CLASSIFICATION OF REDUCED FREE PRODUCTS OF NUCLEAR C*-ALGEBRAS

GEORGE A. ELLIOTT, CHUN GUANG LI, AND ZHUANG NIU

Abstract. Consider the countably infinite reduced free product D of copies of $(\mathrm{C}([0,1]),\lambda)$, where λ is Lebesgue measure. It is proved that finite and countable reduced free products of unital separable nuclear UCT C*-algebras with respect to a chosen family of faithful traces are classified, after adjoining D, by the Elliott invariant. In other words, an isomorphism between the Elliott invariants of two such algebras is induced by an isomorphism between the algebras.

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1. Introduction

In [26], using techniques from the classification of nuclear C*-algebras, Hirshberg and Phillips show that certain nuclear free factors can be adjoined to infinite reduced free products without changing the isomorphism class. In particular, they identify the infinite reduced free product of Lebesgue interval algebras with that of Jiang–Su algebras. Their proofs combine approximate intertwining with existence and uniqueness results for homomorphisms from nuclear algebras. Motivated by their work (in particular, their circle–annulus isomorphism questions), we extend these techniques further to classify a class of nonnuclear algebras obtained by reduced free products.

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The structural study of reduced free products goes back to Avitzour's simplicity and unique-trace criteria [2]. Dykema and Rørdam developed the theory of projections and ordered K-groups in free products [16], while Dykema, Haagerup and Rørdam proved stable-rank-one results [15]. More recently, Robert's theory of selfless C*-algebras provided a route to strict comparison [41]. Amrutam, Gao, Kunnawalkam Elayavalli and Patchell proved strict comparison for $\mathrm{C}^*_{\mathrm{r}}(\mathbb{F}_n)$, $2 \leq n < \infty$, using this framework [1]. Hayes, Kunnawalkam Elayavalli and Robert developed selflessness and further regularity results for reduced free products [25]. These results provide the structural ingredients for the Elliott invariant in this setting.

On the K-theoretic side, the calculation of Pimsner and Voiculescu [37] gives $\mathrm{K}_1(\mathrm{C}^*_{\mathrm{r}}(\mathbb{F}_n)) \cong \mathbb{Z}^n$, distinguishing reduced free group algebras of different finite ranks. The K- and KK-theory of amalgamated free products developed by Fima and Germain [20] provides the corresponding calculations for general factors.

In this paper, we consider

$$(D, \tau_D) = *_{\mathrm{r}, j=1}^{\infty}(\mathrm{C}([0,1]), \lambda),$$

where λ is Lebesgue measure. For a unital algebra E with a faithful selected tracial state τ_E, we call the operation

$$(E, \tau_E) \longmapsto (E, \tau_E) *_{\mathrm{r}} (D, \tau_D)$$

its *D-stabilization*. We say that E is D-stable if $E *_{\mathrm{r}} D \cong E$ by a trace-preserving isomorphism. Reindexing the interval factors gives $D *_{\mathrm{r}} D \cong D$, so every D-stabilized algebra is D-stable.

The absorption results of [26] motivate viewing this operation as a free-product analogue of tensorial $\mathcal{Z}$-stabilization. Separable simple unital nuclear UCT C*-algebras that absorb the Jiang–Su algebra tensorially are classified by the Elliott invariant, in the purely infinite case [28, 36] and in the stably finite case [5, 19, 22, 23, 50].

The following Theorem A gives a free-product counterpart: after adjoining D, finite and countable reduced free products of nuclear UCT algebras are classified by their Elliott invariants. The factors may be nonsimple or multitracial, and only their selected traces enter the construction. The classification and morphism-realization assertions are restated and proved in Theorem 9.11.

Write Ell for the Elliott invariant, consisting of ordered K_0 with its distinguished unit, K_1, the tracial state simplex, and the pairing of traces with K_0. A unital morphism of Elliott invariants consists of homomorphisms on the two K-groups, positive and unit preserving in degree zero, and a continuous affine map of trace spaces in the reverse direction, compatible with the pairing. For a nonempty finite or countable family $\mathcal{A} = ((A_i, \tau_i))_{i \in S}$ of unital algebras with faithful selected tracial states, write

$$P_{\mathcal{A}} = *_{\mathrm{r} i \in S}(A_i, \tau_i).$$

Theorem A (Nuclear factors: classification, range and models)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty finite or countable families of separable unital nuclear UCT algebras equipped with faithful tracial states. Put $A = P_{\mathcal{A}} *_{\mathrm{r}} D$ and $B = P_{\mathcal{B}} *_{\mathrm{r}} D$. Then each of A and B is isomorphic to $F *_{\mathrm{r}} D$ for a nuclear UCT core F carrying its aggregate selected pointed data. In particular,*

$$A \cong B \quad \Longleftrightarrow \quad \mathrm{Ell}(A) \cong \mathrm{Ell}(B).$$

Every unital morphism $\mathrm{Ell}(A) \to \mathrm{Ell}(B)$ *is induced by a unital embedding* $A \to B$*. If the invariant morphism is an isomorphism, the realizing map may be chosen to be a* **-isomorphism. If each family contains two distinct factors admitting Haar unitaries, the same conclusion holds for* $P_{\mathcal{A}}$ *and* $P_{\mathcal{B}}$ *without adjoining* D*.*

The class of D*-stabilized products in this theorem is closed under strong Morita equivalence among unital* C^**-algebras. More precisely, every nonzero corner* $p\mathrm{M}_n(A)p$ *is isomorphic to* $E *_{\mathrm{r}} D$ *for a simple, infinite-dimensional, unital, monotracial, nuclear UCT,* $\mathcal{Z}$*-stable core* E *(Corollary 9.12).*

Moreover, the range of the D*-stabilized products consists exactly of monotracial Elliott invariants with countable groups* $\mathrm{K}_0, \mathrm{K}_1$*, a distinguished unit* u*, and a homomorphism* $\rho : \mathrm{K}_0 \to \mathbb{R}$ *satisfying*

$$\rho(u) = 1, \qquad \mathrm{K}_0^+ = \{0\} \cup \{g : \rho(g) > 0\}.$$

Every such invariant is realized by $F *_{\mathrm{r}} \mathcal{Z}$*, where* F *is simple, infinite-dimensional, unital, monotracial, nuclear, UCT and* $\mathcal{Z}$*-stable. Every product* A *also has an inductive-limit model by finite reduced free products of unital type I algebras with faithful traces, with unital trace-preserving injective connecting homomorphisms (Theorem 10.11). If* $\mathrm{K}_0(A)$ *is finitely generated with free rank* r*, at most* $\max\{2, r\}$ *type I factors suffice, even when* $\mathrm{K}_1(A)$ *is not finitely generated (Corollary 10.12). When both groups are finitely generated, a finite product of the commutative standard blocks and homogeneous Moore-space corners also suffices. The commutative subclass has the sharper range and commutative standard models described in Corollary B.*

In the commutative factor case, the invariant range and the standard models admit a more explicit description. For a nonempty finite or countable family $\mathcal{X} = ((X_i, \mu_i))_{i \in S}$ of measured compact metrizable spaces, put

$$P_{\mathcal{X}} = *_{\mathrm{r}\, i \in S}(\mathrm{C}(X_i), \tau_{\mu_i}), \qquad \tau_{\mu_i}(f) = \int_{X_i} f \, d\mu_i.$$

All spaces are nonempty and all measures have full support; atoms are allowed. Evaluation at a point splits every commutative unit class.

Corollary B (Commutative factors: range and standard models)**.** *Let* $\mathcal{X}$ *and* $\mathcal{Y}$ *be nonempty finite or countable families of nonempty compact metrizable spaces with full-support Borel probability measures, and put* $A = P_{\mathcal{X}} *_{\mathrm{r}} D$ *and* $B = P_{\mathcal{Y}} *_{\mathrm{r}} D$*. Then*

$$A \cong B \quad \Longleftrightarrow \quad \mathrm{Ell}(A) \cong \mathrm{Ell}(B).$$

Every unital morphism $\mathrm{Ell}(A) \to \mathrm{Ell}(B)$ *is induced by a unital embedding* $A \to B$*, which preserves the unique traces. If the invariant morphism is an isomorphism, the realizing map may be chosen to be a* **-isomorphism.*

The range consists exactly of monotracial Elliott invariants with trace pairing $\rho : \mathrm{K}_0 \to \mathbb{R}$ *and*

$$\mathrm{K}_0 = F \oplus H, \qquad [1] = u \in F, \qquad \rho(u) = 1, \qquad \rho(H) = 0,$$

where F *is a countable free abelian group with a basis containing* u*,* H *and* K_1 *are arbitrary countable abelian groups, and* $\mathrm{K}_0^+ = \{0\} \cup \{g : \rho(g) > 0\}$*. The restriction of* ρ *to* F *is otherwise arbitrary. Every such invariant is realized in this class.*

Moreover, each of A and B is isomorphic to an inductive limit of finite reduced free products of measured commutative algebras with full-support probability measures and spectra from the list

$$[0,1], \quad [0,1]\sqcup[0,1], \quad S^1, \quad S^2, \quad T_{\mathrm{II},m}, \quad T_{\mathrm{III},m} \quad (m\geq 2).$$

Here $T_{\mathrm{II},m}$ and $T_{\mathrm{III},m}$ are the two- and three-dimensional generalized projective spaces defined in Section 10. The connecting maps are unital trace-preserving embeddings. If both K*-groups of either algebra are finitely generated, that algebra is isomorphic to a single such finite free product. With arbitrary compact metrizable spectra, finite generation of* K_0 *alone suffices for a finite commutative free-product model, by Corollary 10.12. If each family contains two distinct factors admitting Haar unitaries, the same assertions hold for $P_{\mathcal{X}}$ and $P_{\mathcal{Y}}$ without adjoining D. In particular, this last conclusion applies when each family contains at least two nonatomic measures.*

The classification and morphism-realization assertions are restated and proved in Corollary 9.13. The range and model assertions of Theorem A and Corollary B are proved in Section 10.

The preceding existence statements do not give uniqueness of embeddings up to approximate unitary equivalence. Example 4.8 gives two automorphisms of the interval reservoir inducing the same full totally compatible triple that are not approximately unitarily equivalent. The uniqueness theorem for embeddings below requires the source to be nuclear.

For the commutative subclass there is no dimension, connectedness, torsion, or finite-generation restriction on the spaces or their K-groups. The algebras A and B in Corollary B are simple and monotracial, with strict comparison. In particular,

$$\mathrm{K}_0(A)^+ = \{0\}\cup\{g\in\mathrm{K}_0(A) : \tau_{A*}(g)>0\}, \qquad \mathrm{T}(A)=\{\tau_A\}.$$

Thus a unital morphism of their Elliott invariants is precisely a pair of group homomorphisms $\chi_i : \mathrm{K}_i(A)\to\mathrm{K}_i(B)$, $i=0,1$, with

$$\chi_0([1_A])=[1_B], \qquad \tau_{B*}\chi_0=\tau_{A*}.$$

Positivity follows from the strict trace cones. Such a morphism is an isomorphism exactly when both group maps are isomorphisms. The invariant is intrinsic to the product algebra. Its calculation from the individual factors is given in Proposition 2.5. The Haar-unitary condition has a separate role: it permits removal of D by Shlyakhtenko's free-absorption theorem [45]. Without the Haar-unitary hypothesis, these classification statements retain D.

Shlyakhtenko [45] proved that the C*-algebras generated by free semicircular systems of cardinality $n\in\{2,3,\dots,\infty\}$ are mutually isomorphic. His absorption theorem shows that a reduced free product of two faithfully tracial algebras, each containing a Haar unitary, absorbs any finite or countable free semicircular family; the factors need not be nuclear. Combined with Hirshberg–Phillips, this also identifies all reduced free powers of the Jiang–Su algebra with at least two factors. The semicircular absorption theorem is the input used here to remove D under the Haar-unitary hypothesis [45, Theorem 5.2]. In particular, two simple infinite-dimensional factors satisfy the Haar-unitary hypothesis by [47, Corollary B].

The classification also allows more general reservoirs

$$(D, \tau_D) = *_{\mathrm{r},j=1}^{\infty}(R_j, \sigma_j). \tag{1.1}$$

Here the blocks are separable unital nuclear UCT algebras with faithful selected traces, the unit maps induce $\mathrm{K}_0(R_j) = \mathbb{Z}[1_{R_j}]$ and $\mathrm{K}_1(R_j) = 0$, and the traces satisfy the atomic nonconcentration condition in Definition 2.1. For example, the blocks may all be Jiang–Su algebras. The nuclear-core argument is proved directly for these reservoirs; Corollary 4.11 subsequently identifies every such reservoir with the interval product. The later fusion constructions use this interval presentation.

The proof of Theorem A begins with the case of a single nuclear core. We establish this case for the general reservoirs above before passing to several nuclear factors by fusion.

Theorem C (D-stabilization of a nuclear core)**.** *Let D be a reservoir in the sense of Definition 2.1, and let E, F be separable unital nuclear UCT algebras with faithful selected tracial states τ_E, τ_F. Put $A = E*_{\mathrm{r}}D$ and $B = F*_{\mathrm{r}}D$. Let*

$$\chi_i : \mathrm{K}_i(E) \longrightarrow \mathrm{K}_i(F), \qquad i = 0, 1,$$

be group homomorphisms satisfying

$$\chi_0([1_E]) = [1_F], \qquad \tau_{F*}\chi_0 = \tau_{E*}.$$

This pair is induced, through the canonical core inclusions, by a unital trace-preserving embedding $A \to B$. If both group maps are isomorphisms, the realizing map may be chosen to be a *-isomorphism. Equivalently, every unital morphism $\mathrm{Ell}(A) \to \mathrm{Ell}(B)$ is realized by a unital embedding, and every isomorphism is realized by a *-isomorphism. In particular, $A \cong B$ if and only if $\mathrm{Ell}(A) \cong \mathrm{Ell}(B)$. No splitting condition on either unit class is required.

Moreover, the invariant range is exactly that described in Theorem A. Every such invariant is realized by $F_0*_{\mathrm{r}}D$, where F_0 is simple, infinite-dimensional, unital, monotracial, nuclear, UCT and $\mathcal{Z}$-stable.

The invariant here is that of the D-stabilized product. It is computed from the graded K-groups of the core, its unit, and its selected trace pairing; the original core may have other traces. The classification assertion follows from the marked D-stabilization theorem, Theorem 4.7. Its marked form realizes every prescribed unital totally compatible full-invariant morphism, with an isomorphism for an invertible morphism, and can fix any identified finite reservoir-block prefix pointwise. The source order and its other traces are not part of the selected data.

The methodological point of [26] is essential for the proof of Theorem C: Hirshberg and Phillips apply uniqueness to nuclear cores and extend the correcting automorphisms over free complements before approximate intertwining. Our marked D-stabilization argument follows this strategy for nuclear UCT cores with arbitrary K-groups, preserving the selected trace pairing and full-invariant markings. Their first absorption theorem also allows nonexact background factors; our reservoirs have nuclear UCT blocks with scalar pointed K-theory. We answer their circle–annulus isomorphism questions in Example 12.1 and, for exact targets, their uniqueness question in Remark 3.17.

The analytic foundation of Theorem C is the following classification theorem for embeddings with nuclear source. It applies to every separable simple unital infinite-dimensional

exact monotracial target with stable rank one and strict comparison. Neither a free-product presentation nor nuclearity or the UCT is required for the target. At this level, ordinary K-groups and traces must be supplemented by total K-theory and Hausdorffized stable unitary algebraic K_1 (see also [4]). We express the full invariant through the totally compatible triples of [24, Section 6], including the coefficient-to-unitary maps ζ_n^A.

Theorem D (Embeddings)**.** *Let E be a separable unital nuclear UCT algebra with a faithful tracial state τ_E. Let T be a separable simple unital infinite-dimensional exact algebra with unique tracial state τ_T, stable rank one, and strict comparison. Every unital totally compatible triple $(\kappa, \kappa_T, \kappa_\gamma)$ from E to T satisfying $\kappa_T(\tau_T) = \tau_E$ is induced by a unital embedding $E \to T$. Two such embeddings are approximately unitarily equivalent if and only if their full-invariant morphisms agree.*

The precise statement is Theorem 3.16. Its proof combines the trace-kernel ideal methods of [29], the invariant and lifting framework of [4], and the KK-uniqueness theorem of [46]. Working in KL accounts for pure-extension classes when the source K-groups are infinitely generated. The maps ζ_n retain the compatibility between coefficient classes and unitary data in the presence of torsion. Coefficient homotopies and commutator witnesses are preserved through descent from the norm ultrapower to the original target.

These existence and uniqueness results supply the embeddings and unitary corrections for marked nuclear-core replacement. The passage from approximation to isomorphism is an instance of Elliott's one-sided approximate intertwining argument. Its inner-automorphism form is already present in the proof of $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ recorded by Rørdam [42]; see also [43, Proposition 2.3.5] and [49, Section 2]. For infinite reduced free products, Hirshberg and Phillips already implement the local automorphism approximation mechanism, using nuclear-source uniqueness and extension of the correcting conjugacies over free complements [26, proof of Theorem 3.1]; they complete their argument by Elliott approximate intertwining [26, Theorem 2.2]. OpenAI's von Neumann algebra argument uses the analogous mechanism [34, Section 5]. Shlyakhtenko explicitly formulates and names CAAP [45, Proposition 1.2 and Definition 1.4]. Remark 4.4 explains these connections. We use the C*-automorphism approximation property (CAAP) and its isomorphism criterion [45, Proposition 1.2]: trace-preserving automorphisms nearly fix a prescribed finite subset of the source image while moving that image close to prescribed target elements. Our marked version requires every correcting automorphism to act identically on the full invariant. Two applications of nuclear-source uniqueness, with the corrections extended over free complements, verify this property for core replacement. The resulting marked isomorphisms give the isomorphism assertion in Theorem C. For a general invariant morphism, replace the source and target separately by stabilizations of simple monotracial nuclear range cores. Their native core inclusions identify the full invariants. Nuclear-source existence realizes the transported morphism between the two range cores; free extension by id_D and conjugation by the replacement isomorphisms give the required embedding.

Fusion then provides the passage from Theorem C to Theorem A: it replaces a finite or countable family of nuclear factors by one nuclear core with the same pointed graded K-groups and selected pairing.

For the fusion of two factors, we first treat the case in which their K_1-groups vanish. A matrix enlargement gives

$$\mathrm{M}_2(A_1 *_{\mathrm{r}} A_2) \cong (A_1 \oplus A_2) *_{\mathbb{C}^2} \mathrm{M}_2,$$

with the canonical traces and expectations. The direct sum $A_1 \oplus A_2$ is nuclear, and the remaining attachment is finite dimensional. The embedding theorem supplies the coefficient maps needed to compare this amalgamated product with a matrix algebra over a nuclear range model.

To turn this comparison into an isomorphism, we construct a map back to the source and a correcting automorphism of the target. The same map must satisfy two estimates: it must approximately invert the comparison embedding on a prescribed source set, and its composition with the corrected embedding must approximate a prescribed target set. Independent parts of the reservoir make these requirements compatible. Marked automorphism approximation then gives the amplified isomorphism; correction and compression of the distinguished corner yield fusion of the original factors.

The split-unit argument of Section 8, including its Gaussian matrix model [13, 32], separates arbitrary K_1 and reduces the general case to the preceding construction. Iteration gives finite fusion. For countably many factors, each successive fusion is chosen to approximate the preceding identification on an increasing finite set. Summable estimates for both the maps and their inverses ensure that the limiting map is onto.

Throughout the construction we retain compatible total K-theory and unitary data and can fix a finite external reservoir head; Theorem 9.10 states this marked form of fusion. Such control of the maps does not enlarge the final algebra invariant, since unit- and trace-preserving graded K-isomorphisms admit compatible full extensions [4]. Section 5 gives a fuller outline of the construction. Concrete AF and standard Moore-space models are discussed in Section 10.

The general range assertions in Theorems A and C are proved in Theorem 10.1. For the commutative subclass, Theorem 10.4 imposes a condition on the pairing, beyond the splitting of the unit: the pairing must factor through a free abelian direct summand containing the unit as a basis element. The type I models and their standard-block refinements are proved in Theorem 10.11 and Corollary 10.12. The building-block spectra have dimension at most three, and the interval-component masses record the trace pairing. For general nuclear factors, homogeneous corners over finite wedges of two-dimensional Moore spaces also record arbitrary unit classes, including their torsion components. The connecting embeddings are obtained by fusion and nuclear existence.

As a further application, the selected trace pairing determines the exponential length of these free products. For a unital algebra B, write $\mathrm{cel}(B)$ for the supremum of the infimal lengths of unitary paths from 1 to elements of $\mathrm{U}_0(B)$, and $\mathrm{cel}_{\mathrm{CU}}(B)$ for the same supremum restricted to the closed commutator subgroup. Exponential rank $1 + \varepsilon$ means that single exponentials are dense in $\mathrm{U}_0(B)$ but do not exhaust it; see Section 11.

Theorem E (Exponential rank and exponential length)**.** *Let $\mathcal{A}$ be any nonempty finite or countable family of separable unital nuclear UCT algebras with faithful selected traces, put*

$A = P_{\mathcal{A}} *_{\mathrm{r}} D$, and let $H = \tau_{A*}(\mathrm{K}_0(A)) \subseteq \mathbb{R}$. Then

$$\mathrm{cel}(A) = \mathrm{cel}_{\mathrm{CU}}(A) = \begin{cases} 2\pi, & H \text{ is discrete,} \\ \pi, & H \text{ is dense in } \mathbb{R}, \end{cases} \qquad \mathrm{cer}(A) = 1 + \varepsilon.$$

These are the only two possibilities for the length. No splitting condition on the unit classes is needed. The same assertions hold for $P_{\mathcal{A}}$ whenever two distinct factors admit Haar unitaries, and all the displayed values are unchanged by matrix amplification.

In the commutative subclass, connected spectra always give length 2π. For disconnected spectra, irrational values of the trace pairing can give length π. The rank remains $1 + \varepsilon$ in both cases. The 2π upper bound follows from the embedding theorem by lifting the spectral distribution of a unitary to an interval and matching its determinant. Dense trace range gives real rank zero and hence the sharper π bound. Two interval factors provide a unitary whose ambient commutant rules out an exact logarithm.

Section 2 fixes the notation and invariants. Sections 3 and 4 establish the embedding and marked replacement theorems. Section 5 outlines the fusion argument. Sections 6 and 7 establish the logarithm comparison and finite-dimensional relative reservoirs. Section 8 proves split-unit fusion independently and establishes Gaussian matrix and full-corner identifications. Section 9 proves general fusion and the classification assertions of Theorem A, including closure under unital strong Morita equivalence. Section 10 gives the general and commutative ranges, models built from AF factors, and standard free-product models. Exponential rank and exponential length are computed in Section 11; examples follow in Section 12.

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2. Preliminaries

All algebras are separable and unital, unless stated otherwise. Selected traces are faithful unless explicitly stated otherwise. Free products are reduced free products with respect to the displayed traces, and are amalgamated over the scalar units unless an amalgamation algebra is indicated. All isomorphisms of traced algebras preserve their selected traces. For a unital algebra B, let $\mathrm{T}(B)$ denote its tracial state space and put

$$\rho_B : \mathrm{K}_0(B) \longrightarrow \mathrm{Aff}\,\mathrm{T}(B), \qquad \rho_B([p] - [q])(\tau) = (\tau \otimes \mathrm{Tr})(p) - (\tau \otimes \mathrm{Tr})(q).$$

In K_0-pairings, matrix traces are unnormalized. For a selected trace τ_B, the selected state on $\mathrm{M}_n(B)$ is $\tau_B \otimes \mathrm{tr}_n$, where $\mathrm{tr}_n = n^{-1}\,\mathrm{Tr}$. Conditional expectations are denoted by $\mathbb{E}$, with

a subscript specifying the range when needed. We use the Elliott invariant

$$\mathrm{Ell}(B) = (\mathrm{K}_0(B), \mathrm{K}_0(B)^+, [1_B], \mathrm{K}_1(B), \mathrm{T}(B), \rho_B).$$

For a faithfully tracial core (E, τ_E), the comparisons below use its pointed groups and selected trace pairing

$$(\mathrm{K}_0(E), [1_E], \mathrm{K}_1(E), \tau_{E*}). \tag{2.1}$$

Proposition 2.5 explains how these data compute $\mathrm{Ell}(P *_{\mathrm{r}} D)$ from its nuclear factors. The free-product trace and its pairing are always the selected ones. The reservoir D is as in (1.1), with its chosen block presentation. Write

$$D_{\le r} = *_{\mathrm{r}\, j=1}^{r}(R_j, \sigma_j), \qquad D_{>r} = *_{\mathrm{r}\, j>r}(R_j, \sigma_j), \qquad D_{\le 0} = \mathbb{C}.$$

A finite block head and its complementary tail always refer to this presentation until the interval convention following Corollary 4.11 is adopted. We also write $I = (\mathrm{C}([0,1]), \lambda)$ and $D_I = I^{*_{\mathrm{r}}\infty}$.

Definition 2.1 (Reservoirs)**.** For a unital algebra R with faithful tracial state σ, let $\mu_{\sigma,a}$ denote the spectral distribution of $a = a^* \in R$, and put

$$\mathrm{At}(R, \sigma) = \inf_{a=a^*\in R} \sup_{t\in \mathrm{sp}(a)} \mu_{\sigma,a}(\{t\}).$$

A reservoir in this paper is the reduced free product (1.1) of separable unital nuclear UCT blocks with faithful selected traces, such that the unit maps $\mathbb{Z} \to \mathrm{K}_0(R_j)$ are isomorphisms, $\mathrm{K}_1(R_j) = 0$, and

$$\sum_j \Big(1 - \mathrm{At}(R_j, \sigma_j)\Big) = \infty. \tag{2.2}$$

The atom condition is that of [26, Definition 2.5 and Lemma 2.6]. It persists after removing finitely many blocks.

For the support-shifting argument we use a splitting

$$\mathrm{K}_0(E) = \mathbb{Z}[1_E] \oplus G_E. \tag{2.3}$$

Equivalently, the unit class admits a group retraction to $\mathbb{Z}$. Evaluation at a point supplies this splitting for every $\mathrm{C}(X)$. The condition concerns a direct summand, and does not require the remaining group to be free or finitely generated. For a unital algebra B, let $\mathrm{DU}(B)$ denote the subgroup generated by commutators in $\mathrm{U}_0(B)$, and let $\mathrm{CU}(B)$ be its norm closure, as in [24, Definition 2.1].

2.1. **Total K-theory and unitary quotients.** We use total K-theory and its Bockstein operations as in [9, Section 4.1]; see also [24, Definition 6.1] for the coefficient groups. For the coefficient algebras, we use the convention of [4, Sections 2.3 and 3.1]. Thus

$$\underline{\mathrm{K}}(B) = \mathrm{K}_0(B) \oplus \mathrm{K}_1(B) \oplus \bigoplus_{m\ge 2} \Big(\mathrm{K}_0(B; \mathbb{Z}/m) \oplus \mathrm{K}_1(B; \mathbb{Z}/m)\Big).$$

A Λ-morphism respects the grading and all coefficient and Bockstein maps. For a separable UCT source A, the universal multicoefficient theorem [10] identifies

$$\mathrm{KL}(A, B) \cong \mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B)). \tag{2.4}$$

When a target ideal is not σ-unital, we use the separable-subalgebra formulation of KK and KL in [4, Appendix B]; the corresponding multicoefficient theorem is [4, Theorem 8.5]. Every class and each of its countably many witnesses is represented inside an appropriate separable subalgebra. We specify the enlargements needed when using this convention.

Give $\mathrm{U}_\infty(B) = \bigcup_m \mathrm{U}(\mathrm{M}_m(B))$ its topological inductive-limit topology, and put

$$\mathrm{DU}_\infty(B) = \bigcup_m \mathrm{DU}(\mathrm{M}_m(B)), \qquad \mathrm{CU}_\infty(B) = \overline{\mathrm{DU}_\infty(B)}.$$

The closure is taken in that topology. After stabilization it agrees with the closure obtained from commutators of connected unitaries [48, Section 1]. We use the Hausdorffized stable unitary algebraic K_1-group $\mathrm{U}_\infty(B)/\mathrm{CU}_\infty(B)$ of [48, Section 3], with the notation of [24, Definitions 2.4 and 2.7], and write $\bar{u}$ for the class of a stable unitary u. A unital homomorphism $\phi : A \to B$ induces the map

$$\phi^\ddagger : \mathrm{U}_\infty(A)/\mathrm{CU}_\infty(A) \longrightarrow \mathrm{U}_\infty(B)/\mathrm{CU}_\infty(B).$$

The determinant exact sequence [48, Corollary 3.3] is

$$0 \longrightarrow \mathrm{Aff}\,\mathrm{T}(B)/\overline{\rho_B(\mathrm{K}_0(B))} \xrightarrow{\mathrm{Th}_B} \mathrm{U}_\infty(B)/\mathrm{CU}_\infty(B) \xrightarrow{\Pi_B^{\mathrm{cu}}} \mathrm{K}_1(B) \longrightarrow 0. \tag{2.5}$$

Here Π_B^{cu} is the ordinary K_1-class map; we also write Th_B for its determinant-kernel inclusion composed with the quotient map from $\mathrm{Aff}\,\mathrm{T}(B)$. If B has stable rank one, the natural map

$$\mathrm{U}(B)/\,\mathrm{CU}(B) \longrightarrow \mathrm{U}_\infty(B)/\mathrm{CU}_\infty(B) \tag{2.6}$$

is bijective, and the same holds for every matrix algebra over B [24, discussion following Definition 2.6]. The fixed matrix size in this observation is important in the descent argument.

Following [24, Section 6], we describe a morphism of the full invariant from A to B by a *totally compatible triple*

$$(\kappa, \kappa_T, \kappa_\gamma).$$

Here $\kappa : \underline{\mathrm{K}}(A) \to \underline{\mathrm{K}}(B)$ is a unital Λ-morphism, $\kappa_T : \mathrm{T}(B) \to \mathrm{T}(A)$ is continuous affine, and

$$\kappa_\gamma : \mathrm{U}_\infty(A)/\mathrm{CU}_\infty(A) \longrightarrow \mathrm{U}_\infty(B)/\mathrm{CU}_\infty(B)$$

is a group homomorphism. We write $\kappa_\sharp : \mathrm{Aff}\,\mathrm{T}(A) \to \mathrm{Aff}\,\mathrm{T}(B)$ for the unital positive map $f \mapsto f \circ \kappa_T$. Let κ_i and $\kappa_{i,m}$ denote the restrictions of κ to ordinary and coefficient K_i-groups, respectively. Total compatibility means

$$\kappa_\sharp \rho_A = \rho_B \kappa_0, \qquad \kappa_\gamma \mathrm{Th}_A = \mathrm{Th}_B \kappa_\sharp,$$
$$\Pi_B^{\mathrm{cu}} \kappa_\gamma = \kappa_1 \Pi_A^{\mathrm{cu}}, \qquad \kappa_\gamma \zeta_m^A = \zeta_m^B \kappa_{0,m} \quad (m \geq 2),$$

where

$$\zeta_m^A : \mathrm{K}_0(A; \mathbb{Z}/m) \longrightarrow \mathrm{U}_\infty(A)/\mathrm{CU}_\infty(A)$$

are the natural coefficient-to-unitary maps of [24, Remark 6.6]. Write $\mu_m^A : \mathrm{K}_0(A) \to \mathrm{K}_0(A; \mathbb{Z}/m)$ for the coefficient map and $\nu_m^A : \mathrm{K}_0(A; \mathbb{Z}/m) \to \mathrm{K}_1(A)$ for the Bockstein map. With our orientation convention, these maps satisfy

$$\Pi_A^{\mathrm{cu}} \zeta_m^A = \nu_m^A, \qquad \zeta_m^A \mu_m^A = \mathrm{Th}_A \circ \left(m^{-1} \rho_A\right); \tag{2.7}$$

see [4, Propositions 3.1 and 3.2]. A compatible κ_γ is automatically continuous [4, Proposition 2.14]. Under (2.4), we identify κ with its class in $\mathrm{KL}(A,B)$ whenever the source satisfies the UCT. For a unital homomorphism $\phi : A \to B$, the induced triple is

$$([\phi], \phi_T, \phi^\ddagger),$$

where $[\phi]$ denotes the induced Λ-morphism, identified with $\mathrm{KL}(\phi)$ when (2.4) applies. Composition of triples uses the usual order on total K-theory and unitary quotients, and the reverse order on trace spaces. Componentwise injectivity or surjectivity refers to the covariant maps on total K-theory, unitary quotients, and affine trace functions.

Remark 2.2 (Total compatibility). In [24, Definition 6.3], the enlarged triple $(\kappa, \kappa_T, \kappa_\Gamma)$ also records the unitary maps for the coefficient algebras. These additional maps are determined by the basic triple under total compatibility [24, Proposition 6.4(3)]. By [24, Remark 6.6], total compatibility is equivalent to ordinary compatibility together with the displayed ζ_m-relations. Thus we use the basic triple with these relations throughout. Our coefficient-algebra convention is that of [4]; the coefficient-to-unitary maps agree with those of [24] after the interval-orientation identification recorded in [4, Remark 3.4].

2.2. **The regularity class of targets.** Call T an admissible target if it is separable, unital, simple, infinite dimensional, exact and monotracial, and has strict comparison and stable rank one. Its unique trace is denoted τ. A homomorphism is trace preserving when its pullback of τ equals the specified source trace.

We use A_ω for the norm ultrapower associated with a free ultrafilter ω on $\mathbb{N}$, and

$$A_\infty = \ell^\infty(\mathbb{N}, A)/c_0(\mathbb{N}, A)$$

for the norm sequence algebra. These two constructions are kept distinct. For an admissible target T, its unique trace is extremal, so its GNS von Neumann algebra is a finite factor, as is its tracial ultrapower. We first work in the norm ultrapower T_ω and its tracial quotient, and use the sequence algebra only for the final reparameterization step.

Lemma 2.3. *For an admissible target T,*

$$\mathrm{K}_0(T)^+ = \{0\} \cup \{g \in \mathrm{K}_0(T) : \tau_*(g) > 0\}.$$

If $0 < \tau_*(g) < k$, the class g is represented by a projection in $\mathrm{M}_k(T)$.

