/c$. This proves the bijection. The restriction map is continuous and affine, so its compact domain makes it an affine homeomorphism onto $\mathrm{T}(D)$.

Let (e_k) be an increasing positive contractive approximate unit obtained by functional calculus of d. The continuous functions $\tau \mapsto \tau(e_k)$ increase pointwise to the constant c; Dini's theorem gives uniform convergence. For $l \ge k$,

$$\|e_l - e_k\|_{2,X}^2 \le \sup_{\tau \in X} \tau(e_l - e_k) \le \sup_{\tau \in X} (c - \tau(e_k)) \longrightarrow 0, \qquad X = \mathrm{T}(A).$$

Hence the bounded sequence has a limit $p \in \mathcal{M}_A$. The same Dini argument applies to $\tau(e_k^2) \uparrow c$, so $\|e_k - e_k^2\|_{2,X}^2 \le \sup_\tau \tau(e_k - e_k^2) \to 0$. Continuity of multiplication on bounded balls gives $p^2 = p$. In each tracial GNS representation this limit is the support of d, and $\tau(p) = c$ at every designated trace. On D the normalized 2-norm is precisely $c^{-1/2}$ times the ambient one. The unit ball of D is uniformly 2-dense in that of $p\mathcal{M}_A p$: if $a_k \in A$ are contraction approximants to $x \in (p\mathcal{M}_A p)_1$, the elements $e_k a_k e_k \in D$ have the same norm bound and converge to x after taking sufficiently fine approximants. This identifies the two completions. □

Remark 2.8 (Projection calculus in the hereditary completion)**.** If the ambient completion is regular, its projection-calculus properties pass to $p\mathcal{M}_A p$ with the normalized designated traces. Bounded-ball completeness passes to this corner, as does real rank zero. An ambient partial isometry whose initial and final projections are bounded by p lies in $p\mathcal{M}_A p$. To realize a normalized profile f in the corner, realize the ambient profile cf and use comparison to move the resulting projection below p. All trace formulas are rescaled by the single

constant c. The next corollary establishes regularity directly for the hereditary completions used below.

Corollary 2.9. *Let A be a separable unital C^*-algebra with nonempty tracial state space and strict comparison by ordinary tracial states. Suppose that A contains a unital copy of $\mathcal{Z}$. For every nonzero $h \in \mathcal{Z}_+$, put $D = \mathrm{Her}_A(h)$. Then $(\overline{D}^{\mathrm{T}(D)}, X_D)$ is regular, factorial, and of type II_1, and its full tracial state space is X_D.*

Proof. The element h is full in A, and $\mathrm{d}_\tau(h) = \mathrm{d}_{\tau_{\mathcal{Z}}}(h) = c > 0$ is constant on $\mathrm{T}(A)$. Lemma 2.7 identifies $\mathrm{T}(D)$ affinely homeomorphically with $\mathrm{T}(A)$ by $\tau \mapsto c^{-1}\tau|_D$. Comparison in a hereditary subalgebra agrees with comparison in the ambient algebra. The trace identification therefore shows that D has strict comparison by ordinary tracial states. The last part of the proof of Proposition 2.5, applied to $[h] \in \mathrm{Cu}(\mathcal{Z})$, shows that the strictly positive class $[h] \in \mathrm{Cu}(D)$ is almost divisible.

There are no nonzero finite-dimensional representations of A, since their restriction to the unital copy of $\mathcal{Z}$ would be a nonzero finite-dimensional representation of $\mathcal{Z}$. We check the same assertion for D using the unitality of A. Choose the finite frame ξ_i from the proof of Lemma 2.7, so $\sum_i \xi_i\xi_i^* = 1_A$ and $\xi_i^*\xi_j \in D$. The matrix $P = (\xi_i^*\xi_j)_{i,j} \in \mathrm{M}_m(D)$ is a projection, and

$$a \longmapsto (\xi_i^* a \xi_j)_{i,j}$$

is a unital isomorphism from A onto $P\mathrm{M}_m(D)P$. If $\sigma : D \to \mathrm{M}_r$ were a nonzero representation, $\sigma^{(m)}(P)$ would be nonzero. Indeed the entries of P generate D as an ideal: for $a, b \in D$,

$$ab = \sum_{i,j} (a\xi_i)(\xi_i^*\xi_j)(\xi_j^* b),$$

and all three factors of each summand belong to D. The linear span of such products ab is dense in D, so a representation annihilating all entries of P is zero. The displayed corner isomorphism would then give a nonzero finite-dimensional representation of A, a contradiction. Thus D has no such representations. Apply Proposition 2.6. □

3. An external AF core with uniform tracial averaging

Let $(\mathcal{M}, X)$ be a unital factorial tracially complete C^*-algebra of type II_1. We assume that X is a metrizable Choquet simplex, that the unit ball of $\mathcal{M}$ is separable for

$$\|x\|_{2,X} = \sup_{\tau \in X} \tau(x^*x)^{1/2},$$

and that $(\mathcal{M}, X)$ is regular. We use the following precise consequences of regularity:

(i) $\mathcal{M}$ has real rank zero;
(ii) every continuous affine $f : X \to [0,1]$ is the profile $\tau \mapsto \tau(p)$ of a projection $p \in \mathcal{M}$;
(iii) projections with equal profiles are Murray–von Neumann equivalent, and profile inequality gives Murray–von Neumann subequivalence.

These are the regular projection conclusions of [16, Theorem 1.3 and Definition 3.2]. In particular, if p is a projection and $0 \le f \le \widehat{p}$, where $\widehat{p}(\tau) = \tau(p)$, there is $q \le p$ with $\widehat{q} = f$. This assertion includes profiles which vanish at some points of X: first realize f in $\mathcal{M}$ and

then move the resulting projection into p by comparison. This non-strict version is needed below.

If $\mathcal{M}$ is the bounded uniform tracial completion of a norm-separable algebra A, separability of its unit ball for $\|\cdot\|_{2,X}$ is automatic. Here the canonical homomorphism from A to its completion need not be injective. No real-rank assumption on A is imposed in this section.

Lemma 3.1 (A dimension group of affine profiles)**.** *There is a countable dense rational vector subspace $F \subseteq \mathrm{Aff}(X)$ containing the constant function 1. With*

$$F_+ = \{0\} \cup \{f \in F : \min_{\tau \in X} f(\tau) > 0\},$$

$(F, F_+, 1)$ is a simple dimension group with order unit, and $X \to S(F,1)$, $\tau \mapsto (f \mapsto f(\tau))$, is an affine homeomorphism. Consequently there is a simple unital infinite-dimensional AF algebra B_0 with this scaled ordered K_0-group and with $\mathrm{T}(B_0)$ identified with X.

Proof. Choose a countable dense subset of $\mathrm{Aff}(X)$ and take its rational linear span together with 1. The strict cone is proper and unperforated, 1 is an order unit, and every nonzero positive element is an order unit. To check interpolation, suppose $a_i \le b_j$ for $i, j \in \{1, 2\}$. If $a_i = b_j$ for one pair, that element interpolates. Otherwise the four differences $b_j - a_i$ are uniformly strictly positive. Strict affine interpolation on the Choquet simplex gives a continuous affine function lying strictly between the two lower and two upper functions; a sufficiently close uniform approximation from F interpolates in F.

An order-unit state s on F satisfies $|s(f)| \le \|f\|_\infty$: for every rational $q > \|f\|_\infty$, one has $q1 \pm f \in F_+$. Thus s extends to a continuous unital linear functional on $\mathrm{Aff}(X)$. It is positive, since a nonnegative continuous affine function can be approximated by strictly positive functions in F. States on $\mathrm{Aff}(X)$ are precisely evaluations at points of X. The evaluation map is continuous and bijective from the compact space X to the Hausdorff state space, and hence is an affine homeomorphism. Dimension-group realization and AF classification supply B_0; see [10, 11, 19]. It is simple because every nonzero positive group element is an order unit. It is infinite dimensional because F contains the constant rational functions and hence is divisible and noncyclic. □

Lemma 3.2 (Profile realization and relative conjugacy)**.** *Let $D = \bigoplus_\alpha \mathrm{M}_{n_\alpha}$ be finite dimensional. If strictly positive continuous affine functions f_α satisfy $\sum_\alpha n_\alpha f_\alpha = 1$, there is a unital embedding $D \to \mathcal{M}$ whose minimal projections in the αth summand have profile f_α.*

Two unital homomorphisms $D \to \mathcal{M}$ with the same minimal projection profiles are unitarily conjugate. Moreover, two finite spectral refinements of a fixed $D \subseteq \mathcal{M}$, with the same spectral labels and corresponding profiles inside every old minimal corner, are conjugate by a unitary in $D' \cap \mathcal{M}$.

Proof. Repeated projection realization inside the remaining projection produces a partition of 1 with the prescribed profiles. Equal profiles give partial isometries and hence matrix units in each summand. Strict positivity ensures that no summand is killed.

For conjugacy, write the homomorphisms as ϕ and ψ. Choose v_α with

$$v_\alpha^* v_\alpha = \phi(e_{11}^\alpha), \qquad v_\alpha v_\alpha^* = \psi(e_{11}^\alpha).$$

Then

$$u = \sum_{\alpha,i} \psi(e_{i1}^{\alpha}) v_{\alpha} \phi(e_{1i}^{\alpha})$$

is a unitary and $u\phi(d)u^* = \psi(d)$ for $d \in D$. For the relative assertion, perform the corresponding partition conjugacies inside one old minimal corner of each matrix summand, and replicate them using the old matrix units. The resulting unitary commutes with D. $\square$

Fix a unital finite-dimensional exhaustion $B_0 = \overline{\bigcup_l B_{0,l}}$. Projection realization and Lemma 3.2 construct a unital embedding $\iota : B_0 \to \mathcal{M}$ preserving the prescribed trace profiles. Indeed, embed successive stages with their F-profiles, and conjugate the next-stage embedding so that it agrees with the preceding one. The union extends to B_0. Its map is injective because B_0 is simple and the map is unital.

Lemma 3.3 (Finite-spectrum approximation with prescribed profiles)**.** *Let $D = \iota(B_{0,l}) \subseteq \mathcal{M}$, and let $c = c^* \in D' \cap \mathcal{M}$ have finite spectrum. Given $\varepsilon > 0$, there is a finite-spectrum selfadjoint $\widetilde{c} \in D' \cap \mathcal{M}$ such that*

$$\|\widetilde{c} - c\|_{2,X} < \varepsilon, \qquad \|\widetilde{c}\| \leq \|c\|,$$

and, inside every minimal corner of D, its spectral partition has strictly positive profiles in F. There are $L > l$ and $c_0 \in D' \cap \iota(B_{0,L})$ with the same spectral labels and corresponding minimal-corner profiles as $\widetilde{c}$.

Proof. It suffices to treat one minimal corner $p = e_{11}^{\alpha}$ and replicate the construction over its matrix summand. In that corner write

$$pcp = \sum_{i=1}^{r} \lambda_i p_i, \qquad \sum_i p_i = p.$$

The nonzero p_i need not have strictly positive profiles. Put $R = \|c\|$. If $R = 0$ the conclusion is immediate. Choose $s \in (0,1)$. Realize projections $p_i' \leq p_i$ with

$$\widehat{p_i'} = (1-s)\widehat{p}_i.$$

For $e = p - \sum_i p_i'$ one has $\widehat{e} = s\widehat{p}$. Partition $e = \sum_i z_i$ with $\widehat{z}_i = (s/r)\widehat{p}$, and put $q_i = p_i' + z_i$. Then $\sum_i q_i = p$ and

$$\widehat{q}_i \geq (s/r)\widehat{p} > 0.$$

The old and new spectral elements agree on $p - e$; their difference is supported on e. Thus

$$\left\| \sum_i \lambda_i p_i - \sum_i \lambda_i q_i \right\|_{2,\tau} \leq 2R\sqrt{s\tau(p)}.$$

Next choose $t \in (0,1)$ and write $\mu = \min_{i,\tau} \tau(q_i) > 0$. Since $\widehat{p} \in F$, density of F supplies functions $g_i \in F$ with $\sum_i g_i = \widehat{p}$ and

$$\|g_i - \widehat{q}_i\|_{\infty} < t\mu/2 \quad (1 \leq i \leq r).$$

For completeness, approximate the first $r-1$ profiles sufficiently closely and define the last by subtraction from $\widehat{p}$. Every g_i is strictly positive and

$$h_i := g_i - (1-t)\widehat{q}_i > 0, \qquad \sum_i h_i = t\widehat{p}.$$

Shrink q_i to $q_i' \le q_i$ with $\widehat{q_i'} = (1-t)\widehat{q_i}$. Partition $p - \sum_i q_i'$ into orthogonal projections w_i with $\widehat{w_i} = h_i$. The projections $\widetilde{p}_i = q_i' + w_i$ then partition p and have profiles $g_i \in F_+ \setminus \{0\}$. This second modification has 2-norm error at most $2R\sqrt{t\tau(p)}$.

Perform this procedure in one minimal corner of each summand and replicate by the matrix units. Choosing the same sufficiently small s, t in all finitely many summands gives a total error at most $2R(\sqrt{s} + \sqrt{t})$, since the replicated old minimal corners partition the unit. The spectral labels are unchanged, proving the norm bound.

In $\mathrm{K}_0(B_0) = F$, the functions g_i form a positive decomposition of the old minimal projection class. This decomposition is realized by projections at a later finite-dimensional stage. More explicitly, represent all the positive classes at a common later stage and pass farther until their sum equality holds there. The equality is then an equality of nonnegative integer multiplicity vectors, so the old projection can be partitioned accordingly inside that stage. Replication over the old matrix units realizes the entire refinement. This supplies c_0 at a common stage $L > l$. □

Theorem 3.4 (External AF core)**.** *Under the standing hypotheses there is a unital simple infinite-dimensional AF algebra*

$$B = \overline{\bigcup_n D_n} \subseteq \mathcal{M},$$

where the D_n are increasing unital finite-dimensional algebras, such that restriction $X \to \mathrm{T}(B)$ is an affine homeomorphism and

$$\operatorname{dist}_{2,X}\Big(\mathbb{E}_{D_n}(x), Z(D_n)\Big) \longrightarrow 0 \qquad (x \in \mathcal{M}). \tag{3.1}$$

Here $\mathbb{E}_D(x) = \int_{\mathcal{U}(D)} uxu^*\,du$. For every $\rho \in X$, if

$$P_\rho = \pi_\rho(\mathcal{M})'', \qquad N_\rho = \pi_\rho(B)'',$$

then

$$N_\rho' \cap P_\rho = Z(N_\rho) = Z(P_\rho). \tag{3.2}$$

The algebra N_ρ is hyperfinite with separable predual and has only type II_1 factor fibres. In particular the inclusion is irreducible when ρ is extreme in X.

Proof. Choose a countable 2-dense sequence (a_j) in the selfadjoint unit ball of $\mathcal{M}$, a schedule $j(n)$ visiting each index infinitely often, and positive errors $\varepsilon_n \to 0$. Starting with the trace-preserving embedding $\iota_1 : B_0 \to \mathcal{M}$ constructed above, build embeddings ι_n and stage indices $l_n \nearrow \infty$ with

$$\iota_{n+1}|_{B_{0,l_n}} = \iota_n|_{B_{0,l_n}}.$$

Put $D_n = \iota_n(B_{0,l_n})$. The algebra $D_n' \cap \mathcal{M}$ is a finite direct sum of corners of $\mathcal{M}$ and hence has real rank zero. Choose a finite-spectrum selfadjoint contraction c there with

$$\|\mathbb{E}_{D_n}(a_{j(n)}) - c\| < \varepsilon_n/2.$$

Apply Lemma 3.3 with error $\varepsilon_n/2$, obtaining $\widetilde{c}$ and c_0 at a later stage L. Relative conjugacy gives $w \in \mathcal{U}(D_n' \cap \mathcal{M})$ with $wc_0w^* = \widetilde{c}$. Set $\iota_{n+1} = \mathrm{Ad}(w)\iota_n$ and choose $l_{n+1} \ge \max\{L, l_n + 1, n+1\}$. This fixes D_n and places $\widetilde{c}$ in D_{n+1}.

Eventually constant values on each abstract stage define a unital embedding $\iota_\infty : B_0 \to \mathcal{M}$. All stages preserve all profiles in F, so restriction identifies X with $\mathrm{T}(B)$ for its range B.

For $m \geq n+1$, averaging absorption $\mathbb{E}_{D_m}\mathbb{E}_{D_n} = \mathbb{E}_{D_m}$ and 2-contractivity imply

$$\|\mathbb{E}_{D_m}(a_{j(n)}) - \mathbb{E}_{D_m}(\widetilde{c})\|_{2,X} < \varepsilon_n.$$

Since $\widetilde{c} \in D_m$, its last average lies in $Z(D_m)$. The repeated dense schedule and 2-contractivity prove (3.1).

Fix ρ and suppress representations from the notation. Let $\mathbb{E}_N : P_\rho \to N_\rho$ be the trace-preserving normal conditional expectation. For $x \in N_\rho' \cap P_\rho$ put $y = x - \mathbb{E}_N(x)$. Then y commutes with N_ρ, and $\rho(yb) = 0$ for every $b \in N_\rho$. For $a \in \mathcal{M}$,

$$\rho(ya) = \rho(y\mathbb{E}_{D_n}(a)).$$

By (3.1), Cauchy–Schwarz makes the right side tend to zero, since its central approximants belong to B. Density in $L^2(P_\rho, \rho)$ gives $y = 0$. Thus $N_\rho' \cap P_\rho = Z(N_\rho)$.

To identify the centres, let $h \in Z(N_\rho)_+$ and $L = \|h\|$. The positive tracial functional $\varphi(b) = \rho(hb)$ on B is dominated by $L\rho|_B$. The affine homeomorphism $X \cong \mathrm{T}(B)$ extends to a homogeneous additive bijection from the cone of designated positive traces on $\mathcal{M}$ to the positive tracial functionals on B. Extend φ and $L\rho|_B - \varphi$ through that bijection. Their extensions sum to $L\rho$, so the extension of φ is dominated by $L\rho$ on $\mathcal{M}$. Finite von Neumann Radon–Nikodym theory therefore gives $k \in Z(P_\rho)$, $0 \leq k \leq L$, with $\widetilde{\varphi}(a) = \rho(ka)$; domination ensures normality in this GNS representation, and traciality makes the density central. The relative commutant equality places k in $Z(N_\rho)$. Restriction to B and L^2 density in N_ρ imply $k = h$. Thus $Z(N_\rho) \subseteq Z(P_\rho)$; the reverse inclusion follows from $N_\rho' \cap P_\rho = Z(N_\rho)$.

Hyperfiniteness follows from the finite-dimensional exhaustion of B, and separability of the predual follows from separability of B. A finite-dimensional factor representation of the simple infinite-dimensional algebra B would be injective and is therefore impossible. Hence its finite factorial fibres are all of type II_1. Finally, extremality of ρ corresponds to extremality of its restriction to B, so both fibre algebras are factors by (3.2) and the stated factoriality. □

Remark 3.5. Put $\mathcal{R} = \overline{B}^{2,X} \subseteq \mathcal{M}$, with completion on bounded balls. Since $X \cong \mathrm{T}(B)$, this is the uniform tracial completion of B. The simple AF algebra B has strict comparison by traces, and its divisible ordered group F gives a divisible unit. Thus its completion has the regular projection calculus of [16, Proposition 1.4 and Theorem 1.3]. For each $\rho \in X$, its von Neumann closure is N_ρ. When $\mathcal{M}$ is the bounded uniform tracial completion of A, bounded-ball density of the canonical image of A makes that image $L^2(P_\rho, \rho)$-dense. Consequently its von Neumann closure is P_ρ, canonically identified with the tracial GNS von Neumann algebra of A at the restricted trace. This identification does not require the canonical map from A to be injective. The averaging conclusion is in uniform 2-norm; all traces used below are the designated extensions in X.

4. Uniform quantization and finite word cancellation

This section records all the uniform quantization arguments used below. In particular, the constants in the small-mean estimates are allowed to depend on the cardinality of the finite coefficient family. No compactness or dimension assumption on the extreme boundary is used.

Standing external AF-completion hypotheses. For this section abbreviate $M = \mathcal{M}$ and $R = \mathcal{R}$. Let (M, X) be the factorial tracially complete algebra from the preceding section. In particular, X is a compact convex face of $\mathrm{T}(M)$, and its traces define a faithful uniform 2-norm. We use the external AF algebra supplied by Theorem 3.4:

$$B = \overline{\bigcup_n D_n} \subseteq M, \qquad R = \overline{B}^{2,X} \subseteq M,$$

where B is unital, simple, and infinite dimensional, the D_n are increasing unital finite-dimensional subalgebras, and the latter completion is taken on bounded balls. Restriction $X \to \mathrm{T}(B)$ is an affine homeomorphism. We write

$$\|x\|_{2,u} = \|x\|_{2,X} = \sup_{\tau \in X} \tau(x^*x)^{1/2}, \qquad \|x\|_{1,\tau} = \tau(|x|).$$

The following inputs have been established above; they are recorded here to specify precisely what the arguments use.

(AF) If $\mathbb{E}_n$ is averaging over the compact group $U(D_n)$, then

$$\mathrm{dist}_{2,X}(\mathbb{E}_n(x), Z(D_n)) \longrightarrow 0 \qquad (x \in M). \tag{AF}$$

(C) The unit balls of M and R are complete for $\|\cdot\|_{2,u}$, and the unit ball of B is dense in that of R in this metric. The uniform 2-norm is faithful on R. Each function $\tau \mapsto \tau(x)$, $x \in M$, is continuous affine on X.

(P) The algebra R has real rank zero. If $e \in R$ is a projection and $f \in \mathrm{Aff}(X)$ satisfies

$$0 < f(\tau) < \tau(e) \qquad (\tau \in X),$$

there is a projection $g \leq e$ in R with $\tau(g) = f(\tau)$ for every $\tau \in X$. Projections in R with identical trace profiles are Murray–von Neumann equivalent in R.