Proof. Write $g = [p] - [q]$, with p, q in matrix algebras over T. If $\tau_*(g) > 0$, strict comparison gives $q \precsim p$. For projections, this gives a partial isometry v in the stabilization with $v^*v = q$ and $vv^* \le p$. Thus $r = p - vv^*$ is a projection with $[r] = g$. If $\tau_*(g) < k$, compare r with $1_{\mathrm{M}_k(T)}$. The resulting subprojection of $1_{\mathrm{M}_k(T)}$ represents g in $\mathrm{M}_k(T)$. Conversely, a nonzero projection has strictly positive trace, since τ is faithful. □

Lemma 2.4 (Regularity and K-theory of reservoir tails)**.** *Let D be a reservoir and let P be a separable unital exact algebra with a faithful tracial state. For every $r \ge 0$, $P *_{\mathrm{r}} D_{>r}$ is an admissible target. The canonical inclusion*

$$P *_{\mathrm{r}} D_{>r} \longrightarrow P *_{\mathrm{r}} D$$

induces graded and total K*-isomorphisms preserving the unit and selected pairing. It therefore induces a full-invariant isomorphism. If P is nuclear, its canonical inclusion into $P *_{\mathrm{r}} D_{>r}$*

also induces graded and total K*-isomorphisms. In particular,*

$$\mathrm{K}_0(D) = \mathbb{Z}[1_D], \qquad \mathrm{K}_1(D) = 0.$$

Proof. Apply [26, Lemma 2.6] to P together with the reservoir blocks remaining after the finite deletion. Condition (2.2) gives simplicity, unique trace, stable rank one and strict comparison; the same lemma gives infinite dimensionality. Exactness follows from reduced free-product permanence [14].

The faithful traces give GNS-faithful scalar expectations. Hence the ordinary reduced free products here are covered by [20, Theorem 3.1 and Corollary 4.12], even when a partial product is nonnuclear. For every finite head $H = D_{\leq r}$, its scalar free-product K-sequence and the assumptions on the blocks give $\mathrm{K}_0(H) = \mathbb{Z}[1_H]$ and $\mathrm{K}_1(H) = 0$. Applying the same sequence to $(P *_{\mathrm{r}} D_{>r}) *_{\mathrm{r}} H$ proves the assertion about the tail inclusion. Here the unit-relation map is injective because the faithful normalized trace forces every unit class to have infinite order. For nuclear P, iteration and inductive-limit continuity give the assertion about the core inclusion. Coefficient exact sequences give the assertions for total K-theory. The tail and full algebras are monotracial, their pairing ranges agree, and the inclusion preserves their unique traces. The Thomsen sequence therefore makes its algebraic unitary map an isomorphism as well; naturality retains the coefficient-to-unitary maps. This proves the full-invariant assertion. □

2.3. **Computation of the Elliott invariant.**

Proposition 2.5. *Let*

$$P = *_{\mathrm{r}i\in S}(A_i, \tau_i)$$

be a nonempty finite or countable reduced free product of separable unital nuclear UCT algebras with faithful selected traces, and put $A = P *_{\mathrm{r}} D$*. No condition on the unit classes is needed for these calculations. The canonical inclusions identify*

$$\mathrm{K}_0(P) \cong \left(\bigoplus_{i\in S} \mathrm{K}_0(A_i)\right) \Big/ \left\langle [1_{A_i}] - [1_{A_j}] : i, j \in S \right\rangle,$$
$$\mathrm{K}_1(P) \cong \bigoplus_{i\in S} \mathrm{K}_1(A_i).$$

All direct sums are algebraic. The common factor-unit class corresponds to $[1_P]$*, and the selected trace pairing is*

$$\tau_{P*}([(g_i)_i]) = \sum_i (\tau_i)_*(g_i). \tag{2.8}$$

The inclusion $P \to A$ *induces isomorphisms on* K_0 *and* K_1*, preserving the unit and this pairing. Moreover,*

$$\mathrm{K}_0(A)^+ = \{0\} \cup \{g : \tau_{A*}(g) > 0\}, \qquad \mathrm{T}(A) = \{\tau_A\}.$$

Together these formulas compute $\mathrm{Ell}(A)$ *from the factors.*

Proof. The faithful factor traces give GNS-faithful scalar expectations. The scalar-amalgamated free-product K-theory sequence [20, Theorem 3.1 and Corollary 4.12] therefore applies at every binary step, including steps whose first factor is a nonnuclear partial

product. Its unit-relation map is

$$\mathbb{Z} \longrightarrow \mathrm{K}_0(P_0) \oplus \mathrm{K}_0(A_i), \qquad n \longmapsto (n[1_{P_0}], -n[1_{A_i}]).$$

Applying a normalized trace shows that this map is injective. Thus each step contributes no additional K_1-term, without any primitivity assumption on the units. Iteration gives the two group formulas for finite products. The canonical finite-prefix inclusions are injective and have dense union, so inductive-limit continuity gives the formulas for countable products. Each factor trace is normalized, so the sum in (2.8) vanishes on the displayed unit relations and gives the free-product pairing. Since $\mathrm{K}_0(D) = \mathbb{Z}[1_D]$ and $\mathrm{K}_1(D) = 0$, the same sequence shows that adjoining D preserves both groups, the unit, and the pairing. Finally, Lemma 2.4 gives a unique trace and strict comparison for A, and Lemma 2.3 gives its positive cone. This supplies every component of $\mathrm{Ell}(A)$. □

2.4. **Haar unitaries.**

Lemma 2.6 (A Haar unitary)**.** *If M is compact metrizable and μ is a nonatomic probability measure, then $\mathrm{C}(M)$ contains a Haar unitary for τ_μ.*

Proof. Choose real-valued continuous functions f_k with $\|f_k\| \leq 1$ which separate points. For $a = (a_k) \in [0,1]^{\mathbb{N}}$ let

$$h_a(x) = \sum_{k=1}^{\infty} 2^{-k} a_k f_k(x).$$

The series converges uniformly in both a and x, and the resulting function of (a, x) is jointly continuous. Give the parameter space its product Lebesgue measure. If $x \neq y$, choose k with $f_k(x) \neq f_k(y)$. Conditioning on all other coordinates shows that the equation $h_a(x) = h_a(y)$ has parameter probability zero. Because μ has no atoms, the diagonal has $(\mu \times \mu)$-measure zero. Fubini's theorem therefore gives a parameter a such that

$$(\mu \times \mu)\{(x, y) : h_a(x) = h_a(y)\} = 0.$$

The pushforward distribution of this continuous function has no atoms: an atom of mass $c > 0$ would contribute at least c^2 to the displayed collision probability. Its cumulative distribution function F is continuous. The probability integral transform shows that the continuous function $g = F \circ h_a$ has uniform distribution on $[0, 1]$. Therefore $v = \exp(2\pi i g) \in \mathrm{C}(M)$ satisfies $\tau_\mu(v^k) = 0$ for every nonzero integer k, as required. □

3. Embeddings into admissible targets

We extend the trace-kernel approach to the admissible targets of Section 2. The source is nuclear and satisfies the UCT; the target may be nonnuclear. The proof uses KL-uniqueness for absorbing representations and retains coefficient and unitary data through descent to the original target.

The final Theorem B of [4] assumes a nuclear, $\mathcal{Z}$-stable codomain and is used only for nuclear range models below. The theorem for possibly nonnuclear admissible targets proved here instead uses the abstract lifting results of [4], the trace-kernel regularity of [29], and the absorbing-pair uniqueness theorem of [46]. See also [4, Addendum 1.5] for the role of these advances.

3.1. **Classification in the norm ultrapower.** Throughout this section, T is a separable, simple, unital, infinite-dimensional, exact C*-algebra with unique tracial state τ, stable rank one, and strict comparison. We use strict comparison with respect to the canonical unnormalized extensions of τ to matrix algebras. Exactness ensures that there are no additional normalized quasitraces to consider. Fix a free ultrafilter ω on $\mathbb{N}$, and put

$$B = T_\omega, \qquad Q = T^\omega, \qquad J = \ker(q : B \longrightarrow Q).$$

The canonical trace on B and its quotient trace on Q are both denoted by τ. Thus the extension under consideration is

$$0 \longrightarrow J \xrightarrow{j} B \xrightarrow{q} Q \longrightarrow 0. \tag{3.1}$$

Here Q is the tracial ultrapower of the finite factor $\pi_\tau(T)''$, and hence is a II_1 factor. The identification of these two descriptions of Q follows from Kaplansky density; see [29, Section 2.3]. In particular,

$$\mathrm{K}_0(Q) = \mathbb{R}, \qquad \mathrm{K}_1(Q) = 0,$$

where the first identification uses the unnormalized trace on matrix projections. The second assertion follows because every unitary in a finite von Neumann algebra has a bounded Borel self-adjoint logarithm.

We use the total K-theory and stabilized unitary quotients of Section 2. The closure defining $\mathrm{CU}_\infty(C)$ is taken in the topological inductive-limit topology. The canonical map and determinant inclusion are

$$\Pi_C^{\mathrm{cu}} : \mathrm{U}_\infty(C)/\mathrm{CU}_\infty(C) \longrightarrow \mathrm{K}_1(C), \qquad \mathrm{Th}_C : \mathrm{Aff\,T}(C) \longrightarrow \mathrm{U}_\infty(C)/\mathrm{CU}_\infty(C).$$

In particular,

$$\mathrm{K}_0(C) \xrightarrow{\rho_C} \mathrm{Aff\,T}(C) \xrightarrow{\mathrm{Th}_C} \mathrm{U}_\infty(C)/\mathrm{CU}_\infty(C) \xrightarrow{\Pi_C^{\mathrm{cu}}} \mathrm{K}_1(C) \longrightarrow 0 \tag{3.2}$$

is exact when the kernel at $\mathrm{Aff\,T}(C)$ is interpreted as $\overline{\rho_C(\mathrm{K}_0(C))}$. We retain totally compatible triples in the notation of Section 2, including compatibility with the maps ζ_n^C. For a monotracial target, a triple is faithful if its trace component pulls the unique target trace back to a faithful source trace.

The ideal regularity established below permits the use of abstract existence and uniqueness for absorbing representations.

Lemma 3.1. *The following assertions hold.*

(i) *B has stable rank one and strict comparison with respect to its canonical trace.*

(ii) *Every tracial state on B vanishes on J. Consequently B has unique tracial state.*

(iii) *J is separably stable and has almost unperforated Cuntz semigroup. Every separable subalgebra of J is contained in a separable, stable subalgebra of J having almost unperforated Cuntz semigroup, and therefore having the corona factorization property.*

(iv) *Every trace-preserving homomorphism from a unital algebra with faithful trace to B is full.*

Proof. Stable rank one passes to norm ultrapowers with a uniform inverse bound. For a contraction $x_n \in T$ and $\varepsilon > 0$, choose an invertible y_n with $\|y_n - x_n\| < \varepsilon/2$, and write

$y_n = v_n|y_n|$ with v_n unitary. Then

$$z_n = v_n(|y_n| + \varepsilon/2)$$

satisfies $\|z_n - x_n\| < \varepsilon$ and $\|z_n^{-1}\| \leq 2/\varepsilon$, uniformly in n. Thus $(z_n)_\omega$ is invertible in B and approximates $(x_n)_\omega$. The same argument works in each fixed matrix size. We spell out the uniform bound needed for strict comparison, as in [29, proof of Proposition 2.5]. Let $a = (a_n)$ and $b = (b_n)$ be positive contractions in a fixed matrix algebra over B, and suppose $d_\tau(a) < d_\tau(b)$. For each $\varepsilon > 0$, continuous spectral cutoffs give $0 < \delta < 1$ such that

$$d_\tau((a_n - \varepsilon)_+) < d_\tau((b_n - \delta)_+)$$

on a set in ω. Indeed, dominate the support of $(a_n - \varepsilon)_+$ by a continuous function of a_n supported away from zero, and minorize the support of $(b_n - \delta)_+$ by a continuous function of b_n whose limiting trace exceeds $d_\tau(a)$. Coordinate strict comparison and the usual cutdown factorization give v_n with $\|v_n\| \leq 1$,

$$v_n^* v_n = (a_n - 2\varepsilon)_+, \qquad v_n v_n^* \in \mathrm{her}((b_n - \delta)_+).$$

Choose a bounded continuous function f_δ equal to $t^{-1/2}$ on $[\delta, 1]$, with $\|f_\delta\| \leq \delta^{-1/2}$. Then $w_n = f_\delta(b_n)v_n$ satisfies $w_n^* b_n w_n = (a_n - 2\varepsilon)_+$ and $\|w_n\| \leq \delta^{-1/2}$; set $w_n = 0$ off the chosen set. These bounded sequences give $(a - 2\varepsilon)_+ \precsim b$ in the ultrapower. Letting $\varepsilon \downarrow 0$ proves $a \precsim b$. The same argument uses unnormalized traces in every matrix size. Nuclearity or Jiang–Su stability of T is not needed.

If $a \in J_+$ is a contraction, then $d_\tau(a) = 0$, since every positive root of a still belongs to J. Strict comparison in B gives $a^{\oplus k} \precsim 1_B$ for every positive integer k. For any tracial state λ on B, it follows that $k d_\lambda(a) \leq 1$. Thus $\lambda(a) = 0$, and λ factors through Q. A finite factor has unique tracial state, by Dixmier averaging, so $\lambda = \tau$.

The assertions about separable stability and almost unperforation are [29, Propositions 2.4 and 2.5]. One alternates the two separable enlargements and passes to their inductive limit; almost unperforation and stability pass to this limit. A separable stable algebra with almost unperforated Cuntz semigroup has the corona factorization property by [33, Proposition 2.17 and Theorem 5.11]; this consequence is also recorded in [29, Section 2.4].

For the final assertion, let $\phi : A \to B$ preserve a faithful trace, and let $a \in A_+$ be nonzero. Then $d_\tau(\phi(a)) > 0$. For sufficiently large m, strict comparison gives $1_B \precsim \phi(a)^{\oplus m}$. Thus $\phi(a)$ is full in B. □

Lemma 3.2. *Put* $H = \overline{\tau_*(\mathrm{K}_0(T))} \subseteq \mathbb{R}$. *Then*

$$\tau_*(\mathrm{K}_0(B)) = H.$$

In particular, this range is closed.

Proof. A closed subgroup of $\mathbb{R}$ containing 1 is either $\mathbb{R}$ or a discrete cyclic group. If H is discrete, a projection in $\mathrm{M}_k(B)$ lifts to a sequence of projections in $\mathrm{M}_k(T)$. Its trace is an ultralimit of elements of the bounded set $H \cap [0, k]$, and therefore belongs to H. This proves one inclusion; the reverse inclusion follows because a subgroup with discrete closure equals its closure, and constant classes realize the original trace image.

Suppose instead that $H = \mathbb{R}$, and fix $s \in \mathbb{R}$. Choose $g_m \in \mathrm{K}_0(T)$ with $\tau_*(g_m) \to s$. Choose fixed positive integers k, l such that $0 < \tau_*(g_m) + k < l$ for all m, after discarding

finitely many terms. Strict comparison implies that every class of strictly positive trace in $\mathrm{K}_0(T)$ is represented by a projection: if $g = [p] - [r]$ and $\tau(p) > \tau(r)$, first subequate r to p and take the complementary projection. A second comparison with $1_{\mathrm{M}_l(T)}$ shows that a class of trace in $(0, l)$ has a projection representative in $\mathrm{M}_l(T)$. Hence choose $p_m \in \mathrm{M}_l(T)$ with $[p_m] = g_m + k[1_T]$. The class $[(p_m)_\omega] - k[1_B]$ has trace s. □

Proposition 3.3. *Let $s : J^\dagger \to \mathbb{C}1_B$ be the scalar map on the forced unitization. There is a natural isomorphism*

$$\Omega_T : \mathrm{K}_1(J) \xrightarrow{\cong} \mathrm{U}_\infty(B)/\mathrm{CU}_\infty(B), \qquad \Omega_T([u]_1) = \overline{u} - \overline{s(u)}.$$

It identifies $\ker \mathrm{K}_1(j)$ *with* $\ker \Pi_B^{cu}$*, both identified with* $\mathbb{R}/H$*.*

Proof. The argument is the proof of [4, Theorem 6.8], with its hypotheses replaced by the conclusions just proved. We include the details needed here. The formula is additive on matrix unitaries. To see that it vanishes on null-homotopic unitaries, it suffices to check exponentials. For $h = h^* \in \mathrm{M}_k(J^\dagger)$, the difference between h and its scalar part has zero value at every trace on B, by Lemma 3.1. The Thomsen map, or the Lie–Trotter formula modulo closed commutators, therefore gives

$$\overline{e^{2\pi i h}} - \overline{e^{2\pi i s(h)}} = 0.$$

This proves that the formula descends to $\mathrm{K}_1(J)$.

Under $\mathrm{K}_0(Q) = \mathbb{R}$, the six-term sequence and (3.2) give the commuting diagram

$$\begin{array}{ccccccccc}
\mathrm{K}_0(B) & \longrightarrow & \mathbb{R} & \xrightarrow{\partial} & \mathrm{K}_1(J) & \xrightarrow{\mathrm{K}_1(j)} & \mathrm{K}_1(B) & \longrightarrow & 0 \\
\| & & \| & & \downarrow \Omega_T & & \| & & \\
\mathrm{K}_0(B) & \xrightarrow{\tau_*} & \mathbb{R} & \xrightarrow{\mathrm{Th}_B} & \mathrm{U}_\infty(B)/\mathrm{CU}_\infty(B) & \xrightarrow{\Pi_B^{cu}} & \mathrm{K}_1(B) & \longrightarrow & 0.
\end{array}$$

The lower row is exact because $\tau_*(\mathrm{K}_0(B)) = H$ is closed. For the middle square, a projection $p \in \mathrm{M}_k(Q)$ has a positive contraction lift $a \in \mathrm{M}_k(B)$ and $\partial[p] = [e^{2\pi i a}]_1$; its image under the displayed formula is $\mathrm{Th}_B(\tau(a))$. The other squares commute directly. Equivalently, this gives a morphism of short exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{R}/H & \longrightarrow & \mathrm{K}_1(J) & \longrightarrow & \mathrm{K}_1(B) & \longrightarrow & 0 \\
 & & \| & & \downarrow \Omega_T & & \| & & \\
0 & \longrightarrow & \mathbb{R}/H & \longrightarrow & \mathrm{U}_\infty(B)/\mathrm{CU}_\infty(B) & \longrightarrow & \mathrm{K}_1(B) & \longrightarrow & 0.
\end{array}$$

The short five lemma proves that Ω_T is an isomorphism. Its restriction to the two kernels is the identity under their identification with $\mathbb{R}/H$. □

We recall explicitly the two abstract tools used below. First, [4, Theorem 5.16(i),(ii)] is an existence theorem for a separable source and a σ-unital stable ideal I. Its part (ii) requires the source to be nuclear; it does not require a nuclear or Jiang–Su stable codomain. Second, [21, Theorem 2.6] says that, for a separable stable ideal with the corona factorization property, a Busby map is nuclearly absorbing exactly when it is unitizably full. For a nuclear source, nuclear absorption is absorption. The multiplier representation is absorbing if and only if its Busby map is absorbing, by [4, Corollary 5.11].

Lemma 3.4. *In* (3.1)*, or in any finite matrix amplification of that extension (with B, J, Q replaced accordingly), one may pass to a unital separable subextension*

$$0 \longrightarrow I \longrightarrow S \xrightarrow{q_S} D \longrightarrow 0, \qquad I = S \cap J,$$

with I stable and $\mathrm{Cu}(I)$ almost unperforated, subject to the following additional requirements whenever the indicated data are present.

(i) *A prescribed unitizably full map $\theta : A \to Q$ with separable source corestricts to a unitizably full map $\theta_D : A \to D$.*

(ii) *If $\widetilde{\kappa} \in \mathrm{KK}(A, B)$ satisfies $q_*(\widetilde{\kappa}) = \mathrm{KK}(\theta)$, one may retain $\widetilde{\kappa}_S \in \mathrm{KK}(A, S)$ with image $\widetilde{\kappa}$ and $(q_S)_*(\widetilde{\kappa}_S) = \mathrm{KK}(\theta_D)$.*

(iii) *Prescribed separable sets of homomorphism images may be included in S. A prescribed class in $\mathrm{KK}(A, J)$ may be retained in $\mathrm{KK}(A, I)$.*

(iv) *If a prescribed Cuntz pair has zero class in $\mathrm{KL}(A, J)$, the subextension may be chosen so that its class already vanishes in $\mathrm{KL}(A, I)$.*

In particular, I has the corona factorization property.

Proof. This is the separabilization argument of [4, Lemma 7.8 and its proof]; we spell out why Jiang–Su stability is unnecessary here. To retain unitizable fullness, choose a countable dense family in $(A^\dagger)_+$ and all its nonzero rational spectral cutdowns. For each such cutdown, adjoin finitely many witnesses that its image under $\theta^\dagger$ generates the quotient unit as a closed ideal. Use the same quotient unit throughout. These witnesses retain fullness of the forced-unitization map, including its complementary scalar summand when A is unital. The factorization of a KK-class and an equality between its quotient and $\mathrm{KK}(\theta)$ are retained by the direct-limit description

$$\mathrm{KK}(A, C) = \varinjlim_{C_0 \subseteq C \text{ separable}} \mathrm{KK}(A, C_0)$$

of [4, Definition 5.4 and Proposition B.6]. The analogous direct-limit description for KL, [4, Proposition 5.7], retains a zero KL-class after a further separable enlargement. These requirements persist under subsequent enlargement of the subextension.

Starting from such a separable subextension, alternately enlarge its intersection with J to a separable stable subalgebra and to one with almost unperforated Cuntz semigroup, using Lemma 3.1. Include this enlargement in the middle algebra, take its intersection with J again, and continue. If S is the closure of the increasing middle algebras, the closure I of the intervening ideal enlargements is exactly $S \cap J$. Indeed, for $x \in S \cap J$ approximate x by s in an earlier middle algebra. The quotient norm formula gives an element in that earlier algebra's intersection with J within $\|q(s)\|$ plus an arbitrarily small error of s. This intersection was included in the next ideal enlargement. Thus $x \in I$. Stability and almost unperforation pass to the inductive limit. This replaces the separable Jiang–Su stable enlargement in the cited proof; no other step of that proof uses it. The final assertion follows from Lemma 3.1. □

Lemma 3.5. *Let A be separable and nuclear, let $\theta : A \to Q$ be unitizably full, and suppose $\widetilde{\kappa} \in \mathrm{KK}(A, B)$ satisfies $q_*(\widetilde{\kappa}) = \mathrm{KK}(\theta)$. There is a homomorphism $\phi : A \to B$ such that*

$q\phi = \theta$ and $\mathrm{KK}(\phi) = \widetilde{\kappa}$. Moreover, given a lift $\psi : A \to B$ of θ and a class $\delta \in \mathrm{KK}(A, J)$, there is another lift ϕ with $[\phi, \psi]_{\mathrm{KK}(A,J)} = \delta$.

Proof. Choose a subextension as in Lemma 3.4, retaining θ_D and $\widetilde{\kappa}_S$. Let $\mu : S \to M(I)$ be the multiplier map and $\overline{\mu} : D \to Q(I)$ its Busby map. The multiplier map need not be faithful on all of S, but the extension is the pullback

$$S \cong \{(m, d) \in M(I) \oplus D : \pi_I(m) = \overline{\mu}(d)\}.$$

Indeed, lift d to $s \in S$ and correct $m - \mu(s) \in I$. Injectivity follows because μ is faithful on I. The Busby map is unital. Since θ_D is already unitizably full, its forced unitization is full, and composition with the unital $\overline{\mu}$ preserves fullness. Hence $\overline{\mu}\theta_D$ is unitizably full. It is absorbing by the corona factorization property and the absorption criterion recalled above. The commuting pullback square shows that its KK-class factors through $\mathrm{KK}(A, M(I))$, which is zero for stable I; see [4, proof of Theorem 5.16(ii)]. Theorem 5.16(ii) therefore lifts it to an absorbing homomorphism into $M(I)$. Pullback gives a homomorphism $\psi : A \to S$ lifting θ_D.

One must still correct its KK-class. In this same separable extension,

$$\widetilde{\kappa}_S - \mathrm{KK}(\psi) \in \ker\Big(\mathrm{KK}(A, q_S)\Big).$$

Half-exactness in the second variable for nuclear A gives a preimage $\delta' \in \mathrm{KK}(A, I)$; see [4, Proposition 5.5(v)]. Since $\mu\psi$ is absorbing, Theorem 5.16(i) gives an absorbing $\overline{\phi} : A \to M(I)$ with $(\overline{\phi}, \mu\psi)$ a Cuntz pair of class δ'. It has the same Busby quotient as $\mu\psi$, and therefore pulls back to $\phi : A \to S$. Naturality of Cuntz-pair classes, [4, Proposition 5.5(iv)], gives $\mathrm{KK}(\phi) - \mathrm{KK}(\psi) = \widetilde{\kappa}_S - \mathrm{KK}(\psi)$. Thus $\mathrm{KK}(\phi) = \widetilde{\kappa}_S$.

For the relative assertion, separabilize while retaining ψ and δ. The representation $\mu\psi$ is absorbing by the same criterion. Apply Theorem 5.16(i) directly and pull back. Naturality gives the specified relative class after inclusion into J. □

Lemma 3.6. *Let A and I be separable, with I stable. If two absorbing representations $\phi, \psi : A \to M(I)$ form a Cuntz pair with zero class in $\mathrm{KL}(A, I)$, then there are unitaries $v_n \in 1 + I$ such that*

$$\|v_n \phi(a) v_n^* - \psi(a)\| \longrightarrow 0 \qquad (a \in A).$$

Proof. This follows from [46, Theorem A] by the formal proof of [4, Definition 5.6 and proof of Theorem 5.17(ii)]; unlike the theorem stated there, the conclusion here has no Jiang–Su tensor factor. For completeness, KL-vanishing supplies $\eta \in \mathrm{KK}(A, \mathrm{C}(\overline{\mathbb{N}}, I))$, where $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$, whose evaluations at finite integers are zero and whose evaluation at infinity is $[\phi, \psi]$. The constant representation $\Psi(a)(n) = \psi(a)$ is absorbing into $M(\mathrm{C}(\overline{\mathbb{N}}, I)) = \mathrm{C}_{\mathrm{strict}}(\overline{\mathbb{N}}, M(I))$ by [8, Proposition 3.2], applied to the absorbing representation ψ and the separable unital nuclear algebra $\mathrm{C}(\overline{\mathbb{N}})$. Abstract KK existence gives an absorbing Φ with Cuntz-pair class η relative to Ψ. Write $\phi_n = \mathrm{ev}_n \Phi$. Since $\Phi(a) - \Psi(a) \in \mathrm{C}(\overline{\mathbb{N}}, I)$, one has $\|\phi_n(a) - \phi_\infty(a)\| \to 0$, not merely strict convergence. Each ϕ_n is absorbing, since its corona map equals that of ψ. Naturality and additivity of Cuntz-pair classes give, for finite n,

$$[\phi_n, \psi]_{\mathrm{KK}(A,I)} = (\mathrm{ev}_n)_*(\eta) = 0, \qquad [\phi, \phi_\infty]_{\mathrm{KK}(A,I)} = [\phi, \psi]_{\mathrm{KK}(A,I)} - (\mathrm{ev}_\infty)_*(\eta) = 0.$$

These two KK-equalities are the inputs to the uniqueness theorem [46, Theorem A], which gives approximate conjugacies by unitaries in $1 + I$ from ϕ_n to ψ for finite n, and from ϕ to ϕ_∞, since the corresponding KK-classes vanish. For any fixed finite set and tolerance, first choose n so that ϕ_n and ϕ_∞ are close, and then compose these two conjugacies. Exhausting the source proves the assertion. □

Proposition 3.7. *Let A be separable and satisfy the UCT. There is a natural isomorphism*

$$\mathcal{R}_{A,T} : \ker\Big(\mathrm{KL}(A, j) : \mathrm{KL}(A, J) \to \mathrm{KL}(A, B)\Big) \xrightarrow{\cong} \mathrm{Hom}\Big(\mathrm{K}_1(A)/\operatorname{Tor}\mathrm{K}_1(A), \mathbb{R}/H\Big). \quad (3.3)$$

For a unital Cuntz pair $\phi, \psi : A \to B$ with relative class λ in this kernel, it satisfies

$$\Big(\phi^\ddagger - \psi^\ddagger\Big)(x) = \mathcal{R}_{A,T}(\lambda)\Big(\Pi_A^{cu}(x) + \operatorname{Tor}\mathrm{K}_1(A)\Big). \quad (3.4)$$

Proof. Apply the UMCT, [4, Theorem 8.5], to identify KL with morphisms of total K-theory, naturally in the second variable. This applies to the nonseparable ideals here through the separable-subalgebra convention of Appendix B of that paper. The quotient Q has $\mathrm{K}_0(Q) = \mathbb{R}$, $\mathrm{K}_1(Q) = 0$, and all its finite-coefficient K-groups vanish. Here is the relevant algebraic kernel calculation explicitly. The six-term sequence gives injectivity of $\mathrm{K}_0(j)$. Since every finite-coefficient group of Q is zero, every finite-coefficient map induced by j is an isomorphism. Moreover

$$V := \ker \mathrm{K}_1(j) \cong \mathbb{R}/H$$

is divisible. A total-K morphism killed by j therefore has zero K_0 component and zero coefficient components; only its ordinary K_1 component can remain, with image in V. Bockstein naturality forces that component to vanish on torsion, since each n-torsion class is in a Bockstein image. Conversely a map $\mathrm{K}_1(A)/\operatorname{Tor}\mathrm{K}_1(A) \to V$, together with zero maps on all other components, respects reduction modulo n because V is divisible, and respects Bocksteins because it kills torsion. All other coefficient squares are zero. Thus the kernel calculation of [4, Lemma 2.18] gives

$$\ker \mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(j)) \cong \mathrm{Hom}\Big(\mathrm{K}_1(A)/\operatorname{Tor}\mathrm{K}_1(A), \ker \mathrm{K}_1(j)\Big),$$

by the ordinary K_1 component. Compose with Proposition 3.3 to obtain (3.3). This is exactly the proof of [4, Theorem 8.7]; the nuclear and Jiang–Su stable codomain in that statement is used there to obtain the quotient K-theory and the isomorphism Ω_T, which have been proved independently here.

For a matrix unitary u in A, the K_1 component of the relative Cuntz-pair class is $[\phi(u)\psi(u)^*]_1 \in \mathrm{K}_1(J)$, using the scalar-unit convention; see [4, Proposition 8.3]. Applying Ω_T gives $\overline{\phi(u)} - \overline{\psi(u)}$. This proves (3.4). □

Theorem 3.8. *Let T be an admissible target, let ω be a free ultrafilter on $\mathbb{N}$, and put $B = T_\omega$. Let A be separable, unital, nuclear, and satisfy the UCT, and let λ be a faithful tracial state on A. Then:*

(i) *Every unital totally compatible triple $(\kappa, \kappa_T, \kappa_\gamma)$ from A to B satisfying $\kappa_T(\tau) = \lambda$ is realized by a unital embedding $A \to B$.*

(ii) *Two unital embeddings $A \to B$ inducing the same triple and the trace λ are unitarily equivalent.*

No vanishing, torsion, or finite-generation assumption is imposed on either K*-group of* A *or* T*.*

Proof. We first record the quotient-level input. The finite von Neumann algebra $\pi_\lambda(A)''$ is hyperfinite, by nuclearity and the hyperfiniteness theorem of [6]. It admits a normal trace-preserving embedding into the II_1 factor Q. One can obtain this by successively embedding its increasing finite-dimensional approximating algebras, using unitary conjugacy of trace-preserving embeddings of finite-dimensional algebras into a factor, and then taking bounded 2-norm limits. Its restriction gives a unital embedding $\sigma : A \to Q$ with trace λ. Moreover, any two embeddings $A \to Q$ inducing λ extend normally to trace-preserving embeddings of $\pi_\lambda(A)''$: the trace identities give L^2 isometries, and bounded L^2 approximation gives the extensions. They need not have factorial domain. The finite-dimensional approximation just described first gives approximate conjugacy in 2-norm on finite sets. Countable saturation of the tracial ultrapower, equivalently the usual 2-norm epsilon test, gives one unitary implementing the conjugacy on all of A.

Existence of total K*-theory and trace.* Via the UMCT, regard κ as a class in $\mathrm{KL}(A, B)$ and choose a representative $\widetilde{\kappa} \in \mathrm{KK}(A, B)$. Its image in $\mathrm{KK}(A, Q)$ equals $\mathrm{KK}(\sigma)$. Indeed, Q is unital and hence σ-unital, so the UCT applies. Since $\mathrm{K}_0(Q) = \mathbb{R}$ is divisible and $\mathrm{K}_1(Q) = 0$, the extension terms vanish and

$$\mathrm{KK}(A, Q) \cong \mathrm{Hom}(\mathrm{K}_0(A), \mathbb{R}).$$

Trace compatibility gives equality in this group, not merely equality after passage to KL. The map σ itself is unital, so it is not unitizably full as a map from the unital source A. To apply Lemma 3.5, use instead

$$\sigma_0(a) = \begin{pmatrix} \sigma(a) & 0 \\ 0 & 0 \end{pmatrix} : A \longrightarrow \mathrm{M}_2(Q), \qquad \widetilde{\kappa}_0 = (e_{11})_*(\widetilde{\kappa}).$$

The forced unitization of σ_0 maps $A \oplus \mathbb{C}$ to $\mathrm{M}_2(Q)$ by $(a, z) \mapsto \mathrm{diag}(\sigma(a), z)$, so it is full. Here and below, the displayed coordinates identify the forced unitization of a unital algebra with its direct sum with $\mathbb{C}$. The lemma gives $\rho : A \to \mathrm{M}_2(B)$ lifting σ_0 with class $\widetilde{\kappa}_0$. Set $p = \rho(1_A)$. Unitality of the prescribed class gives $[p] = [e_{11}] \in \mathrm{K}_0(\mathrm{M}_2(B))$. Stable rank one gives cancellation, hence Murray–von Neumann equivalence of these projections and also of their complements. Thus they are unitarily equivalent in $\mathrm{M}_2(B)$. Conjugate ρ into the e_{11} corner and identify that corner with B. The resulting unital homomorphism $\psi : A \to B$ has total K-theory map κ and trace λ. Faithfulness of λ makes it injective.

Correction of algebraic K_1*.* Put $d = \kappa_\gamma - \psi^\ddagger$. Compatibility with the Thomsen maps shows that d vanishes on $\mathrm{im}(\mathrm{Th}_A)$, and compatibility with Π^{cu} shows that its image is contained in $\ker \Pi^{cu}_B = \mathbb{R}/H$. Hence it factors as a map $r : \mathrm{K}_1(A) \to \mathbb{R}/H$. It vanishes on torsion. Indeed, an element killed by n is in the image of the Bockstein map $\mathrm{K}_0(A; \mathbb{Z}/n) \to \mathrm{K}_1(A)$; the factorization of that Bockstein map through ζ^A_n, and total compatibility of both triples, give $d\zeta^A_n = 0$. This is the algebraic argument in [4, proof of Theorem 8.9(ii)].

Write $\overline{r} : \mathrm{K}_1(A)/\operatorname{Tor}\mathrm{K}_1(A) \to \mathbb{R}/H$ for the induced map. Choose $\eta \in \ker \mathrm{KL}(A, j)$ with $\mathcal{R}_{A,T}(\eta) = \overline{r}$, and choose a representative $\delta \in \mathrm{KK}(A, J)$ of η. Apply the relative assertion of Lemma 3.5 to $\psi_0 = \mathrm{diag}(\psi, 0)$ and the corner image of δ. This gives $\rho_0 : A \to \mathrm{M}_2(B)$ with

the same quotient as ψ_0 and relative class equal to that corner image. Its absolute total K-theory map is unchanged, since η maps to zero in $\mathrm{KL}(A, B)$. In particular, $[\rho_0(1_A)] = [e_{11}]$.

For clarity, the algebraic K_1 calculation does not treat a nonunital image of a unitary as a unitary. Write $p_0 = \rho_0(1_A)$ and extend the two maps to unital maps on $A^\dagger = A \oplus \mathbb{C}$:

$$\rho_0^\dagger(a, z) = \rho_0(a) + z(1 - p_0), \qquad \psi_0^\dagger(a, z) = \mathrm{diag}(\psi(a), z).$$

For $u \in \mathrm{U}_k(A)$, the unitary $(u, 1_k) \in \mathrm{U}_k(A \oplus \mathbb{C})$ is sent to $\rho_0^{(k)}(u) + 1 - p_0^{(k)}$ and $\mathrm{diag}(\psi^{(k)}(u), 1_k)$, respectively. The K_1 component of the relative class of these unitized maps is the K_1 component of δ, under $\mathrm{K}_1(A^\dagger) = \mathrm{K}_1(A)$. The formula defining Ω_T, or the same calculation as (3.4), therefore gives the difference $r([u]_1)$ for their algebraic K_1 classes. Now conjugate p_0 to e_{11} by a unitary in $\mathrm{M}_2(B)$, as in the preceding paragraph, and compress ρ_0 to a unital map $\phi : A \to B$. Under this conjugacy, the first of the displayed unitary images becomes $\mathrm{diag}(\phi^{(k)}(u), 1_k)$. Inner conjugacy and matrix stabilization preserve the algebraic K_1 class. Consequently $\phi^\ddagger = \psi^\ddagger + d = \kappa_\gamma$. Trace and total K-theory are unchanged. This proves existence.

Uniqueness. Let $\phi, \psi : A \to B$ induce the same prescribed triple. Their quotient maps are unitarily conjugate by the quotient-level assertion above. Every unitary in the finite von Neumann algebra Q is an exponential of a bounded self-adjoint element, and therefore lifts to a unitary in B. After conjugating ψ, we may assume $q\phi = q\psi$. Their relative class $\eta = [\phi, \psi]_{\mathrm{KL}(A,J)}$ maps to zero in $\mathrm{KL}(A, B)$, by the UMCT and equality of total K-theory. Equality on algebraic K_1 and (3.4) imply $\mathcal{R}_{A,T}(\eta) = 0$. Thus $\eta = 0$.

Stabilize the two maps to $\phi_0 = \mathrm{diag}(\phi, 0)$ and $\psi_0 = \mathrm{diag}(\psi, 0)$. Their quotient maps are unitizably full, and their relative KL-class is zero. Lemma 3.4 produces a subextension retaining this zero class and their images, whose ideal I is stable and has the corona factorization property. The two multiplier representations are absorbing. Apply Lemma 3.6 to obtain unitaries $w_n \in 1 + I$ giving approximate conjugacy of those representations. This is also approximate conjugacy in the middle algebra: the difference after conjugation belongs to I, and the multiplier map is isometric on I.

Evaluation at 1_A gives $w_n e_{11} w_n^* \to e_{11}$. The standard close-projection unitary, formed by polar decomposition of $e_{11}(w_n e_{11} w_n^*) + (1 - e_{11})(1 - w_n e_{11} w_n^*)$, belongs to $1 + I$ and converges to 1. Multiplying w_n by this correcting unitary makes it commute with e_{11} exactly without changing the limiting conjugacy. Its upper-left corner is then a unitary in B approximately conjugating ϕ to ψ. Finally, a norm-ultrapower epsilon test on a countable dense subset of A turns approximate unitary equivalence into unitary equivalence in B. □

Remark 3.9. Theorem 3.8 is an ultrapower theorem. Obtaining an embedding into T itself additionally requires the coordinate control and reparameterization argument proved separately. In particular, neither coordinate convergence in a naively stabilized norm quotient nor an unproved interchange of total K-theory with ultraproducts is used in the proof above.

3.2. **Coefficient groups and descent to the original target.** Throughout this section, T is an admissible target, with unique tracial state τ. Put

$$G = \mathrm{K}_0(T), \qquad H_1 = \mathrm{K}_1(T), \qquad \sigma = \tau_* : G \longrightarrow \mathbb{R}, \qquad u = [1_T]_0.$$

For a free ultrafilter ω on $\mathbb{N}$, write

$$T_\omega = \ell^\infty(T)/c_\omega(T), \qquad c_\omega(T) = \{(x_m) : \lim_\omega \|x_m\| = 0\}.$$

We use the facts, established above, that T_ω has stable rank one and a unique tracial state τ_ω. We also use the computation

$$\mathrm{K}_1(T_\omega) \;\cong\; (H_1)_\omega \tag{3.5}$$

from [29, Corollary 4.2]. Here and below, an ultrapower of an abelian group is the algebraic ultrapower, with equality on an ω-large set. The uniform homotopy control behind this computation is supplied by [29, Theorem 1.2]. For injectivity, stable rank one represents a class in $\mathrm{K}_1(T_\omega)$ by a unitary $u \in T_\omega$. Lift u to coordinate unitaries $u_m \in T$. If its coordinate class vanishes, replace u_m by 1_T outside an ω-large set, so that $[u_m]_1 = 0$ for every m. The cited theorem gives $h_m, k_m \in \mathrm{M}_2(T)_{\mathrm{sa}}$ with $\|h_m\|, \|k_m\| \le \pi$ and

$$\|u_m \oplus 1_T - e^{ih_m}e^{ik_m}\| < 1/m.$$

Consequently $u \oplus 1_T = e^{ih}e^{ik}$ in $\mathrm{M}_2(T_\omega)$, where $h = (h_m)_\omega$ and $k = (k_m)_\omega$. The path $t \mapsto e^{ith}e^{itk}$ proves injectivity of the coordinate K_1-map. Stable rank one also permits representatives in $\mathrm{U}(T)$ for arbitrary coordinate classes, which gives surjectivity. The uniform estimate is an external input to the present argument.

Lemma 3.10 (Bounded-trace description of K_0)**.** *The coordinate map identifies* $\mathrm{K}_0(T_\omega)$ *with the subgroup*

$$G_b = \left\{ [(g_m)] \in G_\omega : \sup_m |\sigma(g_m)| < \infty \text{ for some representative } (g_m) \right\}.$$

Under this identification,

$$(\tau_\omega)_*([(g_m)]) = \lim_\omega \sigma(g_m).$$

In particular,

$$(\tau_\omega)_*(\mathrm{K}_0(T_\omega)) = \overline{\sigma(G)}. \tag{3.6}$$

Proof. Projection representatives in a fixed matrix algebra over T_ω lift to sequences of projections in the same matrix algebra over T. Consequently, a difference of projection classes in $\mathrm{K}_0(T_\omega)$ has coordinate traces bounded uniformly in absolute value.

The coordinate map is injective. Indeed, suppose that projections $p, q \in \mathrm{M}_d(T_\omega)$ lift to projections $p_m, q_m \in \mathrm{M}_d(T)$ and that $[p_m]_0 = [q_m]_0$ for ω-many m. Cancellation, which follows from stable rank one, gives Murray–von Neumann equivalence of p_m and q_m in $\mathrm{M}_d(T)$ for those m. Choose implementing partial isometries in $\mathrm{M}_d(T)$; their norms are at most one. Their ultrapower therefore implements $p \sim q$. Padding with zero projections first handles differences represented in different matrix sizes.

For surjectivity, let $|\sigma(g_m)| \le C$ and choose an integer $k > C + 1$. Strict comparison of projections implies

$$\{g \in G : \sigma(g) > 0\} \subseteq \mathrm{K}_0(T)_+.$$

For completeness, if $g = [p]_0 - [q]_0$ has strictly positive trace, then $q \precsim p$, and the difference is represented by the complement of a subprojection of p equivalent to q. Thus $g_m + ku$ is represented by a projection r_m in some matrix algebra over T. Its trace is strictly between zero and $2k$. Comparing r_m with 1_{2k}, or with 1_{2k+1} if desired to keep an unused rank, moves

it by equivalence into the fixed matrix algebra $\mathrm{M}_{2k+1}(T)$. The difference $[(r_m)]_0 - [1_k]_0$ consequently represents $[(g_m)]$ in $\mathrm{K}_0(T_\omega)$.

The trace formula follows from projection representatives. Its range is contained in the right-hand side of (3.6), because a bounded sequence of points in the closed set $\overline{\sigma(G)}$ has its ultralimit in that set. Conversely, if $t \in \overline{\sigma(G)}$, choose $g_m \in G$ such that $\sigma(g_m) \to t$. This sequence has bounded trace and represents a class with trace t. □

We next justify all uses of coefficient classes in coordinate maps. For $n \geq 2$, fix the usual nuclear coefficient algebra D_n with $\mathrm{K}_0(D_n) = \mathbb{Z}/n$ and $\mathrm{K}_1(D_n) = 0$, obtained by suspending the mapping cone of the unital inclusion $\mathbb{C} \to \mathrm{M}_n$. Write $\mathrm{K}_i(B; \mathbb{Z}/n) = \mathrm{K}_i(B \otimes D_n)$. This convention gives the usual Bockstein exact sequence.

Lemma 3.11 (Finite coefficients commute with the ultrapower)**.** *For every $n \geq 2$ and $i = 0, 1$, the canonical coordinate map is an isomorphism*

$$\eta_{i,n} : \mathrm{K}_i(T_\omega; \mathbb{Z}/n) \longrightarrow \mathrm{K}_i(T; \mathbb{Z}/n)_\omega. \tag{3.7}$$

The ordinary and coefficient coordinate maps commute with the Bockstein operations and the changes of coefficient.

Proof. We first describe the map rather than assuming that tensor products commute with ultrapowers. Exactness of D_n gives an exact sequence

$$0 \longrightarrow c_\omega(T) \otimes D_n \longrightarrow \ell^\infty(T) \otimes D_n \longrightarrow T_\omega \otimes D_n \longrightarrow 0.$$

Coordinate evaluation induces a $*$-homomorphism from the middle algebra to $\ell^\infty(T \otimes D_n)$. It sends the left-hand ideal into $c_\omega(T \otimes D_n)$: this follows first for finite sums of elementary tensors and then by norm approximation. It therefore induces a canonical $*$-homomorphism

$$T_\omega \otimes D_n \longrightarrow (T \otimes D_n)_\omega. \tag{3.8}$$

We do not assert that this map is surjective as a map of algebras.

For any C*-algebra B, there are natural coordinate maps $\mathrm{K}_i(B_\omega) \to \mathrm{K}_i(B)_\omega$. Here is the required elementary verification. Lift a projection, or a unitary, in a fixed matrix algebra over the appropriate unitization of B_ω to a bounded sequence of self-adjoint elements, or arbitrary elements, over B. The projection or unitary defects tend to zero along ω. Spectral cut-down, or polar decomposition, gives coordinate projections or unitaries on an ω-large set, in the same matrix size. The scalar part can be kept fixed when B is nonunital.

Independence of representatives and homotopy invariance require no uniform homotopy-length assertion. For a given continuous homotopy in a fixed matrix algebra over B_ω, choose a finite subdivision so that successive values differ in norm by less than $1/8$. Lift these finitely many values and round them as above. On the intersection of finitely many ω-large sets, successive coordinate projections, or unitaries, differ by less than $1/2$. They can be joined by the usual short projection homotopies, or short logarithm paths. The endpoints are the chosen coordinate representatives. Stabilizations used in a given K-theory equality have a fixed finite matrix size and are treated in exactly the same way. This proves that the coordinate maps are well defined. Applying this construction after (3.8) defines $\eta_{i,n}$. We spell out the boundary compatibility used below. Let C_{ι_n} be the cone of $\iota_n : \mathbb{C} \to \mathrm{M}_n$, so

$D_n = SC_{\iota_n}$. For every algebra X the cone extension is

$$0 \longrightarrow S\mathrm{M}_n(X) \longrightarrow X \otimes C_{\iota_n} \longrightarrow X \longrightarrow 0.$$

Apply this first with $X = T_\omega$. There is a morphism from this extension to the norm ultraproduct of the corresponding extensions for T, using the same elementary tensor maps as above. The lower extension is exact: surjectivity follows from the uniformly contractive cone lifts $x \mapsto [t \mapsto \chi(t)(x \otimes 1_n)]$, where $\chi(0) = 1$ and χ vanishes near 1. If a bounded coordinate sequence has quotient norms tending to zero along ω, lift it into the coordinate ideals with errors tending to zero and uniformly bounded norms. The exponential and index formulas for the boundary maps commute with these coordinate maps. Each formula for a particular class uses one fixed matrix size, finitely many lifts, and functional calculus with a fixed spectral cutoff, so it may be evaluated coordinatewise. Suspension and the standard Bott identification also commute with these maps; a fixed Bott representative and finite approximation of its paths suffice. Thus the induced diagram of Bockstein sequences commutes. The standard cone diagrams defining changes of coefficient, with their fixed matrix amplifications and suspension identifications, give the other required naturality squares in the same way. This argument does not assert surjectivity of any algebra-level tensor map.

For the group calculation, Lemma 3.10 gives

$$G_b/nG_b \cong (G/nG)_\omega, \qquad \mathrm{Tor}_n(G_b) \cong (\mathrm{Tor}_n G)_\omega. \tag{3.9}$$

For the first surjectivity, replace an arbitrary representative g_m of a class in $(G/nG)_\omega$ by

$$g_m - n \left\lfloor \frac{\sigma(g_m)}{n} \right\rfloor u.$$

Its trace belongs to $[0, n)$. For injectivity, if $g_m = nh_m$ on an ω-large set and $(\sigma(g_m))$ is bounded, then $\sigma(h_m) = \sigma(g_m)/n$ there; modifying the sequence outside that set makes (h_m) a bounded-trace representative. The second isomorphism follows because all torsion classes have trace zero.

For any abelian group L, coordinate selection gives

$$L_\omega/nL_\omega \cong (L/nL)_\omega, \qquad \mathrm{Tor}_n(L_\omega) \cong (\mathrm{Tor}_n L)_\omega.$$

Use this with $L = H_1$, together with (3.5). The canonical maps now give a commutative diagram between the short exact Bockstein sequence

$$0 \longrightarrow \mathrm{K}_i(T_\omega)/nK_i(T_\omega) \longrightarrow \mathrm{K}_i(T_\omega; \mathbb{Z}/n) \longrightarrow \mathrm{Tor}_n(\mathrm{K}_{i-1}(T_\omega)) \longrightarrow 0$$

and the algebraic ultrapower of the corresponding sequence for T. Algebraic ultrapowers preserve exactness of this finite sequence, as preimages can be chosen coordinatewise. The two outside vertical maps are isomorphisms by the preceding calculations. The middle map is consequently an isomorphism. □

Lemma 3.12 (Witnesses in a fixed matrix algebra)**.** *Let $d \geq 1$, and let $w \in \mathrm{U}_d(T_\omega)$. If $\overline{w} = 0$ in $\mathrm{U}_\infty(T_\omega)/\mathrm{CU}_\infty(T_\omega)$, then, for each $k \geq 1$, there are an integer L_k and unitaries*

$a_{k,\ell}, b_{k,\ell} \in \mathrm{U}_d(T_\omega)$, $1 \le \ell \le L_k$, *such that*

$$\left\| w - \prod_{\ell=1}^{L_k} a_{k,\ell} b_{k,\ell} a_{k,\ell}^* b_{k,\ell}^* \right\| < 2^{-k-1}.$$

The matrix size d is independent of k; the number L_k need not be bounded. If $w = [(w_m)]$ with $w_m \in \mathrm{U}_d(T)$, these witnesses can be lifted to coordinate unitaries in $\mathrm{U}_d(T)$ so that the corresponding inequality with 2^{-k} holds on an ω-large set.

Proof. Apply the stable-rank-one identification

$$\mathrm{U}(\mathrm{M}_d(T_\omega))/\overline{\mathrm{DU}(\mathrm{M}_d(T_\omega))} \cong \mathrm{U}_\infty(\mathrm{M}_d(T_\omega))/\mathrm{CU}_\infty(\mathrm{M}_d(T_\omega)) \cong \mathrm{U}_\infty(T_\omega)/\mathrm{CU}_\infty(T_\omega);$$

see [4, Remark 2.8]. Thus w lies in the norm closure of the commutator subgroup of $\mathrm{U}_d(T_\omega)$ itself. This gives all the approximations in the single matrix algebra $\mathrm{M}_d(T_\omega)$. Recall that the stable Hausdorffized unitary quotient kills commutators of arbitrary stable unitaries, consistently with the connected-commutator convention of Section 2. By injectivity of the displayed stable-rank-one identification, the same is true in this fixed matrix algebra modulo $\mathrm{CU}(\mathrm{M}_d(T_\omega))$. Unitaries lift to coordinate unitaries in a norm ultrapower. For fixed k, lift its finitely many witnesses and use the strict norm inequality to obtain the coordinate assertion. The coordinate witnesses need not be connected to the identity; their commutators nevertheless have zero stable unitary-quotient class. □

Remark 3.13. The fixed matrix size in the preceding lemma is essential. Merely giving, for each accuracy, an approximation after some possibly larger matrix amplification does not establish membership in the closure of the commutator subgroup for the topological inductive limit defining $\mathrm{U}_\infty/\mathrm{CU}_\infty$. We retain a separate finite list of witnesses for each accuracy, always in the same matrix algebra. No uniform bound on commutator length is used.

Lemma 3.14 (The diagonal invariant is injective)**.** *For every free ultrafilter ω, the diagonal embedding $\iota_\omega : T \to T_\omega$ induces injective maps on total* K*-theory, affine trace functions, and the stabilized unitary quotient.*

Proof. Injectivity on ordinary K-theory follows from Lemma 3.10 and (3.5). Injectivity with finite coefficients follows from Lemma 3.11. The affine trace map is the identity on $\mathbb{R}$, since both algebras have the unique trace under consideration.

For algebraic K_1, let $v \in \mathrm{U}_d(T)$ have diagonal image of zero class. Lemma 3.12 gives, for each accuracy, finitely many commutator witnesses in $\mathrm{U}_d(T_\omega)$. Lift them and choose one coordinate where the asserted approximation holds. Hence v is approximable by products of commutators in $\mathrm{U}_d(T)$ and $\overline{v} = 0$. □

The following descent argument is deliberately phrased in terms of the ultrapower existence and uniqueness results, rather than an unstated finite-set classification theorem. It explains all diagonal choices, including the coefficient and unitary data.

Proposition 3.15 (Descent of a prescribed totally compatible triple)**.** *Let A be a unital separable nuclear UCT algebra, and let $\Xi = (\kappa, \kappa_T, \kappa_\gamma)$ be a faithful unital totally compatible triple from A to T. Use the following two conclusions of Theorem 3.8:*

(i) *for some free ultrafilter ω, there is a unital $*$-homomorphism $\Phi : A \to T_\omega$ whose induced triple is the composite of Ξ with the triple induced by the diagonal embedding $\iota_\omega : T \to T_\omega$;*

(ii) *for every free ultrafilter ν, unital full maps $A \to T_\nu$ inducing the same triple are unitarily equivalent.*

Then there is a unital embedding $\rho : A \to T$ satisfying $([\rho], \rho_T, \rho^\ddagger) = \Xi$.

Proof. Since T is monotracial, the trace component specifies $s = \kappa_T(\tau) \in \mathrm{T}(A)$, and faithfulness of Ξ says that s is faithful. Nuclearity and the Choi–Effros lifting theorem give ucp lifts $\phi_m : A \to T$ of Φ. A cpc lift can be made unital by adding $s(a)(1 - \phi_m(1))$ if necessary; this does not alter the quotient map. Thus

$$\lim_\omega \|\phi_m(ab) - \phi_m(a)\phi_m(b)\| = 0, \qquad \lim_\omega \tau(\phi_m(a)) = s(a) \quad (a, b \in A). \tag{3.10}$$

Choose countable dense test sets for A and all its finite matrix algebras. Enumerate representatives of the countable total K-theory group $\underline{\mathrm{K}}(A)$. For coefficient classes, these representatives are projections and unitaries over $A \otimes D_n$ with their prescribed scalar parts. Applying $\phi_m \otimes \mathrm{id}_{D_n}$ and rounding gives their partial coordinate classes whenever the relevant finite multiplicativity conditions hold. Lemma 3.11, and the ordinary coordinate maps, show that every finite list of these partial classes has its prescribed value under κ on an ω-large set.

Choose a sequence $u_j \in \mathrm{U}_{d_j}(A)$ whose terms are dense in each $\mathrm{U}_d(A)$. Choose a fixed unitary $v_j \in \mathrm{U}_{e_j}(T)$ representing $\kappa_\gamma(\overline{u_j})$, and put $D_j = \max\{d_j, e_j\}$. Pad both unitaries with identity blocks when needed. The unitary

$$W_j = \Phi^{(D_j)}(u_j)\, \iota_\omega^{(D_j)}(v_j)^* \in \mathrm{U}_{D_j}(T_\omega)$$

has zero algebraic K_1-class. For every pair (j, k), apply Lemma 3.12 to W_j, and fix $L_{j,k}$ lifted commutator witnesses in this single matrix size D_j. The unitary obtained by polar correction of $\phi_m^{(d_j)}(u_j)$ represents $\Phi^{(d_j)}(u_j)$ in the ultrapower. Consequently, on an ω-large set its padded product with v_j^* is within 2^{-k} of a product of $L_{j,k}$ coordinate commutators in $\mathrm{U}_{D_j}(T)$.

Select an increasing sequence $m(n)$ as follows. At stage n, impose multiplicativity and trace errors less than $1/n$ on the first n dense tests; impose the prescribed values on the first n total-K representatives; and retain all the commutator witness inequalities indexed by $j, k \le n$. Each stage has finitely many requirements, each valid on an ω-large set, so their intersection is infinite and one may choose $m(n) > m(n-1)$. Set $\psi_n = \phi_{m(n)}$.

The maps ψ_n are asymptotically multiplicative in the usual sense, and

$$\tau\psi_n(a) \longrightarrow s(a) \qquad (a \in A). \tag{3.11}$$

They define a unital $*$-homomorphism

$$\psi_\infty : A \longrightarrow T_\infty := \ell^\infty(T)/c_0(T).$$

For each fixed j, k, all sufficiently large n retain a 2^{-k} commutator approximation to the difference between the polar image of u_j and v_j, with exactly $L_{j,k}$ factors in the fixed matrix size D_j.

Now fix any map $r : \mathbb{N} \to \mathbb{N}$ such that $r(n) \to \infty$, and any free ultrafilter ν. The sequences ψ_n and $\psi_{r(n)}$ induce unital homomorphisms $\Psi_\nu, \Psi_\nu^r : A \to T_\nu$. Both have trace s by (3.11). Their ordinary and coefficient K-maps equal $[\iota_\nu]\kappa$, because the partial coordinate classes have their prescribed constant values eventually, and all the coordinate maps above are injective.

Their algebraic K_1 maps also agree with the prescribed one. Indeed, for fixed j, k, the retained $L_{j,k}$ witnesses form unitaries in $\mathrm{U}_{D_j}(T_\nu)$ and give the same 2^{-k} approximation there. Letting $k \to \infty$ shows that the image of u_j differs from the constant v_j by an element of $\overline{\mathrm{DU}(\mathrm{M}_{D_j}(T_\nu))}$. The matrix size is D_j for every k. This argument applies to both ψ_n and $\psi_{r(n)}$.

Equality on the dense unitary test sets implies equality on all unitaries by an exact Thomsen-map calculation. For $u \in \mathrm{U}_d(A)$, choose a tested $v \in \mathrm{U}_d(A)$ with $\|u - v\| < 2$ and write $v^*u = e^{ih}$ with $h = h^* \in \mathrm{M}_d(A)$. For each of the two induced maps, compatibility with the Thomsen map sends $\overline{u} - \overline{v}$ to

$$\mathrm{Th}_{T_\nu}\Big(\kappa_{\sharp,\nu}(\widehat{h})/(2\pi)\Big),$$

where $\kappa_{T,\nu} = \kappa_T \circ (\iota_\nu)_T$ is their common map on tracial state spaces, and $\kappa_{\sharp,\nu} = \mathrm{Aff}\,\mathrm{T}(\iota_\nu) \circ \kappa_\sharp$ is its dual map on affine functions. The prescribed map has exactly the same value on this difference. Equality on $\overline{v}$ therefore gives equality on $\overline{u}$, with no limiting argument in the quotient group. We have proved

$$\begin{aligned} [\Psi_\nu] = [\Psi_\nu^r] &= [\iota_\nu]\kappa, \\ (\Psi_\nu)_T = (\Psi_\nu^r)_T &= \kappa_{T,\nu}, \\ \Psi_\nu^\ddagger = (\Psi_\nu^r)^\ddagger &= \iota_\nu^\ddagger \kappa_\gamma. \end{aligned} \tag{3.12}$$

The maps are full by Lemma 3.1(iv). Explicitly, for a nonzero positive $a \in A$ the image has positive canonical trace. Choose $\delta > 0$ so its spectral cut $(\Psi_\nu(a) - \delta)_+$ has positive dimension function, and then choose m with $m d_\tau((\Psi_\nu(a) - \delta)_+) > 1$. Strict comparison puts the unit below the direct sum of these m cutdowns, so the image of a is full. The same argument applies to Ψ_ν^r. The ultrapower uniqueness therefore gives a unitary conjugating Ψ_ν and Ψ_ν^r.

We claim that, for every finite $F \subset A$ and $\varepsilon > 0$, for all sufficiently large n there is $z_n \in \mathrm{U}(T)$ with

$$\|z_n \psi_n(a) z_n^* - \psi_{r(n)}(a)\| < \varepsilon \qquad (a \in F). \tag{3.13}$$

Otherwise, the set of indices where no such unitary exists is infinite. Choose a free ultrafilter ν containing that set. The unitary just obtained from (3.12) lifts to a sequence of unitaries in T. Its coordinate inequalities on F contradict the choice of ν. This proves the claim. Diagonalizing (3.13) over increasing finite dense sets gives a unitary in T_∞ conjugating ψ_∞ and $r^*\psi_\infty$.

This holds for every diverging reparameterization r. Intertwining by reparameterization [4, Proposition 9.2] now gives a homomorphism $\rho : A \to T$ whose constant-sequence map is unitarily equivalent to ψ_∞. That proposition requires only a separable source and a unital target; it does not require nuclearity of T. Since ψ_∞ is unital, so is ρ.

Finally, compose the unitary equivalence in T_∞ with its quotient to any T_ν. Equation (3.12) and Lemma 3.14 give

$$\begin{aligned}[\iota_\nu][\rho] &= [\iota_\nu]\kappa,\\ \rho_T(\tau) &= s,\\ \iota_\nu^\ddagger \rho^\ddagger &= \iota_\nu^\ddagger \kappa_\gamma.\end{aligned}$$

Diagonal injectivity and monotraciality therefore give $([\rho], \rho_T, \rho^\ddagger) = \Xi$. In particular, $\tau\rho = s$ is faithful, so ρ is injective. This last recovery uses the monotracial ultrapower, and requires no assertion about all traces of the sequence algebra T_∞. □

Theorem 3.16 (Embeddings into an admissible target)**.** *Let A be separable, unital, nuclear, and satisfy the UCT, and let T be an admissible target. Every faithful unital totally compatible triple $\Xi = (\kappa, \kappa_T, \kappa_\gamma)$ from A to T is realized by a unital embedding $A \to T$. Two unital embeddings inducing the same triple, whose common induced trace on A is faithful, are approximately unitarily equivalent. No assumption on either* K*-group is needed.*

Proof. For existence, compose Ξ with the triple induced by a diagonal embedding into T_ω. Theorem 3.8 realizes this composite in the ultrapower. That theorem supplies the same uniqueness statement for every free ultrafilter. Proposition 3.15 therefore gives an embedding into T inducing Ξ.

For uniqueness, apply Theorem 3.8 to the two diagonal embeddings into T_ω. Lift an implementing unitary to a sequence of unitaries of T. For a prescribed finite source set and tolerance, one of its coordinates gives the desired approximate conjugacy. Exhaust the separable source. □

Remark 3.17 (A uniqueness question of Hirshberg and Phillips). Let X be a nonempty contractible compact metrizable space, let T be an admissible target with unique trace τ, and let $\phi, \psi : \mathrm{C}(X) \to T$ be unital embeddings. Then

$$\tau \circ \phi = \tau \circ \psi \quad \Longleftrightarrow \quad \phi \text{ and } \psi \text{ are approximately unitarily equivalent.}$$

Taking $X = [0,1]^n$ and $T = D_I$ answers the first part of Hirshberg and Phillips's [26, Question 4.8] affirmatively. The same conclusion answers the exact-target case of their more general target question. Their formulation of that second part does not assume exactness.

Contractibility gives $\mathrm{K}_0(\mathrm{C}(X)) = \mathbb{Z}[1]$ and $\mathrm{K}_1(\mathrm{C}(X)) = 0$. Thus unitality determines the maps on ordinary K-theory and, by the coefficient exact sequences, on total K-theory. Equality of the induced traces determines the trace component. Since the source K_1-group vanishes, the Thomsen sequence shows that its Hausdorffized unitary group is the determinant kernel, so its map is determined by the trace component as well. Hence ϕ and ψ induce the same triple. Their common induced trace is faithful, and $\mathrm{C}(X)$ is nuclear and satisfies the UCT. Theorem 3.16 therefore gives approximate unitary equivalence. The converse follows from traciality and norm continuity. Finally, D_I is admissible by Lemma 2.4, so the stated specialization applies.

Remark 3.18. Theorem 3.16 is the analytic input used for replacement. Its hypotheses concern the nuclearity and UCT of the source, not nuclearity of the target. The target may be any of the free products $\mathrm{C}(M) *_{\mathrm{r}} D$ or $A *_{\mathrm{r}} D$ used below. The entire triple is needed for

uniqueness even when the theorem to be deduced is stated using ordinary K-groups and one trace pairing.

4. Marked replacement of a nuclear core

Throughout this section D is any reservoir of Definition 2.1. The nuclear cores are separable, unital and UCT, and their selected traces are faithful. No finite-generation assumption is made. No splitting of the core's unit class is required. The separate support-shifting construction of Section 8 uses a splitting only for the factor being moved. Write $S_E = E *_{\mathrm{r}} D$, with its selected free-product trace. The targets S_E and their infinite reservoir tails have the regularity required in Theorem 3.16, by Lemma 2.4. Their canonical inclusions $j_E : E \to S_E$ induce isomorphisms on graded and total K-theory. We retain totally compatible triples, including their total K-theory, trace and Hausdorffized unitary maps and the coefficient-to-unitary compatibility. The selected data determine the invariants of the monotracial D-stabilizations; we do not identify the trace simplices of the original cores. In particular, selected traces on multitracial cores are allowed. Whenever $f : L \to L'$ is a trace-preserving unital map between monotracial algebras and f_* is a graded K-isomorphism, the induced triple $([f], f_T, f^{\ddagger})$ is invertible. The coefficient exact sequences prove this on total K-theory. The trace map identifies the one-point trace simplices, and its induced map on affine functions is $\mathrm{id}_{\mathbb{R}}$. The two determinant quotients agree because the pairing ranges agree, so the Thomsen sequence proves invertibility on the unitary quotients as well. We shall use this observation for the reservoir-tail inclusions without further mention.