(F) For $\rho \in X$, set $P_\rho = \pi_\rho(M)''$ and $N_\rho = \pi_\rho(B)''$. Then

$$N_\rho' \cap P_\rho = Z(N_\rho) = Z(P_\rho). \tag{F}$$

In the common central decomposition the factor fibres of N_ρ are hyperfinite II_1 factors.

In (C), density is density of the indicated bounded balls; selfadjoint contractions may therefore be approximated by selfadjoint contractions. The same conclusion for arbitrary bounded elements follows by rescaling. The continuity assertion for $x \in R$ also follows directly from $|\tau(x-y)| \leq \|x-y\|_{2,u}$ and bounded-ball approximation by B; for $x \in M$, it is part of the designated tracial completion structure. The facial property of X ensures that central reweightings of any $\rho \in X$ in its GNS von Neumann algebra still belong to X; this point will be used explicitly in the compression argument. All coefficients below belong to M, and all newly constructed projections and partial isometries belong to R.

We first prove linear projection equidistribution. This also explains why no projection-transfer assertion is an additional hypothesis.

Lemma 4.1 (Global fractional projections)**.** *Let $0 < \lambda < 1$, let $\mathcal{X} \subset M$ and $\mathcal{H} \subset R$ be finite, and let $\eta > 0$. There is a projection $e \in R$ such that*

$$\tau(e) = \lambda, \qquad |\tau(ex) - \lambda\tau(x)| < \eta \quad (x \in \mathcal{X},\ \tau \in X), \qquad \|[e,h]\|_{2,u} < \eta \quad (h \in \mathcal{H}). \tag{4.1}$$

Proof. Approximate the elements of $\mathcal{H}$ in uniform 2-norm by elements of a fixed D_k. For sufficiently large $n \geq k$, (AF) gives approximants in $Z(D_n)$, in uniform 2-norm, for the averages of the finitely many elements of $\mathcal{X}$.

Write $D_n = \bigoplus_l \mathrm{M}_{d_l}$ and choose matrix units (e^l_{ij}) in each summand. By (P), choose a projection $f_l \leq e^l_{11}$ in R with $\tau(f_l) = \lambda\tau(e^l_{11})$ for all $\tau \in X$. The strict inequalities needed in (P) hold because each nonzero projection of B has strictly positive trace at every point of X. Set

$$e = \sum_l \sum_{i=1}^{d_l} e^l_{i1} f_l e^l_{1i}.$$

This is a projection commuting with D_n, and

$$\tau(ez) = \lambda\tau(z) \qquad (z \in Z(D_n),\ \tau \in X). \tag{4.2}$$

In particular $\tau(e) = \lambda$.

For $a \in M$, commutation with D_n and traciality give $\tau(ea) = \tau(e\mathbb{E}_n(a))$ and $\tau(a) = \tau(\mathbb{E}_n(a))$. Consequently, for $z \in Z(D_n)$,

$$|\tau(ea) - \lambda\tau(a)| = |\tau((e - \lambda 1)(\mathbb{E}_n(a) - z))| \leq (1 + \lambda)\|\mathbb{E}_n(a) - z\|_{2,u}. \tag{4.3}$$

The last inequality is tracial Cauchy–Schwarz, uniformly over $\tau \in X$. If $h_0 \in D_k$ approximates $h \in R$, then $\|[e, h]\|_{2,u} \leq 2\|h - h_0\|_{2,u}$. Choosing the approximations sufficiently close proves (4.1). □

Lemma 4.2 (Uniform 2-rounding in a fixed corner)**.** *Let $p \in R$ be a projection. For every positive contraction $a \in pRp$ and every $\epsilon > 0$, there is a projection $r \in pRp$ such that*

$$\|r - a\|_{2,u} \leq 2\|a - a^2\|_{2,u} + \epsilon. \tag{4.4}$$

Proof. By real rank zero, approximate a arbitrarily closely in norm by a finite-spectrum positive contraction $b \in pRp$. Let $r = 1_{[1/2,1]}(b)$. The scalar inequality $|1_{[1/2,1]}(t) - t| \leq 2t(1 - t)$, $0 \leq t \leq 1$, gives $\|r - b\|_{2,u} \leq 2\|b - b^2\|_{2,u}$. Since $\|b - b^2 - (a - a^2)\| \leq 3\|a - b\|$, norm approximation yields (4.4). □

Lemma 4.3 (Linear projection equidistribution in a fixed corner)**.** *Let $p \in R$ be a projection of constant trace $\beta > 0$. Given $0 < t < \beta$, a finite set $\mathcal{X} \subset pMp$, and $\eta > 0$, there is a projection $r \leq p$ in R satisfying*

$$\tau(r) = t, \qquad \left|\tau(rx) - \frac{t}{\beta}\tau(x)\right| < \eta \quad (x \in \mathcal{X},\ \tau \in X). \tag{4.5}$$

Proof. Rescale to suppose the elements of $\mathcal{X}$ are contractions. Choose a small $\zeta > 0$ with $\zeta < t/2$, and put $\lambda = (t - \zeta)/\beta \in (0, 1)$. Apply Lemma 4.1 to $\mathcal{X} \cup \{p\}$ and $\mathcal{H} = \{p\}$, with an arbitrarily small error tolerance. For the resulting projection e, put $a = pep \in pRp$. Then

$$\tau(a) \approx t - \zeta, \qquad \|a - a^2\|_{2,u} = \|pe(1 - p)ep\|_{2,u} \leq \|(1 - p)ep\|_{2,u} \leq \|[e, p]\|_{2,u}. \tag{4.6}$$

By Lemma 4.2, the projection $r_0 \leq p$ obtained by rounding a can be made arbitrarily close to a in uniform 2-norm. Choose the errors in (4.6) and in rounding so small that

$$t - \tfrac{3}{2}\zeta < \tau(r_0) < t - \tfrac{1}{2}\zeta \qquad (\tau \in X). \tag{4.7}$$

The function $f(\tau) = t - \tau(r_0)$ is continuous affine and satisfies

$$0 < f(\tau) < \tau(p - r_0),$$

the last strict inequality being exactly $t < \beta$. Thus (P) gives a projection $g \le p - r_0$ in R with trace profile f. Set $r = r_0 + g$. Then $\tau(r) = t$ and

$$\|r - a\|_{2,u} \le \|r_0 - a\|_{2,u} + \sqrt{3\zeta/2}. \tag{4.8}$$

For $x \in pMp$, traciality gives $\tau(ax) = \tau(ex)$, while

$$\left|\left(\lambda - \frac{t}{\beta}\right)\tau(x)\right| \le \zeta$$

for contractions $x \in pMp$. Use (4.1), (4.8), and tracial Cauchy–Schwarz, first choosing ζ small and then the errors in (4.6) small. This proves (4.5). □

We next establish the pointwise compression estimate using only a finite-matrix second-moment calculation. In particular, no uniform freeness theorem is being invoked.

Lemma 4.4 (A finite-matrix compression calculation)**.** *Let $F \cong \mathrm{M}_d$, $d \ge 2$, be a unital matrix algebra in a finite von Neumann algebra (P, ρ), and let $q_0 \in F$ have matrix rank $1 \le r < d$. Put $\delta = r/d$, and integrate over $u \in U(F)$ with normalized Haar measure. Then*

$$\int uq_0u^*\, x\, uq_0u^*\, du = a_d x + b_d \mathbb{E}_{F'\cap P}(x), \quad a_d = \frac{r^2 - \delta}{d^2 - 1}, \quad b_d = \delta - a_d. \tag{4.9}$$

In particular $0 \le a_d \le \delta^2$. For selfadjoint x,

$$\int \rho((uq_0u^*xuq_0u^*)^2)\, du = a_d\rho(x^2) + b_d\|\mathbb{E}_{F'\cap P}(x)\|_{2,\rho}^2. \tag{4.10}$$

Proof. Identify P with $\mathrm{M}_d \mathbin{\overline{\otimes}} (e_{11}Pe_{11})$. The matrix-factor averaging map is equivariant for conjugation by $U(d)$, hence is scalar on the scalar matrices and on the traceless matrices. Its value at 1 is $\delta 1$. The trace of this linear operator on the d^2-dimensional matrix space is r^2, because the map $X \mapsto q_0Xq_0$ has rank r^2 and is a projection. Thus its traceless eigenvalue is $(r^2 - \delta)/(d^2 - 1)$, giving (4.9), coefficientwise in the second factor. Taking the trace after multiplication by x, and using $\rho(\mathbb{E}_{F'\cap P}(x)x) = \|\mathbb{E}_{F'\cap P}(x)\|_2^2$ for selfadjoint x, gives (4.10). □

Lemma 4.5 (Pointwise small-rank compression)**.** *Let $p \in R$ have constant positive trace. Normalize the traces in the corner by $\tau_p = \tau/\tau(p)$. Let $x_1, \dots, x_m \in (pMp)_{\mathrm{sa}}$ be contractions with $|\tau_p(x_i)| \le \kappa$ for every $\tau \in X$. Fix a rational $0 < \delta < 1$ and a projection $r \le p$ in R with $\tau_p(r) = \delta$ for all τ. For every $\rho \in X$,*

$$\inf_{u \in U(pRp)} \sum_{i=1}^{m} \rho_p(|uru^*x_iuru^*|) \le m\delta\kappa + m\delta^{3/2}. \tag{4.11}$$

Proof. We first justify the full-corner and central-mean assertions, without requiring simultaneous pointwise evaluations of elements of M in a central decomposition. Write $P = P_\rho$, $N = N_\rho$, and $Z_0 = Z(P) = Z(N)$. For $0 \ne h \in (Z_0)_+$, the formula

$$\sigma_h(x) = \frac{\rho(h\pi_\rho(x))}{\rho(h)} \qquad (x \in M)$$

defines a tracial state of M. It belongs to X: it is dominated by $c\rho$, where $c = \|h\|/\rho(h) \geq 1$; if $c > 1$, write $\rho = c^{-1}\sigma_h + (1 - c^{-1})\psi$ with $\psi \in \mathrm{T}(M)$ and use that X is a face, while $c = 1$ gives $\sigma_h = \rho$. On bounded sets this formula is uniform-2 continuous, since

$$\sigma_h(x^*x) \leq \frac{\|h\|}{\rho(h)}\rho(x^*x) \leq \frac{\|h\|}{\rho(h)}\|x\|_{2,u}^2. \tag{4.12a}$$

Constant trace of p therefore implies $\rho(hp) = \beta\rho(h)$ for all $h \in (Z_0)_+$, where $\beta = \tau(p) > 0$. Equivalently $\mathbb{E}_{Z_0}(p) = \beta 1$. Thus p has full central support in both N and P.

The full-corner identifications give

$$Z(pPp) = pZ_0 = Z(pNp), \qquad (pNp)' \cap pPp = pZ_0. \tag{4.12b}$$

Here is an explicit verification of the relative commutant assertion. Finite von Neumann comparison, applied to the constant centre-valued rank β, provides a finite frame $v_i \in N$ with $v_i = v_i p$ and $\sum_i v_i v_i^* = 1$. Indeed, partition 1 into finitely many projections of centre-valued traces at most $\beta 1$, compare each with p, and choose the corresponding partial isometries with those mutually orthogonal range projections. For $y \in (pNp)' \cap pPp$, put $Y = \sum_i v_i y v_i^*$. Inserting the frame once and commuting y with $pv_j^* n v_i p \in pNp$ shows $nY = Yn$ for every $n \in N$. Also $pYp = y$, by the same frame identity. Thus $Y \in N' \cap P = Z_0$, proving the second equality in (4.12b). The same argument with $N = P$ proves the centre identification.

Now work with the normalized corner trace $\rho_p = \rho/\beta$, and write $Z = pZ_0$ for its centre. If $h' = ph \in Z_+$, with $h \in (Z_0)_+$, the small-mean hypothesis applied to σ_h gives

$$|\rho_p(h'x_i)| = \beta^{-1}|\rho(hx_i)| \leq \kappa\rho(h) = \kappa\rho_p(h').$$

Testing all positive central h' proves

$$\|\mathbb{E}_Z(x_i)\| \leq \kappa. \tag{4.12}$$

We use the standard structure of a finite hyperfinite von Neumann algebra with separable predual and only II_1 factor fibres [39, Chapter XIV, Sections 1–2, and Chapter XVI, Section 1]: it is $Z \mathbin{\overline{\otimes}} \mathcal{R}$, where $\mathcal{R}$ is the hyperfinite II_1 factor. Consequently it contains an increasing sequence of full matrix algebras F_j whose matrix sizes are divisible by the denominator of δ, such that

$$\left(\bigcup_j F_j\right)' \cap pP_\rho p = Z. \tag{4.13}$$

For clarity, take full matrix approximants of a copy of $\mathcal{R}$; their commutant in $pP_\rho p$ equals the commutant of $Z \vee \mathcal{R} = pN_\rho p$, because Z is central in $pP_\rho p$. The last commutant is Z by (F).

Write $x_i = a_i + y_i$, where $a_i = \mathbb{E}_Z(x_i)$. Then $\mathbb{E}_Z(y_i) = 0$ and $\rho_p(y_i^2) \leq \rho_p(x_i^2) \leq 1$. On the Hilbert space $L^2(pP_\rho p, \rho_p)$, the inner product is $\langle x, y\rangle = \rho_p(y^*x)$. The trace-preserving conditional expectations onto $F_j' \cap pP_\rho p$ are the orthogonal projections onto the corresponding L^2-subspaces. These projections decrease to the orthogonal projection onto $L^2(Z, \rho_p)$, by (4.13). Thus the expectations converge in $\|\cdot\|_{2,\rho_p}$ to $\mathbb{E}_Z$. Choose a projection of rank fraction δ in F_j, and average its unitary conjugates q within F_j. Equations (4.9)–(4.10), followed by

Cauchy–Schwarz and Jensen, give

$$\limsup_j \int \rho_p(|qy_iq|)\,dq \le \delta^{3/2}. \tag{4.14}$$

Here $\rho_p(q) = \delta$, so $\rho_p(|qy_iq|) \le \sqrt{\delta}\,\|qy_iq\|_{2,\rho_p}$. Since a_i is central and $\|a_i\| \le \kappa$, $\rho_p(|qa_iq|) \le \delta\kappa$. Summing and taking one conjugate with cost at most the average proves the desired bound with an arbitrarily small additional error in $pN_\rho p$.

Both this q and the image of r have constant centre-valued rank δ, so they are unitarily conjugate in $pN_\rho p$. For r, the centre-valued rank assertion follows by repeating the dominated-central-trace argument above with its constant trace profile $\tau(r) = \delta\beta$. The unitaries of pRp are 2-dense in the unitaries of $pN_\rho p$. One way to see the latter assertion is to write a von Neumann unitary as e^{ih} with bounded selfadjoint h, apply Kaplansky density to h, and exponentiate. The cost in (4.11) is 2-continuous on bounded sets. Approximation therefore proves (4.11) for unitaries in pRp. □

The next elementary avoidance fact is used when turning a convex mixture of compressions into a single compression.

Lemma 4.6 (Avoiding a small projection)**.** *Let $a, P \in R$ be projections of constant traces δ and s, where $s < \delta/2$. For every $\nu > 0$, there are a projection $h \le 1 - P$ of constant trace $\delta/2$ and a unitary $U \in R$ such that*

$$h \le UaU^*, \qquad \|U - 1\| < \nu. \tag{4.15}$$

Proof. By real rank zero, the hereditary algebra generated by aPa has an approximate unit of projections. Choose such a projection $e \le a$ with $\|(a-e)aPa(a-e)\|$ arbitrarily small. For every trace, $\tau(e) \le \tau(P)$: in the corresponding finite von Neumann algebra the support of aPa is equivalent to a subprojection of P. Thus $\tau(a-e) \ge \delta - s > \delta/2$. By (P), choose $t \le a - e$ of constant trace $\delta/2$. It satisfies $\|Pt\|^2 = \|tPt\|$ arbitrarily small.

The polar part of $(1-P)t$ is a partial isometry in R with initial projection t and final projection $h \le 1 - P$, and h is arbitrarily close to t in norm. For completeness, the polar part is

$$(1-P)t\,[t(1-P)t]^{-1/2},$$

where the inverse is taken in tRt; it exists when $\|Pt\| < 1$. The standard close-projection unitary, obtained from the polar part of $ht + (1-h)(1-t)$, conjugates t to h and tends to 1 as $\|h - t\| \to 0$. Choose this unitary U with $\|U - 1\| < \nu$. Since $t \le a$, (4.15) follows. □

Proposition 4.7 (Uniform relative L^1 quantization)**.** *Let $p \in R$ have constant positive trace, let $x_1, \dots, x_m \in (pMp)_{\mathrm{sa}}$ be contractions, and suppose*

$$|\tau(x_i)| \le \kappa\tau(p) \qquad (i,\ \tau \in X).$$

For every $\epsilon > 0$, there is a projection $q \le p$ in R of constant positive trace such that

$$\|qx_iq\|_{1,\tau} \le (2m\kappa + \epsilon)\tau(q) \qquad (i,\ \tau \in X). \tag{4.16}$$

In particular the small-mean constant can be taken to be $C_m = 2m$.

Proof. Normalize traces in pMp and write them simply as τ during this proof, so $\tau(p) = 1$. Choose a small rational rank $\delta > 0$ and a projection $r \le p$ in R of constant rank δ, using

(P). For $u \in U(pRp)$ put $q_u = uru^*$ and

$$g_u(\tau) = \sum_{i=1}^{m} \tau(|q_u x_i q_u|).$$

These are continuous affine functions on the compact convex set X. By Lemma 4.5,

$$\sup_{\tau \in X} \inf_u g_u(\tau) \le m\kappa\delta + m\delta^{3/2}.$$

Choose δ so small that $m\sqrt{\delta} < \epsilon_0$. The elementary convex minimax theorem, applied to finite probability mixtures of the g_u, gives unitaries $u_1, \dots, u_L$ and weights $\alpha_j > 0$, $\sum_j \alpha_j = 1$, such that, with $q_j = u_j r u_j^*$,

$$\sum_j \alpha_j \sum_i \tau(|q_j x_i q_j|) \le (m\kappa + 2\epsilon_0)\delta \qquad (\tau \in X). \tag{4.17}$$

Only the trace variable is required to be compact; the other convex variable is the space of finitely supported probability measures. Equivalently, (4.17) follows by separating the convex hull of the continuous affine costs from a strict constant upper bound and using the barycentre of the separating probability measure on X.

Fix $0 < c < 1/4$. We recursively construct orthogonal projections $r_j \in R$ of constant ranks $c\alpha_j\delta$. Let $P_{j-1} = \sum_{k<j} r_k$, whose rank is at most $c\delta$. Apply Lemma 4.6 to q_j and P_{j-1}. It gives a projection $h_j \le p - P_{j-1}$ of rank $\delta/2$, contained in a conjugate $q_j' = U_j q_j U_j^*$, where U_j can be chosen arbitrarily close to p in the corner. Choose the conjugating unitaries so close to the corner unit that, uniformly over τ and i,

$$\left|\tau(|q_j' x_i q_j'|) - \tau(|q_j x_i q_j|)\right| < \zeta, \tag{4.18}$$

where $\zeta > 0$ will be chosen after the finite mixture is fixed. Indeed the difference in the two trace norms is at most the operator norm of $q_j' x_i q_j' - q_j x_i q_j$.

Apply Lemma 4.3 in h_j with ratio $2c\alpha_j$ to the positive coefficients

$$h_j |q_j' x_i q_j'| h_j, \qquad h_j x_i r_k x_i h_j \quad (k < j), \tag{4.19}$$

using error at most ζ in each linear test. The resulting $r_j \le h_j$ has rank $c\alpha_j\delta$ and is orthogonal to all previous r_k. For a selfadjoint H and a projection e,

$$\tau(|eHe|) \le \tau(e|H|), \tag{4.20}$$

as follows by writing $H = H_+ - H_-$ and using the trace-norm triangle inequality. Thus (4.17)–(4.20) give for each i

$$\sum_j \|r_j x_i r_j\|_{1,\tau} \le 2c(m\kappa + 2\epsilon_0)\delta + (2c + L)\zeta. \tag{4.21}$$

The cross requirements in (4.19) give

$$\tau(r_j x_i r_k x_i) \le 2c\alpha_j \tau(r_k) + \zeta \qquad (k < j).$$

Put $q = \sum_j r_j$, of constant rank $c\delta$. By orthogonality of matrix blocks in tracial L^2,

$$\left\| q x_i q - \sum_j r_j x_i r_j \right\|_{2,\tau}^2 = 2 \sum_{k<j} \tau(r_j x_i r_k x_i) \le 2c^2\delta + L(L-1)\zeta. \tag{4.22}$$

Here we used $\sum_{k<j}\alpha_j\alpha_k \le 1/2$. The operator in (4.22) is supported on q, so its 1-norm is at most $\sqrt{\tau(q)}$ times its 2-norm. Combining (4.21)–(4.22) and dividing by $c\delta$ yields

$$\frac{\|qx_iq\|_{1,\tau}}{\tau(q)} \le 2m\kappa + 4\epsilon_0 + \frac{(2c+L)\zeta}{c\delta} + \left(2c + \frac{L(L-1)\zeta}{c\delta}\right)^{1/2}.$$

Choose ϵ_0 and c so that $4\epsilon_0 + \sqrt{2c} < \epsilon/2$, and then choose $\zeta > 0$ small enough to obtain (4.16). □

The following local construction combines quantization with the cancellation of same-sign words. Taking the absolute values *after* the q_0-compression is essential to preserve the mean term when thinning.

Lemma 4.8 (Quantization with analytic cancellation)**.** *Let $p \in R$ have constant trace $\beta > 0$. Let $z_1, \dots, z_m \in pMp$ be contractions satisfying $|\tau(z_i)| \le \mu\beta$ for every $\tau \in X$. Fix finitely many templates*

$$b_0 w^\sigma b_1 \cdots w^\sigma b_k, \qquad 1 \le k \le n, \quad \sigma \in \{1, -1\}, \tag{4.23}$$

whose coefficients are contractions in pMp. For every $\eta, b > 0$ there are a projection $q \le p$ in R of constant trace $0 < \Delta < b$ and a unitary $w \in qRq$ such that

$$\|qz_iq\|_{1,\tau} \le (C\mu + \eta)\Delta, \qquad |\tau(b_0 w^\sigma b_1 \cdots w^\sigma b_k)| \le \eta\Delta, \tag{4.24}$$

where one can take $C = 4C_{2m} = 16m$. If necessary pad the z_i-list by zeros so that $m \ge 1$.