Lemma 4.1 (A monotracial range model)**.** *Let E be a nuclear UCT core with faithful selected trace. There is a simple, infinite-dimensional, unital, monotracial, nuclear UCT, $\mathcal{Z}$-stable algebra A with the same selected graded* K*-groups, unit and trace pairing as E. There is a unital trace-preserving embedding $\theta : E \to A$ inducing graded and total* K*-isomorphisms.*

Proof. Put $G_i = \mathrm{K}_i(E)$, $u = [1_E]$ and $\sigma = \tau_{E*}$, and equip G_0 with the strict trace cone

$$G_0^+ = \{0\} \cup \{g : \sigma(g) > 0\}.$$

This is a simple weakly unperforated scaled ordered group: adding a sufficiently large multiple of u makes any element positive, every nonzero positive element is an order unit, and rational comparisons with u show that its only normalized state is σ. The nuclear range theorem [19, Theorem 2.8] supplies a classifiable monotracial range model for these data, with one-point trace simplex. Choose it infinite dimensional and $\mathcal{Z}$-stable. For clarity, a finite-dimensional simple model would have $(G_0, u, G_1) = (\mathbb{Z}, m, 0)$ with $m \geq 1$. For these data one can instead take the infinite-dimensional model $\mathrm{M}_m(\mathcal{Z})$, with its normalized trace. Thus no splitting of u is needed. The identity on the selected graded data is compatible with the trace map $\mathrm{T}(A) \to \mathrm{T}(E)$ sending the unique trace of A to τ_E. Its induced map on affine functions is evaluation at τ_E. A nonzero source projection has strictly positive selected trace, so its image is positive in the displayed cone. The nuclear embedding theorem [4, Theorem B and Corollary C] therefore gives θ; its version with totally compatible triples permits any chosen compatible extension of these data. Coefficient exact sequences give the asserted total K-isomorphism. □

Lemma 4.2 (Factorization of totally compatible triples)**.** *Let E have faithful selected trace and let $\theta : E \to A$, $j : E \to T$ be trace-preserving unital embeddings inducing graded* K*-isomorphisms, where A, T are monotracial. Their total* K*-theory maps, Hausdorffized unitary maps and induced maps on affine functions are surjective. There is a unique totally compatible isomorphism triple $\lambda = (\lambda_{\mathrm{K}}, \lambda_T, \lambda_\gamma)$ from A to T such that*

$$\lambda([\theta], \theta_T, \theta^\ddagger) = ([j], j_T, j^\ddagger).$$

Proof. The total K-theory maps are isomorphisms; set $\lambda_{\mathrm{K}} = [j][\theta]^{-1}$. Let $\lambda_T : \mathrm{T}(T) \to \mathrm{T}(A)$ be the unique affine map between the one-point trace simplices. Its induced map $\lambda_\sharp : \mathrm{Aff}\,\mathrm{T}(A) \to \mathrm{Aff}\,\mathrm{T}(T)$ is $\mathrm{id}_{\mathbb{R}}$. Put $H = \overline{\tau_{E*}\mathrm{K}_0(E)}$ and use the Thomsen sequence

$$0 \longrightarrow \mathrm{Aff}\,\mathrm{T}(L)/\overline{\rho_L \mathrm{K}_0(L)} \longrightarrow \mathrm{U}_\infty(L)/\mathrm{CU}_\infty(L) \xrightarrow{\Pi_L^{\mathrm{cu}}} \mathrm{K}_1(L) \longrightarrow 0.$$

The determinant-kernel maps induced by θ, j are the same surjection, namely evaluation at τ_E modulo H. Together with the K_1-isomorphisms this makes $\theta^\ddagger$ and $j^\ddagger$ surjective. Their kernels coincide: an element in either kernel has zero ordinary K_1-class, hence is $\mathrm{Th}_E(f)$, and its image vanishes precisely when $f(\tau_E) \in H$. Thus there is a unique unitary isomorphism λ_γ with $\lambda_\gamma \theta^\ddagger = j^\ddagger$. The induced maps on Aff T are the same surjection, namely evaluation at the selected trace. This is not a surjectivity assertion about θ_T or j_T when E has several traces; both maps have the single selected trace as their image.

The compatibility squares follow by cancellation on total K-theory, unitary quotients and affine functions. In particular, for every $m \geq 2$, naturality gives

$$\lambda_\gamma \zeta_m^A \underline{\mathrm{K}}_0(\theta; \mathbb{Z}/m) = \zeta_m^T \lambda_{\mathrm{K}}^{(m)} \underline{\mathrm{K}}_0(\theta; \mathbb{Z}/m),$$

and the last factor is an isomorphism. This proves the coefficient-to-unitary compatibility, including torsion, and proves the lemma. □

Following [45, Definition 1.4], we use CAAP to abbreviate the C*-*automorphism approximation property*. For a trace-preserving unital inclusion $B \subset C$, it asks for trace-preserving automorphisms of C that almost fix a prescribed finite subset of B and move B close to a prescribed element of C. We use the following form of the one-sided Elliott approximate intertwining argument, retaining control of the full invariant; compare [45, Proposition 1.2].

Lemma 4.3 (One-sided approximate intertwining with an exact invariant)**.** *Let $B \subset C$ be a unital inclusion of faithfully tracial separable algebras, with $\tau_B = \tau_C|_B$. Suppose a subgroup $\mathcal{G} \subset \mathrm{Aut}_\tau(C)$ induces the identity on total* K*-theory, the trace simplex and the Hausdorffized unitary quotient of C, and has the following property: given finite $F \subset B$, $z \in C$ and $\eta, \varepsilon > 0$, some $\alpha \in \mathcal{G}$ and $y \in B$ satisfy*

$$\max_{b \in F} \|\alpha(b) - b\| < \eta, \qquad \|z - \alpha(y)\| < \varepsilon. \tag{CAAP}$$

Then there is a trace-preserving isomorphism $\Phi : B \to C$, arbitrarily close to inclusion on any prescribed finite set, whose induced triple is that of the inclusion $\iota : B \hookrightarrow C$:

$$([\Phi], \Phi_T, \Phi^\ddagger) = ([\iota], \iota_T, \iota^\ddagger).$$

Proof. Fix a finite set $F_0 \subset B$ and $\delta > 0$, and choose $\varepsilon_n > 0$ with $\sum_n \varepsilon_n < \delta$. Let (b_j) be dense in B, and let (z_n) be a sequence in C in which each member of a fixed countable

dense subset occurs infinitely often. Put $\gamma_0 = \mathrm{id}_C$ and $\phi_0 = \iota$. Inductively, having chosen γ_{n-1} and $y_1, \dots, y_{n-1} \in B$, put

$$S_n = F_0 \cup \{b_1, \dots, b_n\} \cup \{y_1, \dots, y_{n-1}\}.$$

Apply (CAAP) to S_n and $\gamma_{n-1}^{-1}(z_n)$ to find $\alpha_n \in \mathcal{G}$ and $y_n \in B$ such that

$$\max_{b \in S_n} \|\alpha_n(b) - b\| < \varepsilon_n, \qquad \|\gamma_{n-1}^{-1}(z_n) - \alpha_n(y_n)\| < \varepsilon_n.$$

Set $\gamma_n = \gamma_{n-1}\alpha_n$ and $\phi_n = \gamma_n \iota$. Since γ_{n-1} is isometric, these estimates give

$$\|\phi_n(b) - \phi_{n-1}(b)\| < \varepsilon_n \quad (b \in S_n), \qquad \|z_n - \phi_n(y_n)\| < \varepsilon_n.$$

This is the one-sided approximate intertwining

$$\begin{array}{ccccccccc}
B & \xrightarrow{\mathrm{id}_B} & B & \xrightarrow{\mathrm{id}_B} & B & \longrightarrow & \cdots & \longrightarrow & B \\
\downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \iota} & & & & \downarrow{\scriptstyle \Phi} \\
C & \xrightarrow[\alpha_1^{-1}]{} & C & \xrightarrow[\alpha_2^{-1}]{} & C & \longrightarrow & \cdots & \longrightarrow & C.
\end{array}$$

The square containing α_n^{-1} commutes within ε_n on S_n, because $\|\alpha_n^{-1}(b) - b\| = \|\alpha_n(b) - b\|$. The bottom limit is identified with C by the stage maps γ_n, which satisfy $\gamma_n \alpha_n^{-1} = \gamma_{n-1}$. Under these identifications the vertical maps are ϕ_n.

The summable estimates on the dense sequence (b_j) show that ϕ_n converges pointwise in norm to a unital trace-preserving embedding $\Phi : B \to C$. It is isometric as a limit of isometric maps, and

$$\max_{b \in F_0} \|\Phi(b) - b\| \leq \sum_n \varepsilon_n < \delta.$$

Every chosen preimage y_n belongs to all subsequent sets S_m, $m > n$. Hence

$$\|\Phi(y_n) - z_n\| \leq \|\phi_n(y_n) - z_n\| + \sum_{m>n} \|\phi_m(y_n) - \phi_{m-1}(y_n)\| < \sum_{m \geq n} \varepsilon_m.$$

The right side tends to zero. Since every member of the chosen dense subset of C occurs infinitely often among the z_n, the range of Φ is dense. It is also norm closed, so Φ is surjective.

Finally, every $\gamma_n \in \mathcal{G}$ acts identically on the full invariant, so each ϕ_n induces the inclusion triple. For any fixed projection or unitary representative over B or $B \otimes D_m$, the images converge in norm; eventual closeness gives equality of its K-class. Tensoring by a fixed coefficient algebra preserves point-norm convergence, so this also treats total K-theory. For $v \in \mathrm{U}_d(B)$, the images converge in the single matrix algebra $\mathrm{M}_d(C)$, and their common algebraic unitary class passes to the limit because $\mathrm{U}_d(C) \to \mathrm{U}_\infty(C)/\mathrm{CU}_\infty(C)$ is continuous and $\mathrm{U}_\infty(C)/\mathrm{CU}_\infty(C)$ is Hausdorff. Every trace of C gives the inclusion trace on B at each stage, and this equality passes to the limit as well. Thus $([\Phi], \Phi_T, \Phi^\ddagger) = ([\iota], \iota_T, \iota^\ddagger)$. This argument treats each individual class and requires neither finite generation nor a uniform matrix bound across different classes. □

Remark 4.4 (Approximate intertwining and the history of CAAP). The limiting argument in Lemma 4.3 is the one-sided Elliott approximate intertwining argument. A direct antecedent of its local approximation condition is Elliott's proof of $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$, as presented in [42].

That presentation uses two-sided approximate intertwining. To see the corresponding inner-automorphism condition, suppose $i : A \hookrightarrow B$ and $j : B \to A$ are unital *-homomorphisms with $ji \approx_{\mathrm{u}} \mathrm{id}_A$ and $ij \approx_{\mathrm{u}} \mathrm{id}_B$. Given finite $F \subset A$, $z \in B$ and $\eta, \varepsilon > 0$, first conjugate j by a unitary of A to obtain j' with $\|j'i(a) - a\| < \eta/2$ for $a \in F$. Since $ij' \approx_{\mathrm{u}} \mathrm{id}_B$, there is a unitary $u \in B$ such that

$$\| \mathrm{Ad}(u)(ij'(i(a))) - i(a)\| < \eta/2 \quad (a \in F), \qquad \| \mathrm{Ad}(u)(ij'(z)) - z\| < \varepsilon.$$

Thus $\alpha = \mathrm{Ad}(u)$ almost fixes $i(F)$ within η and moves $i(j'(z))$ within ε of z. These are precisely the local estimates in (CAAP), with inner automorphisms and with the trace requirement omitted. The omission matters here, since $\mathcal{O}_2$ has no tracial state. The general inner-automorphism criterion is recorded in [43, Proposition 2.3.5]; [49, Section 2 and Proposition 2.6] explicitly connects this absorption method with Elliott's proof.

Hirshberg and Phillips already implement the same local automorphism approximation mechanism for infinite reduced free-product absorption [26, proof of Theorem 3.1]. Under their hypotheses, their construction verifies CAAP for the canonical inclusion $D \subset A *_{\mathrm{r}} D$. Indeed, after their regrouping, write $D = *_{\mathrm{r}\, n \geq 1} C_n$. Given finite $S \subset D$, $z \in A *_{\mathrm{r}} D$ and tolerances $\eta, \varepsilon > 0$, approximate S and z by word polynomials using $C_1, \ldots, C_r$ and finitely many coefficients from A. Choose $n > r$. They construct a trace-preserving embedding $\phi_n : A \to C_n$ and, by nuclear-source uniqueness, a unitary $u \in A *_{\mathrm{r}} C_n$ such that $u\phi_n(a)u^*$ is as close to a as needed on those coefficients. Put

$$D_{\neq n} = \underset{k \neq n}{*_{\mathrm{r}}} C_k, \qquad \alpha = \mathrm{Ad}(u) *_{\mathrm{r}} \mathrm{id}_{D_{\neq n}} \in \mathrm{Aut}_\tau\Big((A *_{\mathrm{r}} C_n) *_{\mathrm{r}} D_{\neq n}\Big).$$

Under this identification, α is a trace-preserving automorphism of $A *_{\mathrm{r}} D$ fixing the first r blocks pointwise. Its image $\alpha(D)$ approximately contains the required coefficients from A. Choosing the polynomial and coefficient errors sufficiently small gives

$$\max_{b \in S} \|\alpha(b) - b\| < \eta, \qquad \mathrm{dist}(z, \alpha(D)) < \varepsilon.$$

Thus their proof yields exactly the local condition (CAAP). They do not use the name CAAP; their printed proof uses the two-sided Elliott approximate intertwining of [26, Theorem 2.2]. The extension over the free complement is also emphasized in their introduction; it need not be inner on the entire free product.

The name CAAP and the trace-preserving norm formulation used here are from [45, Proposition 1.2 and Definition 1.4]. Its von Neumann algebra counterpart, AAP, describes the mechanism of [34, Proposition 5.1 and Section 5.2]: automorphisms approximately fix the existing generators and move elements of their algebra close to the additional generator, followed by a summable iteration. The original argument does not use the names AAP or CAAP, and its von Neumann algebra automorphisms are not asserted to preserve the canonical reduced group C*-algebra. Our use of the classical intertwining argument additionally records the prescribed full invariant: the subgroup in Lemma 4.3 acts trivially on that invariant, so the limiting isomorphism induces the inclusion triple. The subsequent core-replacement and fusion arguments verify the required local approximation condition in the settings of this paper.

Proposition 4.5 (Replacement with the inclusion marking)**.** *Let $\theta : E \to A$ be as in Lemma 4.2, with E, A nuclear UCT and A monotracial. Then there is a trace-preserving isomorphism $E *_{\mathrm{r}} D \to A *_{\mathrm{r}} D$ inducing the same totally compatible triple as $\theta *_{\mathrm{r}} \mathrm{id}_D$.*

Proof. Regard $B = \mathrm{C}^*(\theta(E), D)$ as a subalgebra of $C = A *_{\mathrm{r}} D$. For every finite reservoir-block head $D_{\le r}$, the tail inclusion $k : C_0 = A *_{\mathrm{r}} D_{>r} \to C$ is a total K-isomorphism. Its induced triple is invertible, since both algebras are monotracial with the same pairing, by the Thomsen sequence. Consequently every automorphism $\beta = \mathrm{id}_{D_{\le r}} *_{\mathrm{r}} \mathrm{Ad}(u)$, $u \in \mathrm{U}(C_0)$, induces the identity triple on C: the relation $\beta k = k\,\mathrm{Ad}(u)$ and triviality of the triple of an inner automorphism permit cancellation of the invertible triple of k.

We verify (CAAP) using precisely these automorphisms and their inverses. Approximate finite $F \subset B$ and $z \in C$ by polynomials with a common finite reservoir-block head. Put $T = E *_{\mathrm{r}} D_{>r}$ and let $j : E \to T$, $j_A : A \to C_0$ be canonical. Lemma 4.2 gives a totally compatible isomorphism triple λ from A to T with

$$\lambda([\theta], \theta_T, \theta^{\ddagger}) = ([j], j_T, j^{\ddagger}).$$

Theorem 3.16 realizes it by $\rho : A \to T$. Nuclear-source uniqueness gives $\rho\theta \approx_{\mathrm{u}} j$; conjugate ρ to make this approximation as accurate as needed on the finitely many E-coefficients of F. Set $L = \theta *_{\mathrm{r}} \mathrm{id}_{D_{>r}}$ and $\chi = L\rho : A \to C_0$. Since $Lj = j_A\theta$, the triples induced by $\chi\theta$ and $j_A\theta$ agree. Surjectivity of $[\theta]$ and $\theta^{\ddagger}$ permits cancellation on total K-theory and unitary quotients; the trace maps agree because A is monotracial. Thus

$$([\chi], \chi_T, \chi^{\ddagger}) = ([j_A], (j_A)_T, j_A^{\ddagger}).$$

A second use of nuclear-source uniqueness gives $u \in \mathrm{U}(C_0)$ with $\mathrm{Ad}(u)j_A$ close to χ on the A-coefficients of z and the θ-images of the selected E-coefficients of F. For every selected coefficient $e \in E$, the two uniqueness estimates give

$$\|\beta(\theta(e)) - \theta(e)\| \le \|u\theta(e)u^* - \chi\theta(e)\| + \|L\rho\theta(e) - Lj(e)\| < 2\delta.$$

The head is fixed exactly. Replacing each A-coefficient of the polynomial for z by its image under χ gives an element of B, because $\chi(A) \subset B$; its distance from $\beta(z)$ is arbitrarily small. Polynomial continuity gives the specified errors, first choosing the polynomial approximations and then δ. Taking $\alpha = \beta^{-1}$ gives the orientation in Lemma 4.3. The subgroup generated by these automorphisms induces the identity triple on C, so that lemma gives the isomorphism with exactly the inclusion marking. □

Remark 4.6 (Comparison with Hirshberg–Phillips: replacement). Proposition 4.5 uses the same CAAP mechanism as Hirshberg–Phillips's absorption argument, as explained in Remark 4.4. In both arguments, nuclear-source uniqueness supplies correcting conjugacies, which are extended by the identity over free complements. Their uniqueness input [26, Corollary 3.4] treats unital one-dimensional NCCW algebras and their unital countable inductive limits with vanishing K_1; they apply it inside an exact intermediate product with $\mathcal{Z}$ and transfer the conjugacies through their diagram (3.4). The additional input here is Theorem 3.16, which permits general nuclear UCT sources with arbitrary K-groups and prescribed total K-theory and algebraic unitary data. Cancellation through the invertible triple induced by k verifies that each correction preserves the full invariant exactly, so Lemma 4.3 yields the prescribed inclusion marking.

Theorem 4.7 (Marked nuclear-core D-stabilization)**.** *Let D be a reservoir. Let E, F be nuclear UCT cores with faithful selected traces. Every prescribed unital totally compatible morphism ξ between the full invariants of $E*_{\mathrm{r}}D$ and $F*_{\mathrm{r}}D$ is realized by a unital trace-preserving embedding. If ξ is an isomorphism of full invariant triples, the realizing map may be chosen to be a *-isomorphism.*

Every unit-preserving homomorphism of graded K*-groups that preserves the selected pairing admits a compatible full extension and is therefore realized. If both group maps are isomorphisms, the realizing map may be chosen to be an isomorphism. The prescribed ξ may be any unital totally compatible extension, as provided by* [4, Theorem 3.9]*; no canonical extension is asserted. No preservation of the original ordered* K_0 *or of the other source traces is required. The realizing map may fix any identified finite reservoir-block prefix $H = D_{\le r}$ pointwise and carry $E*_{\mathrm{r}}D_{>r}$ into $F*_{\mathrm{r}}D_{>r}$, and onto it in the isomorphism case. In particular the old core stays in the complementary tail. We call H a* protected head*: it consists of the first r reservoir blocks, and the realizing map fixes every element of H. Only the number of blocks is finite; H need not be finite dimensional.*

Proof. We first suppose that ξ is an isomorphism of full invariant triples, that is, an invertible morphism, and take $r = 0$, so $H = D_{\le 0} = \mathbb{C}1$. Choose a monotracial range model A for F and an embedding $\theta_F : F \to A$ as in Lemma 4.1. Factorization gives a totally compatible isomorphism triple λ_F from A to $F*_{\mathrm{r}}D$ with

$$\lambda_F([\theta_F], (\theta_F)_T, \theta_F^{\ddagger}) = ([j_F], (j_F)_T, j_F^{\ddagger}).$$

The faithful totally compatible triple

$$\lambda_F^{-1}\xi([j_E], (j_E)_T, j_E^{\ddagger})$$

from E to A is realized by a trace-preserving embedding $\theta_E : E \to A$, using the nuclear embedding theorem [4, Theorem B]. It induces graded K-isomorphisms. Proposition 4.5 gives isomorphisms $\Phi_E : E*_{\mathrm{r}}D \to A*_{\mathrm{r}}D$ and $\Phi_F : F*_{\mathrm{r}}D \to A*_{\mathrm{r}}D$. Writing $j_A : A \to A*_{\mathrm{r}}D$, their exact markings are

$$([\Phi_F], (\Phi_F)_T, \Phi_F^{\ddagger}) = ([j_A], (j_A)_T, j_A^{\ddagger})\lambda_F^{-1},$$
$$([\Phi_E], (\Phi_E)_T, \Phi_E^{\ddagger}) = ([j_A], (j_A)_T, j_A^{\ddagger})\lambda_F^{-1}\xi.$$

For the first identity, compose on the right with the triple of j_F; for the second, use that of j_E. The resulting equalities are exactly the two inclusion markings of Proposition 4.5 and the definition of θ_E. The total K-theory and unitary maps induced by j_E, j_F are surjective by Lemma 4.2, so cancellation proves these two components of each identity. All stabilizations are monotracial, so the trace components agree as well. Hence $\Phi_F^{-1}\Phi_E$ realizes ξ exactly.

Now let ξ be an arbitrary unital totally compatible morphism. Choose simple monotracial nuclear range models A_E, A_F by Lemma 4.1. Choose compatible isomorphism triples between their stabilizations and the respective original stabilizations by [4, Theorem 3.9]. Apply the invertible case just proved separately to the two matching pairs of cores, obtaining marked trace-preserving isomorphisms

$$\Theta_E : E*_{\mathrm{r}}D \longrightarrow A_E*_{\mathrm{r}}D, \qquad \Theta_F : F*_{\mathrm{r}}D \longrightarrow A_F*_{\mathrm{r}}D.$$

Let Λ_E, Λ_F be their induced triples, and let J_E, J_F be the triples of the native inclusions

$$j_E : A_E \longrightarrow A_E *_{\mathrm{r}} D, \qquad j_F : A_F \longrightarrow A_F *_{\mathrm{r}} D.$$

Both J_E and J_F are invertible: they are total-K isomorphisms, both their source and target are monotracial, and the selected trace ranges agree, so the Thomsen sequence gives the unitary isomorphisms as well. The triple

$$\zeta = J_F^{-1} \Lambda_F \xi \Lambda_E^{-1} J_E$$

is a unital totally compatible morphism from A_E to A_F. Its trace map sends the unique trace to the unique faithful trace. Theorem 3.16 therefore realizes ζ by a unital trace-preserving embedding $\theta : A_E \to A_F$. Reduced free-product functoriality gives an actual embedding

$$L = \theta *_{\mathrm{r}} \mathrm{id}_D : A_E *_{\mathrm{r}} D \longrightarrow A_F *_{\mathrm{r}} D.$$

Write $\mathrm{Inv}(L)$ for its induced triple. The identity $L j_E = j_F \theta$ gives

$$\mathrm{Inv}(L) J_E = J_F \zeta = \Lambda_F \xi \Lambda_E^{-1} J_E.$$

Cancel J_E to see that

$$\Theta_F^{-1} L \Theta_E : E *_{\mathrm{r}} D \longrightarrow F *_{\mathrm{r}} D$$

induces exactly ξ. No invertibility of ξ was used in this construction.

For the assertion about graded group homomorphisms, transport the given unit- and pairing-preserving maps to A_E, A_F. They are positive for the strict trace cones: a nonzero positive class has strictly positive selected trace, and the pairing is preserved. The unique trace map is faithful. The compatible-extension theorem [4, Theorem 3.9] therefore supplies a full morphism, which is realized as above. For group isomorphisms the extension is invertible and the first part of the proof supplies an isomorphism.

For a prescribed head $H = D_{\leq r}$, write $D = H *_{\mathrm{r}} D_{>r}$. The inclusions of $E *_{\mathrm{r}} D_{>r}$ and $F *_{\mathrm{r}} D_{>r}$ into $E *_{\mathrm{r}} D$ and $F *_{\mathrm{r}} D$, respectively, induce invertible triples. Transport ξ through these triples, apply the appropriate preceding construction to $D_{>r}$, and extend the resulting tail embedding by id_H using reduced free products. If ξ is an isomorphism of invariants, choose the tail isomorphism. Cancellation through the invertible triple of either tail inclusion verifies its marking. This fixes H pointwise and has the stated containment, or equality, of complementary tails. □

However, uniqueness up to approximate unitary equivalence fails in general for embeddings between these D-stabilized free products, even when the full invariant maps agree.

Example 4.8 (Failure of uniqueness for free-product embeddings)**.** Let

$$D_I = *_{\mathrm{r}, j=1}^{\infty} (\mathrm{C}([0,1]), \lambda),$$

where λ is Lebesgue measure, and let α exchange the first two free factors while fixing all later factors pointwise. Then α and id_{D_I} induce the same full totally compatible triple, but are not approximately unitarily equivalent. Thus uniqueness fails even for automorphisms in the class of Theorem 4.7, with both nuclear cores equal to $\mathbb{C}$. Such counterexamples can also fix any prescribed finite interval prefix pointwise.

Proof. Permutation of the identically traced factors gives a unital trace-preserving automorphism α. The reservoir has

$$\mathrm{K}_0(D_I) = \mathbb{Z}[1], \qquad \mathrm{K}_1(D_I) = 0, \qquad \mathrm{T}(D_I) = \{\tau\}.$$

Hence unitality determines the identity map on ordinary K-theory. For every $m \geq 2$, the coefficient exact sequences give

$$\mathrm{K}_0(D_I; \mathbb{Z}/m) = \mathbb{Z}/m, \qquad \mathrm{K}_1(D_I; \mathbb{Z}/m) = 0.$$

The coefficient map from $\mathrm{K}_0(D_I)$ is surjective, so α acts identically on total K-theory as well. The determinant sequence (2.5) identifies

$$\mathrm{U}_\infty(D_I)/\mathrm{CU}_\infty(D_I) \cong \mathbb{R}/\mathbb{Z}.$$

This identification is natural with respect to the trace, so $\alpha^\ddagger = \mathrm{id}$. Its trace map is also the identity. Thus all components of its totally compatible triple agree with those of id_{D_I}.

For the failure of approximate unitary equivalence, put $M = \pi_\tau(D_I)''$, and let $N \subset M$ be the von Neumann algebra generated by the factors with indices $j \geq 3$. The tracial free-product construction gives

$$M = \Big(\mathrm{L}^\infty([0,1], \lambda) * \mathrm{L}^\infty([0,1], \lambda)\Big) * N.$$

Here N is a type II_1 factor, M is full, and $N' \cap M = \mathbb{C}$, by [51, Corollary 3.2 and Theorems 3.4 and 3.7]. The normal extension $\bar{\alpha}$ fixes N pointwise. An implementing unitary for $\bar{\alpha}$ would therefore belong to $N' \cap M = \mathbb{C}$, forcing $\bar{\alpha} = \mathrm{id}$. This is impossible because α exchanges two distinct free factors. Thus $\bar{\alpha}$ is outer.

If $\alpha \approx_{\mathrm{u}} \mathrm{id}_{D_I}$, there would be unitaries $u_n \in D_I$ with $u_n a u_n^* \to \alpha(a)$ in norm for every $a \in D_I$. Density of D_I in M for $\| \cdot \|_2$ and trace preservation would then give

$$\|u_n x u_n^* - \bar{\alpha}(x)\|_2 \longrightarrow 0 \qquad (x \in M).$$

For a full finite factor, the inner automorphism group is closed in this topology. Hence $\bar{\alpha}$ would be inner, a contradiction. Finally, exchanging two factors after a prescribed finite prefix and fixing all other factors gives the same counterexample with that prefix fixed pointwise. □

Remark 4.9 (Comparison with Hirshberg–Phillips: the nuclear model). The common nuclear model A used in the invertible case of Theorem 4.7 plays the role of the Jiang–Su model in [26, Proposition 2.13 and proof of Theorem 3.1]. Its pointed graded K-groups and selected pairing are chosen to match the given core data, and the two embeddings into it retain the prescribed totally compatible triple.

Remark 4.10 (The role of K-theory and of the reservoir). Theorem 4.7 classifies the products using the pointed graded K-groups of their nuclear cores and the selected trace pairing. The order on the original core and its full trace simplex need not be preserved. Strict comparison and monotraciality recover the order and trace simplex of the product from these selected data. Totally compatible triples are needed in the embedding and uniqueness steps, and compatible extensions of the selected group isomorphisms exist.

The proof uses four properties of the reservoir: the core inclusions and complementary-tail inclusions preserve K-theory; every infinite tail gives an admissible target; finite sets are approximated using finitely many reservoir blocks; and a trace-preserving inner automorphism of the complementary tail extends by the identity on the finite head. The first two properties are supplied by Lemma 2.4. The last two follow from the free-product presentation. No interval coordinate, semicircular distribution or polynomial-flow construction is used in this proof.

Corollary 4.11 (Independence of the reservoir presentation)**.** *Every reservoir D of Definition 2.1 is trace-preservingly isomorphic to $D_I = I^{*_{\mathrm{r}}\infty}$. Consequently any two such reservoirs give the same class of D-stabilized algebras, and $D*_{\mathrm{r}}D \cong D$.*

Proof. We use the elementary approximate-intertwining fact that, if the connecting inclusions $A_n \hookrightarrow A_{n+1}$ are point-norm limits of isomorphisms, then $A_0 \cong \varinjlim A_n$. To see this, compare the inclusion system with the system of isomorphisms, choosing the latter with summable errors on increasing finite sets containing dense subsets and the relevant images from all earlier stages. The usual approximate intertwining identifies their limits; the isomorphism system has limit A_0. The same argument preserves selected traces.

Apply Theorem 4.7 to the nuclear cores $\mathbb{C}$ and I with reservoir D. It gives isomorphisms $D \to I*_{\mathrm{r}}D$ fixing any prescribed finite block prefix of D. Since such prefixes have dense union, the canonical inclusion is a point-norm limit of these isomorphisms. Extending over an external finite interval head proves the same assertion for the inclusions

$$I^{*_{\mathrm{r}}n}*_{\mathrm{r}}D \longrightarrow I^{*_{\mathrm{r}}(n+1)}*_{\mathrm{r}}D.$$

The preceding fact gives $D \cong D_I*_{\mathrm{r}}D$.

Conversely, D_I itself satisfies Definition 2.1. Apply Theorem 4.7, with reservoir D_I, to $\mathbb{C}$ and each individual nuclear block R_j. It gives isomorphisms $D_I \to R_j*_{\mathrm{r}}D_I$ approximating the canonical inclusion by fixing arbitrarily large interval prefixes. Extend them over the preceding finite free head $R_1*_{\mathrm{r}}\cdots*_{\mathrm{r}}R_{j-1}$. Thus the inclusions in the system

$$D_I \longrightarrow R_1*_{\mathrm{r}}D_I \longrightarrow R_1*_{\mathrm{r}}R_2*_{\mathrm{r}}D_I \longrightarrow \cdots$$

are point-norm limits of isomorphisms. Its limit is $D*_{\mathrm{r}}D_I$, so $D_I \cong D*_{\mathrm{r}}D_I$. Associativity and interchange of the two free factors now give $D \cong D_I$. No fusion theorem is used. Finally, reindexing interval factors gives $D_I*_{\mathrm{r}}D_I \cong D_I$. □

Remark 4.12 (Finite old-core alignment). Suppose additionally that a unital embedding $\kappa : E \to F*_{\mathrm{r}}D_{>r}$ preserves the selected trace and has the same induced triple as the restriction of the constructed tail isomorphism. Theorem 3.16 makes the two restrictions approximately unitarily equivalent inside $F*_{\mathrm{r}}D_{>r}$. For any finite core packet, postcompose by $\mathrm{id}_H *_{\mathrm{r}} \mathrm{Ad}(u)$ for the resulting tail unitary u. This attains the desired approximation, fixes H exactly, preserves the complementary tail, and changes no component of ξ. Thus exact invariant marking and finite old-core coherence are available whenever the comparison embedding already lies in the same chosen complementary tail. This statement does not prepare an arbitrary core which is merely scalar free from a reservoir.

Remark 4.13 (An exactly protected common matrix frame). Suppose the two nuclear-source embeddings being compared agree on a unital frame $C \cong \mathrm{M}_d$ inside the same complementary

tail. Choose an approximate conjugator u in that tail, including its matrix units among the uniqueness tests. Then

$$y = \frac{1}{d}\sum_{i,j=1}^{d} e_{ij}ue_{ji} \in C'$$

satisfies $\|y-u\| \le d\max_{i,j}\|[e_{ij},u]\|$. For a sufficiently accurate initial conjugator, y is invertible. Its polar unitary lies in C' and remains close to u. Thus the frame can be fixed exactly, while extending by the identity on an external protected reservoir head.

The invariant must mark the minimal-projection class of this frame. Preservation of the total unit alone does not imply preservation of $[e_{11}]$ in the presence of d-torsion. For instance, in $\mathbb{Z}g \oplus \mathbb{Z}/2\,t$ with unit $2g$, the automorphism $g \mapsto g+t$, $t \mapsto t$ preserves the unit and the trace pairing but changes the chosen minimal class g.

Remark 4.14 (The geometric scope of replacement). The preceding results classify nuclear-source embeddings and replace abstract nuclear cores after adjoining a general reservoir of Definition 2.1. They do not assert that approximate conjugating unitaries protect an additional reservoir. The alignment in Remark 4.12 requires the compared maps already to lie in the same ambient complementary tail; scalar freeness from the reservoir alone does not give this containment. The finite-base construction in Section 9 supplies the additional relative compatibility needed for fusion. Its correcting maps are constructed in the chosen complementary tail, so this geometric requirement is verified explicitly.

Remark 4.15 (Interval coordinates for the subsequent constructions). From the next section onward, fix a trace-preserving identification $D \cong D_I$ from Corollary 4.11 and use D for this interval presentation. Finite heads and complementary tails in the matrix, fusion and exponential-length constructions refer to these interval coordinates. A continuous change of variables also gives the free standard semicircular presentation used there. The extension $\mathrm{id}_P *_{\mathrm{r}} (D \cong D_I)$ transfers the conclusions back to every reservoir of Definition 2.1; it preserves the canonical P-inclusion and its selected invariant data. This choice does not identify an original block prefix with an interval prefix. Protection of original block prefixes is the direct conclusion of Theorem 4.7.

5. Outline of the fusion argument

The nuclear-core theorem of Section 4 classifies algebras of the form $E*_{\mathrm{r}}D$. To apply it to several nuclear factors, we must first identify their D-stabilized free product with $E*_{\mathrm{r}}D$ for a nuclear core E carrying the same aggregate selected data. We outline this fusion construction before developing its analytic tools. Section 6 treats logarithm comparison and interval automorphisms; Section 7 supplies the finite-dimensional relative reservoirs, and Section 9 assembles the general argument. The independent split-unit construction in Section 8 supplies the reduction for arbitrary K_1.