Proof. Apply Proposition 4.7 to $\operatorname{Re} z_i, \operatorname{Im} z_i$, with error $\epsilon_0 > 0$. It gives a projection $q_0 \le p$ of constant trace $r > 0$ such that, for every one of these selfadjoint contractions x,

$$\tau(|q_0xq_0|) \le (C_{2m}\mu + \epsilon_0)r. \tag{4.25}$$

Set $N = n + 1$, and choose $0 < c < 1/2$ small enough that $cr < b$. We shall select N orthogonal projections $r_j \le q_0$, each of constant trace $d_0 = cr/N$.

For every template (4.23), put the coefficients $b_1, \dots, b_{k-1}, b_kb_0$ in a list $\mathcal{B}$. In particular the cyclic wrap coefficient is included. At step j put

$$h_j = q_0 - \sum_{i<j} r_i, \qquad \theta_j = \frac{d_0}{r - (j-1)d_0} \le \frac{2c}{N}.$$

Use linear projection equidistribution in h_j, with ratio θ_j, for the finite positive coefficient list

$$h_j|q_0xq_0|h_j, \quad h_jbr_ib^*h_j, \quad h_jb^*r_ibh_j \quad (b \in \mathcal{B},\ i < j), \tag{4.26}$$

and tolerance $\zeta > 0$. It gives $r_j \le h_j$ of rank d_0. Put $q = \sum_j r_j$, so $\Delta = cr$.

For $H = q_0xq_0$, (4.26) and (4.20) imply

$$\tau(|qxq|) \le \tau(q|H|) \le 2c\tau(|H|) + N\zeta.$$

Splitting z_i into real and imaginary parts, (4.25) yields

$$\frac{\|qz_iq\|_{1,\tau}}{\Delta} \le 4C_{2m}\mu + 4\epsilon_0 + \frac{2N\zeta}{cr}. \tag{4.27}$$

This is the step that prevents amplification of the mean by $1/c$.

For $b \in \mathcal{B}$, put $D_b = qbq - \sum_j r_j b r_j$. The two cross orientations in (4.26) imply

$$\|D_b\|_{2,\tau}^2 \le 2c^2 r + N^2\zeta, \qquad \frac{\|D_b\|_{1,\tau}}{\Delta} \le \left(2c + \frac{N^2\zeta}{cr}\right)^{1/2}. \tag{4.28}$$

For the first inequality sum the orthogonal blocks $r_i b r_j$, $i \ne j$; for each unordered pair, (4.26) bounds the two squared 2-norms by $2\theta_j d_0 + 2\zeta$. The second inequality follows because D_b is supported on q.

By (P), the equal-rank r_j admit matrix units (e_{ij}) in R with $e_{jj} = r_j$. Let $w = e_{21} + e_{32} + \cdots + e_{N,N-1} + e_{1N}$. It is a unitary in qRq that cyclically permutes the r_j. After cyclically moving b_0 to the end of (4.23), replace each qbq by its block diagonal $\sum_j r_j b r_j$. The block-diagonal map $b \mapsto \sum_j r_j b r_j$ is contractive, so all replacement factors are contractions. The resulting product has degree σk modulo N in the cyclic matrix-block grading. As $1 \le k < N$, its diagonal blocks and its trace vanish. Telescoping the replacements gives a trace error at most

$$n\left(2c + \frac{N^2\zeta}{cr}\right)^{1/2}\Delta. \tag{4.29}$$

First choose ϵ_0 small, then c small, and finally ζ small, to make the residual terms in (4.27) at most η and (4.29) at most $\eta\Delta$. All choices are finite, and Δ can be smaller than any prescribed positive b. This proves (4.24). □

Proposition 4.9 (Finite weakened-word cancellation)**.** *Fix finitely many formal words*

$$W(t) = a_0 t^{\epsilon_1} a_1 \cdots t^{\epsilon_k} a_k, \qquad 1 \le k \le n, \quad \epsilon_i \in \{1, -1\}, \tag{4.30}$$

whose coefficients are contractions in M. For every $\varepsilon > 0$ there is $\kappa_0 > 0$, depending only on ε and the finite word patterns (including their number), such that the following implication holds:

$$\sup_{\tau\in X}|\tau(a_i)| \le \kappa_0 \quad \textit{whenever } \epsilon_i \ne \epsilon_{i+1} \quad \Longrightarrow \quad \exists t \in U(R)\ \sup_{\tau\in X}|\tau(W(t))| < \varepsilon \tag{4.31}$$

for every prescribed word. No condition is imposed on the outer coefficients, or on coefficients between equal signs.

Proof. We give the finite closure, constants, maximal-patching argument, and limit argument explicitly.

For every original word (4.30), include every interval of letter positions $i \le j$, with either inherited or unit boundary coefficients:

$$b_L t^{\epsilon_i} a_i \cdots a_{j-1} t^{\epsilon_j} b_R, \qquad b_L \in \{1, a_{i-1}\}, \quad b_R \in \{1, a_j\}. \tag{4.32}$$

Call the resulting finite indexed family $\mathcal{F}$, retaining repeated templates even when some coefficients coincide. It contains the original family and is closed under the same interval operation. Each of its internal coefficients is an original internal coefficient at the same pair of signs. Its indexed sign patterns therefore depend only on those of the original words.

Let $m \ge 1$ be the number of pairs of positions $i < j$ with opposite signs, summed over all words in $\mathcal{F}$, using a dummy zero entry if this number is zero. This is an upper bound for all gap-list cardinalities below, independent of any subsequently chosen partial unitary. Pad by

zeros to keep this cardinality fixed. Let $C = 4C_{2m}$ be the constant in Lemma 4.8, and put

$$B_0 = 2^n, \qquad D = B_0 C, \qquad K = \lceil 2D \rceil + 1. \tag{4.33}$$

It suffices to consider $0 < \varepsilon \leq 1$. Set

$$\gamma = \frac{\varepsilon}{4n}, \quad S = 1 - \gamma, \quad L = \gamma^{-K} - 1, \quad a = \kappa_0 = \frac{\varepsilon}{8L}, \quad \eta = \frac{a}{2B_0}. \tag{4.34}$$

Suppose the opposite-sign coefficient traces are bounded by $\kappa \leq \kappa_0$. Define the increasing convex budget

$$F(s) = (\kappa + a)((1 - s)^{-K} - 1), \qquad 0 \leq s \leq S. \tag{4.35}$$

Consider normal partial unitaries $v \in R$ whose support $f = v^*v = vv^*$ has constant trace $s \leq S$, and satisfying

$$|\tau(W(v))| \leq F(s) \qquad (W \in \mathcal{F},\ \tau \in X). \tag{4.36}$$

Here v^{-1} means v^*. Order them by orthogonal extension: $v \preceq u$ if $u = v + w$, where w is a normal partial unitary with support orthogonal to that of v. The zero partial unitary belongs to this partially ordered set.

Every chain has an upper bound. To verify this, let s be the supremum of its support ranks. If a member attains this rank, it is an upper bound: comparable members of equal rank differ by a partial unitary of zero uniform 2-norm and therefore agree. Otherwise choose an increasing sequence v_j in the chain with support ranks $s_j \uparrow s$. Orthogonal extension gives

$$\|v_j - v_i\|_{2,u}^2 = s_j - s_i \qquad (j \geq i). \tag{4.37}$$

By (C), v_j converges in the unit ball of R to v. Adjoints and multiplication on bounded sets are uniformly 2-continuous. Passing the partial-unitary identities to the limit shows that v is a normal partial unitary of constant support rank s. Uniform tracial continuity of each finite word and continuity of F show that (4.36) passes to the limit. For any chain member w of rank less than s, some v_j extends it; passing $v_j \operatorname{supp}(w) = w = \operatorname{supp}(w)v_j$ to the limit shows that v extends w. This proves the upper-bound assertion. Zorn's lemma therefore gives a maximal v.

Suppose its rank is $s < S$, and put $p = 1 - f$ and $\beta = 1 - s$. For a word in $\mathcal{F}$ and opposite-sign positions $i < j$, form

$$G = a_i v^{\epsilon_{i+1}} a_{i+1} \cdots v^{\epsilon_{j-1}} a_{j-1}. \tag{4.38}$$

If $j = i + 1$, this means $G = a_i$. When G contains a variable letter, G is an interval word in $\mathcal{F}$, and so $|\tau(G)| \leq F(s)$. Otherwise $|\tau(G)| \leq \kappa$, because the two endpoint signs are opposite. Furthermore $v^{\epsilon_i} G v^{\epsilon_j}$ is the larger interval word from position i to j, with unit boundaries, and belongs to $\mathcal{F}$. The orientation matters: because $\epsilon_j = -\epsilon_i$,

$$\tau(fG) = \tau(v^{\epsilon_i} G v^{\epsilon_j}).$$

Consequently

$$|\tau(pGp)| = |\tau(G) - \tau(fG)| \leq \kappa + 2F(s). \tag{4.39}$$

The finite coefficient list for quantization consists of all these pGp, padded by zeros, and its normalized mean bound is

$$\mu = \frac{\kappa + 2F(s)}{1-s}. \tag{4.40}$$

For the analytic list, formally expand each $W(v+w)$. For each nonempty set of new-w positions whose signs all agree, write its trace as

$$\tau(w^{\sigma} B_1 \cdots w^{\sigma} B_l), \qquad l \le n. \tag{4.41}$$

The $B_j \in pMp$ are the compressed gaps of old-v letters and coefficients. The last B_j includes the cyclic wrap through both outer coefficients. They are contractions, and there are only finitely many such templates. Their values depend on v, but their number and degrees are bounded by the fixed word patterns.

Apply Lemma 4.8 with local error η, mean bound (4.40), and rank bound $S-s$. It gives $q \le p$ of constant rank $0 < \Delta < S-s$ and a unitary $w \in qRq$. Put $u = v + w$, a normal partial unitary of rank $s + \Delta$.

Fix a degree-k word in $\mathcal{F}$ and expand at $v+w$. Apart from its old term, there are at most $2^k - 1$ terms. If the new letters in a term all have one sign, (4.41) bounds its trace by $\eta\Delta$, including the case of a single new letter. Otherwise choose two consecutive new letters with opposite signs. Their intervening old gap is some G in (4.38). The term contains $w^{\epsilon_i} G w^{-\epsilon_i}$, whose trace norm equals $\|qGq\|_{1,\tau}$; all surrounding factors are contractions. It follows that its trace has absolute value at most $(C\mu + \eta)\Delta$. Thus for every word in $\mathcal{F}$,

$$|\tau(W(u))| \le F(s) + \left(\frac{D\kappa + 2DF(s)}{1-s} + B_0\eta \right) \Delta. \tag{4.42}$$

The derivative of the budget is $F'(s) = K(F(s) + \kappa + a)/(1-s)$, and hence

$$F'(s) - \frac{D\kappa + 2DF(s)}{1-s} = \frac{(K-2D)F(s) + (K-D)\kappa + Ka}{1-s} \ge a. \tag{4.43}$$

Since $B_0\eta = a/2$, convexity of F and (4.42)–(4.43) imply $|\tau(W(u))| \le F(s+\Delta)$. Thus u is an admissible strict extension, contradicting maximality. The maximal rank must be S.

Finally set $t = v + (1-f) \in U(R)$. Telescoping over the at most n variable positions, using $t - v = t^* - v^* = 1 - f$, gives

$$|\tau(W(t)) - \tau(W(v))| \le n\tau(1-f) = n\gamma. \tag{4.44}$$

This is a telescoping estimate, not a 2^n-term expansion. By (4.34)–(4.35), $F(S) \le 2aL = \varepsilon/4$ and $n\gamma = \varepsilon/4$. Thus $|\tau(W(t))| \le \varepsilon/2 < \varepsilon$ uniformly in τ, as required. □

5. Comparison storage and finite capture

We write $\mathrm{C}^*(S)$ without adjoining a unit. If D is nonunital, $\widetilde{D}$ denotes its minimal unitization. The uniform tracial completion is always the completion of bounded balls. Its designated traces are the extensions of the specified traces on D. The uniform tracial seminorm on D need not be faithful. All series, operator norms, and functional calculi in this section are taken in D or its minimal unitization before applying the canonical map to the completion.

Proposition 5.1 (Comparison storage without a unit)**.** *Let D be a σ-unital* C*-*algebra with a nonempty family X of tracial states. Suppose that*

$$\mathrm{d}_\tau(a) < \mathrm{d}_\tau(b) \quad (\tau \in X) \quad \Longrightarrow \quad a \precsim b, \qquad a, b \in D_+.$$

Let $a_i \in D_+$ have norm at most 2, and let $k_i, \ell_i \in D_+$ be contractions satisfying

$$\ell_i a_i = a_i, \qquad \ell_i k_i = \ell_i, \qquad \mathrm{d}_\tau(k_i) \le \alpha_i \quad (\tau \in X), \qquad \sum_{i\ge 1} \alpha_i < \tfrac{1}{4}.$$

For prescribed $0 < \rho_i \le 1$, there are $H, K \in D_{\mathrm{sa}}$ and $V_i \in \mathrm{C}^*(H, K)$ such that

$$\|V_i^* V_i - a_i\| < \rho_i \qquad (i \ge 1).$$

In particular, the decoding algebra in this statement has no artificially adjoined unit.

Proof. Fix a strictly positive contraction $h \in D$. Every $\tau \in X$ extends to a tracial state on $\widetilde{D}$. In its nondegenerate tracial GNS representation, h has support 1. Consequently, for every positive contraction $f \in \widetilde{D}$,

$$\mathrm{d}_\tau(fhf) = \mathrm{d}_\tau(f). \tag{5.1}$$

Indeed, $\ker(h^{1/2}f) = \ker(f)$ in this representation; the positive operators $fhf = (h^{1/2}f)^*(h^{1/2}f)$ and f therefore have the same support. This argument uses the norm-one extension of τ to $\widetilde{D}$, and does not introduce any other trace on the unitization.

Put $R = 2$ and $\eta_i = \rho_i/[4(2R+1)]$. Choose positive numbers $w_i, \mu_i, \lambda_i, \delta_i$ recursively. First choose $w_i \le 2^{-i}$ such that, for $j < i$,

$$w_i < 2^{-(i-j)} w_j \eta_j/(6R). \tag{5.2}$$

Next choose $\mu_i \le 2^{-i}$ such that, for $j < i$,

$$2\mu_i < 2^{-(i-j)} \lambda_j \eta_j/(6R). \tag{5.3}$$

Choose $0 < \lambda_i \le \mu_i$ sufficiently small that

$$R\sqrt{3\lambda_i/\mu_i} + \frac{R}{w_i} \sum_{j<i} w_j \left(\sqrt{3\lambda_i/\lambda_j} + \sqrt{3\lambda_i/\mu_j} \right) < \eta_i/3. \tag{5.4}$$

Finally choose $0 < \delta_i < \lambda_i \eta_i/(6R)$. These choices depend only on prescribed errors and the preceding numerical choices.

We construct contractions $e_i \in D_+$ and elements $r_i \in D$ with

$$e_i r_i = r_i, \quad r_i k_i = r_i, \quad \|r_i\| < R, \quad \mathrm{d}_\tau(e_i) \le \alpha_i, \quad \|r_i^* r_i - a_i\| < \rho_i/2. \tag{5.5}$$

At stage i, set

$$S_i = \sum_{j<i} (\lambda_j e_j + \mu_j k_j) + \mu_i k_i, \qquad f_i = (1 - S_i/\delta_i)_+ \in \widetilde{D}, \qquad b_i = f_i h f_i \in D_+.$$

Support subadditivity and the inductive bounds give

$$\mathrm{d}_\tau(S_i) \le \sum_{j<i} \alpha_j + \sum_{j\le i} \alpha_j < \tfrac{1}{2}.$$

Since $f_i = 1$ on the kernel of S_i, (5.1) gives

$$\mathrm{d}_\tau(b_i) = \mathrm{d}_\tau(f_i) \ge 1 - \mathrm{d}_\tau(S_i) > \tfrac{1}{2} > \alpha_i \ge \mathrm{d}_\tau(a_i). \tag{5.6}$$

The last inequality follows from the two mask identities in the statement: $s(a_i) \le 1_{\{1\}}(\ell_i) \le s(\ell_i) \le 1_{\{1\}}(k_i) \le s(k_i)$ in the bidual. Comparison gives $a_i \precsim b_i$. Choose $y_i \in D$ with $\|y_i^* b_i y_i - a_i\| < \rho_i/4$ and set $r_i^0 = b_i^{1/2} y_i \ell_i$. Then

$$r_i^0 k_i = r_i^0, \qquad \|(r_i^0)^* r_i^0 - a_i\| < \rho_i/4, \qquad \|r_i^0\| < R,$$

and

$$\mathrm{d}_\tau(r_i^0 (r_i^0)^*) = \mathrm{d}_\tau((r_i^0)^* r_i^0) \le \mathrm{d}_\tau(\ell_i) \le \alpha_i.$$

Here comparison witnesses can be chosen in D even when Cuntz subequivalence was defined in the stabilization: compress a witness to its $(1,1)$ matrix entry.

Put $c_i = r_i^0 (r_i^0)^*$. Choose continuous functions $\chi_i, \gamma_i : [0,\infty) \to [0,1]$, zero near zero, with $\gamma_i \chi_i = \chi_i$ and

$$\sup_{t \ge 0} t(1 - \chi_i(t)^2) < \rho_i/4.$$

Set $r_i = \chi_i(c_i) r_i^0$ and $e_i = \gamma_i(c_i)$. Functional calculus gives all of (5.5). For the Gram estimate, use

$$(r_i^0)^* \chi_i(c_i)^2 r_i^0 = \big[t\chi_i(t)^2\big]\big((r_i^0)^* r_i^0\big).$$

The range support of b_i is contained in that of f_i, hence in $1_{[0,\delta_i)}(S_i)$. Therefore

$$\|S_i r_i\| \le \delta_i R. \tag{5.7}$$

This completes the induction. Notice that replacing the unital carrier $(1 - S_i/\delta_i)_+$ by $f_i h f_i$ preserves both its rank bound and its range control, which are precisely the two required properties.

Define norm-convergent series

$$H = \sum_{i \ge 1} (\lambda_i e_i + \mu_i k_i), \qquad K = \sum_{i \ge 1} w_i (r_i + r_i^*).$$

The numerical choices imply

$$\sum_{j > i} w_j < w_i \eta_i/(6R), \qquad \sum_{j > i} (\lambda_j + \mu_j) < \lambda_i \eta_i/(6R). \tag{5.8}$$

Choose a continuous $\chi : [0,\infty) \to [0,1]$ equal to one on $[0,2]$ and zero on $[3,\infty)$, and put $F_i = \chi(H/\lambda_i) \in \widetilde{D}$. The continuous function

$$\theta_i(t) = \frac{1 - \chi(t/\lambda_i)}{t - \lambda_i},$$

defined as zero near λ_i, has norm at most $1/\lambda_i$. Since

$$(1 - F_i) r_i = \theta_i(H)(H - \lambda_i) r_i,$$

(5.7) and (5.8) give

$$\|F_i r_i - r_i\| \le \frac{R}{\lambda_i} \left(\delta_i + \sum_{j > i} (\lambda_j + \mu_j) \right) < \eta_i/3. \tag{5.9}$$

For the earlier witnesses, let p_j, q_j be the left and right support projections of r_j. The identities $e_j r_j = r_j$ and $r_j k_j = r_j$ give $e_j \ge p_j$ and $k_j \ge q_j$. Thus $H \ge \lambda_j p_j$ and $H \ge \mu_j q_j$.

Since $F_i H F_i \leq 3\lambda_i 1$, we obtain

$$\|F_i r_j\| \leq R\sqrt{3\lambda_i/\lambda_j}, \qquad \|F_i r_j^*\| \leq R\sqrt{3\lambda_i/\mu_j}. \tag{5.10}$$

No commutation between H and the witnesses or masks is used.

Set $V_i = w_i^{-1} F_i K$. This element belongs to $\mathrm{C}^*(H, K)$, because

$$F_i K = K + (F_i - 1)K,$$

and $F_i - 1$ is a continuous function of H vanishing at zero. Splitting its defining series into the main term, its own adjoint, earlier terms and later terms gives

$$\begin{aligned}\|V_i - r_i\| &\leq \|F_i r_i - r_i\| + R\sqrt{3\lambda_i/\mu_i} \\ &\quad + \frac{R}{w_i}\sum_{j<i} w_j \left(\sqrt{3\lambda_i/\lambda_j} + \sqrt{3\lambda_i/\mu_j}\right) + \frac{2R}{w_i}\sum_{j>i} w_j < \eta_i.\end{aligned}$$

The three groups are bounded by $\eta_i/3$ using, respectively, (5.9), (5.4), and (5.8). Finally

$$\|V_i^* V_i - r_i^* r_i\| \leq (2R + \eta_i)\eta_i \leq (2R+1)\eta_i = \rho_i/4.$$

Together with (5.5), this proves the asserted error bound. □

Proposition 5.2 (Finite capture by actual elements)**.** *Let D be σ-unital and let X be a nonempty family of tracial states such that D has comparison by X as in Proposition 5.1. Let $\mathcal{M}$ be the bounded uniform tracial completion of D for X, and write $j_D : D \to \mathcal{M}$ for its canonical homomorphism. Given $m \geq 1$ and selfadjoint contractions $x_1, \ldots, x_m \in \mathcal{M}$, there is a sub-C*-algebra $C_0 \subseteq D$ generated by $m+2$ selfadjoint elements such that each x_j is a uniform 2-norm limit of the images under j_D of selfadjoint contractions in C_0. The map j_D need not be injective.*

Proof. Fix summable positive sequences $\epsilon_n \leq 1$ and $\rho_{j,n}^{\pm} \leq 1$, with $\sum_n(\rho_{j,n}^{+} + \rho_{j,n}^{-}) < \infty$ for every j. Choose positive support budgets $\alpha_{j,n}^{\pm}$ whose total sum is less than $1/4$. Put $\beta_n = \min_{1\leq j\leq m,\ \sigma\in\{+,-\}} \alpha_{j,n}^{\sigma}$. Choose numbers $\gamma_n > 0$ tending to zero such that

$$\gamma_n + \gamma_{n+1} < \tfrac{1}{2}\epsilon_n\sqrt{\beta_n}.$$

For example, it suffices to choose γ_n smaller than 2^{-n} and than one eighth of each of $\epsilon_n\sqrt{\beta_n}$ and, when $n \geq 2$, $\epsilon_{n-1}\sqrt{\beta_{n-1}}$. Choose selfadjoint contractions $a_{j,n} \in D$ satisfying $\|j_D(a_{j,n}) - x_j\|_{2,X} < \gamma_n$, and put

$$d_{j,n} = a_{j,n+1} - a_{j,n}, \qquad a_{j,n}^{\pm} = (\pm d_{j,n} - \epsilon_n)_+.$$

These contractive approximants exist by bounded-ball density, or by applying selfadjoint clipping to a representing sequence, even when j_D has a kernel. We have

$$4\epsilon_n^{-2}\|d_{j,n}\|_{2,X}^2 < \beta_n. \tag{5.11}$$

Choose continuous $f_n, g_n : \mathbb{R} \to [0,1]$ such that $f_n = 0$ below $\epsilon_n/2$ and $f_n = 1$ above $3\epsilon_n/4$, whereas $g_n = 0$ below $3\epsilon_n/4$ and $g_n = 1$ above ϵ_n. Put $k_{j,n}^{\pm} = f_n(\pm d_{j,n})$ and $\ell_{j,n}^{\pm} = g_n(\pm d_{j,n})$. These elements belong to D; both functions vanish at zero. They have the required mask identities, and spectral Markov inequality gives

$$\mathrm{d}_\tau(k_{j,n}^{\pm}) \leq 4\epsilon_n^{-2}\|d_{j,n}\|_{2,\tau}^2 < \alpha_{j,n}^{\pm}.$$

Apply Proposition 5.1 to this single countable family. It gives $H, K \in D_{\mathrm{sa}}$ and positive $E_{j,n}^{\pm} \in \mathrm{C}^*(H,K)$ with $\|E_{j,n}^{\pm} - a_{j,n}^{\pm}\| < \rho_{j,n}^{\pm}$.

Define

$$r_{j,n} = d_{j,n} - a_{j,n}^{+} + a_{j,n}^{-}, \qquad e_{j,n} = (a_{j,n}^{+} - E_{j,n}^{+}) - (a_{j,n}^{-} - E_{j,n}^{-}).$$

Scalar functional calculus gives $\|r_{j,n}\| \le \epsilon_n$ and $\|e_{j,n}\| \le \rho_{j,n}^{+} + \rho_{j,n}^{-}$. Therefore

$$c_j = a_{j,1} + \sum_{n \ge 1} (r_{j,n} + e_{j,n}) \in D_{\mathrm{sa}}$$

is a norm-convergent series. Let $C_0 = \mathrm{C}^*(c_1, \ldots, c_m, H, K)$ and

$$P_{j,N} = c_j + \sum_{n<N} (E_{j,n}^{+} - E_{j,n}^{-}) \in C_0.$$

The identity $r_{j,n} + e_{j,n} = d_{j,n} - E_{j,n}^{+} + E_{j,n}^{-}$ gives the exact telescoping formula

$$P_{j,N} - a_{j,N} = \sum_{n \ge N} (r_{j,n} + e_{j,n}).$$

Its right side tends to zero in the operator norm of D. Consequently $j_D(P_{j,N}) \to x_j$ uniformly in 2-norm, and the $P_{j,N}$ are uniformly norm bounded in D. Clip them to $[-1,1]$. Clipping does not increase the 2-norm distance to a selfadjoint contraction, separately at every trace; the clipped elements remain in C_0 because the clipping function vanishes at zero. They give the asserted contractive approximants. □

Remark 5.3 (Location of captured elements)**.** Proposition 5.2 applies to the bounded uniform completion of the diagonal algebra. It does not assert that an arbitrary element of the uniform tracial ultrapower is a uniform 2-norm limit of diagonal elements. Such an ultrapower element need not admit the uniformly 2-Cauchy sequence used in the proof.

6. Uniform relative generation from finite word estimates

The argument in this section converts a finite uniform word estimate into relative generation in uniform tracial norm. The fixed algebra C_0 may be nonunital and need not contain the auxiliary AF algebra used to obtain that estimate. All unitary witnesses, however, must belong to $1 + C_0$. We give the finite perturbation and the simultaneous Baire argument in full.

6.1. **Setting and the precise finite input.** Let A be a separable, possibly nonunital C*-algebra, and let $X = \mathrm{T}(A)$ be its nonempty compact space of normalized bounded tracial states. Compactness is an explicit assumption when A is nonunital. Let $C_0 \subseteq A$ be a fixed sub-C*-algebra. Write $\widetilde{A}$ for the minimal unitization of A, with $\widetilde{A} = A$ when A is unital, and put

$$\widetilde{C}_0 = \mathrm{C}^*(1, C_0) \subseteq \widetilde{A},$$
$$\mathcal{U}_1(A) = \{u \in \mathcal{U}(\widetilde{A}) : u - 1 \in A\},$$
$$\mathcal{U}_1(C_0) = \{v \in \mathcal{U}(\widetilde{C}_0) : v - 1 \in C_0\}.$$

Thus $\mathcal{U}_1(A) = \mathcal{U}(A)$ when A is unital. Both groups carry the operator norm topology. The group $\mathcal{U}_1(A)$ is a closed subset of $\widetilde{A}$, and hence is a complete separable metric space.

Let $\mathcal{M}$ be a unital tracial completion equipped with a unital $*$-homomorphism $j : \widetilde{A} \to \mathcal{M}$. Every $\tau \in X$ is understood to have a designated tracial state on $\mathcal{M}$ whose pullback by j is the normalized extension of τ to $\widetilde{A}$. We assume that, for each fixed $x \in \mathcal{M}$, the function $\tau \mapsto \tau(x)$ on X is continuous and affine. We write

$$\|x\|_{2,\tau} = \tau(x^*x)^{1/2}, \qquad \|x\|_{2,X} = \sup_{\tau \in X} \|x\|_{2,\tau}.$$

The map j need not be injective. For elements of $\widetilde{A}$, these formulas denote the corresponding tracial seminorms; inside products with elements of $\mathcal{M}$ we implicitly apply j. All operator norms, algebra memberships, and functional calculi in A or $\widetilde{A}$ refer to the actual algebras, before applying j. The compact trace space throughout is X, not the full trace space of $\mathcal{M}$. In particular, the scalar quotient trace on a unitization is not added to X.

Here is the exact finite input needed in this section. A word in a new unitary letter t is written in expanded form as

$$x_0 t^{\epsilon_1} x_1 \cdots t^{\epsilon_m} x_m, \qquad \epsilon_j \in \{-1, 1\}, \quad m \geq 1. \tag{6.1}$$

Only gaps between successive, opposite signs are required to have small trace. The two outside coefficients, and all gaps between equal signs, are arbitrary contractions.

Hypothesis 6.1 (Finite weakened-word estimate)**.** For every pair of positive integers q, n and every $\varepsilon > 0$ there is a number $\kappa(q, n, \varepsilon) > 0$ with the following property. Consider at most q words of the form (6.1), with at most n occurrences of $t^{\pm 1}$ in each word and with all coefficients in the unit ball of $\mathcal{M}$. Suppose that every internal coefficient x_j lying between opposite signs satisfies

$$\sup_{\tau \in X} |\tau(x_j)| \leq \kappa(q, n, \varepsilon).$$

Then some $t \in \mathcal{U}_1(C_0)$, evaluated in each word as $j(t)$, makes the trace of every prescribed word have absolute value at most ε, simultaneously for all $\tau \in X$. The number $\kappa(q, n, \varepsilon)$ is independent of the actual coefficients.

If the finite-word lemma is initially formulated for a fixed list of sign patterns, the displayed formulation follows by taking the minimum of its moduli over the finitely many lists with these size bounds. Cardinality dependence must be retained. No linear dependence on ε is assumed.

On uniformly norm-bounded sets, multiplication and taking adjoints are continuous for $\|\cdot\|_{2,X}$. In particular, for contractions,

$$\|a_1 \cdots a_m - b_1 \cdots b_m\|_{2,X} \leq \sum_{j=1}^{m} \|a_j - b_j\|_{2,X}. \tag{6.2}$$

Consequently, any finite collection of strict moment inequalities is retained, with an arbitrarily small additional error, when its unitary letters are replaced sufficiently closely in uniform 2-norm.

Lemma 6.2 (Transferring external AF witnesses)**.** Let $B \subseteq \mathcal{R} \subseteq \mathcal{M}$ be unital sub-C*-algebras with the same unit as $\mathcal{M}$, where B is AF. Suppose that the unit ball of B is dense in the unit ball of $\mathcal{R}$ for $\|\cdot\|_{2,X}$ and that $B \subseteq \overline{j(C_0)}^{\|\cdot\|_{2,X}}$. Then every unitary of $\mathcal{R}$ is a uniform 2-norm limit of images of elements of $\mathcal{U}_1(C_0)$. Consequently, a finite weakened-word

estimate with witnesses in $\mathcal{U}(\mathcal{R})$ implies Hypothesis 6.1, after decreasing its prescribed output error before transferring the witness.

Proof. Fix $v \in \mathcal{U}(\mathcal{R})$. Approximate v in uniform 2-norm by an element of B, and then approximate that element in norm by an element of a finite-dimensional unital subalgebra of B. Invertibles are dense in this finite-dimensional algebra, so we obtain an invertible $b \in B$ arbitrarily close to v in uniform 2-norm. Write $b = w|b|$, where $w \in \mathcal{U}(B)$. For every $\tau \in X$,

$$\operatorname{Re}\tau(v^*w|b|) \le \tau(|b|)$$

and expansion of squared norms gives $\|b - w\|_{2,\tau} \le \|b - v\|_{2,\tau}$. Thus

$$\|w - v\|_{2,X} \le 2\|b - v\|_{2,X}.$$

This proves that $\mathcal{U}(B)$ is uniformly 2-norm dense in $\mathcal{U}(\mathcal{R})$.

The same finite-dimensional approximation and polar decomposition approximate a unitary of B in norm by a finite-stage unitary. Every finite-dimensional unitary has the form e^{ih} for a self-adjoint element h of that stage. By the capture assumption, choose self-adjoint $h_m \in C_0$ such that $j(h_m)$ converges to h in uniform 2-norm. These approximants may be chosen with $\|h_m\| \le \|h\|$. Indeed, if $R = \|h\| > 0$, clip any self-adjoint approximant x to $q = \operatorname{clip}_R(x) \in C_0$. With $p_+ = (x - R)_+$ and $p_- = (-x - R)_+$, traciality gives

$$\tau((x - q)(q - h)) = \tau(p_+(R - h)) + \tau(p_-(R + h)) \ge 0,$$

so $\|q - h\|_{2,\tau} \le \|x - h\|_{2,\tau}$. The case $h = 0$ is immediate. The Duhamel formula and traciality now give

$$\|j(e^{ih_m}) - e^{ih}\|_{2,X} \le \|j(h_m) - h\|_{2,X}.$$

Hence $e^{ih_m} \in \mathcal{U}_1(C_0)$ proves the asserted density. For a finite family of word tests, apply the estimate in $\mathcal{R}$ with output error $\varepsilon/2$, then approximate its witness closely enough that (6.2) changes each trace by less than $\varepsilon/2$. Its input modulus is unchanged. □

6.2. **A finite uniform perturbation.**

Proposition 6.3. Assume Hypothesis 6.1. Given $u, z \in \mathcal{U}_1(A)$, $\xi \in \widetilde{A}$, and $0 < \alpha < 1/2$, there are $u_\ell \in \mathcal{U}_1(A)$ and $W_\ell \in \mathcal{U}(\mathrm{C}^*(1, C_0, u_\ell))$, for $\ell \ge 1$, such that

$$\|u_\ell - u\| \le \frac{2\sqrt{2}\alpha}{\sqrt{\ell}} \tag{6.3}$$

and

$$\left|\tau(\xi^* W_\ell) - \frac{\alpha}{2}\tau(\xi^* z)\right| \le \frac{4\alpha^2}{1 - 2\alpha}\|\xi\|_{2,\tau} + r_\ell \qquad (\tau \in X), \tag{6.4}$$

where $r_\ell \ge 0$ and $r_\ell \to 0$.

Proof. Fix ℓ. All accuracies used below are finite accuracies, chosen before the unitary letters are constructed.

Formal labels. Let G be the abstract free group on $s_1, \dots, s_\ell$, and set

$$g_j = s_1 \cdots s_j, \qquad p_{kj} = g_k g_j^{-1}, \qquad P = \{e\} \cup \{g_j^{-1} : 1 \le j \le \ell\} \cup \{p_{kj} : 1 \le k, j \le \ell\}.$$

The set P consists of labels, with repetitions removed. We use a second free group F_P, whose free generators are V_p, $p \in P$. The symbol V_e is a free generator, not the identity of

F_P. Moreover, $V_{p^{-1}}$ and V_p^{-1} are different symbols whenever both labels occur; no relation between them is imposed.

The diagonal labels p_{kk} are all e. If $k \neq j$, the reduced word for p_{kj} has k positive letters followed by j negative letters, with no cancellation at the change of sign. Thus all off-diagonal labels p_{kj} are distinct and different from e. The labels g_j^{-1} are also pairwise distinct. Every nonidentity $p^{-1}q$, with $p, q \in P$, has reduced G-length at most 4ℓ.

Write φ for the canonical trace on $C_r^*(F_P)$, and let

$$c = \frac{\alpha}{2\ell}, \qquad A_k = c\sum_{j=1}^{\ell}(V_{p_{kj}} - V_{p_{kj}}^{-1}), \qquad T = c\sum_{k,j=1}^{\ell}(V_{p_{kj}} + V_{p_{kj}}^{-1}).$$

If C_d denotes total degree d in $\prod_{k=1}^{\ell}\exp(A_k)$, then the absolute values of its reduced group coefficients are coefficientwise bounded by the coefficients of T^d. To see this before group reduction, a term in C_d has a nondecreasing sequence of row indices and coefficient $1/(m_1!\cdots m_\ell!) \leq 1$. Every such term occurs in T^d; the latter also contains all other row-index sequences. Taking absolute values and then reducing words preserves this domination.

The label multiplicities just checked give

$$\|T\|_2^2 = c^2\{2\ell^2 + 2\ell(\ell-1)\} = \alpha^2\left(1 - \frac{1}{2\ell}\right) \leq \alpha^2.$$

The length-one case of Haagerup's inequality [20, Lemma 1.4] gives $\|T\| \leq 2\alpha$. Therefore

$$\|C_d\|_{2,\varphi} \leq \|T^d\|_{2,\varphi} \leq (2\alpha)^d. \tag{6.5}$$

Also, the sum of the absolute group coefficients of C_d is at most $(\alpha\ell)^d/d!$.

Choose $0 < \delta \leq \ell^{-1}$, and then choose $N \geq 2$ such that

$$\sum_{d>N}\frac{(\alpha\ell)^d}{d!} \leq \delta, \qquad L_N = \sum_{d=2}^{N}\frac{(\alpha\ell)^d}{d!}. \tag{6.6}$$

Set

$$b_\ell = \frac{\sqrt{2}\alpha}{\sqrt{\ell}}, \qquad c_\ell = 2b_\ell.$$

Choose $k \geq 1$ so that

$$c_\ell^2\, 2^{-2k} \leq \tfrac{1}{2}\ell^{-4}, \qquad D = \max\{2N, 2k\}. \tag{6.7}$$

Finally choose a positive moment tolerance η, as small as needed below. In particular, we require

$$\sqrt{\eta}L_N \leq \delta, \qquad \eta\alpha^{2k}c_\ell^{-2k+2} \leq \tfrac{1}{2}\ell^{-4}. \tag{6.8}$$

The finite tests and their order of construction. We construct $t_0, \ldots, t_\ell \in \mathcal{U}_1(C_0)$, and now evaluate the formal G-words by

$$s_1 = t_0, \qquad s_{j+1} = ut_j\ (1 \leq j < \ell), \qquad a = g_\ell u, \quad W = at_\ell, \quad y = zW^*, \quad Y_p = pyp^{-1}.$$

Let $S = \{p_{kj} : 1 \le k, j \le \ell\}$ and put $M_\xi = \max\{1, \|\xi\|\}$. The final requirements are

$$|\tau(Q(Y))| \le \eta \qquad \text{for every nonidentity reduced } F_P\text{-word } Q \text{ of length} \le D, \tag{6.9}$$

$$|\tau(\xi^* W)| \le M_\xi \eta, \tag{6.10}$$

$$|\tau(\xi^* Y_p W)| \le M_\xi \eta \qquad (p \in S \setminus \{e\}), \tag{6.11}$$

$$|\tau(\xi^* Y_p^{-1} W)| \le M_\xi \eta \qquad (p \in S), \tag{6.12}$$

simultaneously for all $\tau \in X$.

Here are all the centering conditions needed for the final letter t_ℓ. Group a reduced F_P-word into powers $Y_{p_1}^{m_1} \cdots Y_{p_r}^{m_r}$ with successive labels different. Since $y = z t_\ell^{-1} a^{-1}$, expansion into $t_\ell^{\pm 1}$ has opposite-sign gaps only of the forms

$$a^* p_i^{-1} p_{i+1} a \quad \text{or} \quad z^* p_i^{-1} p_{i+1} z. \tag{6.13}$$

Their traces are those of nonidentity G-words of length at most 4ℓ. Outside coefficients are unrestricted in Hypothesis 6.1, so no cyclic end-gap condition is needed. The special tests have the expansions

$$\begin{aligned} \xi^* W &= \xi^* a t_\ell, \\ \xi^* Y_p W &= \xi^* p z \, t_\ell^{-1} [a^* p^{-1} a] t_\ell, \\ \xi^* Y_p^{-1} W &= \xi^* p a \, t_\ell [z^* p^{-1} a] t_\ell. \end{aligned}$$

The first and third are analytic words. In the second, $p \ne e$, so its single opposite-sign gap again has the trace of a nonidentity G-word of length at most 4ℓ. For $p = e$ the omitted test is instead the exact identity $Y_e W = z$.

We specify the earlier tolerances backwards, before choosing any letters. The final test count is at most $\sum_{m=1}^{D} (2|P|)^m + 2\ell^2 + 1$, and each final test has at most $\max\{D, 2\}$ occurrences of $t_\ell^{\pm 1}$. After dividing the special tests by M_ξ, all coefficients are contractions. Choose $\theta_\ell > 0$ below the input-gap modulus from Hypothesis 6.1 for this final list and output error $\eta/2$.

For $r = \ell, \dots, 2$, consider all nonidentity reduced words of length at most $L = 4\ell$ on $s_1, \dots, s_r$. For the words involving s_r, substitute $s_r = u t_{r-1}$ and expand its powers. Every opposite-sign gap is c_0 or $u^* c_0 u$, where c_0 is a nonidentity reduced word on the earlier generators of length at most L. The number of tests, their lengths, and their coefficient norm bounds are bounded solely in terms of r, L. Let $\kappa_r(\theta_r) > 0$ be the corresponding structural input modulus, and choose

$$0 < \theta_{r-1} \le \min\{\theta_r, \kappa_r(\theta_r)\}. \tag{6.14}$$

At the first stage, the tests are the nonzero powers of $s_1 = t_0$ of length at most L; all are analytic and require no centered gaps. Choose t_0 with these traces at most θ_1. Applying the finite-word hypothesis successively constructs the remaining earlier letters and ensures that all nonidentity words on $s_1, \dots, s_r$ of length at most L have trace at most θ_r. Words not involving the new generator retain the preceding bound because $\theta_{r-1} \le \theta_r$. The final application constructs t_ℓ and gives (6.9)–(6.12) with the smaller tolerance $\eta/2$ in place of η. This order of choices uses no coefficient-dependent modulus and no accuracy-independent linear modulus.

All letters already lie in $\mathcal{U}_1(C_0)$, and all evaluated unitary words lie in $\mathcal{U}_1(A)$. In particular, $Y_p - 1 \in A$ and $\mathrm{Im}(Y_p) \in A$ for every label p.

Remainder and first-order correlation. Define

$$h = \frac{\alpha}{\ell} \sum_{j=1}^{\ell} \mathrm{Im}(Y_{g_j^{-1}}), \qquad u' = e^{ih}u, \qquad H_k = g_k h g_k^{-1}.$$

Then $h = h^* \in A$, $\|h\| \le \alpha$, and direct multiplication gives

$$W' = t_0 \prod_{k=1}^{\ell} (u' t_k) = \left(\prod_{k=1}^{\ell} e^{iH_k} \right) W, \qquad iH_k = c \sum_j (Y_{p_{kj}} - Y_{p_{kj}}^{-1}). \tag{6.15}$$

All products are ordered by increasing indices from left to right.