For the binary step, let A_1, A_2 be nuclear UCT algebras with faithful selected traces, and initially assume $\mathrm{K}_1(A_1) = \mathrm{K}_1(A_2) = 0$. The goal is an isomorphism $(A_1 *_{\mathrm{r}} A_2) *_{\mathrm{r}} D \cong E_0 *_{\mathrm{r}} D$, where E_0 is nuclear UCT and carries the aggregate pointed K-groups and selected trace pairing.

(i) *Matrix reduction.* The exact reduced identity

$$\mathrm{M}_2(A_1 *_{\mathrm{r}} A_2) \cong (A_1 \oplus A_2) *_{\mathbb{C}^2} \mathrm{M}_2$$

puts the two original factors into the nuclear coefficient algebra $A_1 \oplus A_2$. The remaining attachment is M_2, over the common labelled diagonal $\mathbb{C}^2$. The trace on $A_1 \oplus A_2$ is half-weighted, and the selected traces on the matrix algebras are normalized. Thus the problem becomes a comparison of a nuclear coefficient algebra with a finite-dimensional attachment.

(ii) *Choice of a nuclear model.* Choose a simple monotracial nuclear UCT range core E_0 with the aggregate selected data and put $E = \mathrm{M}_2(E_0)$. The embedding theorem supplies maps $a_i : A_i \to E_0$ with the canonical selected markings. Their diagonal sum gives $a : A_1 \oplus A_2 \to E$, explicitly $a(x_1 \oplus x_2) = \mathrm{diag}(a_1(x_1), a_2(x_2))$, and the standard matrix frame gives $b : \mathrm{M}_2 \to E$. These maps agree on $\mathbb{C}^2$. The core E_0 records the aggregate unit and pairing, while the labelled frame in E records the corner to be compressed at the end.

(iii) *The relative interval comparison.* Let $I_h = (\mathrm{C}([0,1]), \lambda)$, with coordinate h, and put

$$S = \Big((A_1 \oplus A_2) *_{\mathbb{C}^2} \mathrm{M}_2\Big) *_{\mathrm{r}} D,$$
$$T = \Big(E *_{\mathbb{C}^2} (\mathbb{C}^2 \otimes I_h)\Big) *_{\mathrm{r}} D, \qquad q = e^{2\pi i h}.$$

The relative Haar conjugation rule gives a faithful embedding $\eta : S \to T$ defined by

$$\eta|_{A_1 \oplus A_2} = a, \qquad \eta|_{\mathrm{M}_2} = \mathrm{Ad}(q) \circ b, \qquad \eta|_D = \mathrm{id}_D\,.$$

The last prescription concerns the common external reservoir; the interval h is a separate relative coordinate commuting with $\mathbb{C}^2$. Proposition 9.7 verifies that η has invertible full selected marking and that the working targets satisfy the admissibility and zero-K_1 hypotheses of the return construction.

(iv) *One return with source and target control.* For prescribed finite sets $F \subset S$ and $L \subset T$ and $\varepsilon > 0$, construct an actual trace-preserving embedding $\beta : T \to S$ and an actual trace-preserving automorphism α of T such that

$$\|\beta\eta(s) - s\| < \varepsilon \quad (s \in F), \qquad \|\alpha\eta\beta(t) - t\| < \varepsilon \quad (t \in L).$$

The return has the inverse full selected marking, and α acts identically on the full invariant. The first estimate comes from nuclear-source uniqueness and relative free transport. The second requires compatible corrections of the nuclear coefficients and the whole finite matrix frame, using independent parts of the reservoir. Relative logarithm comparison, lifts of circle automorphisms to interval automorphisms, and normalized corner determinants make the two estimates hold for the same β. These are the principal analytic ingredients developed in Sections 6, 7 and 9.

(v) *Isomorphism and compression.* Include $\eta(F)$ in L. The two estimates then make α almost fix $\eta(F)$, while placing each target test element near $\alpha\eta(S)$. More precisely,

$$\begin{aligned} \|\alpha\eta(s) - \eta(s)\| &\le \|s - \beta\eta(s)\| + \|\alpha\eta\beta(\eta(s)) - \eta(s)\| < 2\varepsilon \quad (s \in F), \\ \mathrm{dist}(t, \alpha\eta(S)) &\le \|t - \alpha\eta\beta(t)\| < \varepsilon \qquad\qquad\qquad\qquad\quad (t \in L). \end{aligned}$$

Since the tolerances are arbitrary, Lemma 4.3 gives $S \cong T$ with the marking of η. Finite-dimensional tag absorption identifies T with $E*_{\mathrm{r}}D$, fixing E. Together with the matrix identities of Proposition 9.4, this yields

$$\mathrm{M}_2\Big((A_1*_{\mathrm{r}}A_2)*_{\mathrm{r}}D\Big) \cong \mathrm{M}_2(E_0*_{\mathrm{r}}D).$$

The marking tracks the exact class of the standard projection e_{11}, including possible two-torsion: $\eta(e_{11}) = e_{11}$, because $e_{11} \in \mathbb{C}^2$ and q commutes with $\mathbb{C}^2$. Stable-rank-one cancellation for this projection and its complement gives a unitary correction carrying it to the standard target corner. Compression therefore gives the desired binary fusion.

(vi) *Finite and countable families and arbitrary* K_1. Iteration treats finite zero-K_1 families. For a countable zero-K_1 family, each new fusion is chosen to approximate the previous identification on increasing finite subsets of the old D-stabilized product, using Proposition 9.8. The two-sided dense exhaustions and summable estimates make both the stage isomorphisms and their inverses converge, giving mutually inverse limiting maps as in Proposition 9.9. The split-unit argument of Section 8 then separates arbitrary K_1 and completes the general fusion theorem. Marked nuclear-core replacement realizes any prescribed compatible global marking after the fusion isomorphism has been constructed.

6. Relative logarithms and interval automorphisms

We now develop the logarithm comparison needed for the simultaneous estimates in the outline of Section 5. The coefficient algebra need not be nuclear: proper unitary comparison uses a nuclear difference ideal and a vanishing actual relative KK class. We also lift suitable circle automorphisms to actual interval automorphisms. These results will later be applied to a global nuclear core together with finitely many distinguished intervals, and extended over a finite diagonal in Section 9.

6.1. **The endpoint state and one-turn interval relations.** Throughout this subsection, all reduced free products use their specified faithful tracial states. We write

$$R = (F, \tau_F)*_{\mathrm{r}}(\mathrm{C}(\mathbb{T}), m), \qquad Q = (F, \tau_F)*_{\mathrm{r}}(\mathrm{C}([0,1]), \lambda),$$

and denote the circle and interval coordinates by w and h. The canonical inclusion $j : R \to Q$ sends w to $e^{2\pi i h}$ and fixes F. No nuclearity assumption on F is made in the next three lemmas.

Lemma 6.1 (The endpoint state)**.** *The state $\omega = \mathrm{ev}_1 \circ \mathbb{E}_{\mathbb{T}}$ is the unique state on R taking value one at w. For every $a \in R$,*

$$\sup_f \|f(w)(a - \omega(a)1)f(w)\| \longrightarrow 0 \qquad (\delta \downarrow 0),$$

where the supremum runs over positive contractions f supported in $\{z : |z-1| \le \delta\}$. In any representation of R, the spectral projection $P = 1_{\{1\}}(w)$ satisfies $PaP = \omega(a)P$.

Proof. Put $F_n = w^n F w^{-n}$ and $B = \mathrm{C}^*(F_n : n \in \mathbb{Z})$. Expanding centered words proves that the F_n are free. Finite sums $\sum_k x_k w^k$, with each x_k supported in finitely many F_n, are dense in R, and $\mathbb{E}_{\mathbb{T}}(xw^k) = \tau_B(x)w^k$ for $x \in B$. For centered elements $y_1, \dots, y_N$ in

free subalgebras, the creation, annihilation and diagonal decomposition on the reduced free-product Hilbert space gives

$$\Big\|\sum_j y_j\Big\| \le \max_j \|y_j\| + \Big(\sum_j \tau(y_j^* y_j)\Big)^{1/2} + \Big(\sum_j \tau(y_j y_j^*)\Big)^{1/2}.$$

If $x \in B$ is centered and has finite integer support of width less than L, its translates by w^{jL} lie in free subalgebras. Consequently

$$\Big\|N^{-1}\sum_{j=1}^{N} w^{jL} x w^{-jL}\Big\| \le (N^{-1} + 2N^{-1/2})\|x\|.$$

A state φ with $\varphi(w) = 1$ has w in its multiplicative domain. It is invariant under these translates and satisfies $\varphi(aw^k) = \varphi(a)$. Averaging and then using the displayed dense algebra gives $\varphi(\sum_k x_k w^k) = \sum_k \tau_B(x_k) = \omega(\sum_k x_k w^k)$. This proves uniqueness.

For the asserted uniform estimate, $\|(w^n - 1)f(w)\| \le |n|\delta$ gives

$$\|f(w)xf(w)\| \le (N^{-1} + 2N^{-1/2} + 2LN\delta)\|x\|.$$

First choose N large and then δ small. For a local sum $\sum_k x_k w^k$, apply this to $x_k - \tau_B(x_k)1$; the remaining scalar error is at most $\delta \sum_k |k|\,|\tau_B(x_k)|$. Density proves the statement for all a. Finally, every unit vector in $P\mathcal{H}$ defines a state taking value one at w. Uniqueness and polarization give the compression formula. □

Lemma 6.2 (The reduced interval pushout)**.** *The canonical surjection*

$$R *^{\max}_{\mathrm{C}(\mathbb{T})} \mathrm{C}([0,1]) \longrightarrow Q$$

is an isomorphism.

Proof. For $0 < \delta < 1/4$, define continuous circle functions by

$$g^0_\delta(e^{2\pi i t}) = \begin{cases} t & 0 \le t \le 1-\delta, \\ (1-\delta)(1-t)/\delta & 1-\delta \le t \le 1, \end{cases}$$

and

$$g^1_\delta(e^{2\pi i t}) = \begin{cases} 1-(1-\delta)t/\delta & 0 \le t \le \delta, \\ t & \delta \le t \le 1. \end{cases}$$

Their Haar laws are exactly normalized Lebesgue measure on $[0, 1-\delta]$ and $[\delta, 1]$, respectively: the two linear branches give the constant density $1+\delta/(1-\delta) = 1/(1-\delta)$ on the corresponding interval. Each $g^s_\delta(w)$ is free from F. Equality of the joint free laws and faithfulness of the specified traces therefore give, for every joint polynomial with finitely many coefficients in F,

$$\|p(F, g^0_\delta(w))\| = \|p(F, (1-\delta)h)\|_Q,$$
$$\|p(F, g^1_\delta(w))\| = \|p(F, \delta + (1-\delta)h)\|_Q.$$

Both affine functions of h converge to h in norm. Polynomial continuity consequently gives

$$\|p(F, g^s_\delta(w))\| \longrightarrow \|p(F, h)\|_Q. \tag{6.1}$$

Represent the full pushout on $\mathcal{H}$, and write W, H for its circle and interval coordinates. Set $P_s = 1_{\{s\}}(H)$ and $P = P_0 + P_1 = 1_{\{1\}}(W)$. By Lemma 6.1, $PaP = \omega(a)P$ for $a \in R$.

The subspaces $\mathcal{H}_s = \overline{RP_s\mathcal{H}}$ are orthogonal, reduce R, and satisfy $P\mathcal{H}_s = P_s\mathcal{H}$. Off $P\mathcal{H}$, the equation $e^{2\pi iH} = W$ uniquely determines H as a Borel argument of W. On $P_s\mathcal{H}$, it equals s. Hence $\mathcal{H}_s$ also reduces H. Their orthogonal complement $\mathcal{H}_*$ reduces R, H and has no 1-eigenvectors for W. The spectral theorem gives $g^s_\delta(W) \to H$ strongly on $\mathcal{H}_s$, and either branch converges strongly on $\mathcal{H}_*$. Strong lower semicontinuity of the norm, contractivity of the representation of R, and (6.1) now give $\|p(F,H)\| \le \|p(F,h)\|_Q$ on each reducing subspace. The canonical quotient gives the opposite inequality. □

Corollary 6.3 (Strong one-turn logarithms)**.** *Let L have a faithful tracial state, let $F \subset L$, and let $w \in L$ be Haar and free from F. If positive contractions $H_n \in L$ satisfy $\|e^{2\pi iH_n} - w\| \to 0$, then*

$$\|p(F,H_n)\| \longrightarrow \|p(F,h)\|_Q$$

for every joint polynomial, and all joint moments converge. In particular, $h \mapsto (H_n)_n$ and the constant inclusion of F define a faithful homomorphism $\Theta : Q \to L_\infty$, where $L_\infty = \ell^\infty(L)/c_0(L)$, and $\Theta j = \iota_L|_R$.

Proof. In every norm ultrapower of every subsequence, $H = (H_n)_\omega$ has $e^{2\pi iH} = w$. Lemma 6.2 gives the upper bound for each joint polynomial. Failure of that bound along the original sequence would persist along a subsequence and contradict this observation.

Put $w_n = e^{2\pi iH_n}$. Its law converges to Haar measure. The difference $H_n - g^0_\delta(w_n)$ is bounded by one and is supported where $H_n \in [1-\delta, 1]$. The latter set maps into a circle arc of Haar measure δ. Weak convergence of the circle laws therefore gives $\limsup_n \|H_n - g^0_\delta(w_n)\|_2^2 \le \delta$. For fixed δ, $g^0_\delta(w_n) \to g^0_\delta(w)$ in norm. Letting $\delta \downarrow 0$ proves convergence of H_n in 2-norm to the Borel argument of w. This argument is uniform on bounded words and proves joint moment convergence. Faithfulness of the trace on Q then gives

$$\|p(F,h)\| = \lim_{k\to\infty} \tau_Q((p^*p)^k)^{1/(2k)} \le \liminf_n \|p(F,H_n)\|.$$

This supplies the lower bound and the asserted faithful map. Every ultralimit trace on its image is the specified trace. □

6.2. **The actual two-logarithm comparison ideal.**

Proposition 6.4 (The induced two-projection ideal)**.** *Let R be separable and unital, and let $w \in R$ have a unique state ω taking value one at w. Suppose $T = \mathrm{C}^*(R, H_0, H_1)$, where H_s are positive contractions with $e^{2\pi iH_s} = w$. Put*

$$J = \mathrm{Ideal}_{\mathrm{C}^*(1,H_0,H_1)}(H_1 - H_0), \qquad I = \mathrm{Ideal}_T(H_1 - H_0).$$

Then, with $\mathcal{H}_\omega$ the GNS space of ω,

$$I \cong \mathcal{K}(\mathcal{H}_\omega) \otimes J, \quad axb^* \longleftrightarrow |\pi_\omega(a)\Omega\rangle\langle\pi_\omega(b)\Omega| \otimes x. \tag{6.2}$$

Moreover, J is a quotient of

$$J_{\mathrm{pair}} = \{f \in \mathrm{C}([0,1], \mathrm{M}_2) : f(0) = 0,\ f(1) \text{ is diagonal}\}.$$

Thus I is separable, nuclear and type I, with nuclear dimension at most one. If $\dim \mathcal{H}_\omega = \infty$, then I is stable and has the corona factorisation property.

Proof. Represent T faithfully on $\mathcal{H}$ and put $P = 1_{\{1\}}(w)$. Each H_s commutes with w; off $P\mathcal{H}$ both are the same Borel argument, while on $P\mathcal{H}$ they are projections. Hence $(w-1)J = J(w-1) = 0$. Uniqueness of ω gives $PaP = \omega(a)P$, and therefore

$$xay = \omega(a)xy \qquad (x, y \in J, a \in R). \tag{6.3}$$

Let $\mathcal{V} = \overline{J\mathcal{H}}$ and $\mathcal{M} = \overline{R\mathcal{V}}$. The map

$$V : \mathcal{H}_\omega \otimes \mathcal{V} \to \mathcal{M}, \qquad \pi_\omega(a)\Omega \otimes \xi \mapsto a\xi$$

is unitary, since its inner products are $\omega(a^*b)\langle\xi, \eta\rangle$. Here R acts as $\pi_\omega(R) \otimes 1$, and $x \in J$ acts as $|\Omega\rangle\langle\Omega| \otimes x|_{\mathcal{V}}$. The representation of J on $\mathcal{V}$ is faithful. It follows that the closed span K of axb^* acts as the right side of (6.2) on $\mathcal{M}$ and as zero on $\mathcal{M}^\perp$.

We check invariance under the two logarithms. The 1-eigenspace of $\pi_\omega(w)$ is exactly $\mathbb{C}\Omega$: the endpoint compression gives $1_{\{1\}}(\pi_\omega(w))\pi_\omega(a)\Omega = \omega(a)\Omega$, and these vectors are dense. Also $P\mathcal{M} = \mathcal{V}$, and $\mathcal{M}$ reduces each H_s because its complement to $\mathcal{V}$ carries the common argument of w, whereas $H_sJ \subset J$. In the tensor coordinates above,

$$H_s = \arg_0(\pi_\omega(w)) \otimes 1 + |\Omega\rangle\langle\Omega| \otimes m_s,$$

where $m_s \in \mathcal{M}(J)$ is the multiplier induced by H_s. Both terms normalize $\mathcal{K}(\mathcal{H}_\omega) \otimes J$. Thus K is an ideal of T containing J, and equals I.

The element w is central in $D = \mathrm{C}^*(1, H_0, H_1)$. Since J annihilates $\overline{(w-1)D}$, its intersection with that ideal is zero, so J embeds as an ideal in the fiber at $w = 1$. This fiber is generated by two projections. The universal algebra of two projections is

$$\{f \in \mathrm{C}([0,1], \mathrm{M}_2) : f(0), f(1) \text{ are diagonal}\}.$$

Choose its usual generators p, q with $p = q$ at zero and $q = 1 - p$ at one. The ideal generated by $p - q$ is exactly J_{pair}, proving the quotient assertion. Its one-dimensional subhomogeneous description, passage to quotients and matrix stability give the nuclear dimension bound. Infinite-dimensional $\mathcal{H}_\omega$ gives stability; finite nuclear dimension gives the corona factorisation property [40]. □

Lemma 6.5 (Proper comparison)**.** *Let A be separable and unital, and let I be separable, stable, nuclear and have the corona factorisation property. For unital full homomorphisms $\varphi_0, \varphi_1 : A \to \mathcal{M}(I)$ forming a Cuntz pair, assume $[\varphi_0, \varphi_1] = 0$ in $\mathrm{KK}(A, I)$. Then there are unitaries $v_m \in 1 + I$ such that $\mathrm{Ad}(v_m)\varphi_0(a) \to \varphi_1(a)$ for every $a \in A$.*

Proof. Every compression $a \mapsto b^*\varphi_s(a)b$, $b \in I$, takes values in the nuclear algebra I, so is nuclear. Both maps are weakly nuclear. The same argument for $\mathrm{C}([0,1]) \otimes I$ makes every ordinary Cuntz-pair homotopy weakly nuclear: this homotopy ideal is nuclear as well. Thus both the pairs and their homotopy relations agree, identifying $\mathrm{KK}_{\mathrm{nuc}}(A, I) = \mathrm{KK}(A, I)$. Fullness and the corona factorisation property give unital nuclear absorption [27, Theorem 1.11]. The corona relative commutant is K_1-injective by [46, Corollary 2.9]. Thus [27, Theorem 1.12(i)] gives asymptotic conjugacy by unitaries in the unitization of I. Removing their scalar parts puts them in $1 + I$ without changing conjugation. □

Proposition 6.6 (Reparameterization descent)**.** *Let F be separable, unital and faithfully tracial, with $\dim L^2(F, \tau_F) = \infty$, and suppose $Q = F *_{\mathrm{r}} \mathrm{C}([0,1])$ is simple. Let L be unital with a specified faithful trace, and suppose the native $R = F *_{\mathrm{r}} \mathrm{C}(\mathbb{T})$ embeds unitally and*

trace preservingly in L. If positive contractions $H_n \in L$ satisfy $e^{2\pi i H_n} \to w$ in norm, there are a faithful unital $f : Q \to L$ and unitaries $u_n \in L$ such that

$$\mathrm{Ad}(u_n) f j(x) \longrightarrow x \qquad (x \in R).$$

The map f preserves the specified trace.

Proof. Corollary 6.3 gives $\Theta : Q \to L_\infty$ with $\Theta j = \iota_L|_R$. For any diverging $r : \mathbb{N} \to \mathbb{N}$, form $T = \mathrm{C}^*(R, H_0, H_1) \subset L_\infty$, with $H_0 = (H_n)_n$ and $H_1 = (H_{r(n)})_n$. Their exponentials equal the constant w. By Proposition 6.4, the difference ideal I is stable and nuclear with the corona factorisation property: the GNS space contains $L^2(F, \tau_F)$ isometrically. If $I = 0$ the two maps coincide. Otherwise their multiplier maps form a unital Cuntz pair and are full, since Q is simple. They agree exactly on F.

The inclusion $F \to Q$ is a KK-equivalence. Indeed the full free product with the interval retracts onto F by evaluation at zero, and $h \mapsto th$ gives the required homotopy. The full-to-reduced quotient is a KK-equivalence by [20, Proposition 2.15 and Theorem 3.1]: the faithful expectations identify the vertex-reduced and ordinary reduced products. Restriction to F therefore annihilates the actual KK-pair injectively. Lemma 6.5 gives proper approximate unitary comparison of Θ and $r^*\Theta$.

These multiplier estimates are estimates in T itself: their errors belong to I, on which the multiplier action is isometric. Essentiality of I is unnecessary. Separability and diagonal selection improve approximate unitary equivalence in L_∞ to unitary equivalence. Explicitly, lift successively chosen implementing unitaries to coordinate unitaries in L, and let the finite packet index increase sufficiently slowly with the coordinate. Such coordinate lifts exist by polar decomposition of an asymptotically unitary representative in $\ell^\infty(L)$. No homomorphic or completely positive lift of Θ is used. Intertwining by reparameterization [4, Proposition 9.2] now gives $\Theta = \mathrm{Ad}(U)\iota_L f$ for a homomorphism $f : Q \to L$. It is faithful and unital because Θ is. Lifting U to coordinate unitaries proves the displayed limit. Applying any ultralimit of the specified trace to this equality and using joint moment convergence proves trace preservation. □

Corollary 6.7 (Two logarithms in a sequence algebra)**.** *Let F be separable, unital and faithfully tracial, with $\dim L^2(F) = \infty$, and suppose $Q = F *_{\mathrm{r}} \mathrm{C}([0,1])$ is simple. Put $q = e^{2\pi i h}$ and $R = \mathrm{C}^*(F, q) \subset Q$. For a unital algebra L, let $\rho_0, \rho_1 : Q \to L_\infty$ be faithful unital homomorphisms agreeing on R. Then there is $V \in \mathrm{U}(L_\infty)$ with $\mathrm{Ad}(V)\rho_0 = \rho_1$.*

Proof. Identify R with its common faithful image and form the separable algebra generated by this image and $\rho_0(h), \rho_1(h)$. Lemma 6.1 and Proposition 6.4 give the difference ideal $I \cong \mathcal{K}(\mathcal{H}_\omega) \otimes J$. The GNS space contains $L^2(F)$, so I is stable, nuclear and has the corona factorisation property. If $I = 0$ the maps coincide. Otherwise the two multiplier maps are full because their source Q is simple and their common unit is the multiplier unit. They agree on F. As in Proposition 6.6, $F \hookrightarrow Q$ is a KK-equivalence, so their actual Cuntz-pair class is zero. Lemma 6.5 gives proper approximate conjugacies. Their errors lie in I, where the multiplier action is isometric, so these are norm estimates in L_∞. Lifting the implementing unitaries to coordinate unitaries and diagonalizing on a countable dense subset of Q gives the asserted single unitary V. □

Corollary 6.8 (Actual marked interval models)**.** *Let L be an admissible target containing a uniform interval coordinate k. Let $F \subset L$ be separable and unital with $\dim L^2(F) = \infty$, and let $w \in L$ be Haar and free from F. Suppose $Q = F *_{\mathrm{r}} \mathrm{C}([0,1])$ is simple and the two embeddings of $\mathrm{C}(\mathbb{T})$ with coordinate images w and $e^{2\pi i k}$ have the same full invariant. Then there are a trace-preserving embedding $f : Q \to L$ and unitaries $u_n \in L$ such that*

$$\mathrm{Ad}(u_n) f(a) \longrightarrow a \quad (a \in F), \qquad \mathrm{Ad}(u_n) f(e^{2\pi i h}) \longrightarrow w.$$

In particular $f|_F$ has the full invariant of its native inclusion.

Proof. Theorem 3.16, applied to the nuclear source $\mathrm{C}(\mathbb{T})$, supplies unitaries z_n with $z_n e^{2\pi i k} z_n^* \to w$. The positive uniform coordinates $H_n = z_n k z_n^*$ therefore satisfy the hypotheses of Proposition 6.6. Its conclusion gives the embedding and unitaries. Approximate unitary equivalence on F retains ordinary and coefficient K-theory, traces and the continuous algebraic-unitary invariant. □

6.3. **Lifting circle automorphisms to actual interval automorphisms.**

Theorem 6.9 (Relative interval lifting)**.** *Let F be separable, unital and faithfully tracial, with $\dim L^2(F, \tau_F) = \infty$. Suppose $Q = F *_{\mathrm{r}} \mathrm{C}([0,1])$ is an admissible target, and put $q = e^{2\pi i h}$ and $R = \mathrm{C}^*(F, q) \subset Q$. Let $\gamma \in \mathrm{Aut}(R)$ preserve the specified trace and satisfy $\gamma(F) = F$. Assume that the three embeddings of $\mathrm{C}(\mathbb{T})$ into Q with coordinate images $q, \gamma(q), \gamma^{-1}(q)$ induce the same totally compatible triple. Then there are trace-preserving automorphisms $\alpha_n \in \mathrm{Aut}(Q)$ such that*

$$\alpha_n(x) \longrightarrow \gamma(x) \qquad (x \in R).$$

They may be chosen as inner conjugates of one fixed automorphism α. For every n, the restriction $\alpha_n|_F$ is approximately unitarily equivalent to $\gamma|_F$, and therefore induces exactly its full compatible invariant on F. No convergence on the interval coordinate h is asserted.

Proof. Nuclear-source uniqueness for $\mathrm{C}(\mathbb{T})$, Theorem 3.16, gives positive uniform contractions H_n with $e^{2\pi i H_n} \to \gamma(q)$: conjugate the native h by unitaries comparing q with $\gamma(q)$. The coefficient algebra is now $\gamma(F)$, and $\gamma(q)$ is Haar and free from it. Applying Proposition 6.6 with the identification of F supplied by γ gives a fixed trace-preserving embedding $f : Q \to Q$ and unitaries $u_n \in Q$ with $f_n = \mathrm{Ad}(u_n) f$ satisfying $f_n|_R \to \gamma$. Applying the same construction to γ^{-1} gives a fixed embedding g and $g_n = \mathrm{Ad}(v_n) g$ with $g_n|_R \to \gamma^{-1}$.

Contractivity and the two limits imply $f_n g_n|_R \to \mathrm{id}_R$ and $g_n f_n|_R \to \mathrm{id}_R$. For example, for $x \in R$,

$$\|f_n g_n(x) - x\| \le \|g_n(x) - \gamma^{-1}(x)\| + \|f_n(\gamma^{-1}(x)) - x\| \longrightarrow 0.$$

Define faithful unital homomorphisms

$$\rho_0(x) = (f_n g_n(x))_n, \qquad \rho_1(x) = \iota_Q(x) \quad (x \in Q).$$

The preceding convergence gives $\rho_0|_R = \rho_1|_R$. Corollary 6.7, with $L = Q$, supplies $V \in \mathrm{U}(Q_\infty)$ such that $\mathrm{Ad}(V)\rho_0 = \rho_1$. Lift V to coordinate unitaries $V_n \in Q$. Since

$$f_n g_n = \mathrm{Ad}(u_n f(v_n))\, fg,$$

we have $\mathrm{Ad}(V_n u_n f(v_n)) fg(x) \to x$ for every $x \in Q$. Thus the fixed composition fg is approximately unitarily equivalent to id_Q. The same argument gives $gf \approx_{\mathrm{u}} \mathrm{id}_Q$. The

two-sided approximate intertwining argument [4, Proposition 9.7] gives an isomorphism $\alpha : Q \to Q$ approximately unitarily equivalent to f. For each successive finite packet in R, choose n making f_n close to γ, and then an inner conjugate of α close to f_n on that packet. A diagonal choice gives the asserted automorphisms. Their trace preservation follows either from approximate unitary equivalence to f or from uniqueness of the trace on Q. Finally, $f|_F \approx_{\mathrm{u}} \gamma|_F$ by the original sequence u_n, and $\alpha \approx_{\mathrm{u}} f$. Approximate unitary equivalence is unchanged by an inner conjugacy, so each $\alpha_n|_F$ has this same approximate-unitary class. Its ordinary and coefficient K-maps and its continuous algebraic-unitary map consequently agree with those of $\gamma|_F$; its trace map agrees by trace preservation. □

Remark 6.10 (Finite packets and moving coefficients). The theorem applies separately at every coordinate. Thus, for a sequence γ_n satisfying its hypotheses, finite packets $G_n \subset R$, and positive tolerances ε_n, one may choose actual automorphisms α_n with $\|\alpha_n(x) - \gamma_n(x)\| < \varepsilon_n$ for $x \in G_n$. The packets may include moving coefficients or multipliers belonging to R. This conclusion supplies precisely finite packet control on R; it adds no control on h. The full marked hypothesis in the theorem must be verified. Vanishing of the ordinary K_1 class alone does not replace it. An infinite free interval tail in F supplies the infinite GNS space and simplicity required above; the admissible-target and marked uniqueness hypotheses remain part of the assertion.

6.4. **Comparison of finitely many common free circles.** We first extend the logarithm comparison in the preceding one-circle construction to finitely many independent circle coordinates. The independence is essential: it separates their endpoint defects. Throughout this subsection, F is separable, unital and faithfully tracial, and

$$R = F *_{\mathrm{r}} \overset{r}{\underset{i=1}{*_{\mathrm{r}}}} (\mathrm{C}(\mathbb{T}_{w_i}), m), \qquad Q = F *_{\mathrm{r}} \overset{r}{\underset{i=1}{*_{\mathrm{r}}}} (\mathrm{C}([0,1]_{h_i}), \lambda), \qquad 1 \le r < \infty.$$

The native inclusion $j : R \to Q$ fixes F and sends w_i to $e^{2\pi i h_i}$. Put

$$\omega_i = \mathrm{ev}_1 \circ \mathbb{E}_{\mathbb{T}_{w_i}}, \qquad (\pi_i, \mathcal{H}_i, \Omega_i) = \mathrm{GNS}(\omega_i).$$

Applying Lemma 6.1 with all the other free factors included in the coefficient algebra shows that ω_i is the unique state of R taking value one at w_i.

Lemma 6.11 (Separation of endpoint representations)**.** *If $i \neq j$, then*

$$\pi_i(w_j)^n \longrightarrow 0 \quad \textit{weakly as } |n| \longrightarrow \infty.$$

In particular $\pi_i(w_j)$ has no 1-eigenvectors. In any representation of R, put $P_i = 1_{\{1\}}(w_i)$ and $\mathcal{M}_i = \overline{RP_i\mathcal{H}}$. The spaces $\mathcal{M}_i$ are pairwise orthogonal and reduce R.

Proof. Let a, b belong to the dense algebra of finite reduced words, using Laurent polynomials in the circle coordinates. For sufficiently large $|n|$, reduction of $a^* w_j^n b$ leaves a nonzero power of w_j: the fixed words can cancel only finitely many powers at its two boundaries. Since $j \neq i$, its expectation onto $\mathrm{C}(\mathbb{T}_{w_i})$ is zero. Thus

$$\langle \pi_i(w_j)^n \pi_i(b)\Omega_i, \pi_i(a)\Omega_i \rangle = \omega_i(a^* w_j^n b) = 0$$

eventually. Density and boundedness give weak convergence on $\mathcal{H}_i$, and therefore absence of 1-eigenvectors.

The endpoint compression formula gives $P_i a P_i = \omega_i(a) P_i$. Consequently the representation of R on $\mathcal{M}_i$ is an amplification of π_i, under the unitary

$$\mathcal{H}_i \otimes P_i\mathcal{H} \longrightarrow \mathcal{M}_i, \qquad \pi_i(a)\Omega_i \otimes \xi \longmapsto a\xi.$$

The first assertion implies $P_j\mathcal{M}_i = 0$ for $j \neq i$. Since $\mathcal{M}_i$ reduces R, it is then orthogonal to $\overline{RP_j\mathcal{H}} = \mathcal{M}_j$. □

Proposition 6.12 (The finite logarithm difference ideal)**.** *Let*

$$\mathcal{T} = \mathrm{C}^*(R, H_i^s : 1 \leq i \leq r, s = 0, 1), \qquad 0 \leq H_i^s \leq 1, \qquad e^{2\pi i H_i^s} = w_i.$$

Write

$$J_i = \mathrm{Ideal}_{\mathrm{C}^*(1, H_i^0, H_i^1)}(H_i^1 - H_i^0), \qquad I = \mathrm{Ideal}_{\mathcal{T}}(H_i^1 - H_i^0 : 1 \leq i \leq r).$$

Then

$$I = \bigoplus_{i=1}^{r} I_i, \qquad I_i \cong \mathcal{K}(\mathcal{H}_i) \otimes J_i. \tag{6.4}$$

Each J_i is a quotient of

$$J_{\mathrm{pair}} = \{f \in \mathrm{C}([0,1], \mathrm{M}_2) : f(0) = 0,\ f(1) \text{ is diagonal}\}.$$

Thus I is separable, nuclear and type I, of nuclear dimension at most one. If $\dim L^2(F, \tau_F) = \infty$, it is stable and has the corona factorisation property.