For any group polynomial Q of degree at most N, the moment tests give

$$|\tau(Q(Y)^*Q(Y)) - \varphi(Q(V)^*Q(V))| \le \eta \|Q\|_{\ell^1(F_P)}^2. \tag{6.16}$$

Indeed, every nonidentity group word in Q^*Q has length at most $2N$; identity coefficients contribute no error. Applying this to $R_N = \sum_{d=2}^{N} C_d$, and using (6.5)–(6.8), yields

$$\|R_N(Y)\|_{2,X} \le \frac{4\alpha^2}{1-2\alpha} + \delta.$$

In the actual algebra, each row in (6.15) has norm at most α. Thus (6.6) bounds the operator norm of its total-degree tail by δ. We obtain

$$\left\| \prod_k e^{iH_k} - 1 - c \sum_{k,j} (Y_{p_{kj}} - Y_{p_{kj}}^{-1}) \right\|_{2,X} \le \frac{4\alpha^2}{1-2\alpha} + 2\delta. \tag{6.17}$$

The label e occurs ℓ times in the positive first-order sum, and $Y_e W = z$. All remaining first-order correlations are among (6.11)–(6.12). Therefore, for every $\tau \in X$,

$$\left| \tau(\xi^* W') - \frac{\alpha}{2} \tau(\xi^* z) \right| \le \left(\frac{4\alpha^2}{1-2\alpha} + 2\delta \right) \|\xi\|_{2,\tau} + M_\xi \eta (1 + \alpha\ell). \tag{6.18}$$

Norm-small clipping. The free polynomial

$$h_{\mathrm{f}} = \frac{\alpha}{2i\ell} \sum_j (V_{g_j^{-1}} - V_{g_j^{-1}}^{-1})$$

has 2ℓ distinct length-one terms. Haagerup's inequality gives $\|h_{\mathrm{f}}\| \le b_\ell$. Its coefficient ℓ^1-norm is α, so the moment tests through degree $2k$ imply

$$\tau(h^{2k}) \le \varphi(h_{\mathrm{f}}^{2k}) + \eta\alpha^{2k} \le b_\ell^{2k} + \eta\alpha^{2k}.$$

Let $\widehat{h}$ be the continuous spectral clipping of h to $[-c_\ell, c_\ell]$. The scalar estimate

$$|s - \mathrm{clip}_{c_\ell}(s)|^2 \le c_\ell^{-2k+2} |s|^{2k} \qquad (s \in \mathbb{R})$$

and (6.7)–(6.8) give

$$\|h - \widehat{h}\|_{2,X}^2 \le c_\ell^2 \, 2^{-2k} + \eta\alpha^{2k} c_\ell^{-2k+2} \le \ell^{-4}. \tag{6.19}$$

Put

$$u_\ell = e^{i\widehat{h}} u, \qquad W_\ell = t_0 \prod_{k=1}^{\ell} (u_\ell t_k).$$

The clipping function vanishes at zero, so $\widehat{h} \in A$. These elements belong to $\mathcal{U}_1(A)$ and $\mathcal{U}(\mathrm{C}^*(1, C_0, u_\ell))$, respectively. The norm estimate (6.3) follows from $\|\widehat{h}\| \leq c_\ell$ in A. Since h and $\widehat{h}$ commute, scalar functional calculus gives $\|e^{ih} - e^{i\widehat{h}}\|_{2,X} \leq \ell^{-2}$. Telescoping the ℓ occurrences of the perturbed unitary yields

$$\|W_\ell - W'\|_{2,X} \leq \ell^{-1}.$$

In choosing η before constructing the letters, impose also $M_\xi \eta(1+\alpha\ell) \leq \ell^{-1}$. Equation (6.18) now proves (6.4), for example with $r_\ell = (3\|\xi\| + 1)/\ell$, uniformly over τ. □

6.3. **Hypographs and moving traces.** We give the topological step explicitly, because pointwise Baire arguments in separate trace fibres would not suffice.

Lemma 6.4. Let S be a complete metric space, Y a nonempty compact metric space, and $f : S \times Y \longrightarrow [0, 1]$ an upper semicontinuous function. For $s \in S$ set

$$H(s) = \{(y, t) \in Y \times [0, 1] : 0 \leq t \leq f(s, y)\}.$$

There is a dense G_δ subset of S at every point of which H is continuous in the Hausdorff metric.

Proof. Each $H(s)$ is nonempty and compact. Joint upper semicontinuity of f gives a closed graph for H. Compactness of $Y \times [0, 1]$ then gives upper semicontinuity of the set-valued map H. For every fixed $w \in Y \times [0, 1]$, the function

$$d_w(s) = \operatorname{dist}(w, H(s))$$

is bounded and lower semicontinuous. A bounded lower semicontinuous real-valued function on a complete metric space has a dense G_δ set of continuity points. For completeness, in any nonempty open set choose a point whose value is within ε of the supremum on that set; lower semicontinuity supplies a smaller open set on which the oscillation is at most 2ε. The Baire category theorem, applied for a sequence of ε tending to zero, proves the assertion.

Take a countable dense set of w's and intersect their continuity sets. All functions $w \mapsto \operatorname{dist}(w, H(s))$ are 1-Lipschitz. A finite net in the compact target shows that, at a common continuity point, their convergence is uniform in w. For nonempty compact sets K, L in a metric space,

$$d_{\mathrm{H}}(K, L) = \sup_w |\operatorname{dist}(w, K) - \operatorname{dist}(w, L)|.$$

Thus H is Hausdorff-continuous at every such point. □

Theorem 6.5. Assume Hypothesis 6.1. The set of $u \in \mathcal{U}_1(A)$ for which the unit ball of $\mathrm{C}^*(1, C_0, u)$ is dense in the unit ball of $\widetilde{A}$ for $\| \cdot \|_{2,X}$ contains a dense G_δ subset of $\mathcal{U}_1(A)$, with its norm topology.

Proof. For $z \in \mathcal{U}_1(A)$ define

$$f_z(u, \tau) = \inf_{b \in \mathrm{C}^*(1, C_0, u)} \|z - b\|_{2,\tau}^2. \tag{6.20}$$

This function takes values in $[0, 1]$. Expressing the infimum using all fixed finite $*$-polynomials in u, u^* with coefficients in $\widetilde{C}_0$ shows that f_z is jointly upper semicontinuous on $\mathcal{U}_1(A) \times X$, where the first space has its norm topology and the second its weak-$*$ topology. For each

fixed polynomial, the squared error is jointly continuous, and an infimum of continuous functions is upper semicontinuous.

Both $\mathcal{U}_1(A)$ and X are metrizable; the former is complete for its norm metric and the latter is compact. Apply Lemma 6.4 to f_z. Intersect its dense G_δ sets for a countable norm-dense family $\{z_j\} \subset \mathcal{U}_1(A)$. Fix u in the resulting dense G_δ set.

We claim that $f_z(u,\tau) = 0$ for every test $z = z_j$ and every $\tau \in X$. Suppose instead that $f_z(u,\tau_0) = d^2 > 0$. Choose

$$0 < \alpha < \min\{1/8, d/128\}, \qquad 0 < \varepsilon < \alpha d^2/64.$$

There is a fixed polynomial $b = b(u)$ with coefficients in $\widetilde{C}_0$ such that

$$\|z - b\|_{2,\tau_0}^2 \le d^2 + \varepsilon^2. \tag{6.21}$$

Put $\xi = z - b$. In the GNS von Neumann algebra of τ_0, let $N_0 = \pi_{\tau_0}(\mathrm{C}^*(1, C_0, u))''$ and let $\mathbb{E}_{N_0}$ be the trace-preserving conditional expectation. The distance in (6.20) equals the distance to $L^2(N_0,\tau_0)$. Pythagoras therefore gives

$$\|\xi\|_{2,\tau_0} \le \sqrt{d^2+\varepsilon^2}, \qquad |\tau_0(\xi^* z) - d^2| \le \varepsilon. \tag{6.22}$$

For the second inequality, write $b = \mathbb{E}_{N_0} z + e$, where $\|e\|_{2,\tau_0} \le \varepsilon$, and use $\|\mathbb{E}_{N_0} z\|_{2,\tau_0} \le 1$.

Apply Proposition 6.3 to these fixed u, z, ξ, α. Since $u_\ell \to u$ in norm and u is a Hausdorff-continuity point of the hypograph, the point (τ_0, d^2) in that hypograph has approximants (τ_ℓ, t_ℓ) in the hypographs at u_ℓ satisfying

$$\tau_\ell \longrightarrow \tau_0, \qquad f_z(u_\ell,\tau_\ell) \ge t_\ell \longrightarrow d^2. \tag{6.23}$$

Joint upper semicontinuity also gives $\limsup_\ell f_z(u_\ell,\tau_\ell) \le d^2$, so these defects converge to d^2. Define $b_\ell = b(u_\ell)$ using the same polynomial. Then $b_\ell \to b$ in norm. In the GNS algebra of τ_ℓ, put $N_\ell = \pi_{\tau_\ell}(\mathrm{C}^*(1, C_0, u_\ell))''$. Pythagoras and (6.21)–(6.23) give

$$\begin{aligned} \|\mathbb{E}_{N_\ell} z - b_\ell\|_{2,\tau_\ell}^2 &= \|z - b_\ell\|_{2,\tau_\ell}^2 - f_z(u_\ell,\tau_\ell) \\ &\le \varepsilon^2 + o(1). \end{aligned}$$

The first term on the right converges because $b_\ell \to b$ in norm and $\tau_\ell \to \tau_0$ weak-$*$. Furthermore,

$$\mathbb{E}_{N_\ell}\xi = (\mathbb{E}_{N_\ell} z - b_\ell) + \mathbb{E}_{N_\ell}(b_\ell - b),$$

so contractivity gives $\|\mathbb{E}_{N_\ell}\xi\|_{2,\tau_\ell} \le \varepsilon + o(1)$. Since $W_\ell \in N_\ell$ is unitary,

$$|\tau_\ell(\xi^* W_\ell)| \le \varepsilon + o(1). \tag{6.24}$$

On the other hand, the *uniform* estimate (6.4) can be evaluated at these moving traces. Using (6.22) gives

$$\begin{aligned} \liminf_\ell |\tau_\ell(\xi^* W_\ell)| &\ge \frac{\alpha}{2}(d^2 - \varepsilon) - \frac{4\alpha^2}{1-2\alpha}\sqrt{d^2+\varepsilon^2} \\ &> \frac{\alpha d^2}{8} > \varepsilon. \end{aligned}$$

For the displayed numerical bound, use $4\alpha^2/(1-2\alpha) \le 8\alpha^2$, $\sqrt{d^2+\varepsilon^2} \le 2d$, and the chosen bounds on α, ε. This contradicts (6.24). Thus the claim holds.

Norm density of the test unitaries extends this conclusion to every element of $\mathcal{U}_1(A)$. Every unitary of $\widetilde{A}$ is a scalar multiple of one in $\mathcal{U}_1(A)$: in the nonunital case, divide by its scalar image in $\widetilde{A}/A$. Since every element of a unital C*-algebra is a linear combination of unitaries, for every $a \in \widetilde{A}$ we have

$$\inf_{b \in \mathrm{C}^*(1, C_0, u)} \|a - b\|_{2,\tau}^2 = 0 \qquad (\tau \in X). \tag{6.25}$$

Let $C = \mathrm{C}^*(1, C_0, u)$ and regard C as a real locally convex vector space with its norm topology. The function $(\tau, b) \mapsto \|a - b\|_{2,\tau}^2$ is continuous and affine in τ and continuous and convex in b. Sion's minimax theorem [37, Theorem 4.2], with the compact convex variable X, therefore gives

$$\inf_{b \in C} \sup_{\tau \in X} \|a - b\|_{2,\tau}^2 = \sup_{\tau \in X} \inf_{b \in C} \|a - b\|_{2,\tau}^2 = 0. \tag{6.26}$$

No compactness of the approximating space C is required.

For clarity, bounded approximation follows from (6.26); it is not an additional assumption. If $\|a\| \leq 1$ and $b \in C$, set $\widehat{b} = bf(|b|)$, where $f(t) = 1$ for $0 \leq t \leq 1$ and $f(t) = 1/t$ for $t \geq 1$. Then $\widehat{b} \in C$ is a contraction. For each trace, write $b = v|b|$ in the corresponding finite von Neumann algebra and put $P = (|b| - 1)_+$. Since $b - \widehat{b} = vP$,

$$\operatorname{Re} \tau((b - \widehat{b})^*(\widehat{b} - a)) = \tau(P) - \operatorname{Re} \tau(Pv^*a) \geq 0.$$

Expanding $\|b - a\|_{2,\tau}^2$ now gives $\|\widehat{b} - a\|_{2,\tau} \leq \|b - a\|_{2,\tau}$. Thus the contraction ball of C is dense in that of $\widetilde{A}$ in the uniform 2-norm, as asserted. □

Corollary 6.6 (Removing the adjoined unit)**.** Assume Hypothesis 6.1, and assume that the unit of $\mathcal{M}$ belongs to the uniform 2-norm closure of $j(C_0)$. There is $h = h^* \in A$ such that the unit ball of the possibly nonunital algebra $C = \mathrm{C}^*(C_0, h) \subseteq A$ is dense in the unit ball of A for $\|\cdot\|_{2,X}$. If C_0 is generated by m self-adjoint elements, then C is generated by $m + 1$ self-adjoint elements.

Proof. Choose a unitary u from the dense set in Theorem 6.5 such that $\|u - 1\| < 1$. Continuous functional calculus with the principal logarithm gives $h = -i \log u = h^* \in A$, because the logarithm vanishes at 1. Moreover,

$$\mathrm{C}^*(1, C_0, u) = \mathrm{C}^*(1, C_0, h) = \mathrm{C}^*(1, C).$$

The uniform 2-norm closure of the linear space $j(C)$ contains $1_{\mathcal{M}}$ by the hypothesis on C_0. It therefore contains $j(C + \mathbb{C}1)$, and hence $j(\mathrm{C}^*(1, C))$, since j is contractive for operator norms. The theorem now implies that every $a \in A$ is a uniform 2-norm limit of elements of C. For a contraction a, clip each approximant $b \in C$ to $bf(|b|)$ using the function in the proof of the theorem. Although $f(0) = 1$, the product belongs to C, because $f(|b|) \in C + \mathbb{C}1$ and $b \in C$. The same trace inequality shows that clipping decreases the error. This gives the stated density of unit balls. The generator count is immediate from the definition of C. □

Remark 6.7 (Dependency boundary). The only structural input in this section is the finite-word estimate with actual witnesses in $\mathcal{U}_1(C_0)$. An external AF algebra inside the uniform tracial closure of $j(C_0)$ can supply those witnesses by the transfer argument above. If that AF algebra has unit 1, the extra unit-capture hypothesis in Corollary 6.6 holds automatically.

The proof uses neither the topology of $\partial\mathrm{T}(A)$ nor nuclearity. The passages from this tracially dense algebra to norm generation and then to single generation require separate compression and encoding arguments.

7. A uniform finite generator bound

Proposition 7.1 (Seven selfadjoint generators from an external AF core)**.** *Let D be separable and σ-unital with compact nonempty $\mathrm{T}(D)$, and assume comparison by these tracial states for positive elements of D. Suppose its bounded uniform tracial completion $\mathcal{M}_D$ is unital and has a unital AF subalgebra B with the following two properties:*

(i) *B has two selfadjoint generators, without adjoining its unit.*
(ii) *The conclusion of Proposition 4.9 holds with coefficients in $\mathcal{M}_D$, witnesses in the uniform 2-closure of B, and designated traces $\mathrm{T}(D)$, with moduli independent of coefficient values.*

Then D has at most seven selfadjoint norm generators.

Proof. The canonical map $D \to \mathcal{M}_D$ need not be injective. Whenever a closure in $\mathcal{M}_D$ is mentioned below, it is the closure of the canonical image. All generators and comparison witnesses are actual elements of D.

Scale the two selfadjoint AF generators to contractions. By Proposition 5.2, a subalgebra $C_0 \subseteq D$ generated by four selfadjoint elements has bounded uniform 2-closure containing B. It also contains the external unit in its bounded uniform 2-closure, because the AF generators generate that unit without adjoining it. Concretely, evaluate nonunital $*$-polynomials in bounded approximants and then apply the radial clipping $b \mapsto bf(|b|)$ from the proof of Corollary 6.6. This preserves C_0 and does not increase the uniform 2-distance to a contraction.

By Lemma 6.2, finite-word witnesses in the completion of B transfer to exact unitaries in $1 + C_0$. Thus Hypothesis 6.1 holds for this fixed $C_0 \subseteq D$.

Apply Theorem 6.5 to the complete norm-metric group $\mathcal{U}(1+D)$. The perturbation stays in this group: its selfadjoint perturbing term is a real linear combination of $(Y - Y^*)/i$ with $Y \in 1 + D$, so lies in D, and the clipping function vanishes at zero. The good unitaries are norm dense; choose one, u, with $\|u - 1\| < 1$. Its principal logarithm $z = -i \log u$ belongs to D_{sa}, and $\mathrm{C}^*(1, u) = \mathrm{C}^*(1, z)$ in $\widetilde{D}$.

The relative-generation conclusion gives uniform 2-density of $\mathrm{C}^*(1, C_0, u)$ in $\mathcal{M}_D$. Its adjoined unit is itself a bounded uniform 2-limit from C_0. Hence, by Corollary 6.6, $C = \mathrm{C}^*(C_0, z) \subseteq D$ has selfadjoint contraction ball uniformly 2-dense in that of D. It has five selfadjoint generators. Clipping here does not adjoin a unit, because clipping to $[-1, 1]$ vanishes at zero.

Choose a norm-dense sequence (a_j) in the selfadjoint unit ball of D, and put $\epsilon_k = 2^{-k}$. Give each sign of each pair (j, k) the budget

$$\alpha^{\pm}_{j,k} = 2^{-(j+k+4)}, \qquad \sum_{j,k \geq 1,\, \sigma \in \{+,-\}} \alpha^{\sigma}_{j,k} = \tfrac{1}{8}.$$

By the selfadjoint contraction-ball density just proved, choose $c_{j,k} \in C_{\mathrm{sa}}$ with $\|c_{j,k}\| \leq 1$ so that, for $x_{j,k} = a_j - c_{j,k}$,

$$4\epsilon_k^{-2}\|x_{j,k}\|_{2,\mathrm{T}(D)}^2 < \min\{\alpha_{j,k}^+, \alpha_{j,k}^-\}.$$

The positive cuts $a_{j,k}^{\pm} = (\pm x_{j,k} - \epsilon_k)_+$ have norm at most 2. Apply to $\pm x_{j,k}$ the same mask functions as in Proposition 5.2: the outer mask is zero on $(-\infty, \epsilon_k/2]$ and one on $[3\epsilon_k/4, \infty)$, and the inner mask is zero on $(-\infty, 3\epsilon_k/4]$ and one on $[\epsilon_k, \infty)$. These masks lie in D, satisfy the two exact mask identities, and their outer support ranks are less than $\alpha_{j,k}^{\pm}$ by spectral Markov inequality. Thus a single application of Proposition 5.1 gives two selfadjoint elements $H, K \in D$ and positive elements $E_{j,k}^{\pm} \in \mathrm{C}^*(H, K)$ with $\|E_{j,k}^{\pm} - a_{j,k}^{\pm}\| < \epsilon_k$. The scalar identity

$$\|x - (x - \epsilon_k)_+ + (-x - \epsilon_k)_+\| \leq \epsilon_k \qquad (x = x^*)$$

shows, in the actual operator norm of D, that

$$\|a_j - (c_{j,k} + E_{j,k}^+ - E_{j,k}^-)\| < 3\epsilon_k.$$

Letting $k \to \infty$ gives $D = \mathrm{C}^*(C, H, K)$, which proves the asserted seven-selfadjoint bound. In particular, this last step also encodes elements annihilated by every designated trace; no injectivity of $D \to \mathcal{M}_D$ is required. □

8. Compression using a matrix cone and a reserved matrix position

We use $\mathrm{Her}_A(a) = \overline{aAa}$ for $a \in A_+$. All C*-algebras generated in this section are generated without automatically adjoining a unit. A positive invertible element will recover the ambient unit explicitly.

Lemma 8.1 (Full hereditary bimodules)**.** *Let A be a C*-algebra, let $B, E \subseteq A$ be nonzero hereditary sub-C*-algebras, and put*

$$X = \overline{\mathrm{span}}\, BAE.$$

If $v \in X$ and v^*v is full in E, then

$$X = \overline{\mathrm{span}}\, BvE.$$

Consequently, if $v \in A$ satisfies $vv^* \in B$ and v^*v is full in E, then $\mathrm{C}^*(B, E, v)$ contains $\overline{\mathrm{span}}BAE$ and $\overline{\mathrm{span}}EAB$.

Proof. The closed linear span

$$J = \overline{\mathrm{span}}\, Ev^*BvE$$

is a closed two-sided ideal of E. An approximate unit (b_λ) of B satisfies $b_\lambda v \to v$ in norm. Multiplying $v^*b_\lambda v$ on both sides by an approximate unit of E shows that $v^*b_\lambda v \in J$, and hence $v^*v \in J$. Fullness of v^*v in E gives $J = E$. For $x \in X$, one has $xEv^*B \subseteq B$: this follows by approximating x and v by finite sums in BAE and using $BAB \subseteq B$. Hence

$$x(Ev^*BvE) \subseteq BvE.$$

The right action of E on X is nondegenerate. Approximating a right approximate unit from $J = E$ therefore gives $x \in \overline{\mathrm{span}}BvE$. The reverse inclusion is immediate. For the last assertion, approximate units of B and E show that $v \in \overline{BAE}$ whenever $vv^* \in B$ and $v^*v \in E$; then apply the first assertion and take adjoints. □

Proposition 8.2 (Compression by a matrix cone)**.** *Let A be a unital* C*-*algebra with* $\mathrm{T}(A) \neq \varnothing$*. Assume comparison by ordinary tracial states for positive elements of A:*

$$\mathrm{d}_\tau(a) < \mathrm{d}_\tau(b) \quad (\tau \in \mathrm{T}(A)) \quad \Longrightarrow \quad a \precsim b.$$

Let $N \geq 1$ and $n \geq 3$ satisfy $n - 1 \geq N$. Suppose there is a completely positive contractive order-zero map

$$\phi\colon \mathrm{M}_n \longrightarrow A, \qquad t = \phi(1_n),$$

such that, for some $0 < \varepsilon < 1/(64n)$,

$$\tau(t) > 1 - \varepsilon \qquad (\tau \in \mathrm{T}(A)).$$

Put $\delta = 1/16$ and suppose that

$$D = \mathrm{Her}_A\Big((\phi(e_{11}) - \delta)_+\Big)$$

is generated, without adjoining a unit, by at most N selfadjoint elements. Then A is singly generated.

Proof. Write $H_t = \mathrm{Her}_A(t)$. The supporting $*$-homomorphism of ϕ from [48, Theorem 2.3] gives matrix units, denoted again by e_{ij}, in the multiplier algebra of H_t, with

$$\phi(e_{ij}) = e_{ij}t = te_{ij}.$$

Their sum $p = \sum_i e_{ii}$ is the support of t in A^{**}. For completeness, they act as multipliers on H_t: on the dense subspace tAt, multiplication on either side stays in H_t, since $e_{ij}t = \phi(e_{ij}) \in H_t$. These matrix units need not belong to A. Whenever $f \in \mathrm{C}([0,1])$ satisfies $f(0) = 0$, however,

$$e_{ij}f(t) \in H_t \subseteq A. \tag{8.1}$$

For example, this follows first when $f(s) = sg(s)$, and then by uniform approximation for arbitrary continuous f vanishing at zero. The matrix units also act as multipliers on $H_\delta = \mathrm{Her}_A((t-\delta)_+)$. Thus

$$H_\delta \cong \mathrm{M}_n(D), \qquad D = e_{11}H_\delta e_{11},$$

with the isomorphism given by $(x_{ij}) \mapsto \sum_{ij} e_{i1}x_{ij}e_{1j}$. Every such expression belongs to A, since its entries belong to H_δ and the e_{ij} are multipliers of that algebra.

Put $r_0 = n - 1$, $\gamma = 1/4$, and

$$d = (t-\delta)_+, \qquad d_i = e_{ii}d, \qquad w = e_{1n}(t-\gamma)_+.$$

$$B_0 = \sum_{i,j<n} e_{i1}De_{1j} \cong \mathrm{M}_{r_0}(D).$$

Here B_0 is a hereditary subalgebra of A, and the displayed finite matrix sum is closed. Let

$$E_0 = \mathrm{Her}_A\Big(e_{nn}(t-\gamma)_+\Big).$$

We claim that

$$B := \mathrm{C}^*(B_0, w) = \mathrm{Her}_A\left(\sum_{i<n} d_i + e_{nn}(t-\gamma)_+\right). \tag{8.2}$$

Indeed, $ww^* = e_{11}(t-\gamma)_+^2 \in B_0$, and

$$\overline{w^*B_0w} = \mathrm{Her}_A(w^*w) = E_0.$$

The first equality follows because B_0 is hereditary and contains the hereditary algebra generated by ww^*. The standard identity

$$\overline{w^* \operatorname{Her}(ww^*)w} = \operatorname{Her}(w^*w)$$

follows directly from functional calculus. Thus $E_0 \subseteq \mathrm{C}^*(B_0, w)$. Both B_0 and E_0 are nonzero: $\tau(t) > 1 - \varepsilon > \gamma$ forces $(t-\gamma)_+ \neq 0$, and the supporting matrix units make all its diagonal slices simultaneously nonzero. Moreover, w^*w is strictly positive in E_0. Lemma 8.1 recovers their two cross bimodules from w. Their support projections are orthogonal, and the hereditary algebra on the sum of those supports is the closed span of the two diagonal algebras and the two cross bimodules. This proves (8.2).

Choose continuous functions $g_0, g_1 \colon [0,1] \to [0,1]$ such that

	zero on	one on
g_0	$[0, \delta]$	$[2\delta, 1]$
g_1	$[0, \gamma]$	$[2\gamma, 1]$.

Set

$$s = \sum_{i<n} e_{ii} g_0(t) + e_{nn} g_1(t) \in B, \qquad r = 1 - s,$$
$$b = e_{nn}(t - 3/4)_+ \in B.$$

All the terms belong to A by (8.1); membership in B follows from (8.2). The element s is a positive contraction. In every tracial GNS closure, r vanishes on the spectral subspace where $t \geq 1/2$. Hence

$$\mathrm{d}_\tau(r) \leq \tau(1_{[0,1/2)}(t)) \leq 2\tau(1-t) < 2\varepsilon. \tag{8.3}$$

Traciality and the supporting matrix units give

$$n\mathrm{d}_\tau(b) = \tau(1_{(3/4,1]}(t)) \geq \tau(t) - \tfrac{3}{4} > \tfrac{1}{4} - \varepsilon.$$

One can justify the first equality using continuous cutoff functions vanishing at zero and then monotone convergence in the finite GNS algebra. Our choice of ε implies

$$2\varepsilon < \frac{1/4 - \varepsilon}{n}.$$

Comparison therefore gives $r \precsim b$.

Use the elementary exact-cutdown consequence of Cuntz subequivalence: if $a \precsim b$ and $\eta > 0$, then there is $v \in A$ with $v^*v = (a-\eta)_+$ and $vv^* \in \operatorname{Her}_A(b)$. Indeed, choose $x \in A$ with $\|a - x^*bx\| < \eta$. The norm-perturbation cutdown factorization gives $y \in A$ such that $(a-\eta)_+ = y^*x^*bxy$; put $v = b^{1/2}xy$. Although the intermediate factors need not have a uniform norm bound, this final witness satisfies $\|v\|^2 = \|(a-\eta)_+\|$. With $\eta = 1/4$ we obtain

$$v^*v = q := (r - \tfrac{1}{4})_+, \qquad vv^* \in \operatorname{Her}_A(b) \subseteq B. \tag{8.4}$$

Notice that $br = 0$, hence $bq = 0$. The initial and final support projections of v are orthogonal, so $v^2 = 0$. Also

$$s + q = s + (1 - s - \tfrac{1}{4})_+ \geq \tfrac{3}{4}\, 1_A. \tag{8.5}$$

Choose selfadjoint contraction generators $x_1, \ldots, x_N$ of D, rescaling and padding with zero elements if necessary. Choose distinct numbers $\lambda_1, \ldots, \lambda_n$ in $[1,2]$ and a continuous

function $F\colon [0,1] \to [0,1]$ which is zero near zero and one on $[\delta, 1]$. Define

$$L = \sum_{i=1}^{n} \lambda_i e_{ii} F(t), \qquad h = 1_A + L,$$

and

$$k = \sum_{i=1}^{N} e_{i1} x_i e_{1i} + \sum_{i=1}^{n-2} \Big(e_{i,i+1} d + e_{i+1,i} d \Big) + (w + w^*) + (v + v^*).$$

These are selfadjoint elements of A. In particular, $e_{ij} d \in A$ by (8.1), and the payload terms belong to H_δ by the matrix identification above. Put $C = \mathrm{C}^*(h, k)$. Since $h \geq 1_A$, continuous functional calculus with a function vanishing at zero and equal to one on $[1, \|h\|]$ gives $1_A \in C$. The construction also has bounds independent of n and N:

$$1_A \leq h \leq 3\, 1_A, \qquad \|k\| \leq 1 + 2\|d\| + \|w\| + \|v\| \leq 1 + \tfrac{15}{8} + \tfrac{3}{4} + \sqrt{\tfrac{3}{4}} < 5.$$

Here the diagonal payload has norm at most one, the consecutive matrix edges have norm at most $2\|d\|$, and the orthogonal initial and final supports give $\|w + w^*\| = \|w\|$ and $\|v + v^*\| = \|v\|$.

Let $\mu_i = 1 + \lambda_i$, and choose real Lagrange polynomials ℓ_i satisfying $\ell_i(\mu_j) = \delta_{ij}$. Although the spectrum of h may contain intervals between these labels, its action on the hereditary carrier has exact labels: for $x \in e_{i1} D e_{1j}$,

$$hx = \mu_i x, \qquad xh = \mu_j x. \tag{8.6}$$

The same identities hold on the relevant corners involving w, since its support lies where $t > \gamma > \delta$. Moreover, the final range condition in (8.4) implies

$$hv = \mu_n v, \qquad v^* h = \mu_n v^*.$$

For example, approximate v on the left by an approximate unit of $\mathrm{Her}_A(b)$ and use that h has the constant label μ_n on this hereditary algebra.

It follows that for $i, j < n$ the cuts $\ell_i(h) k \ell_j(h)$ kill both v and v^*: the former is killed on the left, and the latter on the right. They also kill w and w^*, since w has right label μ_n and w^* has left label μ_n. Consequently C contains

$$X_i = e_{i1} x_i e_{1i} \quad (1 \leq i \leq N), \qquad z_i = e_{i,i+1} d \quad (1 \leq i \leq n-2).$$

The roots of $z_i z_i^*$ and $z_i^* z_i$ give d_i and d_{i+1}, so all d_i, $i < n$, belong to C. For $m \geq 1$ put $f_m(u) = u/(u + m^{-1})$ and

$$z_{i,m} = z_i (d_{i+1} + m^{-1} 1_A)^{-1} = e_{i,i+1} f_m(d) \in C.$$

For $2 \leq i < n$ their products give

$$R_{i,m} = z_{1,m} \cdots z_{i-1,m} = e_{1i} f_m(d)^{i-1} \in C.$$

Since d_1 is strictly positive in D,

$$R_{i,m} X_i R_{i,m}^* \longrightarrow x_i \quad \text{in norm} \qquad (2 \leq i \leq N).$$

Together with $X_1 = x_1$, this shows $D \subseteq C$ in the first corner. Set $R_{1,m} = f_m(d_1)$. For every $x \in D$ and $i, j < n$, the approximate-unit property then gives

$$R_{i,m}^* x R_{j,m} \longrightarrow e_{i1} x e_{1j} \quad \text{in norm.}$$

Thus

$$B_0 \subseteq C. \tag{8.7}$$

Choose $a_0 \colon [0,1] \to [0,1]$ continuous, zero on $[0, 2\delta]$, and one on $[\gamma, 1]$, and put $a = e_{11} a_0(t)$. This is a continuous function, vanishing at zero, of d_1; hence $a \in B_0 \subseteq C$. On the first matrix position, q can be nonzero only where $g_0(t) < 3/4$, which is contained in $\{t < 2\delta\}$. Therefore $aq = 0$, while $aw = w$. In particular,

$$\|av^*\|^2 = \|av^* va\| = \|aqa\| = 0.$$

Using the exact labels, $a\ell_1(h) = a$, and these two identities, we obtain

$$a\ell_1(h) k \ell_n(h) = w. \tag{8.8}$$

Indeed, the core terms contribute only w; the v term is killed by the left label, and the v^* term is killed by a. Equations (8.7) and (8.8) give $B = \mathrm{C}^*(B_0, w) \subseteq C$. Subtracting the now-recovered core terms from k yields $v + v^* \in C$.

Let (u_m) be the functional-calculus approximate unit of the hereditary algebra generated by b. Then $u_m \in B \subseteq C$, $u_m v \to v$ in norm by $vv^* \in \mathrm{Her}_A(b)$, and $u_m v^* = 0$ because $bq = 0$. Hence

$$u_m(v + v^*) \longrightarrow v \quad \text{in norm,}$$

and $v \in C$.

It remains to verify that B and v generate A; the argument also records why no support projection must be recovered in norm. If $q = 0$, (8.5) makes s an invertible positive element of the hereditary algebra B, so $B = A$. Suppose $q \neq 0$, and put $E = \mathrm{Her}_A(q)$. The identity

$$\overline{v^* \,\mathrm{Her}_A(vv^*) v} = \mathrm{Her}_A(v^* v) = E$$

shows $E \subseteq \mathrm{C}^*(B, v)$. Since $v^* v = q$ is strictly positive in E, Lemma 8.1 shows that this algebra also contains $\overline{BAE}$ and $\overline{EAB}$. Put $z = s + q$. By (8.5), z is invertible. Heredity gives

$$zAz \subseteq B + \overline{BAE} + \overline{EAB} + E \subseteq \mathrm{C}^*(B, v).$$

Functional calculus, using functions vanishing at zero, gives $1_A, z^{-1} \in \mathrm{C}^*(B, v)$, and thus

$$A = z^{-1}(zAz)z^{-1} \subseteq \mathrm{C}^*(B, v).$$

Therefore $A = \mathrm{C}^*(B, v) \subseteq C = \mathrm{C}^*(h, k)$. The reverse inclusion is automatic, so $A = \mathrm{C}^*(h, k) = \mathrm{C}^*(h + ik)$. □

Remark 8.3 (Scope of the compression proposition)**.** The proposition uses actual elements of A throughout its encoder. The supporting matrix units are used only as multiplier notation; every occurrence in a generator or a recovered element is multiplied by an element of an appropriate hereditary algebra. No support projection is claimed to lie in A. The remaining input for a global application is a uniform finite selfadjoint generator bound for the selected hereditary coefficient algebra D.

9. Proof of the single-generation theorem

We apply the preceding construction to a matrix cone in a unital copy of the Jiang–Su algebra.

Theorem 9.1. *Let A be a separable unital* C*-*algebra with* $\mathrm{T}(A) \neq \varnothing$ *and strict comparison by ordinary tracial states. If $\mathcal{Z}$ embeds unitally in A, then A is singly generated.*

Proof of Theorems 1.2 and 9.1. Fix a unital copy of $\mathcal{Z}$ in A. Choose $n = 8$ and $0 < \varepsilon < 1/(64n)$. By Proposition 2.5, there is a c.p.c. order-zero map $\phi : \mathrm{M}_n \to \mathcal{Z} \subseteq A$ such that, writing $t = \phi(1_n)$,

$$\tau(t) > 1 - \varepsilon \qquad (\tau \in \mathrm{T}(A)).$$

Set $\delta = 1/16$ and

$$d = (\phi(e_{11}) - \delta)_+, \qquad D = \mathrm{Her}_A(d).$$

The element d is nonzero and full in A, and has the same strictly positive rank c_δ at every trace. Lemma 2.7 gives a compact nonempty normalized trace space for D, affinely identified with $\mathrm{T}(A)$. Comparison of positive elements in D agrees with comparison in A, so D inherits comparison by these tracial states.

Proposition 2.5 gives almost divisibility of the strictly positive class $[d] \in \mathrm{Cu}(D)$. Corollary 2.9 shows that the normalized uniform tracial completion of D is regular, factorial, and of type II_1. Its designated traces form a metrizable Choquet simplex by the trace identification above, and its unit ball is separable in the uniform 2-norm because D is norm separable.

Apply Theorem 3.4 to this completion. It supplies an external unital simple AF algebra B with the required common-centre and averaging properties. Proposition 4.9 supplies finite weakened-word estimates for coefficients in that completion. The algebra B has two selfadjoint generators [40, Theorem 7.3]. Shifting one of them by a sufficiently large scalar multiple of the unit makes it positive and invertible; functional calculus then recovers the unit without adjoining it. Proposition 7.1 therefore gives at most seven selfadjoint norm generators for D.

Proposition 8.2 now applies with $N = 7$ and $n = 8$. Its coefficient algebra is precisely the D chosen above, and $n - 1 = N$. Thus there are $h, k \in A_{\mathrm{sa}}$ such that $A = \mathrm{C}^*(h + ik)$. □

Corollary 9.2. *Let A be a separable unital pure* C*-*algebra of stable rank one. Suppose that every lower semicontinuous* 2*-quasitrace on A is an ordinary trace. Then A is singly generated.*

Proof. The zero algebra is immediate, so suppose $A \neq 0$. Lemma 2.3 gives $\mathrm{T}(A) \neq \varnothing$ and strict comparison by ordinary tracial states. Pureness gives almost unperforation of $\mathrm{Cu}(A)$ and almost divisibility of $[1_A]$. Proposition 2.4 gives a unital embedding of $\mathcal{Z}$ into A. Apply Theorem 9.1. □

Corollary 9.3. *Let A be a simple separable unital* C*-*algebra with* $\mathrm{T}(A) \neq \varnothing$ *and strict comparison by ordinary tracial states. Then A is singly generated.*

Proof. If A is finite dimensional, it is isomorphic to M_m. A diagonal matrix with distinct positive diagonal entries and the selfadjoint sum of adjacent matrix units generate M_m; for $m = 1$ the unit alone suffices.

Suppose that A is infinite dimensional. Lemma 2.2 gives strict comparison by the full ordinary trace cone, which is the convention in [42, Corollary B]. That result makes A pure. Its tracial states are faithful, so A is not purely infinite. Lin's stable rank theorem [28, Corollary 4.5] gives stable rank one. Proposition 2.4 supplies a unital copy of $\mathcal{Z}$, and Theorem 9.1 applies. □

Corollary 9.4. *Let A be a separable unital nowhere scattered C^*-algebra of stable rank one. If A has strict comparison by the full cone of lower semicontinuous ordinary traces, then A is singly generated.*

Proof. Ng–Robert's comparison theorem [30, Theorem 3.6 and Remark 3.7] gives strict comparison by 2-quasitraces and shows that every lower semicontinuous 2-quasitrace is an ordinary trace. Strict comparison implies controlled comparison, so [5, Theorem 5.12] shows that A is pure. Corollary 9.2 applies. □

Remark 9.5. Corollary 9.3 also recovers the finite-factor conclusion of [32]. Indeed, let M be a II_1 factor with separable predual and normalized trace τ. A countable witness construction gives a weakly dense separable simple unital sub-C^*-algebra $A \subseteq M$ with real rank zero, unique trace $\tau|_A$, and strict comparison by ordinary traces.

Here are the ingredients of that construction. Start with 1 and a countable 2-norm-dense subset of the unit ball of M. At each stage, adjoin finite-spectrum norm approximants, unitaries witnessing norm Dixmier averaging [2, Theorem 6.4.1 and Corollary 6.4.2], and bounded approximate minimizers for

$$F_R(a,b) = \inf_{\substack{x \in \mathrm{M}_m(M) \\ \|x\| \le R}} \|a - x^* b x\|.$$

For these choices use every accuracy $1/k$, $k \geq 1$. For the last step take positive pairs from countable norm-dense lists at every matrix level and integers $R \geq 1$. Adjoin the matrix entries of the witnesses and repeat these choices for norm-dense lists in each enlarged algebra. The estimate

$$|F_R(a,b) - F_R(a',b')| \leq \|a - a'\| + R^2 \|b - b'\|$$

ensures that, in the resulting algebra A, the infimum restricted to $\mathrm{M}_m(A)$ equals the infimum in $\mathrm{M}_m(M)$. For positive $a, b \in \mathrm{M}_m(A)$ with $\mathrm{d}_\tau(a) < \mathrm{d}_\tau(b)$, choose $\delta > 0$ so that $\mathrm{d}_\tau(a) < \mathrm{d}_\tau((b-\delta)_+)$. Put $p_\delta = 1_{(\delta,\infty)}(b)$. Projection comparison in the factor gives a partial isometry w with w^*w equal to the support of a and $ww^* \leq p_\delta$. Then $x = (b|_{p_\delta})^{-1/2} w a^{1/2} \in \mathrm{M}_m(M)$ is bounded and satisfies $x^* b x = a$. Choose an integer $R > \|x\|$. Equality of the two F_R infima gives $a \precsim b$ in $\mathrm{M}_m(A)$. For stabilization, put $p_j = 1_M \otimes \sum_{i=1}^{j} e_{ii}$. For $a, b \in (A \otimes \mathcal{K})_+$ with $\mathrm{d}_\tau(a) < \mathrm{d}_\tau(b)$, one has

$$\mathrm{d}_\tau(p_j a p_j) \leq \mathrm{d}_\tau(a), \qquad \mathrm{d}_\tau(p_j b p_j) \uparrow \mathrm{d}_\tau(b).$$

The second assertion follows by replacing $p_j b p_j$ with the Cuntz-equivalent increasing sequence $b^{1/2} p_j b^{1/2}$ and using normality of the matrix-amplified trace. Choose l with $\mathrm{d}_\tau(p_l b p_l) > \mathrm{d}_\tau(a)$. Comparison at finite matrix levels, after padding with zero blocks, gives $p_j a p_j \precsim p_l b p_l \precsim b$ for every j. Since $p_j a p_j \to a$ in norm, this proves $a \precsim b$. The spectral approximants give real rank zero. Dixmier averaging gives both uniqueness of the trace and simplicity: every nonzero $a \in A_+$ averages in norm to the nonzero scalar $\tau(a)1$.

Corollary 9.3 now gives $A = \mathrm{C}^*(g)$ and therefore $M = \mathrm{W}^*(g)$. This observation uses the C*-algebra theorem directly, while the relative-generation argument in its proof adapts the method of [32].

Remark 9.6. In the nonsimple setting, pointwise comparison by normalized tracial states and uniform-ratio comparison by the full lower semicontinuous trace cone must be distinguished. Theorem 9.1 assumes the former. Corollary 9.2 obtains it from the latter using stable rank one and the existence of traces on nonzero unital quotients, as proved in Lemma 2.3. No faithfulness assumption is imposed on the individual tracial states in these nonsimple results.

10. Dense generators in real rank zero

We first remove simplicity from the spectral-refinement argument. The point is to choose a connecting element that generates the whole rectangular bimodule, rather than merely a nonzero connecting element. As before, $\mathrm{C}^*(S)$ denotes generation without automatically adjoining a unit.

Lemma 10.1 (Full rectangular bridges)**.** *Let A be a separable unital* C*-*algebra of real rank zero, and let $p, q \in A$ be projections. For every $x \in pAq$ and $\varepsilon > 0$, there is $z \in pAq$ such that*

$$\|z - x\| < \varepsilon, \qquad pAq = \overline{\operatorname{span}}\,(pAp)z(qAq).$$

The assertion includes the case $pAq = 0$, when $z = 0$.

Proof. We first check that closed-range elements are dense in every rectangular corner PAQ, where P, Q are projections in A. Fix $u \in PAQ$. Given small numbers $0 < \kappa < \delta$, choose a positive finite-spectrum element $d \in QAQ$ with $\|d - u^*u\| < \kappa$. This is possible because hereditary subalgebras have real rank zero [8]. The spectral projection $e = 1_{(\delta,\infty)}(d)$ belongs to QAQ, and

$$eu^*ue \geq (\delta - \kappa)e, \qquad \|u - ue\|^2 = \|(Q - e)u^*u(Q - e)\| \leq \delta + \kappa.$$

If $e = 0$, put $y = 0$ and $v = 0$. Otherwise set

$$y = ue, \qquad v = ue(eu^*ue)^{-1/2},$$

where the inverse is taken in the unital corner eAe. Then $y = v|y|$, $v^*v = e$, and $f = vv^* \leq P$ is a projection. Moreover, $|y|$ is invertible in eAe. Letting δ, κ tend to zero proves the asserted density. We shall use this construction also when one of P, Q is zero.