Proof. Represent $\mathcal{T}$ faithfully on $\mathcal{H}$. For a fixed i, Proposition 6.4, applied first to $\mathrm{C}^*(R, H_i^0, H_i^1)$, identifies the closed span

$$K_i = \overline{\mathrm{span}}\{axb^* : a, b \in R, x \in J_i\} \cong \mathcal{K}(\mathcal{H}_i) \otimes J_i.$$

Its support space is $\mathcal{N}_i = \overline{RJ_i\mathcal{H}} \subset \mathcal{M}_i$. The proof of that proposition shows that $\mathcal{N}_i$ reduces R, H_i^0, H_i^1, and that these two logarithms normalize K_i. We verify the additional invariance required here.

For $j \neq i$, Lemma 6.11 shows that $w_j|_{\mathcal{N}_i}$ has no 1-eigenvectors. Both H_j^s therefore restrict to the same Borel argument of w_j on $\mathcal{N}_i$. More explicitly, this subspace reduces w_j, its spectral projection at one is zero there, and off that projection the relation $e^{2\pi i H_j^s} = w_j$ determines H_j^s uniquely. Thus $\mathcal{N}_i$ reduces both logarithms, and in the tensor representation they act as

$$\arg_0(\pi_i(w_j)) \otimes 1.$$

These bounded operators normalize $\mathcal{K}(\mathcal{H}_i) \otimes J_i$. On $\mathcal{N}_i^\perp$ the ideal K_i is zero, so this proves that K_i is an ideal of the whole algebra $\mathcal{T}$. It consequently equals the ideal I_i generated there by $H_i^1 - H_i^0$.

The spaces $\mathcal{N}_i \subset \mathcal{M}_i$ are pairwise orthogonal. Hence $I_i I_j = 0$ for $i \neq j$, and the finite sum of the I_i is closed and equals I. The description of J_i and its nuclear dimension are those of Proposition 6.4. Moreover $L^2(F, \tau_F)$ embeds isometrically in every $\mathcal{H}_i$, since $\omega_i|_F = \tau_F$. Infinite dimensionality of this space therefore gives stability of every nonzero I_i. The finite direct sum is stable, and its finite nuclear dimension gives the corona factorisation property [40]. □

Proposition 6.13 (Simultaneous strong one-turn logarithms)**.** *The canonical map from the iterated full pushout*

$$R *^{\max}_{\mathrm{C}(\mathbb{T}_{w_1})} \mathrm{C}([0,1]_{h_1}) *^{\max}_{\mathrm{C}(\mathbb{T}_{w_2})} \cdots *^{\max}_{\mathrm{C}(\mathbb{T}_{w_r})} \mathrm{C}([0,1]_{h_r}) \longrightarrow Q \tag{6.5}$$

is an isomorphism, where every amalgamation sends w_i to $e^{2\pi i h_i}$.

Let L be unital with a faithful specified trace, and suppose the native R embeds unitally and trace preservingly in L. If positive contractions $H_{i,n} \in L$ satisfy

$$\|e^{2\pi i H_{i,n}} - w_i\| \longrightarrow 0 \qquad (1 \le i \le r),$$

then, for every joint polynomial with coefficients in F,

$$\|p(F, H_{1,n}, \ldots, H_{r,n})\| \longrightarrow \|p(F, h_1, \ldots, h_r)\|_Q,$$

and all joint moments converge to those in Q. In particular the constant inclusion of F and the assignments $h_i \mapsto (H_{i,n})_n$ define a faithful unital homomorphism $\Theta : Q \to L_\infty$ with $\Theta j = \iota_L|_R$.

Proof. We establish the full pushout assertion first. At the first step, Lemma 6.2 applies with coefficient algebra $F *_{\mathrm{r}} *_{\mathrm{r}\,i=2}^{r} \mathrm{C}(\mathbb{T}_{w_i})$. Its conclusion replaces just the first circle by its uniform interval. Repeat the argument, putting the intervals already obtained and all the remaining circles into the coefficient algebra at each step. These are reduced products with faithful specified traces. After r steps the resulting reduced product is Q, proving (6.5).

In any norm ultrapower of any subsequence, the classes $H_i = (H_{i,n})_\omega$ satisfy $e^{2\pi i H_i} = w_i$. The full pushout just proved gives the upper bound for every polynomial norm. A failure of this upper bound on the original sequence would persist on a subsequence and contradict that conclusion.

For the moments, use the continuous argument functions g^0_δ from the proof of Lemma 6.2. As in Corollary 6.3, the circle laws converge to Haar measure and

$$\limsup_n \|H_{i,n} - g^0_\delta(e^{2\pi i H_{i,n}})\|_2^2 \le \delta.$$

For fixed δ the continuous functions of the exponentials converge in norm to $g^0_\delta(w_i)$. Letting δ tend to zero shows that $H_{i,n}$ converges in 2-norm to the Borel argument of w_i in the tracial von Neumann algebra of L. There are only finitely many coordinates. Telescoping bounded words therefore gives simultaneous joint moment convergence. The Borel arguments have uniform laws and remain free from each other and from F, so these are exactly the moments of Q. Faithfulness of its trace gives the lower norm bound through

$$\|p(F, h_1, \ldots, h_r)\| = \lim_{k\to\infty} \tau_Q((p^*p)^k)^{1/(2k)} \le \liminf_n \|p(F, H_{1,n}, \ldots, H_{r,n})\|.$$

This proves the norm assertion and the faithful sequence-algebra map. Every ultralimit of the specified trace induces the native trace on its image. □

Corollary 6.14 (Finite common-circle comparison)**.** *Suppose $\dim L^2(F, \tau_F) = \infty$ and Q is simple. Let L be unital, and let $\rho_0, \rho_1 : Q \to L_\infty$ be faithful unital homomorphisms agreeing on $j(R)$. Then there is $V \in \mathrm{U}(L_\infty)$ with*

$$\mathrm{Ad}(V)\rho_0 = \rho_1.$$

Proof. Identify R with its common faithful image, and form the separable algebra generated by that image and the finitely many pairs $\rho_0(h_i), \rho_1(h_i)$. Proposition 6.12 gives its difference ideal I, which is stable and nuclear and has the corona factorisation property. If $I = 0$, the maps coincide. Otherwise their multiplier maps are unital and full: the simple source Q makes the image of every nonzero positive element generate the multiplier unit. Their differences lie in I, so they form a Cuntz pair.

The inclusion $F \to Q$ is a KK-equivalence. Indeed, the full free product of F with finitely many intervals retracts onto F by evaluation at zero, and simultaneous contraction of the interval coordinates gives a homotopy inverse. Its full-to-reduced quotient is a KK-equivalence by [20, Proposition 2.15 and Theorem 3.1], using the faithful expectations to identify the reduced models. Since the two multiplier maps agree exactly on F, restriction annihilates their actual Cuntz-pair class in $\mathrm{KK}(Q, I)$ injectively. That class is therefore zero. Lemma 6.5 gives proper approximate unitary comparison by unitaries in $1 + I$.

These are norm estimates in the generated algebra, not merely in its multiplier image: their errors belong to I, where the multiplier action is isometric. Lift the implementing unitaries to coordinate unitaries in L, and diagonalize on a countable dense subset of Q. The resulting single unitary in L_∞ gives the asserted equality. Essentiality of I is not needed. □

Corollary 6.15 (Actual finite interval models)**.** *Suppose* $\dim L^2(F, \tau_F) = \infty$ *and* Q *is simple. Let* L *be an admissible target containing the native* R *unitally and trace preservingly. Suppose that* L *contains a uniform interval coordinate* k *such that, for every* i*, the circle embeddings with coordinate images* w_i *and* $e^{2\pi i k}$ *induce the same totally compatible triple. Then there are a trace-preserving embedding* $f : Q \to L$ *and unitaries* $u_n \in L$ *such that*

$$\mathrm{Ad}(u_n) f(a) \longrightarrow a \quad (a \in F), \qquad \mathrm{Ad}(u_n) f(e^{2\pi i h_i}) \longrightarrow w_i \quad (1 \le i \le r).$$

In particular, the restriction $f|_F$ *is approximately unitarily equivalent to its native inclusion and has exactly its full classification invariant.*

Proof. Nuclear-source uniqueness for $\mathrm{C}(\mathbb{T})$, Theorem 3.16, gives unitaries $z_{i,n} \in L$ with $z_{i,n} e^{2\pi i k} z_{i,n}^* \to w_i$. Put $H_{i,n} = z_{i,n} k z_{i,n}^*$. Proposition 6.13 gives a faithful unital map $\Theta : Q \to L_\infty$ agreeing with the native inclusion on R.

For every diverging map $s : \mathbb{N} \to \mathbb{N}$, the reparameterization $s^*\Theta$ also agrees with Θ on R. Corollary 6.14 therefore gives unitary equivalence of these two maps. Intertwining by reparameterization [4, Proposition 9.2], exactly as in Proposition 6.6, yields a homomorphism $f : Q \to L$ and a unitary $U \in L_\infty$ with

$$\Theta = \mathrm{Ad}(U) \iota_L f.$$

The map f is faithful and unital because Θ is. Lifting U to coordinate unitaries gives the displayed limits. Joint moment convergence in Proposition 6.13, applied to any ultralimit of the specified trace, gives trace preservation. Approximate unitary equivalence on F preserves ordinary and coefficient K-theory, traces, and the continuous algebraic-unitary invariant. Thus it preserves the complete classification marking on that coefficient algebra. □

Remark 6.16. The vanishing used in Corollary 6.14 concerns the actual $\mathrm{KK}(Q, I)$ class of a specified Cuntz pair; it follows from exact agreement on F and the KK-equivalence

$F \to Q$. The marking assertion in Corollary 6.15 instead follows from approximate unitary equivalence of the coefficient restrictions. It does not identify arbitrary KK classes from their induced K-group maps when F is nonnuclear.

The finite-family construction below requires only one-circle automorphism lifting. At each such lift, all other whole interval factors belong to the coefficient algebra in Theorem 6.9. Its full marked hypothesis is checked before applying the theorem; the resulting coefficient restriction retains its global approximate-unitary class.

7. Finite-dimensional relative reservoirs

All traces in this section are faithful. All conditional expectations onto finite-dimensional subalgebras preserve the specified trace, and all amalgamated free products use these expectations. We write $I = (\mathrm{C}([0,1]), \int_0^1 \cdot\, dt)$ and $D_\infty = *_{j\geq 1} I_j$. A uniform interval coordinate can equivalently be replaced by a standard semicircular coordinate: the semicircle distribution function is a homeomorphism from $[-2,2]$ onto $[0,1]$. We use this identification when calculating operator-valued moments. In particular, all Gaussian calculations below concern actual elements of reduced C*-algebras.

If

$$F = \bigoplus_{s=1}^{r} \mathrm{M}_{d_s} \subset H,$$

we fix matrix units $e_{ij}^{(s)}$ and put

$$p_s = \sum_{i=1}^{d_s} e_{ii}^{(s)}, \qquad q_s = e_{11}^{(s)}, \qquad \alpha_s = \tau(p_s), \qquad \lambda_s = \tau(q_s) = \alpha_s/d_s.$$

The p_s need not be central in H. The trace-preserving expectation onto F is

$$\mathbb{E}_F(b) = \sum_{s=1}^{r} \frac{d_s}{\alpha_s} \sum_{i,l=1}^{d_s} \tau(e_{li}^{(s)} b) e_{il}^{(s)}. \tag{7.1}$$

A label-preserving map between two such inclusions preserves the chosen matrix units as well as the traces; consequently it preserves all the numbers α_s and λ_s.

Lemma 7.1 (An actual finite-dimensional relative tag)**.** *Let $F \subset H$ be a unital finite-dimensional subalgebra of a unital faithfully tracial* C*-*algebra. There is a unital trace-preserving embedding*

$$H *_F (F \otimes I) \longrightarrow H * I$$

which fixes H. Its interval coordinate commutes with F. The construction is natural for label-preserving trace-preserving embeddings and uses only the displayed fresh interval. Additional external scalar-free factors are therefore unchanged.

Proof. Let s be the standard semicircular generator of the fresh interval, free from H, and set

$$Y = \sum_{s_0=1}^{r} \alpha_{s_0}^{-1/2} \sum_{i,j=1}^{d_{s_0}} e_{ij}^{(s_0)} s e_{ji}^{(s_0)}.$$

The index s_0 denotes a summand and is distinct from the semicircular variable s. Matrix multiplication gives $[Y, F] = 0$. With $\mathbb{E}_H : H * I \to H$ the canonical expectation, the second

operator-valued cumulant is

$$\mathbb{E}_H(YbY) = \sum_{s_0=1}^{r} \frac{d_{s_0}}{\alpha_{s_0}} \sum_{i,l=1}^{d_{s_0}} \tau(e_{li}^{(s_0)} b) e_{il}^{(s_0)} = \mathbb{E}_F(b).$$

For example, the scalar contraction of two terms from summands s_0, t_0 contains $\tau(e_{ji}^{(s_0)} b e_{lr}^{(t_0)})$; cyclicity of the trace and the outside matrix product force $s_0 = t_0$ and give the factor d_{s_0}. Multilinearity of operator-valued cumulants shows that every cumulant of Y of order different from two vanishes. Thus Y is H-valued semicircular with covariance $\mathbb{E}_F$.

The central standard semicircle in $F \otimes I$, regarded in $H *_F (F \otimes I)$, has exactly this H-valued Wick formula. In particular its scalar law is the standard semicircle law, since $\mathbb{E}_F(1) = 1$. Equality of all scalar mixed moments, together with faithfulness of both reduced traces, identifies the generated C*-algebras isometrically. Applying the semicircle distribution function gives the claimed uniform interval. The formula for Y commutes with every label-preserving embedding, proving naturality. The assertion about external factors follows by reduced free-product functoriality. □

Lemma 7.2 (Reassociation and decomposition into block tags)**.** *Let $F_1, \dots, F_m$ be unital finite-dimensional subalgebras of a faithfully tracial algebra R. Relative interval extensions over these subalgebras can be performed in any order, with canonical trace-preserving isomorphisms fixing R and all the labelled interval coordinates.*

For a single algebra $F = \bigoplus_s \mathrm{M}_{d_s}$, its F-central interval extension is also the successive extension by one block-supported interval for each s. The sth extension is over

$$F_s^+ = \mathrm{M}_{d_s} \oplus \mathbb{C}(1 - p_s)$$

and adjoins

$$G_s = (\mathrm{M}_{d_s} \otimes I_s) \oplus \mathbb{C}(1 - p_s).$$

Zero complementary summands are omitted. The added interval is supported on p_s; every other block tag belongs to the remaining coefficient algebra when this tag is placed last.

Proof. For the first assertion replace the intervals by their semicircular coordinates. Over R the joint Wick formula has covariance maps $E_{F_i}^R$ and zero mixed cumulants. To check this in an iterated extension, use at each step the expectation identity

$$E_{F_i}^{R'} = E_{F_i}^R \mathbb{E}_R^{R'},$$

where R' is the algebra already constructed. This identity follows from trace preservation and uniqueness of the expectation onto F_i. Inserting it in the Wick recursion gives the same R-valued formula in every order. Equivalently, the recursion is the Fock formula for the direct sum of the covariance correspondences. Faithfulness of the reduced traces upgrades equality of moments to the asserted C*-isomorphisms.

For the second assertion let t be the F-central standard semicircle and put $t_s = p_s t$. Its block variables have covariances

$$\eta_s(b) = p_s \mathbb{E}_F(b) p_s,$$

zero mixed cumulants, and vanishing higher cumulants. The supported semicircle in G_s has precisely covariance η_s; the complementary scalar summand contributes no variable. The

preceding Wick recursion therefore identifies the successive construction with the original extension. The block tags generate the original factor because $t = \sum_s t_s$ and $t_s = p_s t$. Transforming back to uniform intervals uses functional calculus separately in the unital corners $p_s R p_s$. □

Lemma 7.3 (The native relative-extension class)**.** *Let R be separable and faithfully tracial. The native inclusion*

$$R \longrightarrow R *_F (F \otimes I)$$

is a KK*-equivalence for every unital finite-dimensional $F \subset R$. The same is true for a block-supported extension and for any finite iteration of these extensions. Consequently the native inclusion, rather than just an unmarked abstract absorption isomorphism, gives the canonical identification of total* K*-theory.*

Proof. First use the full amalgamated free product. Evaluation of the interval coordinate at zero gives a retraction onto R. The homotopy $h \mapsto th$, $0 \le t \le 1$, in the uniform coordinate, fixes R and the amalgamated F and joins the retracted map to the identity. Thus the full native inclusion is a homotopy equivalence. For a supported extension the identical homotopy is used in the supported matrix block and is the identity in the complementary constant block.

The full-to-reduced comparison is a KK-equivalence by [20, Proposition 2.15 and Theorem 3.1]: the specified trace-preserving expectations are faithful, since the traces are faithful, and hence are GNS-faithful. Proposition 2.15 identifies the vertex-reduced and ordinary reduced products, and Theorem 3.1 applies to the canonical quotient. That quotient commutes with the two native inclusions of R. Therefore the reduced native inclusion is a KK-equivalence as well. Composition proves the assertion for finite iterations. Applying coefficient K-theory to these KK-equivalences preserves all Bockstein operations and proves the final assertion. □

When both algebras in a native inclusion above are admissible targets, the inclusion also identifies their full compatible invariants for the selected traces. Indeed its total K-theory map is an isomorphism, its normalized trace map is the identity, and the trace ranges of K_0 therefore agree. The determinant exact sequence

$$0 \longrightarrow \mathbb{R}/\overline{\tau_*(\mathrm{K}_0(\,\cdot\,))} \longrightarrow \mathrm{U}_\infty(\,\cdot\,)/\mathrm{CU}_\infty(\,\cdot\,) \longrightarrow \mathrm{K}_1(\,\cdot\,) \longrightarrow 0$$

then shows that its Hausdorffized algebraic unitary map is an isomorphism. The same argument applies in compatible normalized full corners. This observation does not assume that either K_1 group vanishes.

Lemma 7.4 (Onto absorption with all finite-dimensional labels fixed)**.** *Let $R = H * D_\infty$ and let $F_1, \dots, F_m \subset H$ be unital finite-dimensional subalgebras with fixed matrix units. There is a trace-preserving isomorphism*

$$J_R : R *_{F_1} (F_1 \otimes I_1) \cdots *_{F_m} (F_m \otimes I_m) \longrightarrow R$$

which fixes H pointwise. It can fix any prescribed finite ordinary reservoir prefix. The choices are natural for label-preserving trace-preserving embeddings of H that use the same ordinary reservoir coordinates. The same conclusion holds for block-supported tags: in any

one relative addition, interval coordinates may be added on only some of the summands of the finite-dimensional base, with the other summands left constant.

Proof. We first treat one algebra $F = \bigoplus_s \mathrm{M}_{d_s}$. Let

$$p = \sum_s q_s, \qquad v_{s,i} = e_{i1}^{(s)}.$$

Use a single letter $a = (s, i)$ for a pair of indices, and write $s(a) = s$. Then

$$v_a^* v_b = \delta_{ab} q_{s(a)}, \qquad \sum_a v_a v_a^* = 1. \tag{7.2}$$

In particular p is full. Let s_j be the standard free semicircular coordinates of D_∞, and put

$$z_{ab}^{(j)} = v_a^* s_j v_b \in q_{s(a)} R q_{s(b)}.$$

Together with pHp, these entries generate pRp. This follows by inserting the second identity in (7.2) on either side of every reservoir letter and between successive coefficient letters.

The operator-valued Wick recursion gives, for every $x \in pHp$,

$$\kappa_2\Big(z_{ab}^{(j)}, x z_{cd}^{(l)}\Big) = \delta_{jl}\delta_{ad}\delta_{bc}\tau(q_{s(b)}x)q_{s(a)}. \tag{7.3}$$

All higher cumulants vanish. For clarity the identity is valid for offdiagonal coefficients as well: its scalar contraction is

$$\tau(v_b x v_c^*) = \tau(v_c^* v_b x) = \delta_{bc}\tau(q_{s(b)}x).$$

The remaining outside product is $v_a^* v_d = \delta_{ad} q_{s(a)}$. This also proves the Wick recursion directly by contracting innermost pairs; no centrality of the p_s in H has been used.

Consequently the self-adjoint diagonal entries

$$\lambda_s^{-1/2} z_{(s,i),(s,i)}^{(j)} \quad (j \geq 1,\ 1 \leq i \leq d_s)$$

are mutually free over pHp, are free over pHp from the entire offdiagonal entry family, and have covariance

$$b \longmapsto \lambda_s^{-1}\tau(q_s b)q_s.$$

Here freeness refers to the operator-valued Gaussian law; the offdiagonal entries retain their prescribed adjoint relations and their joint covariances.

In $R^+ = R *_F (F \otimes I)$ let t be the new central standard semicircle and set $t_s = q_s t q_s$. Its compressed covariances are

$$\kappa_2(t_s, b t_l) = \delta_{sl}\lambda_s^{-1}\tau(q_s b)q_s.$$

All higher cumulants vanish, and these variables are free over pHp from the preceding entry family. One way to see the last assertion without a compression assumption is to use the joint H-valued Wick formula of s_j, t: its covariances are $b \mapsto \tau(b)1$ and $\mathbb{E}_F^H$, with zero mixed cumulants, by the expectation tower in Lemma 7.2. Compressing this recursion gives exactly the displayed formulas.

For each s reindex the countably infinite diagonal family indexed by (j, i) together with the one additional variable t_s onto the native diagonal family. Fix pHp and all offdiagonal entries. This prescription preserves all joint Wick moments, has an inverse given by the inverse reindexing, and therefore gives a trace-preserving isomorphism

$$\phi : pR^+p \longrightarrow pRp.$$

The corner is generated by the listed entries and the t_s, because $t = \sum_{s,i} v_{s,i} t_s v_{s,i}^*$.

Reconstruct the full algebra by the finite frame:

$$J_R(x) = \sum_{a,b} v_a \phi(v_a^* x v_b) v_b^*. \tag{7.4}$$

Using (7.2) proves that this map is unital and multiplicative. The same formula with ϕ^{-1} is its inverse. It fixes H, since ϕ fixes pHp, and trace preservation follows by applying cyclicity to the diagonal terms. Thus this is an onto isomorphism, rather than merely a relative embedding.

For finitely many tags partition the unused reservoir into m infinite banks. Absorb the tags successively, using one bank for each tag and treating all preceding banks and tag images as coefficients. Lemma 7.2 permits this order and leaves the other banks as external free factors. A finite ordinary reservoir prefix is first included in H. Finally every entry formula, normalization, partition and reindexing can be chosen with the same labels on the two sides of a label-preserving embedding. Formula (7.4) then commutes with that embedding, proving the exact naturality assertion. For a block-supported tag the same proof simply adds a variable t_s at each selected vertex and adds no variable at the other vertices. The native diagonal families at unselected vertices are left unchanged. This proves the stated supported version as well. □

Lemma 7.5 (A free implementing circle for a finite-dimensional frame)**.** *Let N be faithfully tracial, let $F_0, B \subset N$ be scalar-free unital subalgebras, and let $C \subset F_0$ be unital and finite dimensional. Suppose $w_0 \in \mathrm{U}(N)$ conjugates the labelled frame of C onto a labelled frame of B, so in particular $B = w_0 C w_0^*$. In*

$$\widetilde{N} = N *_C (C \otimes \mathrm{C}(\mathbb{T}))$$

let v be the C-central Haar generator. Then

$$W = w_0 v$$

is scalar Haar, is free from F_0, and satisfies $WcW^* = w_0 c w_0^*$ for every $c \in C$. Hence $\mathrm{C}^*(F_0, W) \cong F_0 * \mathrm{C}(\mathbb{T})$ with the specified faithful trace. Multiplying W by a scalar phase preserves these conclusions.

Proof. Centrality of v proves the frame identity. Put $\theta = \mathrm{Ad}(w_0)|_C : C \to B$ and $H_0 = \mathrm{C}^*(F_0, B) = F_0 * B$. Substituting $W = w_0 v$ in a word with coefficients in N gives the reduced HNN normal-form rules for θ: an unpinched word has expectation zero onto N, and the two pinches have expectations

$$\mathbb{E}_N(WxW^*) = w_0 \mathbb{E}_C(x) w_0^*, \qquad \mathbb{E}_N(W^* x W) = w_0^* \mathbb{E}_B(x) w_0.$$

These formulas follow directly from the amalgamated reduced-word rule for the nonzero powers of v. The second uses uniqueness of the trace-preserving expectations. They remain valid on H_0, where

$$\mathbb{E}_B(f) = \tau(f)1 \quad (f \in F_0), \qquad \mathbb{E}_C|_{H_0} = \mathbb{E}_C|_{F_0} E_{F_0}|_{H_0}.$$

For comparison, in $F_0 * \mathrm{C}(\mathbb{T})$ let z be the free Haar generator. The algebras F_0 and zCz^* are scalar-free, so the assignment $\theta(c) \mapsto zcz^*$ identifies H_0 with $\mathrm{C}^*(F_0, zCz^*)$. Expanding

its coefficients into alternating centered F_0- and zCz^*-words gives precisely the same normal-form rules: a z–z^* pinch keeps the $\mathbb{E}_C$ component, the reverse pinch keeps the expectation onto zCz^*, and every unpinched term has zero trace. This can also be checked inductively on the number of stable letters: remove a pinch by its displayed expectation, while the remaining centered term is an amalgamated reduced word; if there is no pinch the whole word is reduced. It follows that every scalar word moment in H_0, W equals the corresponding moment in $\mathrm{C}^*(F_0, zCz^*), z$. The relations eliminate $B = WCW^*$, so the generated algebras are exactly $\mathrm{C}^*(F_0, W)$ and $F_0 * \mathrm{C}(\mathbb{T})$. Faithfulness of the traces proves the isometric identification. No step uses simplicity of C. □

We will also use retained-frame liberation for direct sums. If $A, B \subset M$ are scalar-free unital subalgebras, A is finite dimensional, $F_0 \subset M$ contains A, and u is an A-central Haar unitary free from M over A, then uF_0u^* is scalar-free from B and contains the original A. Indeed uMu^* and M are free over A by the amalgamated reduced-word rule. Their restricted product is

$$(uF_0u^*) *_A (A * B) \cong (uF_0u^*) * B,$$

since $\mathbb{E}_A(b) = \tau(b)1$ on B. This proves the claim with its reduced norm.

Exact finite-dimensional frame conjugacy also requires all summands to be retained. If two unital embeddings of $F = \bigoplus_s \mathrm{M}_{d_s}$ into a unital stable-rank-one algebra have the same K_0 class on each labelled q_s, cancellation gives partial isometries between the corresponding q_s. Writing these as a_s, with initial projection $\varphi(q_s)$ and final projection $\psi(q_s)$, the unitary

$$\sum_{s,i} \psi(e_{i1}^{(s)}) a_s \varphi(e_{1i}^{(s)})$$

conjugates the entire labelled φ-frame onto the ψ-frame. Equality of a single ambient unit class would not suffice.

Lemma 7.6 (Relative Nielsen moves as actual ambient automorphisms)**.** *Let $T = H * D_\infty$ be an admissible target, and let $F_1, \dots, F_m \subset H$ be unital finite-dimensional subalgebras. Form the finitely enlarged algebra*

$$T^+ = T *_{F_1} (F_1 \otimes I_{H_1}) \cdots *_{F_m} (F_m \otimes I_{H_m}).$$

Fix $a \in \{1, \dots, m\}$, write $F_a = \bigoplus_s \mathrm{M}_{d_s}$, and put $Z = \exp(2\pi i H_a)$. Suppose $u \in \mathrm{U}(F_a' \cap T)$ and, for every labelled minimal projection $q_s \in F_a$,

$$q_s u q_s \in \mathrm{CU}(q_s T q_s). \tag{7.5}$$

There are actual trace-preserving automorphisms of T^+ which, on arbitrary prescribed finite packets, approximate the identity on T and on every other tag logarithm, and approximate the prescription

$$Z \longmapsto Zu.$$

They induce the identity on the full compatible invariant. All packets may contain moving coefficient elements and are specified before the approximation is chosen. No approximation assertion is made on H_a.

In (7.5) *one may equivalently impose zero K_1 class and zero Hausdorffized determinant in each normalized corner. Ambient determinant zero alone is not the hypothesis. No*

assumption $\mathrm{K}_1(T) = 0$ *is needed: the multiplier classes in* (7.5) *are required to vanish individually.*

Proof. Split the selected tag into its block-supported tags by Lemma 7.2. We perform a move on one block at a time, putting that block outermost and including all other block tags among the coefficients. Fix a block and abbreviate its support by p_0, its minimal projection by q, its matrix units by e_{ij}, and $v_i = e_{i1}$. Let M denote the full remaining coefficient algebra, including every other tag. The whole algebra, in this ordering, is

$$R = M *_{\mathrm{M}_d \oplus \mathbb{C}(1-p_0)} ((\mathrm{M}_d \otimes I) \oplus \mathbb{C}(1-p_0)).$$

Write h for the supported interval and $h_q = qhq$, with functional calculus in the unital corner qRq. Then

$$qRq \cong (qMq) * I_{h_q}. \tag{7.6}$$

To prove generation, use $h = \sum_i v_i h_q v_i^*$ to expand every occurrence of the supported tag in a corner word. The coefficients between consecutive h_q's then lie in qMq. For the free law, a centered element of qMq has expectation zero onto $\mathrm{M}_d \oplus \mathbb{C}(1-p_0)$: its expectation is its normalized corner trace times q. A centered function of h_q has the same property. The amalgamated reduced-word rule therefore gives scalar freeness in the normalized corner trace. Faithfulness proves (7.6). This argument uses the outermost block, rather than compressing an algebra in which other block tags are still treated as added variables.

Set $z = \exp(2\pi i h_q)$ and $u_q = quq$. The corner circle free product has the trace-preserving Nielsen automorphism

$$\gamma|_{qMq} = \mathrm{id}, \qquad \gamma(z) = zu_q,$$

whose inverse sends z to zu_q^*. The multiplier u_q belongs to the coefficient corner. Its class in the Hausdorffized algebraic unitary group is zero by (7.5), also after inclusion in qMq. Thus z, zu_q, zu_q^* have identical full marked circle invariants: their K_1 classes and their algebraic unitary classes agree, and all three have Haar spectral measure. The unit determines their K_0 maps; since both ordinary K-groups of $\mathrm{C}(\mathbb{T})$ are free cyclic, the coefficient maps are determined by the ordinary maps as well. This checks the full marking required by Theorem 6.9.

All corner hypotheses of that theorem hold here. By Lemma 7.4, M and R are isomorphic to T, with the original H fixed. The nonzero projection $q \in T$ is full in the original simple algebra T, hence in both M and R. Admissibility passes to full corners. The coefficient corner has infinite-dimensional GNS space since it contains qTq. Theorem 6.9 and its moving-packet formulation in Remark 6.10 therefore supply actual trace-preserving corner automorphisms θ approximating γ on any chosen finite coefficient-and-circle packet.

We now extend this finite-packet assertion to the whole ambient algebra without asserting an ordinary matrix amplification by q. Fullness in T gives finitely many $x_j \in Tq \subset Mq$ such that

$$\sum_{j=1}^{N} x_j x_j^* = 1.$$

For instance choose a finite positive sum of terms $y_j q y_j^*$ which is invertible and normalize the $y_j q$ by its inverse square root. Set

$$P = (x_i^* x_j)_{i,j} \in \mathrm{M}_N(qMq), \qquad \Psi(r) = (x_i^* r x_j)_{i,j}.$$

Then P is a projection and

$$\Psi : R \longrightarrow P\mathrm{M}_N(qRq)P$$

is a unital isomorphism; its inverse is $(b_{ij}) \mapsto \sum_{i,j} x_i b_{ij} x_j^*$. Include every entry of P in the protected corner coefficient packet. The projection $Q = \theta_N(P)$ is then as close to P as desired. When $\|P - Q\| < 1$, the polar unitary of

$$PQ + (1-P)(1-Q)$$

is a unitary w satisfying $wQw^* = P$, with $\|w - 1\| \to 0$ as $\|P - Q\| \to 0$. Thus

$$\beta = \Psi^{-1} \operatorname{Ad}(w)\theta_N \Psi \tag{7.7}$$

is an actual automorphism of R.

For each protected coefficient $r \in M$, include all entries $x_i^* r x_j$ in the corner packet. Equation (7.7) then makes $\beta(r)$ as close to r as desired. To control the selected supported circle

$$Z_0 = \sum_{k=1}^{d} v_k z v_k^*, \qquad U_0 = p_0 u p_0 = \sum_{k=1}^{d} v_k u_q v_k^*,$$

also include the finitely many coefficients $x_i^* v_k$ and $v_k^* x_j$. Since

$$x_i^* Z_0 x_j = \sum_k (x_i^* v_k) z (v_k^* x_j),$$

the corner estimate $\theta(z) \approx z u_q$ then gives $\beta(Z_0) \approx Z_0 U_0$. Every other block tag belongs to M and is protected, including its logarithm. These estimates involve only finitely many elements and finite sums; choosing the corner tolerance sufficiently small gives any specified ambient tolerance.

The map (7.7) preserves trace: with $\tau_q = \tau(q)^{-1}\tau|_{qRq}$,

$$\tau(r) = \tau(q)(\mathrm{Tr}_N \otimes \tau_q)(\Psi(r)),$$

and both θ_N and $\operatorname{Ad}(w)$ preserve the matrix trace. It also preserves the full marking. The corner coefficient inclusion in (7.6) is a KK- equivalence, and hence a total-K-theory isomorphism, by Lemma 7.3 with scalar amalgamation. Its trace range on K_0 is unchanged. The determinant exact sequence therefore shows that its map on Hausdorffized algebraic unitary groups is also an isomorphism. The corner lifts can be chosen with the identity marking on this coefficient inclusion. More precisely, the proof of Theorem 6.9 first constructs a fixed embedding whose coefficient restriction has the prescribed full approximate-unitary class; it then uses inner conjugates of a fixed isomorphism approximately unitarily equivalent to that embedding. Consequently each chosen automorphism, and not only its point-norm limit on finite packets, has this full coefficient marking. It therefore has identity full marking on the corner. Matrix stabilization, the full-corner identification and the inner correction in (7.7) preserve that conclusion.