Put $X = pAq$, and suppose $X \neq 0$. Choose a closed-range $y_0 \in X$ with $\|y_0 - x\| < \varepsilon/2$. Write $y_0 = v_0|y_0|$ and let $e_0 = v_0^*v_0$, $f_0 = v_0v_0^*$; if $y_0 = 0$, take $v_0 = e_0 = f_0 = 0$. Choose a sequence $(x_n)_{n\geq 1}$ in the closed unit ball of X obtained by repeating every member of a countable norm-dense subset infinitely often. Inductively, once e_j, f_j are chosen for $0 \leq j < n$, put

$$P_{n-1} = p - \sum_{j<n} f_j, \qquad Q_{n-1} = q - \sum_{j<n} e_j, \qquad r_n = P_{n-1}x_nQ_{n-1}.$$

Choose a closed-range element $y_n \in P_{n-1}AQ_{n-1}$ with $\|y_n - r_n\| < 1/n$. Its polar decomposition in A gives $y_n = v_n|y_n|$, with

$$e_n = v_n^* v_n \le Q_{n-1}, \qquad f_n = v_n v_n^* \le P_{n-1}.$$

For a zero y_n all three elements are taken to be zero. Thus the e_n are pairwise orthogonal, as are the f_n. Choose strictly positive scalars c_n with $\sum_{n\ge1} c_n < \varepsilon/2$, and define the norm-convergent sum

$$z = y_0 + \sum_{n\ge1} c_n v_n.$$

Then $z \in X$ and $\|z - x\| < \varepsilon$.

Let I be a closed two-sided ideal of A containing z. The orthogonality on both sides gives

$$z^* z = y_0^* y_0 + \sum_{n\ge1} c_n^2 e_n.$$

Consequently $e_n \in I$ for $n \ge 1$, since each positive summand is dominated by z^*z. Also $y_0^* y_0 \in I$; if $e_0 \ne 0$, invertibility of $|y_0|$ in $e_0 A e_0$ gives a number $\alpha > 0$ with $y_0^* y_0 \ge \alpha e_0$, so $e_0 \in I$ as well. It follows that $v_n, f_n, y_n \in I$ for every $n \ge 0$. If $\pi : A \to A/I$ is the quotient map, then $\pi(r_n) = \pi(x_n)$, and hence

$$\|\pi(x_n)\| = \|\pi(r_n)\| \le \|r_n - y_n\| < 1/n.$$

Every member of the chosen dense subset occurs at arbitrarily large indices, so $\pi(X) = 0$. Applying this observation to $I = \mathrm{Ideal}_A(z)$ proves

$$\mathrm{Ideal}_A(z) = \mathrm{Ideal}_A(X). \tag{10.1}$$

We finish by passing from ideals to the rectangular bimodule. Set $B = pAp$, $E = qAq$, and $K = \mathrm{Ideal}_A(X)$. The hereditary ideal correspondence gives

$$J_X := \overline{\mathrm{span}} X^* X = E \cap K.$$

Indeed, J_X is an ideal of E and generates K as an ideal of A. The same correspondence and (10.1) show that

$$J := \overline{\mathrm{span}} E z^* B z E = \mathrm{Ideal}_E(z^* z) = E \cap K = J_X.$$

For the first equality with the generated ideal, use $p \in B$ and $z^* b z \le \|b\| z^* z$ for $b \in B_+$. The right action of J_X on X is nondegenerate: if (a_λ) is an approximate unit of J_X, then $\|u - u a_\lambda\|^2 = \|(q - a_\lambda) u^* u (q - a_\lambda)\| \to 0$ for $u \in X$. Since $XEz^*B \subseteq B$, we have

$$XJ \subseteq \overline{\mathrm{span}} BzE.$$

Thus $X \subseteq \overline{\mathrm{span}} BzE$. The reverse inclusion is immediate. □

Proposition 10.2. *Let A be a separable unital C^*-algebra of real rank zero. Suppose that pAp is singly generated for every nonzero projection $p \in A$. Then*

$$\mathrm{Gen}(A) = \{g \in A : \mathrm{C}^*(g) = A\}$$

is a norm-dense G_δ subset of A. Equivalently, A has generator rank at most one.

Proof. If $A = 0$, the conclusion is immediate, so assume $A \ne 0$. Fix $a, b \in A_{\mathrm{sa}}$ and $\varepsilon > 0$, and choose $0 < \eta < \varepsilon/6$. Real rank zero gives a finite partition $1 = \sum_{i=1}^m p_i$ by nonzero

projections and real numbers λ_i such that

$$\left\| a - \sum_i \lambda_i p_i \right\| < \eta.$$

Approximate $p_i b p_i$ within η by a finite-spectrum selfadjoint element of the real-rank-zero corner $p_i A p_i$. Relabelling its spectral projections and discarding zero projections gives a refinement $1 = \sum_{\ell=1}^{L} q_\ell$, with $q_\ell \le p_{i(\ell)}$, and real numbers μ_ℓ satisfying

$$\|q_\ell b q_\ell - \mu_\ell q_\ell\| < \eta.$$

The block-diagonal element

$$\Delta = \sum_{\ell=1}^{L} (\mu_\ell q_\ell - q_\ell b q_\ell)$$

has norm less than η.

Choose pairwise distinct nonzero real numbers α_ℓ with $|\alpha_\ell - \lambda_{i(\ell)}| < \eta$. By hypothesis there are selfadjoint contractions $H_\ell, K_\ell \in q_\ell A q_\ell$ such that $q_\ell A q_\ell = \mathrm{C}^*(H_\ell, K_\ell)$. Choose $0 < \delta < \eta$ so that the intervals $[\alpha_\ell - \delta, \alpha_\ell + \delta]$ are pairwise disjoint and avoid zero, and put

$$h = \sum_{\ell=1}^{L} (\alpha_\ell q_\ell + \delta H_\ell).$$

Continuous functional calculus, with functions vanishing at zero, recovers 1 and every q_ℓ from h. Consequently

$$H_\ell = \delta^{-1}(q_\ell h q_\ell - \alpha_\ell q_\ell) \in \mathrm{C}^*(h).$$

For every unordered pair $\ell < j$, use Lemma 10.1 to choose $z_{\ell j} \in q_\ell A q_j$ such that

$$\|z_{\ell j} - q_\ell b q_j\| < \eta / L^2, \qquad q_\ell A q_j = \overline{\mathrm{span}}(q_\ell A q_\ell) z_{\ell j} (q_j A q_j).$$

When $q_\ell A q_j = 0$, choose $z_{\ell j} = 0$. Set

$$w = \sum_{\ell < j} \Big(z_{\ell j} - q_\ell b q_j + z_{\ell j}^* - q_j b q_\ell \Big),$$

$$k = b + \Delta + \delta \sum_{\ell=1}^{L} K_\ell + w, \qquad C = \mathrm{C}^*(h, k).$$

The sum defining w has $L(L-1)$ terms, so $\|w\| < \eta$; for $L = 1$ it is zero. Its diagonal compressions vanish, and hence

$$K_\ell = \delta^{-1}(q_\ell k q_\ell - \mu_\ell q_\ell) \in C.$$

Thus C contains every $q_\ell A q_\ell$. Also $q_\ell k q_j = z_{\ell j} \in C$ for $\ell < j$. The full-bridge property now puts every $q_\ell A q_j$ and its adjoint in C. Since $\sum_\ell q_\ell = 1$, we obtain $C = A$. Finally,

$$\|h - a\| < 3\eta, \qquad \|k - b\| < 3\eta,$$

so $h + ik$ generates A and $\|(h + ik) - (a + ib)\| < 6\eta < \varepsilon$.

For the G_δ assertion, choose a norm-dense sequence (a_j) in A. Then

$$\mathrm{Gen}(A) = \bigcap_{j,m \ge 1} \left\{ g \in A : \mathrm{dist}(a_j, \mathrm{C}^*(g)) < 1/m \right\}.$$

Each set in braces is open: its strict inequality is witnessed by a fixed $*$-polynomial in g, g^* with zero constant term, whose evaluation is norm continuous. The generator-rank formulation is [40, Theorem 3.4 and Remark 3.7]. □

Lemma 10.3. *Let A be a unital pure C^*-algebra of stable rank one. Suppose every lower semicontinuous 2-quasitrace on A is a trace. Every nonzero projection corner pAp has these same three properties.*

Proof. Let $I = \mathrm{Ideal}_A(p)$. The full hereditary inclusion $pAp \subseteq I$ identifies $\mathrm{Cu}(pAp)$ with the ideal $J = \mathrm{Cu}(I)$ of $\mathrm{Cu}(A)$. Almost unperforation passes to this ideal. For almost divisibility, take $x' \ll x$ in J and $n \geq 1$. An ambient witness z satisfies $nz \leq x$ and $x' \leq (n+1)z$; since $z \leq nz \leq x$ and J is hereditary, $z \in J$. Thus pAp is pure.

For completeness, stable rank one passes to this corner by a Schur-complement argument. If $p = 1$ there is nothing to prove. For $a \in pAp$, approximate $a + (1-p)$ by invertible elements $u \in A$. The block $(1-p)u(1-p)$ is invertible in $(1-p)A(1-p)$ for sufficiently close approximants. The Schur complement

$$pup - pu(1-p)\Big((1-p)u(1-p)\Big)^{-1}(1-p)up$$

is invertible in pAp and converges to a.

Let ρ be a lower semicontinuous 2-quasitrace on pAp. The quasitrace–functional correspondence [15, Proposition 4.2 and Theorem 4.4] transports d_ρ to a functional $\lambda : J \to [0, \infty]$. Extend it to $\mathrm{Cu}(A)$ by

$$\widetilde{\lambda}(x) = \begin{cases} \lambda(x), & x \in J, \\ \infty, & x \notin J. \end{cases}$$

This extension is additive and order preserving because J is an ideal. It preserves increasing suprema because J is closed under such suprema: an increasing sequence with supremum outside J must already have a term outside J. Thus $\widetilde{\lambda}$ is a functional on $\mathrm{Cu}(A)$. By the same correspondence it defines a lower semicontinuous 2-quasitrace $\widetilde{\rho}$ on A whose restriction to pAp is ρ. In fact the inverse formula

$$\widetilde{\rho}(a) = \int_0^\infty \widetilde{\lambda}([(a-t)_+])\,\mathrm{d}t$$

shows the restriction assertion directly. The hypothesis makes $\widetilde{\rho}$ a trace, so ρ is a trace. The extension is allowed to be unbounded or not densely finite; this is why the full lower semicontinuous quasitrace hypothesis is used. □

Theorem 10.4. *Let A be a separable unital pure C^*-algebra with real rank zero and stable rank one. Suppose that every lower semicontinuous 2-quasitrace on A is an ordinary trace. Then $\mathrm{Gen}(A)$ is a norm-dense G_δ subset of A. Equivalently, A has generator rank at most one.*

Proof. The zero algebra is immediate. Every nonzero projection corner pAp satisfies the intrinsic hypotheses by Lemma 10.3, and is separable and unital. Corollary 9.2 makes it singly generated. Apply Proposition 10.2. □

Corollary 10.5. *Let A be a simple separable unital* C*-*algebra with* $\mathrm{T}(A) \neq \varnothing$*, real rank zero, and strict comparison by tracial states. Then* $\mathrm{Gen}(A)$ *is a norm-dense* G_δ *subset of* A*, and* A *has generator rank at most one.*

Proof of Corollaries 1.6 and 10.5. If A is finite dimensional, it is a full matrix algebra. Every nonzero projection corner is again a full matrix algebra and is singly generated, so Proposition 10.2 applies. Otherwise Lemma 2.2 gives comparison by the full ordinary trace cone and the quasitrace hypothesis. Hence A is pure by [42, Corollary B] and has stable rank one by [28, Corollary 4.5]. Theorem 10.4 now applies. □

Remark 10.6. The density theorem uses single generation of all projection corners. A prescribed unital $\mathcal{Z}$-embedding into A does not by itself supply unital $\mathcal{Z}$-embeddings into nonfull corners, and traces on such corners need not extend boundedly to A. Thus the argument above does not establish density under the hypotheses of Theorem 1.2 and real rank zero alone. Under the stronger assumption of $\mathcal{Z}$-stability, density for arbitrary separable real-rank-zero algebras is already known from [41, Theorem 5.3], without a comparison hypothesis.

11. Polar generators and topological free entropy dimension

The matrix-cone construction also controls the topological free entropy dimension of its generator. The control comes from the constant spectral labels of the first selfadjoint element and the sparse matrix positions occupied by the second. We first specify the microstate convention and the change of coordinates used below.

Let $x = (x_1, \dots, x_\ell)$ be a selfadjoint tuple in a unital C*-algebra, and fix $R > \max_j \|x_j\|$. For a finite set $\mathcal{F}$ of complex noncommutative polynomials in ℓ selfadjoint variables and $\sigma > 0$, put

$$\Gamma_R^{\mathrm{top}}(x; q, \mathcal{F}, \sigma) = \{X \in (\mathrm{M}_q^{\mathrm{sa}})^\ell : \max_j \|X_j\| \le R, \\ \big|\|P(X)\| - \|P(x)\|\big| < \sigma \quad (P \in \mathcal{F})\}.$$

For a bounded set $\Omega \subseteq (\mathrm{M}_q^{\mathrm{sa}})^\ell$, let $N_\infty(\Omega, \omega)$ be its covering number for balls of radius ω in the maximum operator norm. We allow the centres to belong to the ambient matrix space. Requiring centres in Ω changes the covering radius by at most a factor of two and gives the same dimension. With $\log 0 = -\infty$, set

$$\mathbb{K}_R(x; \omega) = \inf_{\mathcal{F}, \sigma > 0} \limsup_{q \to \infty} \frac{1}{q^2} \log N_\infty\Big(\Gamma_R^{\mathrm{top}}(x; q, \mathcal{F}, \sigma), \omega\Big). \tag{11.1}$$

Then

$$\delta_{\mathrm{top}}(x) = \limsup_{\omega \downarrow 0} \frac{\mathbb{K}_R(x; \omega)}{|\log \omega|}. \tag{11.2}$$

The value is independent of the choice of $R > \max_j \|x_j\|$. Indeed, the coordinate polynomials already bound every sufficiently accurate microstate by any such R. These are the operator-norm microstate definitions of [46]. For a possibly nonselfadjoint element g, we use the notation

$$\delta_{\mathrm{top}}(g) = \delta_{\mathrm{top}}(\mathrm{Re}\, g, \mathrm{Im}\, g), \qquad \chi_{\mathrm{top}}(g) = \chi_{\mathrm{top}}(\mathrm{Re}\, g, \mathrm{Im}\, g).$$

Here $\chi_{\mathrm{top}}(x)$ is defined using the same norm microstates by replacing the covering expression in (11.1) with

$$\limsup_{q\to\infty}\left(\frac{1}{q^2}\log\operatorname{vol}\Gamma_R^{\mathrm{top}}(x;q,\mathcal{F},\sigma)+\frac{\ell}{2}\log q\right),$$

and then taking the infimum over $\mathcal{F},\sigma$. The volume is Lebesgue measure for the real Hilbert space structure given by the unnormalized Hilbert–Schmidt inner product.

We shall also use the elementary implication

$$\delta_{\mathrm{top}}(x)<\ell \quad\Longrightarrow\quad \chi_{\mathrm{top}}(x)=-\infty. \tag{11.3}$$

Indeed, a ball of radius ω in the maximum operator norm is contained in a Hilbert–Schmidt ball of radius $\sqrt{\ell q}\,\omega$ in real dimension ℓq^2. The Euclidean ball volume formula and Stirling's estimate give

$$\limsup_{q\to\infty}\left(\frac{1}{q^2}\log\operatorname{vol}B_\infty(0,\omega)+\frac{\ell}{2}\log q\right)\le \ell\log\omega+C_\ell.$$

Covering a norm-microstate set by these balls and then taking the infimum over its tests gives $\chi_{\mathrm{top}}(x)\le\mathbb{K}_R(x;\omega)+\ell\log\omega+C_\ell$. If $\delta_{\mathrm{top}}(x)<d<\ell$, then for all sufficiently small ω the right side is at most $(\ell-d)\log\omega+C_\ell$, which tends to $-\infty$.

A separable unital C*-algebra is MF if it admits a unital embedding into a quotient $\prod_j \mathrm{M}_{d_j}/\bigoplus_j \mathrm{M}_{d_j}$, where the product consists of bounded sequences and the direct sum consists of sequences converging to zero in norm. Such an algebra has a tracial state, obtained by taking an ultralimit of normalized matrix traces. If a unital MF algebra has no nonzero finite-dimensional representations, then every finite selfadjoint generating tuple satisfies

$$\delta_{\mathrm{top}}(x)\ge 1. \tag{11.4}$$

This is [26, Theorem 5.3(i)]. A unital embedding of $\mathcal{Z}$ into A excludes nonzero finite-dimensional representations of A: restricting one to $\mathcal{Z}$ would give a nonzero finite-dimensional representation of the simple infinite-dimensional algebra $\mathcal{Z}$. Thus (11.4) applies to every MF algebra in Theorem 9.1. The MF assumption is needed for this lower bound; the upper estimate below does not require matricial approximation.

Lemma 11.1 (Polar coordinates and topological free entropy dimension)**.** *Let A be a unital C*-algebra, and let $h,k\in A_{\mathrm{sa}}$ satisfy $h\ge 1_A$ and $A=\mathrm{C}^*(h,k)$, where the generated algebra is not assumed to contain a unit in advance. Then $g=h+ik$ is invertible, its polar decomposition*

$$g=u|g|,\qquad u=g(g^*g)^{-1/2},$$

has $u\in\mathcal{U}(A)$, and $A=\mathrm{C}^*(u,|g|)$.

For every $\alpha>0$ with $\alpha\|k\|<\pi$, the element

$$G_\alpha=e^{i\alpha k}h$$

satisfies

$$|G_\alpha|=h,\qquad \operatorname{phase}(G_\alpha)=e^{i\alpha k},\qquad \mathrm{C}^*(G_\alpha)=A. \tag{11.5}$$

Moreover,

$$\delta_{\mathrm{top}}(\operatorname{Re}G_\alpha,\operatorname{Im}G_\alpha)=\delta_{\mathrm{top}}(h,k). \tag{11.6}$$

Here the same convention for empty norm-microstate spaces is used on both sides.

Proof. The selfadjoint element $a = h^{-1/2}kh^{-1/2}$ gives the factorization

$$g = h^{1/2}(1_A + ia)h^{1/2}.$$

The middle factor is invertible by continuous functional calculus, and $\|(1_A + ia)^{-1}\| \leq 1$. Hence g is invertible and $\|g^{-1}\| \leq 1$. In particular, $|g| \geq 1_A$, and the displayed formula for u defines a unitary in A. Functional calculus in $\mathrm{C}^*(g)$ recovers $|g|$, 1_A, $|g|^{-1}$, and u. Conversely $g = u|g|$, while h and k are its real and imaginary parts. This proves the first assertion, including the assertion about generation without adjoining a unit.

For the alternative generator, direct multiplication gives $G_\alpha^* G_\alpha = h^2$. Thus its modulus and unitary phase are as in (11.5). Its modulus again recovers the unit and its inverse. Since $\alpha\sigma(k) \subset (-\pi, \pi)$, the principal argument is continuous on the spectrum of $e^{i\alpha k}$ and gives

$$k = \alpha^{-1} \operatorname{Arg}(e^{i\alpha k}).$$

It follows that $\mathrm{C}^*(G_\alpha) = \mathrm{C}^*(h, k) = A$.

We prove (11.6) by comparing norm-microstate covering numbers under these explicit coordinate maps. In particular, no invariance under an arbitrary change of generators is used. Choose constants

$$0 < c < 1, \qquad M > \|h\|, \qquad \alpha\|k\| < \theta < \pi.$$

For every matrix size q, put

$$\mathcal{D}_q = \{(H, K) \in (\mathrm{M}_q(\mathbb{C})_{\mathrm{sa}})^2 : c1_q \leq H \leq M1_q,\ \|\alpha K\| \leq \theta\},$$
$$\mathcal{E}_q = \{Z \in \mathrm{M}_q(\mathbb{C}) : c1_q \leq |Z| \leq M1_q,\ \sigma(U_Z) \subseteq \{e^{it} : |t| \leq \theta\}\}, \qquad U_Z = Z|Z|^{-1}.$$

The maps

$$\Phi_q(H, K) = e^{i\alpha K}H, \qquad \Psi_q(Z) = \left(|Z|, \alpha^{-1} \operatorname{Arg}(U_Z)\right) \tag{11.7}$$

are inverse bijections between $\mathcal{D}_q$ and $\mathcal{E}_q$. We identify a complex matrix Z with its selfadjoint pair $(\operatorname{Re} Z, \operatorname{Im} Z)$; the corresponding norms satisfy

$$\max\{\|\operatorname{Re} Z\|, \|\operatorname{Im} Z\|\} \leq \|Z\| \leq 2\max\{\|\operatorname{Re} Z\|, \|\operatorname{Im} Z\|\}.$$

Both maps in (11.7) are Lipschitz for these operator-norm metrics, with constants independent of q. For Φ_q, the integral formula for the difference of two exponentials of selfadjoint matrices gives

$$\|e^{i\alpha K}H - e^{i\alpha K'}H'\| \leq \|H - H'\| + \alpha M\|K - K'\|.$$

For completeness, if $Z, Z' \in \mathcal{E}_q$ and $P = |Z|$, $P' = |Z'|$, the identity

$$P(P - P') + (P - P')P' = Z^*Z - Z'^*Z'$$

and its integral solution using e^{-sP} and $e^{-sP'}$ give

$$\|P - P'\| \leq \frac{1}{2c}\|Z^*Z - Z'^*Z'\| \leq \frac{M}{c}\|Z - Z'\|.$$

The identity $P^{-1} - P'^{-1} = P^{-1}(P' - P)P'^{-1}$ then shows that $Z \mapsto U_Z = ZP^{-1}$ is uniformly Lipschitz on these domains. Finally, the principal logarithm is holomorphic on a fixed neighborhood of the compact arc $\{e^{it} : |t| \leq \theta\}$. A fixed contour in that neighborhood, the resolvent identity, and the normality of U_Z show that $U_Z \mapsto \operatorname{Arg}(U_Z) = -i \log U_Z$ is

Lipschitz with a constant depending only on θ. All these estimates are independent of the matrix size.