Finally perform these moves for the finitely many summands. At each step protect the other tags and all preceding moving images needed in the finite request. Their individual errors can be made arbitrarily small, so the finite composition fixes the requested coefficient packets and sends $Z = \sum_s p_s Z$ as close as desired to Zu. Each factor of the composition is an actual trace-preserving automorphism with identity full marking. Stable rank one and

the determinant exact sequence identify (7.5) with the equivalent K_1 and normalized-corner determinant conditions stated in the lemma. □

8. Split-unit fusion

The support-shifting argument below gives a direct proof of fusion when only one unit class may be nonsplit. It is also used in the main construction to separate the root's K_1-group and to incorporate the split-unit leaves.

Theorem 8.1 (Fusion with split-unit leaves)**.** *Let $\mathcal{A} = ((A_i, \tau_i))$ be a nonempty finite or countable family of separable unital nuclear UCT algebras with faithful selected traces, and suppose all but at most one factor have split unit class. If F is a nuclear UCT core with faithful selected trace and the same selected pointed graded* K*-groups and pairing as $P_{\mathcal{A}}$, then, for every reservoir D,*

$$P_{\mathcal{A}} *_{\mathrm{r}} D \cong F *_{\mathrm{r}} D.$$

Every prescribed compatible full marking may be realized, and a finite reservoir prefix may be fixed pointwise. Such products are classified by their Elliott invariants, with realization of every invariant isomorphism. For two such families, if each contains two distinct factors admitting Haar unitaries, the classification also holds without adjoining D.

We first prove the matrix-completion and support-shifting lemmas; the countable fusion argument then proves the theorem.

8.1. **Matrix completion.** For a faithfully tracial algebra (P, τ_P) define

$$\mathcal{F}(P) = (P \oplus \mathbb{C}, \tfrac{1}{2}\tau_P \oplus \tfrac{1}{2}\,\mathrm{id}) *_{\mathrm{r}} D, \qquad r = (1_P, 0), \quad q = 1 - r, \quad R = \mathbb{C}r \oplus \mathbb{C}q. \tag{8.1}$$

The trace on R is the uniform two-point trace. All expectations onto R are trace preserving.

Lemma 8.2 (Mixed associativity)**.** *Let A_1, A_2 contain a common finite-dimensional tracial algebra R, with trace-preserving expectations onto R, and let Q be faithfully tracial. Then*

$$A_1 *_R (A_2 *_{\mathrm{r}} Q) \cong (A_1 *_R A_2) *_{\mathrm{r}} Q \tag{8.2}$$

by an isomorphism respecting the canonical copies of A_1, A_2, Q. The expectation from $A_2 *_{\mathrm{r}} Q$ to R is the composition of its canonical expectation onto A_2 with the expectation onto R.

Proof. Work inside $C = (A_1 *_R A_2) *_{\mathrm{r}} Q$, and put $B = \mathrm{C}^*(A_2, Q)$. Its trace and expectation onto R have the stated form. We verify that A_1 and B are R-free.

Expand an R-centered algebraic coefficient of B as a sum of an R-centered coefficient from A_2 and scalar-reduced A_2–Q words containing centered Q letters. In an alternating word from $\ker \mathbb{E}_R|_{A_1}$ and $\ker \mathbb{E}_R|_B$, either no Q letter occurs, in which case use R-freeness of A_1, A_2, or Q letters occur. Each block between successive centered Q letters is scalar centered: inside one B coefficient this follows from scalar reduction; across coefficients the block contains an R-centered alternating A_1–A_2 word. Testing the resulting word against R shows that its R-expectation is zero, by scalar freeness of Q. The indicated algebras generate C. Equality of their faithful tracial joint laws identifies the reduced GNS completions. □

Lemma 8.3 (Completing a support projection)**.** *For every faithfully tracial P,*

$$(P \oplus \mathbb{C}) *_R \mathrm{M}_2(\mathbb{C}) \cong \mathrm{M}_2(P), \tag{8.3}$$

where $r = e_{11}$, $q = e_{22}$, and the copy of $P \oplus \mathbb{C}$ is $\operatorname{diag}(P, \mathbb{C}1_P)$.

Proof. The two algebras on the right generate $\mathrm{M}_2(P)$: the diagonal pe_{11} and the constant matrix units generate all pe_{ij}. The R-centered elements of $P \oplus \mathbb{C}$ have the form $\operatorname{diag}(a, 0)$, $\tau_P(a) = 0$, while those of $\mathrm{M}_2(\mathbb{C})$ are off diagonal. An alternating centered word with at least two $P \oplus \mathbb{C}$ letters vanishes because

$$\operatorname{diag}(a, 0)\, m\, \operatorname{diag}(b, 0) = 0 \quad \text{for off-diagonal } m.$$

With one such letter, its R-expectation is zero: multiplication by at most two off-diagonal letters gives either an off-diagonal matrix or a diagonal entry proportional to a. This proves R-freeness. The faithful normalized matrix trace then gives exactly the reduced completion. □

Lemma 8.4 (Gaussian matrix amplification)**.** *For every faithfully tracial P there is an isomorphism*

$$(\mathrm{M}_2(P), \operatorname{tr}_2 \otimes \tau_P) *_{\mathrm{r}} D \cong \mathrm{M}_2(P *_{\mathrm{r}} D) \tag{8.4}$$

fixing the canonical $\mathrm{M}_2(P)$.

Proof. In $P *_{\mathrm{r}} D_{\mathrm{ent}}$ choose standard semicirculars a_n, b_n, x_n, y_n, $n \geq 1$, all freely independent and free from P. Set

$$c_n = (x_n + iy_n)/\sqrt{2}, \qquad S_n = \frac{1}{\sqrt{2}} \begin{pmatrix} a_n & c_n \\ c_n^* & b_n \end{pmatrix}. \tag{8.5}$$

Here $D_{\mathrm{ent}} = \mathrm{C}^*(a_n, b_n, x_n, y_n : n \geq 1) \cong D$.

We check the full distribution, including coefficients from the possibly nonnuclear algebra P. Put $B = \mathrm{M}_2(P)$, $\tau_B = \operatorname{tr}_2 \otimes \tau_P$, and $\mathbb{E}_B = \mathrm{id}_{\mathrm{M}_2} \otimes \mathbb{E}_P$. Scalar semicircularity and freeness from P give P-valued cumulants zero except at order two, where

$$\kappa_2^P(z_\alpha, pz_\beta) = \delta_{\alpha\beta} \tau_P(p) 1_P. \tag{8.6}$$

This is the cumulant factorization criterion in [32, Theorem 3.5]. Entrywise matrix amplification preserves the cumulant formula. Equivalently, write S_n as the linear combination of its four real Gaussian entries with coefficients

$$C_1 = e_{11}/\sqrt{2}, \quad C_2 = e_{22}/\sqrt{2}, \quad C_3 = (e_{12} + e_{21})/2, \quad C_4 = i(e_{12} - e_{21})/2.$$

These coefficients give exactly (8.5). All higher B-valued cumulants are zero, and

$$\begin{aligned} \kappa_2^B(S_n, bS_m) &= \delta_{nm} \sum_{\alpha=1}^{4} C_\alpha (\mathrm{id}_{\mathrm{M}_2} \otimes \tau_P)(b) C_\alpha \\ &= \delta_{nm} \tau_B(b) 1_B. \end{aligned} \tag{8.7}$$

Thus the B-valued Wick recursion is precisely that of a standard scalar semicircular family free from B. In particular the family (S_n) has that joint law with B. The scalar matrix version is the classical Gaussian matrix construction in [13, Section 3].

Consequently, the canonical B and the S_n generate a copy of $B *_{\mathrm{r}} D$. It is the entire $\mathrm{M}_2(P *_{\mathrm{r}} D_{\mathrm{ent}})$: constant matrix units recover every matrix entry of S_n, and their real and imaginary parts recover a_n, b_n, x_n, y_n. This proves surjectivity at the C^*-algebraic level.

Faithfulness proves equality of the C*-norms, not only equality of tracial moments. For any z in a faithfully tracial C*-algebra,

$$\|z\| = \lim_{k\to\infty} \tau((z^*z)^k)^{1/(2k)}.$$

Finally reindex the countable entry family to identify D_{ent} with D. □

Lemma 8.5 (Full-corner Gaussian lemma)**.** *Let N be a separable unital nuclear* C*-*algebra with a faithful tracial state τ, let $p \in N$ be a full projection, and let D be a reservoir. Put $t = \tau(p)$ and $\tau_p = t^{-1}\tau|_{pNp}$. There is a trace-preserving isomorphism*

$$\Big(p(N *_{\mathrm{r}} D)p, t^{-1}(\tau *_{\mathrm{r}} \tau_D)|_{p(N*_{\mathrm{r}}D)p}\Big) \cong (pNp, \tau_p) *_{\mathrm{r}} D$$

which fixes pNp pointwise.

Proof. By Corollary 4.11, we may take D to be the countable free semicircular reservoir. The case $p = 1$ is immediate. Suppose $p \neq 1$, put $F = \mathbb{C}p \oplus \mathbb{C}(1-p)$, and form the supported extension

$$S = N *_F G, \qquad G = (\mathbb{C}p \otimes D) \oplus \mathbb{C}(1-p).$$

The trace of G is $t\tau_D \oplus (1-t)\,\mathrm{id}_{\mathbb{C}}$, and the expectations onto F preserve the specified traces. If h_j are the supported standard semicircular coordinates of G, their N-valued joint Wick formula has zero higher cumulants and covariances

$$\kappa_2^N(h_j, bh_k) = \delta_{jk}\, t^{-1}\tau(bp)p \qquad (b \in N).$$

Moreover,

$$pSp \cong (pNp, \tau_p) *_{\mathrm{r}} D. \tag{8.8}$$

Indeed every h_j is supported on p, so compression of a word leaves coefficients in pNp. Centered corner coefficients and centered elements of pGp have expectation zero onto F. The amalgamated reduced-word rule proves scalar freeness for the normalized corner trace, and faithfulness proves the asserted C*-isomorphism. This is the supported-corner argument of (7.6), with countably many coordinates.

We prove that $S \cong N *_{\mathrm{r}} D$ by an isomorphism fixing N. Use the coarse correspondence and its commuting coefficient algebra

$$H = L^2(N,\tau) \otimes N, \qquad B = N^{\mathrm{op}} \otimes_{\min} N,$$

The right Hilbert N-module inner product is $\langle a\otimes b, c\otimes d\rangle = \tau(a^*c)b^*d$, and N acts on the left tensor factor from the left. The coefficient action is $(a^{\mathrm{op}} \otimes b)(c\otimes d) = ca \otimes bd$. These commuting representations give an adjointable action of the maximal tensor product; nuclearity identifies it with the displayed minimal tensor product. The action commutes with the outer N-bimodule actions. Set

$$P = p^{\mathrm{op}} \otimes p, \qquad \zeta = 1\otimes 1, \qquad \xi = t^{-1/2}p \otimes p.$$

The closed bimodule generated by ξ is PH, and

$$\langle \xi, b\xi\rangle = t^{-1}\tau(bp)p.$$

Thus the tracial Gaussian constructions for $H^{\oplus\infty}$ with seeds ζ_j, and for $(PH)^{\oplus\infty}$ with seeds ξ_j, are respectively $N*_{\mathrm{r}}D$ and S. These identifications follow from their joint Wick formulas and the faithful reduced traces.

On the algebraic coarse correspondence put $J(c\otimes d)=d^*\otimes c^*$, and on B put

$$\sigma(a^{\mathrm{op}}\otimes b)=(b^*)^{\mathrm{op}}\otimes a^*.$$

Then σ is a conjugate-linear *-automorphism, $\sigma(P)=P$, and $J(b\eta)=\sigma(b)J\eta$ on algebraic vectors. Both seeds are J-symmetric. Since p is full in N, P is full in B. It is full in the real algebra $B_{\mathbb{R}}=B^{\sigma}$ as well. To see this explicitly, choose a finite complex fullness witness, average its positive sum under σ, and write each coefficient as $a+ib$ with $a,b\in B_{\mathbb{R}}$. The averaged sum is then a finite sum of terms aPa^* and bPb^*. Normalizing by its positive inverse square root gives $a_1,\dots,a_m\in B_{\mathbb{R}}$ with

$$\sum_{i=1}^{m}a_iPa_i^*=1.$$

For completeness, this finite witness gives the required real stabilization. Write $E=PB_{\mathbb{R}}$. The map $b\mapsto(Pa_i^*b)_{i=1}^m$ embeds $B_{\mathbb{R}}$ as an orthogonally complemented summand of E^m, so $E^m\cong B_{\mathbb{R}}\oplus Q$ for a finitely generated projective real module Q. Since E^m is a summand of $B_{\mathbb{R}}^m$, so is Q. Hence

$$E^{\oplus\infty}\cong B_{\mathbb{R}}^{\oplus\infty}\oplus Q^{\oplus\infty}\cong B_{\mathbb{R}}^{\oplus\infty}.$$

For the last isomorphism, choose Q' with $Q\oplus Q'\cong B_{\mathbb{R}}^m$, decompose the standard countable module as $Q^{\oplus\infty}\oplus(Q')^{\oplus\infty}$, and reindex the two countable copies of Q. Complexification and the action on H give an N-bimodule unitary

$$U:(PH)^{\oplus\infty}\longrightarrow H^{\oplus\infty}$$

whose coefficient matrix has entries fixed by σ.

We check that second quantization gives an isomorphism of the Gaussian C*-algebras. For a symmetric seed e and an algebraic vector $\eta=\sum_r a_reb_r$, the field operator is

$$s(\eta)=\ell(\eta)+\ell(J\eta)^*=\sum_r a_r[\ell(e)+\ell(e)^*]b_r.$$

In particular, if $J\eta=\eta$, then $\ell(\eta)+\ell(\eta)^*$ belongs to the algebra generated by N and the seed field. The same observation applies to finite sums over different seeds. Each column of the real module unitary is a norm-convergent column over B. Truncate it to finitely many coordinates and approximate its entries in norm by algebraic tensor coefficients fixed by σ, using symmetrization. Applying these columns to ξ_j produces algebraic vectors η_l with $J\eta_l=\eta_l$ and $\eta_l\to U\xi_j$ in the correspondence norm. Their Gaussian operators belong to $N*_{\mathrm{r}}D$ and converge in norm, since

$$\|[\ell(\eta_l)+\ell(\eta_l)^*]-[\ell(\eta_k)+\ell(\eta_k)^*]\|\le 2\|\eta_l-\eta_k\|.$$

Here ℓ is the creation operator on the full Fock module. The same argument applies to U^* and each ζ_j, multiplying the symmetrized algebraic approximating coefficients on the left by P. Since $\sigma(P)=P$, this preserves their symmetry. Its algebraic approximants lie in PH and are finite sums of vectors $a\xi b$, so their Gaussian operators belong to S. Consequently the Fock unitary induced by U carries S onto $N*_{\mathrm{r}}D$, with its inverse carrying the latter algebra

onto S. It fixes the vacuum coefficient algebra N and preserves the vacuum expectation and the trace. This norm argument uses only symmetric algebraic approximants and makes no boundedness claim for J in the correspondence norm.

Compress this N-fixing isomorphism by p and use (8.8). □

Theorem 8.6 (Matrix completion removes the scalar summand)**.** *Let P, Q be arbitrary faithfully tracial unital algebras. If*

$$\Phi : \mathcal{F}(P) \longrightarrow \mathcal{F}(Q) \tag{8.9}$$

is a trace-preserving isomorphism fixing r, then

$$P *_{\mathrm{r}} D \cong Q *_{\mathrm{r}} D. \tag{8.10}$$

No nuclearity or K*-theory assumption on P, Q is needed here.*

Proof. Trace preservation and $\Phi(r) = r$ imply that Φ preserves the expectation onto R. Explicitly, that expectation is

$$\mathbb{E}_R(a) = 2\tau(rar)r + 2\tau(qaq)q.$$

Reduced amalgamated functoriality therefore gives an isomorphism

$$\widehat{\Phi} = \Phi *_R \mathrm{id}_{\mathrm{M}_2} : \mathcal{F}(P) *_R \mathrm{M}_2 \longrightarrow \mathcal{F}(Q) *_R \mathrm{M}_2$$

fixing every added matrix unit. Lemmas 8.2–8.4 give identifications

$$\begin{aligned} \mathcal{F}(P) *_R \mathrm{M}_2 &\cong ((P \oplus \mathbb{C}) *_R \mathrm{M}_2) *_{\mathrm{r}} D \\ &\cong \mathrm{M}_2(P) *_{\mathrm{r}} D \cong \mathrm{M}_2(P *_{\mathrm{r}} D), \end{aligned} \tag{8.11}$$

and the corresponding identifications for Q. They all preserve the matrix units and take r to e_{11}. Compressing the resulting isomorphism by e_{11} proves (8.10). □

Remark 8.7 (Invariant and reservoir control). The completed isomorphism fixes all matrix units, so it equals $\mathrm{M}_2(\Theta)$ for its corner isomorphism Θ. The prescribed total K-theory and matrix-stabilized unitary data therefore descend through the canonical corner maps. For $u \in \mathrm{U}_0(P *_{\mathrm{r}} D)$, using a path from 1 to u and its diagonal amplification,

$$\Delta_{\mathrm{tr}_2 \otimes \tau}(\mathrm{diag}(u, 1)) = \tfrac{1}{2}\Delta_\tau(u);$$

normalizing the corner trace restores the original determinant. This formula concerns the determinant kernel. The ordinary K_1 component of a general unitary is retained by the same corner identification.

The class $[q] - [r]$ is killed by the actual matrix-unit relation $r \sim q$. Other specified reduced classes survive. This is a functor applied to the exact isomorphism (8.9).

Finite reservoir control survives as well. If Φ fixes finitely many decorated semicircular generators and r, then $\widehat{\Phi}$ fixes these generators and all matrix units. It consequently fixes every Gaussian entry extracted from them. By choosing the entry enumeration, any specified finite collection of ordinary reservoir interval factors can be protected pointwise.

8.2. **Finite support shifting and fusion.**

Proposition 8.8 (Support shifting at equal weights)**.** *Let E_1, E_2 be nuclear UCT cores with faithful selected traces. Suppose that E_2 has a chosen splitting* $\mathrm{K}_0(E_2) = \mathbb{Z}[1_{E_2}] \oplus G_2$.

There is a trace-preserving isomorphism fixing r,

$$\mathcal{F}(E_1 *_{\mathrm{r}} E_2) \cong (E_1 \oplus E_2, \tfrac{1}{2}\tau_1 \oplus \tfrac{1}{2}\tau_2) *_{\mathrm{r}} D. \tag{8.12}$$

No splitting of $[1_{E_1}]$ is required.

Proof. Put $u_2 = [1_{E_2}]$. Compare the nuclear cores $E_2 \oplus \mathbb{C}$ and $\mathbb{C} \oplus E_2$, with half weights on both components. On K_0 use the isomorphism

$$(ku_2 + g, \ell) \longmapsto (k, \ell u_2 + g), \qquad g \in G_2,$$

and on K_1 move the E_2 summand to the second component. This fixes the two component-unit classes r, q and preserves the selected pairing: both pairings on G_2 equal $\frac{1}{2}\tau_{2*}$. Theorem 4.7 gives an isomorphism after adjoining D. Stable-rank-one cancellation adjusts it by a unitary to fix r exactly. It then preserves the expectation onto R.

Amalgamate this isomorphism with $\mathrm{id}_{E_1 \oplus \mathbb{C}}$ over R. Mixed associativity gives

$$(E_1 \oplus \mathbb{C}) *_R ((E_2 \oplus \mathbb{C}) *_{\mathrm{r}} D) \cong ((E_1 *_{\mathrm{r}} E_2) \oplus \mathbb{C}) *_{\mathrm{r}} D,$$
$$(E_1 \oplus \mathbb{C}) *_R ((\mathbb{C} \oplus E_2) *_{\mathrm{r}} D) \cong (E_1 \oplus E_2) *_{\mathrm{r}} D.$$

The first line places both nonscalar cores in the r-corner; the second places them in opposite corners. This proves (8.12), and the construction fixes the canonical $E_1 \oplus \mathbb{C}$ at this stage. All replacements may be carried out in a complementary reservoir tail before extending by the identity on a protected prefix. □

Theorem 8.9 (Connected commutative fusion)**.** *Let $E_i = (\mathrm{C}(X_i), \mu_i)$, $i = 1, 2$, where X_i are connected compact metrizable spaces and the measures are faithful. No assumption on either* K*-group is made. Choose points $x_i \in X_i$, and let $X_1 \vee X_2$ denote the wedge obtained by identifying them. For any faithful selected probability on $X_1 \vee X_2$,*

$$(E_1 *_{\mathrm{r}} E_2) *_{\mathrm{r}} D \cong \mathrm{C}(X_1 \vee X_2) *_{\mathrm{r}} D. \tag{8.13}$$

The specified reduced K*-classes can be preserved. The result holds for any finite number of such cores.*

Proof. Let $W = \mathrm{C}(X_1 \vee X_2)$. Use Proposition 8.8. The pullback description

$$W = \{(f_1, f_2) \in E_1 \oplus E_2 : f_1(x_1) = f_2(x_2)\}$$

gives the split evaluation extension

$$0 \longrightarrow \mathrm{C}_0(X_1 \setminus \{x_1\}) \oplus \mathrm{C}_0(X_2 \setminus \{x_2\}) \longrightarrow W \longrightarrow \mathbb{C} \longrightarrow 0.$$

The splitting is the constant functions. Consequently the reduced K-groups are the direct sums of the two wedge summands; no CW or cofibration hypothesis on the spaces is needed. The nuclear cores $E_1 \oplus E_2$ and $W \oplus \mathbb{C}$ have isomorphic selected data: map the first component unit to $1_W \oplus 0$, the second to $0 \oplus 1$, and map their reduced K-groups to the wedge summands. All reduced K_0-classes have trace zero by connectedness, so the trace pairing agrees for any faithful selected wedge law. Apply Theorem 4.7, then adjust by a unitary to fix r. We obtain $\mathcal{F}(E_1 *_{\mathrm{r}} E_2) \cong \mathcal{F}(W)$ over R. Theorem 8.6 proves (8.13). Repeat for finitely many factors. □

Proposition 8.10 (Fusion into a nuclear range model)**.** *Let E_1, E_2 satisfy Proposition 8.8, with the chosen splitting $\mathrm{K}_0(E_2) = \mathbb{Z}[1_{E_2}] \oplus G_2$. Put $P = E_1 *_{\mathrm{r}} E_2$, and give $G_0 = \mathrm{K}_0(P)$ the*

order

$$(G_0)_+ = \{g : \rho(g) > 0\} \cup \{0\}, \qquad \rho = \tau_{P*}, \qquad u = [1_P]. \tag{8.14}$$

There is a simple, infinite-dimensional, monotracial nuclear UCT $\mathcal{Z}$-stable model F with pointed K*-groups and pairing $(G_0, u, \mathrm{K}_1(P), \rho)$, and*

$$P *_{\mathrm{r}} D \cong F *_{\mathrm{r}} D. \tag{8.15}$$

Any prescribed compatible isomorphism of the selected pointed data may be used. The unit of F need not split.

Proof. Write $u_i = [1_{E_i}]$. The free-product K-sequence [20] gives the identifications

$$G_0 = \mathrm{K}_0(E_1) \oplus G_2, \qquad u = (u_1, 0), \qquad \mathrm{K}_1(P) = \mathrm{K}_1(E_1) \oplus \mathrm{K}_1(E_2).$$

Indeed, the unit relation in $\mathrm{K}_0(E_1) \oplus (\mathbb{Z}u_2 \oplus G_2)$ eliminates the second unit coordinate: $(x, ku_2 + g)$ represents $(x + ku_1, g)$. On this group the selected pairing is $\rho(x, g) = \tau_{1*}(x) + \tau_{2*}(g)$.

The group (8.14) is simple, weakly unperforated and directed, and u is an order unit. Here weak unperforation means that $ng > 0$ for $n \geq 1$ implies $g > 0$; torsion is permitted. Every normalized state equals ρ: rational upper and lower bounds on g using $mu - ng$ force its state value to be $\rho(g)$. The range theorem [19, Theorem 2.8] therefore supplies F. It can be chosen infinite dimensional. In the cyclic case with pointed data $(\mathbb{Z}, m, 0)$, one can use $\mathrm{M}_m(\mathcal{Z})$.

By Proposition 8.8, $\mathcal{F}(P)$ is isomorphic over R to $(E_1 \oplus E_2) *_{\mathrm{r}} D$. Compare that nuclear core with $F \oplus \mathbb{C}$ by the explicit map

$$\mathrm{K}_0(E_1) \oplus (\mathbb{Z}u_2 \oplus G_2) \longrightarrow (\mathrm{K}_0(E_1) \oplus G_2) \oplus \mathbb{Z}, \qquad (x, ku_2 + g) \longmapsto ((x, g), k),$$

and use the corresponding identity on the two K_1 summands. This sends the two support classes $(u_1, 0)$ and $(0, u_2)$ to $(u_F, 0)$ and $(0, 1)$, respectively. At half weights it preserves the selected trace, since both values are $\frac{1}{2}(\tau_{1*}(x) + k + \tau_{2*}(g))$. Theorem 4.7 applies directly to these decorated cores and gives an isomorphism to $\mathcal{F}(F)$; it requires no splitting of their unit classes. The splitting needed in this argument was used in Proposition 8.8. Adjust it to fix r, then apply Theorem 8.6. The exact marking descends through matrix completion by Remark 8.7. Composing the first coordinate of the displayed map with a prescribed pointed isomorphism, and using its K_1 map, gives the stated freedom.

The core K_0-map need not preserve the positive cone of the core algebra $F \oplus \mathbb{C}$. The marked replacement theorem applies to the simple D-stabilized targets, where positivity is controlled by the faithful selected trace. No embedding into the original core $F \oplus \mathbb{C}$ is asserted. □

Remark 8.11 (Two nonsplit units). The split-unit hypothesis on the shifted factor has an algebraic role. For $E_2 = \mathrm{M}_n(\mathbb{C})$, $n > 1$, its support class in $\mathrm{K}_0(E_2 \oplus \mathbb{C})$ is n-divisible, whereas the intended scalar support in $\mathrm{K}_0(\mathbb{C} \oplus E_2)$ is primitive. Hence the support-preserving isomorphism used in Proposition 8.8 cannot exist in this case. The unit relation can also

create torsion for infinite-dimensional simple nuclear factors. If $\mathcal{U}$ is the CAR algebra, then

$$\begin{aligned}\mathrm{K}_0(\mathcal{U}*_{\mathrm{r}}\mathcal{U}) &\cong \frac{\mathbb{Z}[1/2]\oplus\mathbb{Z}[1/2]}{\mathbb{Z}(1,-1)}\\ &\cong \mathbb{Z}[1/2]\oplus\Big(\mathbb{Z}[1/2]/\mathbb{Z}\Big), \qquad \mathrm{K}_1(\mathcal{U}*_{\mathrm{r}}\mathcal{U})=0.\end{aligned}$$

The second isomorphism is $[(x,y)]\mapsto(x+y,y+\mathbb{Z})$; the unit is $(1,0)$ and the selected trace is the first coordinate. Products with two nonsplit unit classes therefore require an additional fusion argument. Section 9 provides it for arbitrary nuclear UCT factors, including this CAR example. The support-shifting proof remains sufficient when all but at most one factor have split unit class.

8.3. **Countable fusion and invariant control.** We now prove the split-unit fusion theorem, retaining the full markings through finite fusion and passage to a countable family.

Proof of Theorem 8.1. Put $P=P_{\mathcal{A}}$ and $A=P*_{\mathrm{r}}D$. To compare two such products, also write $Q=P_{\mathcal{B}}$ and $B=Q*_{\mathrm{r}}D$, and fix the prescribed isomorphism $\Theta:\mathrm{Ell}(A)\to\mathrm{Ell}(B)$. Proposition 2.5 identifies its group maps with $\alpha=(\chi_0,\chi_1):\mathrm{K}_*(P)\to\mathrm{K}_*(Q)$, preserving the unit and selected trace pairing. In each family place the possibly nonsplit factor first; if all units split, choose any first factor. Write $E_j=A_j$ for the first family. Choose splittings $\mathrm{K}_0(E_j)=\mathbb{Z}[1_{E_j}]\oplus G_{E_j}$ for $j\geq 2$. No splitting of $[1_{E_1}]$ is assumed.

For a finite family with at least two factors, iterating Proposition 8.10 gives $P*_{\mathrm{r}}D\cong F*_{\mathrm{r}}D$, where F is a simple monotracial nuclear UCT $\mathcal{Z}$-stable range model for the pointed groups and selected pairing of P, with the strict trace cone. At each step the newly fused factor has split unit; the old nuclear core need not have split unit. For one factor use Theorem 4.7 with its monotracial range model directly. Nuclear classification [19, Theorems 2.7–2.8] identifies models with isomorphic data, realizing a prescribed isomorphism.

Here is the countable construction, including the invariant alignment for unrestricted groups. Choose simple monotracial nuclear UCT $\mathcal{Z}$-stable models F_n, $n\geq 1$, for the data of the first n factors. The free-product K-sequence gives the actual decompositions

$$\mathrm{K}_0(F_n)=\mathrm{K}_0(E_1)\oplus\bigoplus_{2\leq j\leq n}G_{E_j}, \qquad \mathrm{K}_1(F_n)=\bigoplus_{j\leq n}\mathrm{K}_1(E_j),$$

with unit $([1_{E_1}],0)$ and the aggregate selected pairing. Give these groups the strict trace cone and choose trace-preserving embeddings $\iota_n:F_n\to F_{n+1}$ inducing the natural inclusions, using [4]. Choose compatible full-invariant extensions when realizing these embeddings. The limit $F_\infty=\varinjlim(F_n,\iota_n)$ is simple, monotracial, nuclear, UCT and $\mathcal{Z}$-stable. Its groups and pairing are the aggregate data of P, and its order is the strict trace cone: a class of strictly positive trace already has that property at a finite stage.

Put, for $n\geq 1$,

$$A_n=F_n*_{\mathrm{r}}\Big(*_{\mathrm{r}j>n}E_j\Big)*_{\mathrm{r}}D, \qquad B_n=F_n*_{\mathrm{r}}D, \qquad c_n=\iota_n*_{\mathrm{r}}\mathrm{id}_D\,.$$

Theorem 4.7, followed by extension over the later free factors, gives an initial trace-preserving isomorphism $\eta_1:A\to A_1$ with the prescribed first-factor marking. In particular $\eta_1(E_1)$ and $\eta_1(D)$ lie in B_1, and the later factors are fixed. Finite fusion on F_n,E_{n+1}, extended identically over the later factors, gives isomorphisms $\theta_n:A_n\to A_{n+1}$ with $\theta_n(B_n)\subseteq B_{n+1}$. We can arrange that $b_n=\theta_n|_{B_n}$ approximates c_n on any prescribed finite packet. First

approximate its elements by free-product polynomials with error smaller than the prescribed tolerance. Only finitely many reservoir intervals occur in these polynomials. Protect a finite interval prefix containing them and run fusion in its complementary tail. Contractivity controls the error on the original packet. Then compare only the nuclear old core F_n in that tail and apply Theorem 3.16. The inner correction is made in the tail and extended by the identity on the protected prefix.

To justify that comparison, it is necessary to align the full KL-class, equivalently total K-theory, and the unitary map on F_n. One may choose the nuclear replacement between decorated cores with the following KK block. Let $u : \mathbb{C} \to F_{n+1}$ be the unit map. The chosen splitting of the new factor gives a retraction $\epsilon_* : \mathrm{K}_0(E_{n+1}) \to \mathbb{Z}$ with $\epsilon_*([1_{E_{n+1}}]) = 1$. UCT surjectivity lifts it, together with the zero K_1-map, to $\epsilon \in \mathrm{KK}(E_{n+1}, \mathbb{C})$. Choose $\kappa \in \mathrm{KK}(E_{n+1}, F_{n+1})$ inducing the unit-preserving group map that includes the new reduced summand and the new K_1 summand. Use

$$\begin{array}{c|cc} & F_{n+1} & \mathbb{C} \\ \hline F_n & [\iota_n] & 0 \\ E_{n+1} & \kappa - u\epsilon & \epsilon \end{array}. \tag{8.16}$$

UCT surjectivity supplies κ for arbitrary countable groups. Since the composition of ϵ with the unit of E_{n+1} is $1 \in \mathrm{KK}(\mathbb{C}, \mathbb{C})$, this block fixes the two component units r, q. It induces an ordinary K-isomorphism and retains the entire old class $[\iota_n]$. At half weights its pairing on the new factor is $\frac{1}{2}(\rho\kappa - \epsilon + \epsilon) = \frac{1}{2}\tau_{E_{n+1},*}$. The block is realized in the simple D-stabilized target; its separate component $\kappa - u\epsilon$ need not be positive in F_{n+1}. Theorem 4.7 retains the chosen total-K map exactly. It need not retain a chosen absolute KK representative: pure extensions are irrelevant to the KL comparison used here.

Choose the compatible unitary map with this total-K extension using [4, Theorem 3.9 and Remark 3.10]. Any discrepancy on the old core factors through $\mathrm{K}_1(F_n)/\operatorname{Tor}\mathrm{K}_1(F_n)$ into the determinant kernel, by [4, Lemma 3.11]. For this application to supported maps, adjoin the complementary scalar summand: compare the two unital maps from $F_n \oplus \mathbb{C}$ which send the added scalar unit to q. They have equal total K-theory and the same half-weight trace. Their K_1 is just $\mathrm{K}_1(F_n)$, so the lemma gives exactly the stated discrepancy. It extends by zero along the actual direct sum

$$\frac{\mathrm{K}_1(F_n \oplus E_{n+1})}{\mathrm{Tor}} = \frac{\mathrm{K}_1(F_n)}{\mathrm{Tor}} \oplus \frac{\mathrm{K}_1(E_{n+1})}{\mathrm{Tor}}.$$

Adding this correction changes neither total K-theory nor the trace and preserves every coefficient-to-unitary square. No freeness of either quotient and no real-valued lift are used. The full-KL local theorem realizes the corrected data. Corner equivalence and Remark 8.7 retain them through matrix completion. Thus the old-core maps have equal full invariant, as required for the preceding nuclear-source uniqueness step.