We explain why the coordinate maps apply to the norm-microstate spaces in the definition of δ_{top}. Fix norm cutoffs larger than the actual coordinate norms for each tuple; coordinate norm tests make their precise values immaterial. Sufficiently accurate finite norm-polynomial tests for (h, k) force its matrix microstates into $\mathcal{D}_q$. Indeed, norm tests for H, K, and $H - S1_q$, with a fixed $S > \|h\|$, enforce the upper bounds and the positive lower bound, using the strict margins in the choices of c, M, and θ.

The corresponding assertion for $(\operatorname{Re} G_\alpha, \operatorname{Im} G_\alpha)$ also follows from finitely many norm-polynomial tests. First test Z and $1_q - Z^*Z/S^2$, where $S > \|G_\alpha\|$. Since $G_\alpha^* G_\alpha = h^2 \geq 1_A$, these tests, with sufficiently small tolerance, force $c1_q \leq |Z| \leq M1_q$. On this bounded positive-gap set, $Z \mapsto U_Z$ is uniformly approximable by noncommutative $*$-polynomials in Z, Z^*, by approximating $s \mapsto s^{-1/2}$ on a fixed positive interval. The strict inequality

$$\|e^{i\alpha k} - 1_A\| < 2\sin(\theta/2)$$

therefore allows one further finite polynomial norm test to force $\|U_Z - 1_q\| \leq 2\sin(\theta/2)$. Since U_Z is unitary, this is precisely the required phase-arc condition. Thus these microstates lie in $\mathcal{E}_q$.

On $\mathcal{D}_q$ the exponential in Φ_q is uniformly approximable by polynomials, independently of q. On $\mathcal{E}_q$, the square root, inverse square root, and principal argument in Ψ_q are likewise uniformly approximable by $*$-polynomials: use ordinary polynomial approximation on the fixed positive spectral interval and Laurent-polynomial approximation on the fixed unitary arc. Consequently, any prescribed finite collection of norm-microstate conditions for either tuple pulls back to a finite collection for the other tuple, with any prescribed positive tolerance. This follows by first approximating the coordinate maps, then substituting their polynomial approximants into the finitely many test polynomials. The approximation errors are uniform in q.

Let $L \geq 1$ be a common Lipschitz constant for the two coordinate maps, with the above identification of complex matrices and selfadjoint pairs. Given any finite microstate conditions for one tuple, strengthen the conditions for the other tuple so that its image lies in the prescribed microstate space and the relevant gap conditions hold. Applying the inverse coordinate map to a covering gives a covering with radius at most $2L$ times the original radius. The factor 2 permits the centers of the original covering balls to be replaced by points of the covered set. This yields the covering-number inequalities in both directions, with the covering radius changed by the fixed factor $2L$. Write $x = (h, k)$ and $y = (\operatorname{Re} G_\alpha, \operatorname{Im} G_\alpha)$, and let R_x, R_y be their chosen norm cutoffs. Taking the normalized q^{-2} log covering numbers, the limsup as $q \to \infty$, and the infimum over finite tests gives

$$\mathbb{K}_{R_y}(y; 2L\omega) \leq \mathbb{K}_{R_x}(x; \omega), \qquad \mathbb{K}_{R_x}(x; 2L\omega) \leq \mathbb{K}_{R_y}(y; \omega).$$

Finally let the covering radius tend to zero. Multiplication of that radius by $2L$ does not change the result, because

$$\frac{|\log((2L)\omega)|}{|\log \omega|} \longrightarrow 1.$$

This proves (11.6). The same microstate transfers show that failure of matricial approximation occurs simultaneously for the two tuples. □

Remark 11.2. The polar form above imposes no Haar or freeness condition. In fact, the phase u of the original strictly accretive generator $g = h + ik$ cannot be Haar for any tracial state. To see this, put $p = |g|$ and $B = \|g\|$. If $\lambda \in \sigma(u)$, unit approximate eigenvectors for u in a faithful representation give

$$1 \leq (\operatorname{Re}\lambda)\langle p\xi, \xi\rangle + o(1).$$

Positivity of p first forces $\operatorname{Re}\lambda > 0$; then $\langle p\xi, \xi\rangle \leq B$ gives $\operatorname{Re}\lambda \geq B^{-1}$. Hence $\operatorname{Re} u \geq B^{-1}1_A$, whereas a Haar unitary has zero trace. No atomlessness or free independence of the polar factors is asserted.

We record the trace consequence of norm microstates that will be used in the covering argument. A continuous matrix expression means an expression obtained from noncommutative polynomials by finitely many algebraic operations and continuous functional calculus on selfadjoint elements in fixed bounded intervals. Such expressions are uniformly approximable by noncommutative polynomials on a fixed norm-bounded set of matrix tuples, in all matrix sizes. This follows successively from polynomial approximation of the scalar continuous functions.

Lemma 11.3. *Let $A = \mathrm{C}^*(1, x_1, \ldots, x_\ell)$, where the x_j are selfadjoint, and fix $R > \max_j \|x_j\|$. Suppose that Ψ is a continuous matrix expression whose values are selfadjoint on selfadjoint tuples, and*

$$\tau(\Psi(x)) \geq b \qquad (\tau \in \mathrm{T}(A)).$$

For every $\eta > 0$ there are finite norm-microstate tests $\mathcal{F}, \sigma$ such that, in every matrix size,

$$\operatorname{tr}_q(\Psi(X)) > b - \eta \quad \text{for } X \in \Gamma_R^{\mathrm{top}}(x; q, \mathcal{F}, \sigma).$$

Here $\operatorname{tr}_q = q^{-1}\operatorname{Tr}$. Norms of finitely many continuous matrix expressions can also be prescribed within any given tolerance by finite norm-microstate tests.

Proof. For the last assertion, approximate each expression uniformly by a polynomial on the R-ball and include those polynomials in the tests. To prove the trace assertion, suppose it fails. Enumerate the polynomials with rational complex coefficients and choose R-bounded tuples $X^{(j)}$ of sizes q_j which satisfy the first j norm tests within $1/j$ but have

$$\operatorname{tr}_{q_j}(\Psi(X^{(j)})) \leq b - \eta.$$

Density of the enumerated polynomials and the norm bound give

$$\|P(X^{(j)})\| \longrightarrow \|P(x)\|$$

for every polynomial P. Thus $x_i \mapsto [(X_i^{(j)})_j]$ extends to a unital isometric $*$-homomorphism from A into the norm quotient of the bounded product of the M_{q_j} by the norm-null sequences. An ultralimit of the normalized matrix traces is a tracial state on this quotient. Its pullback τ to A satisfies

$$\tau(\Psi(x)) = \lim_{j \to \mathcal{U}} \operatorname{tr}_{q_j}(\Psi(X^{(j)})) \leq b - \eta.$$

Continuous functional calculus commutes with the quotient map, so the equality applies to the stated expressions. This contradicts the hypothesis. □

Proposition 11.4. *Use the hypotheses and the actual elements h, k constructed in the proof of Proposition 8.2. Write m for the matrix size denoted by n there; thus $m \geq 3$, $1 \leq N \leq m-1$, and $0 < \varepsilon < 1/(64m)$. Then $g = h + ik$ is an invertible generator of A, and*

$$\delta_{\mathrm{top}}(g) \leq 2 - \frac{m^2 - m + 2 - N}{m^2}(1 - 2\varepsilon)^2, \qquad \chi_{\mathrm{top}}(g) = -\infty. \tag{11.8}$$

If A is MF *and has no nonzero finite-dimensional representations, then also $\delta_{\mathrm{top}}(g) \geq 1$. For the polar decomposition $g = u|g|$, one has*

$$A = \mathrm{C}^*(u, |g|) = \mathrm{C}^*(g).$$

Proof. Retain the notation of that construction, including its supporting matrix units e_{ij}, the positive contraction t, and $\delta = 1/16$. Thus

$$h = 1 + \sum_{i=1}^{m} \lambda_i e_{ii} F(t), \qquad \mu_i = 1 + \lambda_i,$$

where the λ_i are distinct positive numbers and $F = 1$ on $[\delta, 1]$. The diagonal terms of k occur only in positions $1, \ldots, N$. Its other prescribed cone terms lie on the edges of the tree

$$\mathcal{E} = \{\{i, i+1\} : 1 \leq i \leq m-2\} \cup \{\{1, m\}\}. \tag{11.9}$$

The remaining term is $v + v^*$, where $v^* v = q$ and q is orthogonal to every continuous function of t supported in $(1/2, 1]$. In particular, this term disappears when it is cut on both sides by such functions of t.

Fix a with $1/2 < a < 1 - \varepsilon$, and choose a continuous function $c : [0, 1] \to [0, 1]$ which is zero on $[0, 1/2]$ and one on $[a, 1]$. Put

$$\beta_a = \frac{1 - \varepsilon/(1-a)}{m} > 0. \tag{11.10}$$

For any $\rho > 0$ small enough that the intervals $(\mu_i - \rho, \mu_i + \rho)$ are pairwise disjoint, choose continuous functions $0 \leq f_i \leq 1$ supported in these intervals and satisfying $f_i(\mu_i) = 1$. The constant labels of h on the support of $c(t)$ give the exact identities

$$a_i := f_i(h) c(t)^2 f_i(h) = e_{ii} c(t)^2 \qquad (1 \leq i \leq m). \tag{11.11}$$

These identities do not require the supporting matrix units to belong to A; the displayed products belong to A by (8.1).

Let $\mathcal{J}$ be the following set of ordered matrix positions:

$$\mathcal{J} = \{(i, j) : i \neq j,\ \{i, j\} \notin \mathcal{E}\} \cup \{(i, i) : N < i \leq m\}.$$

The construction of k implies

$$a_i k a_j = 0 \qquad ((i, j) \in \mathcal{J}). \tag{11.12}$$

Indeed, every cone term occupies one of the listed diagonal positions or tree edges. Also $c(t) q = 0$, so $v c(t) = 0$ and $c(t) v^* = 0$. These equalities kill respectively v and v^* in every product in (11.12).

For every positive integer s and every $\tau \in \mathrm{T}(A)$,

$$\tau(a_i^s) = \frac{1}{m} \tau(c(t)^{2s}) \geq \frac{1}{m} \tau(1_{[a,1]}(t)) \geq \beta_a. \tag{11.13}$$

The trace equalities for the matrix slices follow first for continuous functions vanishing at zero and then for spectral projections by monotone convergence in the finite tracial GNS closure. The last inequality follows from

$$(1-a)\tau(1_{[0,a)}(t)) \le \tau(1-t) < \varepsilon.$$

We now give a uniform covering estimate. Fix $R > \max\{\|h\|, \|k\|\}$ and $0 < \eta < \beta_a/4$, and set $\theta = \beta_a - 2\eta$. Take $b = 1/2$, and choose an integer s with $b^s < \eta$. For a covering radius $0 < \omega < 1$, choose

$$0 < \rho < \frac{\omega}{16m}$$

small enough that the spectral bands above are disjoint, and choose the corresponding f_i. Since $A = \mathrm{C}^*(h,k)$, there is a selfadjoint polynomial p with $p(h,k)$ arbitrarily close to t. Write

$$\zeta(r) = \min\{1, \max\{0, r\}\} \quad (r \in \mathbb{R}), \qquad T = \zeta(p(H,K)), \qquad A_i = f_i(H)c(T)^2 f_i(H).$$

The letters T, A_i in this display denote actual matrix functions of a prospective microstate (H,K), not additional microstate coordinates. They satisfy $0 \le T, A_i \le 1$.

Choose p sufficiently accurately that evaluation at (h,k) makes A_i as close as needed to a_i in (11.11). Lemma 11.3, applied also to the powers A_i^s, and the norm part of that lemma, applied to the finitely many products in (11.12), then give finite tests $\mathcal{F}, \sigma$ for which every

$$(H,K) \in \Gamma_R^{\mathrm{top}}((h,k); q, \mathcal{F}, \sigma)$$

satisfies

$$\mathrm{tr}_q(A_i^s) > \beta_a - \eta, \qquad \|A_i K A_j\| < \frac{b^2\omega}{16m^2} \quad ((i,j) \in \mathcal{J}). \tag{11.14}$$

Here one first makes the errors at (h,k) smaller than half the desired tolerances and then imposes the finite microstate tests. The polynomial p, the functions f_i, and these tests may depend on ω. This is permitted by the infimum in (11.1).

Let

$$R_i = 1_{[b,1]}(A_i), \qquad r_i = \operatorname{rank} R_i \quad (1 \le i \le m), \qquad R_0 = 1 - \sum_{i=1}^m R_i, \quad r_0 = \operatorname{rank} R_0.$$

If $P_i = 1_{(\mu_i-\rho,\mu_i+\rho)}(H)$, then $R_i \le P_i$. Thus the R_i are pairwise orthogonal, and

$$\frac{r_i}{q} \ge \mathrm{tr}_q(A_i^s) - b^s > \theta, \qquad \|(H-\mu_i)R_i\| \le \rho. \tag{11.15}$$

The rank estimate uses only $0 \le A_i \le 1$ and the spectral cut at b. The band estimate does not require R_i to commute with H.

The restriction of A_i to $R_i\mathbb{C}^q$ is invertible with inverse of norm at most b^{-1}. Therefore

$$\|R_i K R_j\| \le b^{-2}\|A_i K A_j\| < \frac{\omega}{16m^2} \qquad ((i,j) \in \mathcal{J}).$$

Define selfadjoint matrices

$$H' = \sum_{i=1}^m \mu_i R_i + R_0 H R_0, \qquad K' = K - \sum_{(i,j)\in\mathcal{J}} R_i K R_j.$$

The set $\mathcal{J}$ is closed under transposition, so K' is selfadjoint. The preceding estimates give

$$\|H - H'\| \leq 2m\rho < \frac{\omega}{8}, \qquad \|K - K'\| < \frac{\omega}{16}. \tag{11.16}$$

For the first estimate, write the difference as

$$\sum_i (H - \mu_i)R_i + \sum_i R_i(H - \mu_i)R_0$$

and use (11.15). Also $\|H'\| \leq R$ and $\|K'\| \leq R + 1$.

For fixed ranks $r_0, r_1, \ldots, r_m$, the flags of orthogonal projections with these ranks form the homogeneous space

$$\mathcal{V}_{\boldsymbol{r}} = \mathcal{U}(q)\Big/\Big(\mathcal{U}(r_0) \times \cdots \times \mathcal{U}(r_m)\Big).$$

Zero-rank factors are omitted. Its real dimension is

$$d_F = q^2 - r_0^2 - \sum_{i=1}^{m} r_i^2.$$

By [38, Corollary 12], its covering numbers in the quotient metric induced by the operator norm are bounded by $(C/\alpha)^{d_F}$, with an absolute constant C. The same upper bound, after changing C, applies if one uses the intrinsic operator-norm metric. Conjugation of a matrix of norm at most $R + 1$ changes it by at most $2(R + 1)$ times the distance of the conjugating unitaries. Consequently a flag net at radius a fixed multiple of $\omega/(R + 1)$ suffices.

For a fixed standard flag, the free residual block of H' has real dimension r_0^2. The allowed selfadjoint matrices K' form a real linear space of dimension

$$d_K = q^2 - \sum_{i=N+1}^{m} r_i^2 - 2 \sum_{\substack{1 \leq i < j \leq m \\ \{i,j\} \notin \mathcal{E}}} r_i r_j.$$

Their norm-bounded balls admit nets of cardinality at most $(C_R/\omega)^{d_K}$ by the elementary volume estimate for a ball in a finite-dimensional normed space. The residual block of H' has a net of cardinality at most $(C_R/\omega)^{r_0^2}$.

These flag and fibre nets combine without adding parameters for the block-diagonal unitary group. Indeed, that group preserves the allowed linear space of K' and the norm balls in both fibres. A block-diagonal adjustment in comparing a flag with a flag-net point can therefore be absorbed into its fibre parameters. We obtain a net for all model pairs with the given ranks of cardinality at most

$$\left(\frac{C_R}{\omega}\right)^{D}, \qquad D = 2q^2 - \sum_{i=1}^{m} r_i^2 - \sum_{i=N+1}^{m} r_i^2 - 2 \sum_{\substack{1 \leq i < j \leq m \\ \{i,j\} \notin \mathcal{E}}} r_i r_j. \tag{11.17}$$

Here and below fixed numerical factors in the covering radius are absorbed into C_R. This constant is independent of q, ω, the widths of the f_i, and the polynomial p. Allowing dependence on the fixed integer m would give the same conclusion.

The tree has $m - 1$ edges. The number of forbidden ordered off-diagonal positions is therefore $(m - 1)(m - 2)$. From (11.15) and (11.17) we get

$$\frac{D}{q^2} \leq 2 - \Big(m + (m - N) + (m - 1)(m - 2)\Big)\theta^2 = 2 - (m^2 - m + 2 - N)\theta^2.$$

There are at most $(q+1)^{m+1}$ rank profiles. Combining their nets with (11.16), and choosing the fixed numerical radii so that the resulting balls have radius ω, gives

$$N_\infty\Big(\Gamma_R^{\mathrm{top}}((h,k);q,\mathcal{F},\sigma),\omega\Big) \le (q+1)^{m+1}\left(\frac{C_R}{\omega}\right)^{\left(2-(m^2-m+2-N)\theta^2\right)q^2}.$$

The polynomial rank-profile factor disappears after division by q^2 and passage to the limit. The constant C_R disappears after division by $|\log\omega|$ and $\omega \downarrow 0$. It follows that

$$\delta_{\mathrm{top}}(h,k) \le 2-(m^2-m+2-N)(\beta_a-2\eta)^2.$$

First let $\eta \downarrow 0$, and then let $a \downarrow 1/2$. These are auxiliary parameters for the fixed pair (h,k). Equation (11.10) gives precisely the upper estimate in (11.8).

Since $N \le m-1$ and $0<\varepsilon<1/(64m)$, the upper bound in (11.8) is strictly less than two. Equation (11.3) therefore gives $\chi_{\mathrm{top}}(h,k)=-\infty$.

The invertibility and the polar assertion follow from Lemma 11.1, since $h \ge 1$. The lower bound under the stated MF hypothesis is (11.4). □

Theorem 11.5. *Let A be a separable unital* MF C*-*algebra with strict comparison by ordinary tracial states. Suppose that $\mathcal{Z}$ embeds unitally in A. Then*

$$\inf\{\delta_{\mathrm{top}}(g) : g \in A,\ \mathrm{C}^*(g)=A\} = 1. \tag{11.18}$$

More precisely, for every $\eta>0$ there is an invertible generator $g=ub$, where $u \in \mathcal{U}(A)$ and $b=|g| \ge 1_A$, such that $\mathrm{C}^*(u,b)=\mathrm{C}^*(g)=A$ and

$$1 \le \delta_{\mathrm{top}}(g) < 1+\eta, \qquad \chi_{\mathrm{top}}(g)=-\infty.$$

Proof. The MF assumption supplies a tracial state, and the unital copy of $\mathcal{Z}$ excludes nonzero finite-dimensional representations. We may replace the choice $n=8$ in the proof of Theorem 9.1 by any integer $m \ge 8$. For completeness, fix the given unital copy of $\mathcal{Z}$ in A. Proposition 2.5 supplies a c.p.c. order-zero map $\phi_m : \mathrm{M}_m \to \mathcal{Z} \subseteq A$ with

$$\tau(\phi_m(1_m)) > 1-\varepsilon_m \qquad (\tau \in \mathrm{T}(A)), \qquad 0<\varepsilon_m<\frac{1}{64m}.$$

Set

$$d_m=(\phi_m(e_{11})-1/16)_+, \qquad D_m=\mathrm{Her}_A(d_m).$$

The element d_m is full in A and has constant positive rank over $\mathrm{T}(A)$. Indeed, it is nonzero in the simple unital copy of $\mathcal{Z}$, and every tracial state on A restricts to the unique trace of that copy. The class $[d_m]$ is almost divisible, as the image of a soft class in $\mathrm{Cu}(\mathcal{Z})$. The argument in Theorem 9.1 therefore applies: Lemma 2.7 gives a compact nonempty normalized trace space, the structural and external AF results apply, and Proposition 7.1 gives seven selfadjoint generators for D_m. This bound is independent of m.

Since $m-1 \ge 7$, Proposition 8.2 now gives an invertible generator $g_m=h_m+ik_m$ with $N=7$. Proposition 11.4 gives

$$1 \le \delta_{\mathrm{top}}(g_m) \le 2-\left(1-\frac{1}{m}-\frac{5}{m^2}\right)(1-2\varepsilon_m)^2, \qquad \chi_{\mathrm{top}}(g_m)=-\infty.$$

The upper bound tends to one as $m \to \infty$. Its lower bound is valid for every generator of A, by (11.4). This proves (11.18). Choosing m sufficiently large gives a cone pair with

$\delta_{\mathrm{top}}(h_m, k_m) < \min\{1+\eta, 2\}$. Choose $\alpha > 0$ with $\alpha\|k_m\| < \pi$ and set

$$g = e^{i\alpha k_m} h_m, \qquad u = e^{i\alpha k_m}, \quad b = h_m.$$

Lemma 11.1 gives $|g| = b$, phase u, generation, and $\delta_{\mathrm{top}}(g) = \delta_{\mathrm{top}}(h_m, k_m)$. Equation (11.3) then gives $\chi_{\mathrm{top}}(g) = -\infty$. □

Remark 11.6. Already the choice $m = 8$, $N = 7$ in the single-generation proof gives an invertible generator with

$$1 \le \delta_{\mathrm{top}}(g) \le 2 - \frac{51}{64}(1-2\varepsilon)^2 < 1.21, \qquad \chi_{\mathrm{top}}(g) = -\infty$$

under the hypotheses of Theorem 11.5. Increasing the matrix size gives the infimum in Theorem 11.5; the argument does not assert that a generator attains that infimum.

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