Fix countable dense subsets of the B_n. Recursively choose finite packets $K_n \subset B_n$ tracking larger and larger finite portions of these subsets from every earlier stage under both connecting systems, and containing $b_{n-1}(K_{n-1}) \cup c_{n-1}(K_{n-1})$ when $n > 1$. Choose the next fusion so that $\|b_n(x) - c_n(x)\| < \varepsilon_n$ for $x \in K_n$, where $\sum_n \varepsilon_n < \infty$. After b_n is chosen, its finitely many required images can be included in K_{n+1}. Thus both images of every retained

element belong to the next packet; inductively, every subsequent mixed composition of the b- and c-maps on such an element is retained.

Write $b_{n,m}$ and $c_{n,m}$ for the connecting compositions, and $c_{m,\infty}$ for the canonical map to the c-limit. For $x \in K_n$ and $m \geq n$,

$$\|c_{m+1,\infty} b_{n,m+1}(x) - c_{m,\infty} b_{n,m}(x)\| < \varepsilon_m.$$

Thus these images are Cauchy, with errors bounded by the summable tails. Reversing b and c gives the reverse limit map. The same tail estimates show that, on a tracked element represented at stage m, either composite differs from the identity by at most $2\sum_{j\geq m} \varepsilon_j$. Its later connecting images remain in the packets, so letting m tend to infinity proves that the two maps are inverse on the tracked dense sets, and hence everywhere. Consequently

$$\varinjlim(B_n, b_n) \cong \varinjlim(B_n, c_n) = F_\infty *_{\mathrm{r}} D. \tag{8.17}$$

Set $\eta_n = \theta_{n-1} \cdots \theta_1 \eta_1 : A \to A_n$ for $n \geq 2$, retaining the preceding definition for $n = 1$. The pulled-back algebras $\eta_n^{-1}(B_n)$ increase, because $\theta_n(B_n) \subseteq B_{n+1}$. Each θ_n carries the whole $F_n *_{\mathrm{r}} E_{n+1} *_{\mathrm{r}} D$ onto B_{n+1} and fixes the later factors; its tail correction preserves these properties. Inductively $\eta_n(E_i) \subset B_n$ for $i \leq n$, the later factors are fixed, and $\eta_n(D) \subset B_n$. The pulled-back union therefore contains every original factor and D, so it is dense in A. This proves $P *_{\mathrm{r}} D \cong F_\infty *_{\mathrm{r}} D$. The stage markings identify the induced pointed groups and pairing with the prescribed aggregate data. For two products with isomorphic data, nuclear classification of their F_∞ models proves the required classification, including realization of the prescribed isomorphism Θ of Elliott invariants. For an arbitrary prescribed compatible full marking $\xi : A \to F *_{\mathrm{r}} D$, let Λ be the triple of the constructed isomorphism $\Phi : A \to F_\infty *_{\mathrm{r}} D$. Apply Theorem 4.7 to $\xi\Lambda^{-1}$ between the two nuclear-core stabilizations, and compose its realizing isomorphism with Φ. This gives the requested marking exactly.

To protect an original finite reservoir-block head H, first write $D = H *_{\mathrm{r}} D_{>r}$. The complementary tail is again a reservoir. Run the whole construction with $D_{>r}$, using an interval identification only inside that tail, and extend by id_H. The native tail inclusions induce invertible full triples by Lemma 2.4; transport the prescribed marking through these triples before the construction and cancel them afterwards, as in Theorem 4.7. This fixes H pointwise without identifying an original block prefix with an interval prefix.

Finally, suppose two distinct factors contain Haar unitaries. Group one as the first factor and all the others as the second. Both grouped algebras contain a Haar unitary. Semicircular absorption [45, Theorem 5.2] gives $P \cong P *_{\mathrm{r}} D$, after identifying D with its interval presentation by Corollary 4.11. The same reasoning applies to Q. Since the D-stabilized algebras are simple, monotracial and strictly comparing, their positive cone is $\{0\} \cup \{g : \tau_*(g) > 0\}$, so their Elliott invariant is determined by the pointed groups and selected trace pairing. To realize an isomorphism $\Theta_0 : \mathrm{Ell}(P) \to \mathrm{Ell}(Q)$ after removing D, choose any trace-preserving absorption isomorphisms $a : P \to P *_{\mathrm{r}} D$ and $b : Q \to Q *_{\mathrm{r}} D$. Realize $\mathrm{Ell}(b)\Theta_0\,\mathrm{Ell}(a)^{-1}$ between the D-stabilized products and then compose with b^{-1} and a. No canonical marking of the absorption isomorphisms is required. The two-factor Haar condition is used only for this removal step; the D-stabilized classification also includes scalar cores and one-factor families. □

9. General fusion and classification

This section proves fusion for every nonempty finite or countable family of separable unital nuclear UCT algebras with faithful selected traces. We first fuse zero-K_1 cores over a finite diagonal. Tail control then gives a two-sided countable exhaustion. The independent split-unit theorem, Theorem 8.1, finally separates and reincorporates arbitrary K_1-groups. In all constructions an originally prescribed finite reservoir head may be kept external. We first use the interval presentation $D = D_I$; Corollary 4.11 supplies the general reservoir at the end.

9.1. **The relative interval pushout over a finite diagonal.** Let $G \subset N$ be a unital inclusion, where G is finite dimensional and commutative. Fix faithful selected traces and the trace-preserving expectation, and put

$$R_G = N *_G (G \otimes \mathrm{C}(\mathbb{T}_v)), \qquad Q_G = N *_G (G \otimes \mathrm{C}([0,1]_h)),$$

with product Haar and uniform expectations.

Proposition 9.1 (Relative one-turn pushout)**.** *The canonical map*

$$R_G *^{\max}_{G\otimes \mathrm{C}(\mathbb{T})} (G \otimes \mathrm{C}([0,1])) \longrightarrow Q_G$$

is an isomorphism. In particular, a faithful representation of R_G together with a positive contraction K commuting with its represented G and satisfying $e^{2\pi iK} = v$ extends to a contractive homomorphism of Q_G. If its selected trace on R_G is the native one, the extension preserves the native trace and is faithful.

Proof. Use the continuous branch functions g^0_δ, g^1_δ from the proof of Lemma 6.2. Their Haar distributions are exactly uniform on $[0, 1-\delta]$ and $[\delta, 1]$, respectively. The reduced amalgamated free law therefore gives

$$\|P(N, g^0_\delta(v))\| = \|P(N, (1-\delta)h)\|_{Q_G},$$
$$\|P(N, g^1_\delta(v))\| = \|P(N, \delta + (1-\delta)h)\|_{Q_G}.$$

Both norms tend to $\|P(N,h)\|_{Q_G}$.

Here is the endpoint compression needed for arbitrary representations. Let

$$\epsilon = \mathrm{ev}_1 \mathbb{E}_{G\otimes \mathrm{C}(\mathbb{T})} : R_G \to G.$$

If $P = 1_{\{1\}}(v)$, then

$$PaP = \epsilon(a)P \qquad (a \in R_G). \tag{9.1}$$

To check it, the conjugates $v^k N v^{-k}$ are free over G. For a finite-support word centered under expectation onto G, widely spaced conjugates have disjoint support and are free over G. The creation, annihilation and diagonal decomposition on the reduced Hilbert G-module bounds the norm of their average by $(m^{-1} + 2m^{-1/2})\|a\|$. Compression by P is invariant under these conjugations. For a finite Laurent sum $\sum_k x_k v^k$, this proves $P(\sum_k x_k v^k)P = (\sum_k \mathbb{E}_G(x_k))P$, which is (9.1); density completes the argument.

In a representation of the full pushout put $P_s = 1_{\{s\}}(K)$, $s = 0, 1$. The subspaces $\overline{R_G P_s \mathcal{H}}$ are orthogonal and reduce R_G, by (9.1) and the fact that the P_s's commute with G. They also reduce K: away from the $v = 1$ spectral subspace, K is the unique Borel argument of v, and on P_s it is s. Thus $g^s_\delta(v) \to K$ strongly on the corresponding subspace; either

branch works on their orthogonal complement. The preceding branch norm limits give the required upper bound for every polynomial in N, K. The canonical reduced quotient gives the reverse bound, proving the isomorphism.

For the trace assertion, a positive one-turn logarithm differs in 2-norm from a fixed branch only on an arc whose Haar measure tends to zero. Its joint moments with N therefore equal those of the native relative interval. Faithfulness of the latter trace gives injectivity. □

9.2. **The finite-base endpoint ideal and comparison.** Continue with R_G, Q_G above and finite-dimensional commutative G. Assume that $N = N_0 *_{\mathrm{r}} D_0$, where $G \subset N_0$ is expected and D_0 is an external infinite scalar interval reservoir. This assumption provides stability; it is separate from the uniform-logarithm criterion.

Let $\mathcal{H}_\epsilon$ be the Hilbert G-module obtained by completing R_G for $\langle a, b\rangle_G = \epsilon(a^*b)$, with ϵ as in (9.1). Its left representation has the useful decomposition

$$\mathcal{H}_\epsilon = G \oplus (\ell^2(\mathbb{Z}) \otimes \mathcal{V}), \qquad \pi_\epsilon(v) = 1_G \oplus (S \otimes 1), \tag{9.2}$$

where S is bilateral shift and $\mathcal{V}$ is a countably generated Hilbert G-module. To see this, identify the endpoint GNS module with $L^2(*_{k\in\mathbb{Z},G} N_k, \mathbb{E}_G)$, where $N_k = v^k N v^{-k}$. The vacuum is G; on nonempty reduced words v shifts all labels simultaneously. Group each orbit by requiring its first label to be zero. Every such orbit is a copy of $\mathbb{Z}$, proving (9.2).

The external reservoir supplies a standard $\ell^2 \otimes G$ orthogonal summand in this module. For example the usual free-product decomposition for $N_0 *_{\mathrm{r}} D_0$ contains $(L^2(D_0) \ominus \mathbb{C}) \otimes G$ as a summand. All remaining summands are countably generated. Hilbert-module stabilization therefore gives $\mathcal{H}_\epsilon \cong \ell^2 \otimes G$.

Proposition 9.2 (The relative two-logarithm ideal)**.** *Represent R_G faithfully, and let H_0, H_1 be positive contractions commuting with the represented G, with $e^{2\pi i H_0} = e^{2\pi i H_1} = v$. In the generated algebra put*

$$J_G = \mathrm{Ideal}_{\mathrm{C}^*(G,H_0,H_1)}(H_1 - H_0), \qquad I = \mathrm{Ideal}_{\mathrm{C}^*(R_G,H_0,H_1)}(H_1 - H_0).$$

Then J_G is a quotient of $G \otimes J_{\mathrm{pair}}$, where

$$J_{\mathrm{pair}} = \{f \in \mathrm{C}([0,1], \mathrm{M}_2) : f(0) = 0,\ f(1) \text{ diagonal}\},$$

and

$$I \cong \mathcal{K}_{J_G}(\mathcal{H}_\epsilon \otimes_G J_G) \cong \mathcal{K} \otimes J_G. \tag{9.3}$$

In particular I is nuclear, stable, type I and of finite nuclear dimension, and has the corona factorization property.

Proof. Off $P = 1_{\{1\}}(v)$, both logarithms are the same Borel argument. On P they are two projections commuting with G. The universal two-projection difference ideal is J_{pair}; nuclearity of G gives the asserted tensor-product quotient. By (9.1),

$$xay = x\epsilon(a)y \quad (x, y \in J_G,\ a \in R_G).$$

Thus the rank-one-operator formula

$$\theta_{a\otimes x, b\otimes y} \longmapsto axy^*b^*$$

identifies the compact operators in (9.3) with the closed span of these products. Faithfulness follows by inducing a faithful representation of J_G; the endpoint compression identity gives the required inner products.

Here is why the span is invariant under the logarithms, without assuming in advance that it is an ideal. In the faithful generated representation put

$$\mathcal{L} = \overline{R_G J_G \mathcal{H}}, \qquad P = 1_{\{1\}}(v).$$

The induced-module identification shows that $\mathcal{L}$ reduces R_G. The shift in (9.2) has no eigenvector at 1, so $P\mathcal{L} = \overline{J_G\mathcal{H}}$ is precisely its vacuum part. This subspace is preserved by each H_s because J_G is an ideal in $\mathrm{C}^*(G, H_0, H_1)$. On $(1-P)\mathcal{L}$, both logarithms are the common Borel argument of v and hence preserve $\mathcal{L}$. Self-adjointness makes $\mathcal{L}$ reducing for both logarithms. In this induced representation, they act on the shift summand of (9.2) as the same bounded Borel argument of S, tensored with the identity. This is an adjointable operator: it is a bounded Hilbert-space operator on $\ell^2(\mathbb{Z})$. On the vacuum summand they act as the two endpoint projections in $\mathcal{M}(J_G)$. Consequently they normalize the compact-operator algebra just identified; on $\mathcal{L}^\perp$ those compact operators vanish. The span is therefore an ideal of the whole generated algebra, and is exactly I, rather than merely a subalgebra containing the original difference.

The second isomorphism follows from standardness of $\mathcal{H}_\epsilon$. The quotient J_G is subhomogeneous of topological dimension at most one; finite nuclear dimension passes to its stabilization. The finite-nuclear-dimension corona-factorisation result [40] applies. □

Theorem 9.3 (Comparison over a fixed relative circle)**.** *Assume Q_G is simple and the preceding external-reservoir hypothesis holds. Two faithful unital homomorphisms $\rho_0, \rho_1 : Q_G \to L_\infty$ that agree on R_G are unitarily equivalent. The same holds in a norm ultrapower.*

Consequently, if the native R_G embeds unitally and trace preservingly in an actual faithfully tracial algebra L and positive contractions $H_n \in L$ satisfy

$$[H_n, b] \longrightarrow 0 \quad (b \in G), \qquad e^{2\pi i H_n} \longrightarrow v$$

in norm, there is an actual faithful trace-preserving embedding $f : Q_G \to L$ whose restriction to R_G is approximately unitarily equivalent to its native inclusion.

Proof. The difference ideal is the stable nuclear ideal in Proposition 9.2. If it is zero, the two maps coincide; suppose it is nonzero. The two multiplier maps form a unital full Cuntz pair, by simplicity of Q_G, and agree on N. The inclusion $N \to Q_G$ is a KK-equivalence: contract the interval in the full amalgamated product and use the full-to-reduced equivalence [20, Theorem 3.1] for the specified GNS-faithful expectations, using [20, Proposition 2.15] to identify the vertex-reduced and ordinary reduced products. Restriction thus annihilates their actual KK-pair injectively. Lemma 6.5 gives approximate conjugacy by unitaries in $1 + I$. The norm estimates hold in the generated algebra because their errors lie in I. The sequence-algebra epsilon test gives one implementing unitary.

For descent, Proposition 9.1 first gives $\Theta : Q_G \to L_\infty$ agreeing with the diagonal native map on R_G; the trace follows from the same branch estimate. Every diverging reparameterization still agrees with Θ on this *diagonal* circle algebra. Apply the comparison just proved, and then reparameterization descent [4, Proposition 9.2], as in Proposition 6.6, to obtain an actual f unitarily equivalent to Θ after diagonal inclusion. This proves the

asserted finite-packet limits and trace preservation. Diagonality of the common circle is an essential hypothesis at this step. □

9.3. **Two exact matrix identities.**

Proposition 9.4. *Let A, B contain the same finite-dimensional tracial algebra F, with faithful selected traces and their trace-preserving expectations. For $M = A *_F B$, there is a trace-preserving reduced isomorphism*

$$\mathrm{M}_2(M) \cong (A \oplus B) *_{F\oplus F} \mathrm{M}_2(F), \tag{9.4}$$

using the half-weighted trace on $A \oplus B$ and $\mathrm{tr}_2 \otimes \tau_F$ on $\mathrm{M}_2(F)$. For every faithfully tracial M and $k \geq 1$ there is also an isomorphism

$$\mathrm{M}_k(M) *_{\mathrm{r}} D \cong \mathrm{M}_k(M *_{\mathrm{r}} D) \tag{9.5}$$

fixing the labelled $\mathrm{M}_k(M)$ and preserving the normalized trace.

Proof. For (9.4) embed the first factor diagonally in $\mathrm{M}_2(M)$ and the second through the common F. A centered letter from $\mathrm{M}_2(F)$ is off diagonal. A centered diagonal coefficient is $\mathrm{diag}(a, b)$ with $\mathbb{E}_F(a) = \mathbb{E}_F(b) = 0$. Alternating centered products therefore follow paths alternating between A and B; the defining F-reduced-word rule makes their trace zero. Thus the displayed copies have exactly the required reduced free law. Their matrix units and diagonal coefficients generate $\mathrm{M}_2(M)$. Faithfulness of the traces proves the reduced norm identity and surjectivity.

For (9.5), put $R = \mathrm{M}_k(M)$, $\mathcal{A} = R *_{\mathrm{r}} D$ and $P = e_{11}$, using the standard matrix units $e_{ab} \in R$. Write $\tau_R = \mathrm{tr}_k \otimes \tau_M$ and let $\tau_{\mathcal{A}}$ be the free-product trace. The canonical trace-preserving expectation $\mathbb{E}_R : \mathcal{A} \to R$ restricts to an expectation $\mathbb{E}_M : P\mathcal{A}P \to PRP \cong M$. The normalized corner trace is $\tau_P = k\tau_{\mathcal{A}}|_{P\mathcal{A}P}$, with $\tau_M \circ \mathbb{E}_M = \tau_P$.

Choose the standard semicircular reservoir coordinates $(S_i)_{i\geq 1}$. Their freeness from R gives the R-valued Wick rule: conditional moments are sums over noncrossing pairings, with each pair evaluated by

$$\mathbb{E}_R(S_i r S_j) = \delta_{ij}\tau_R(r)1_R \qquad (r \in R).$$

In particular their first moments and all R-valued cumulants of order greater than two vanish. Set

$$X_{ab}^{(i)} = \sqrt{k}\, e_{1a} S_i e_{b1} \in P\mathcal{A}P \qquad (1 \leq a, b \leq k).$$

Then $(X_{ab}^{(i)})^* = X_{ba}^{(i)}$ and $\mathbb{E}_M(X_{ab}^{(i)}) = 0$. For $m \in PRP \cong M$, bimodularity and the Wick rule give

$$\begin{aligned}
\mathbb{E}_M\Big(X_{ab}^{(i)} m X_{cd}^{(j)}\Big) &= k e_{1a}\mathbb{E}_R\Big(S_i(e_{b1} m e_{1c})S_j\Big)e_{d1} \\
&= k\delta_{ij}\tau_R(e_{b1} m e_{1c}) e_{1a} e_{d1} \\
&= \delta_{ij}\delta_{bc}\delta_{ad}\tau_M(m)P.
\end{aligned}$$

The last equality uses $\tau_R(e_{b1} m e_{1c}) = \delta_{bc}\tau_M(m)/k$ and $e_{1a}e_{d1} = \delta_{ad}P$; this explains the factor $\sqrt{k}$. For a longer product with coefficients in M, substitute the same matrix-unit expressions into the R-valued Wick rule. Contract the innermost pairs first, using the displayed calculation. Each contraction gives a coefficient in PRP, so the same covariance applies recursively to every outer pair. Only pairings occur. Thus the X's satisfy the

M-valued Wick recursion with this covariance; their higher M-valued cumulants vanish as well.

For each i, take the self-adjoint variables

$$X_{aa}^{(i)}\ (1 \le a \le k), \qquad \frac{X_{ab}^{(i)} + X_{ba}^{(i)}}{\sqrt{2}}, \qquad \frac{X_{ab}^{(i)} - X_{ba}^{(i)}}{\mathrm{i}\sqrt{2}} \quad (a < b).$$

There are $k + 2\binom{k}{2} = k^2$ of them. Denoting this whole countable family by (Y_λ), the covariance calculation gives

$$\mathbb{E}_M(Y_\lambda m Y_\mu) = \delta_{\lambda\mu}\tau_M(m)P.$$

The change of variables preserves the vanishing of higher cumulants. Consequently their mixed moments with M are exactly those of standard scalar semicirculars free from one another and from M. This also follows directly from the Wick recursion, since the displayed covariance is precisely the covariance for that free family.

We next check that these variables generate the entire corner. Let $C = \mathrm{C}^*(M, X_{ab}^{(i)} : i, a, b) \subset P\mathcal{A}P$; the X's and the Y's generate the same algebra. The standard matrix decomposition

$$\Psi : \mathcal{A} \longrightarrow \mathrm{M}_k(P\mathcal{A}P), \qquad \Psi(x) = [e_{1a}xe_{b1}]_{a,b}, \qquad \Psi^{-1}([c_{ab}]) = \sum_{a,b} e_{a1}c_{ab}e_{1b}$$

is a *-isomorphism, as follows by multiplying matrix units and using $\sum_a e_{a1}e_{1a} = 1$. Its values on R have entries in M, and $\Psi(S_i) = k^{-1/2}[X_{ab}^{(i)}]_{a,b}$. Since R and the S_i generate $\mathcal{A}$, it follows that $\Psi(\mathcal{A}) \subset \mathrm{M}_k(C)$. Taking the $(1,1)$-corner gives $P\mathcal{A}P = C$.

Reindexing the k^2 semicircular variables for each i by $\{1, \dots, k^2\} \times \mathbb{N} \cong \mathbb{N}$ identifies this corner with $M *_{\mathrm{r}} D$, fixing M and preserving τ_P. For completeness, equality of the mixed moments gives equality of the reduced polynomial norms: for either faithful trace τ,

$$\|x\| = \lim_{n\to\infty} \tau((x^*x)^n)^{1/(2n)}.$$

Thus the polynomial correspondence extends isometrically to a trace-preserving isomorphism $\theta : P\mathcal{A}P \to M *_{\mathrm{r}} D$. The desired isomorphism and its inverse are explicitly

$$x \longmapsto [\theta(e_{1a}xe_{b1})]_{a,b}, \qquad [y_{ab}] \longmapsto \sum_{a,b} e_{a1}\theta^{-1}(y_{ab})e_{1b}.$$

These maps fix the labelled $\mathrm{M}_k(M)$ entry by entry. Finally, traciality and $\tau_P = k\tau_{\mathcal{A}}|_{P\mathcal{A}P}$ give

$$\tau_{\mathcal{A}}\left(\sum_{a,b} e_{a1}c_{ab}e_{1b}\right) = \frac{1}{k}\sum_a \tau_P(c_{aa}),$$

which proves preservation of the normalized matrix trace and completes (9.5). This statement uses a full standard matrix frame; it asserts no corresponding identity for an arbitrary full projection. □

9.4. **Relative free transport and actual returns.** We use the following finite-base setup. Let B be finite dimensional and $G \subset B$ its labelled full diagonal algebra. Let N and E be unital nuclear UCT algebras with faithful selected traces and trace-preserving embeddings

$$a : N \to E, \qquad b : B \to E$$

agreeing on the specified $G \subset N, B$. Identify all these copies of G. Put

$$S = (N *_G B) *_{\mathrm{r}} D, \qquad T = (E *_G (G \otimes I_h)) *_{\mathrm{r}} D, \qquad q = e^{2\pi i h}. \tag{9.6}$$

The G-Haar conjugation rule gives a faithful reduced embedding

$$\eta|_N = a, \qquad \eta|_B = \mathrm{Ad}(q)b, \qquad \eta|_D = \mathrm{id}\,. \tag{9.7}$$

Indeed E and $q\,b(B)q^*$ are free over G, and the ordinary reservoir is external.

Assume in this setup that $[\eta]$ and $\eta^\ddagger$ are isomorphisms, that η_T is an affine homeomorphism, that $\mathrm{K}_1(N) = \mathrm{K}_1(E) = \mathrm{K}_1(S) = 0$, and that the working free-reservoir targets are admissible. They will be verified below for the binary fusion application. Finite relative tags preserve the relevant KK groups by Lemma 7.3. Whenever tag absorption is used, we reserve a fresh external infinite scalar bank: the working algebra has the form $H *_{\mathrm{r}} D_{\mathrm{fresh}}$, with all labelled finite-dimensional bases contained in H. Thus Lemma 7.4 also preserves admissibility in each such working algebra. In particular $\mathrm{K}_1(T) = 0$ by the marking of η; the finite-tag KK-equivalences and full-corner Morita equivalence retain this vanishing in the working algebras and normalized corners used below.

Proposition 9.5. *There are actual trace-preserving embeddings $\beta_n : T \to S$ with inverse core marking and $\beta_n\eta \to \mathrm{id}_S$ pointwise. They may be chosen to fix G exactly.*

Proof. Fix finite source coefficient packets and a finite ordinary reservoir head D_{head}. Keep this head external throughout the following construction, using independent infinite banks in its complement. Choose an infinite working bank D_0 there and put $S_{\mathrm{work}} = (N *_G B) *_{\mathrm{r}} D_0$. Theorem 3.16 supplies an inverse-marked $\theta : E \to S_{\mathrm{work}}$. Theorem 3.16, applied to the whole nuclear N, gives inner-corrected θ_n with $\theta_n a \to \mathrm{id}_N$. Correct their finite G-frames by unitaries tending to 1, so that $\theta_n|_G = \mathrm{id}_G$ exactly.

Reserve a G-central relative Haar unitary v, realized by the finite-dimensional tag construction in a fresh scalar interval. Conjugate the old B by v, fix N, and move the old ordinary working bank into a fresh external bank D_1. Reduced G-freeness gives an actual embedding σ with these prescriptions. To check the retained-frame assertion directly, put $M = (N *_G B) *_{\mathrm{r}} D_1$ and $A = \mathrm{C}^*(G, D_1)$. Reduced reassociation makes N, B, A free over G. In the fresh G-central Haar extension, M and vMv^* are free over G: alternating centered words expand to reduced words containing nonzero powers of v. Thus N, B, A, vBv^* form a free family over G. Consequently $\mathrm{C}^*(N, vBv^*, D_1)$ has the prescribed source law and is free over G from the retained B. Faithful traces give the asserted actual embedding. Its image core $E' = \sigma\theta_n(E)$ remains root-good and is free over G from the retained old B. On the nuclear E, its inverse marking is unchanged: the K_0 action and trace are unchanged, and both source and ambient K_1 vanish, eliminating the degree-zero UCT Ext terms.

Let $C = \sigma\theta_n b(B) \subset E'$ and write the corresponding labelled matrix units as $c_{ij}^{(\ell)}, b_{ij}^{(\ell)}$. Their diagonal projections already agree exactly in G. Consequently the explicit element

$$w_0 = \sum_{\ell,i} b_{i1}^{(\ell)} c_{1i}^{(\ell)} \in \mathrm{C}^*(B, C)$$

is unitary, commutes with G, and satisfies $w_0 c_{ij}^{(\ell)} w_0^* = b_{ij}^{(\ell)}$. These identities follow directly by multiplying matrix units. No projection cancellation or additional K_0-existence statement is needed for this frame-matching step.

Here the free-implementer lemma has a relative version. Over the whole working algebra containing E', the old B and w_0, take a fresh C-amalgamated extension with C-central Haar z, free from that whole algebra over C, and put $W = w_0 z$. Then

$$W \in G', \qquad WCW^* = B, \qquad \mathrm{C}^*(E', W) \cong E' *_G (G \otimes \mathrm{C}(\mathbb{T})). \tag{9.8}$$

To verify the last reduced norm, set $H_0 = \mathrm{C}^*(E', B) = E' *_G B$. Write $\mathbb{E}$ for the canonical expectation onto H_0 in the reduced subextension

$$H_0 *_C (C \otimes \mathrm{C}(\mathbb{T}_z)).$$

This subextension embeds trace preservingly in the whole working extension: the expectations onto the finite-dimensional C restrict correctly. It contains w_0, z, W. No expectation from the whole working coefficient algebra onto H_0 is assumed. The HNN pinch rules are

$$\mathbb{E}(WxW^*) = w_0 \mathbb{E}_C(x) w_0^*, \qquad \mathbb{E}(W^*xW) = w_0^* \mathbb{E}_B(x) w_0.$$

They agree with the rules for the G-central free Haar U in $E' *_G (G \otimes \mathrm{C}(\mathbb{T}))$, identifying B with UCU^*. This uses $\mathbb{E}_B|_{E'} = \mathbb{E}_G$ and the expectation tower onto C. Induction on stable letters, after expanding coefficients in $E' *_G B$, gives equality of all G-valued moments. Faithful traces give (9.8). The C-central tag is actual, by the finite-dimensional tag embedding, so this does not construct only a sequence-algebra circle.

Independently normalize the normalized determinant of W in every minimal G-corner to $[1/2]$, by replacing W by dW, $d \in Z(G)$. All corner K_1 classes vanish. This preserves the G-Haar free law, while rotating the old labelled B-frame by $\mathrm{Ad}(d)$. That rotation can be undone: the source map which fixes N, D and acts by $\mathrm{Ad}(d)$ on B is an actual trace-preserving automorphism, since it fixes G pointwise. A path of diagonal phases shows it is homotopic to the identity, so its total K-map is the identity. Its trace preservation and $\mathrm{K}_1(S) = 0$ retain the full unitary marking as well.

Reserve an additional external infinite bank D_3. The coefficient $E' *_{\mathrm{r}} D_3$ and the fixed actual dW-circle satisfy the relative actual-model theorem (Theorem 9.3, applied to $G = \mathrm{C}(\{\text{finitely many points}\})$). The native marked G-central interval can be made in a fresh bank by

$$Y = \sum_j \tau(e_j)^{-1/2} e_j s e_j$$

and the semicircle distribution function. Nuclear-source uniqueness for $G \otimes \mathrm{C}(\mathbb{T})$ supplies approximate logarithms for the fixed circle, and the fixed-relative-circle comparison descends them to actual embeddings of the full interval domain. In particular their whole coefficient restrictions are approximately unitarily equivalent to the native one.

Choose the root and frame errors tending to zero, and undo the diagonal source gauge by postcomposing the return with its inverse. The gauge fixes N and the protected reservoir head pointwise, so this retains the native root estimates and restores the old whole B-frame. Use D_3 for the unprotected target tail and extend the actual return by the identity on D_{head}. Thus each return approximates the native map on the chosen N-packet, the whole finite B-frame, and this head. Letting these packets and heads increase gives $\beta_n \eta \to \mathrm{id}_S$ by word approximation. Near-identity finite-frame corrections make G exact, and later make the whole returned finite B-frame exact when adjoining its tags. The inverse core marking is

retained by coefficient approximate unitary equivalence and the homotopically trivial source gauges. □

9.5. **Simultaneous corner phases in target capture.**

Theorem 9.6 (Finite-base finite-dimensional attachment)**.** *In the finite-base setup just specified, with S, T as in* (9.6)*, there is a marked trace-preserving isomorphism*

$$S \cong T \cong E *_{\mathrm{r}} D.$$

Proof. Add a G-central uniform root tag H_Z and a whole B-central uniform leaf tag H_Y to the source; on the target the latter tag is central over $B_0 = q\, b(B) q^*$. Write S^+, T^+, η^+ for the resulting diagrams and $Z = e^{2\pi i H_Z}$, $Y = e^{2\pi i H_Y}$. The actual returns extend, after the near-identity whole B-frame correction, to

$$\beta_n^+ : T^+ \to S^+, \qquad \beta_n^+ \eta^+ \to \mathrm{id}_{S^+}, \qquad \beta_n^+(T) \subset S. \tag{9.9}$$

All finite-frame corrections occur in the original S. The tag extensions send each target H_Z, H_Y to its source counterpart exactly. Hence, with $\phi_n = \eta^+ \beta_n^+$,

$$\phi_n(H_Z) = H_Z, \qquad \phi_n(H_Y) = H_Y, \qquad \phi_n|_{B_0} = \mathrm{id}_{B_0}.$$

Put $q_{1,n} = \phi_n(q)$. The core and circle images in this notation lie in the original T, and $\phi_n|_G = \mathrm{id}_G$.

Theorem 3.16, on the nuclear core E, gives core-alignment unitaries. Correct on the finite diagonal to choose them in G':

$$u_n \phi_n(e) u_n^* \longrightarrow e \quad (e \in E).$$

Averaging $q_{1,n} u_n^* q^*$ over the unitary group of B_0 and taking polar parts gives

$$v_n \in B_0' \cap T, \qquad v_n - q_{1,n} u_n^* q^* \longrightarrow 0. \tag{9.10}$$

The source-good whole leaf and the core alignment justify this averaging. Indeed, for $A_n = q_{1,n} u_n^* q^*$, the exact identity $\phi_n|_{B_0} = \mathrm{id}$ and core alignment imply $\|[A_n, b]\| \to 0$ on the finite matrix frame of B_0. Finite dimensionality makes this uniform for $b \in \mathrm{U}(B_0)$; the average differs from A_n by at most that commutator bound. It is therefore invertible for large n, and its polar unitary lies in $B_0' \cap T$ and has the asserted norm estimate.

Here independent root corner phases cannot be discarded. Write $G = \bigoplus_j \mathbb{C} e_j$ and let b_ℓ be the central supports of B_0. Since G is its full diagonal, each b_ℓ is a sum of some e_j's. Equivalent minimal projections in a block have the same normalized determinant for v_n, since v_n commutes with the whole block. Let $r_{\ell,n}$ represent this common determinant and put

$$c_n = \sum_\ell e^{2\pi i r_{\ell,n}} b_\ell \in Z(B_0) \subset G.$$

Because q commutes with G, these block supports are also the central supports of $C_B = b(B)$; hence $c_n \in Z(C_B)$. Every normalized B_0-corner determinant of $v_n' = v_n c_n^*$ is exactly zero.

Both q and $q_{1,n}$ are exponentials of actual uniform intervals in each normalized G-corner. Their corner determinants are $[1/2]$. Equation (9.10) therefore gives

$$\Delta_{e_j}(u_n) + \Delta_{e_j}(v_n) \longrightarrow 0.$$

Thus every normalized G-corner determinant of $c_n u_n$ tends to zero. Choose diagonal phases $d_n \in Z(G)$, $d_n \to 1$, so that

$$u_n'' = d_n c_n u_n$$

has exactly zero determinant in every G-corner. This uses only that a closed trace-range subgroup is either discrete or all of $\mathbb{R}$. Then

$$u_n'' \phi_n(e)(u_n'')^* - c_n e c_n^* \longrightarrow 0, \qquad v_n' q - q_{1,n}(u_n'')^* \longrightarrow 0. \tag{9.11}$$

For completeness, with $\epsilon_n = \|v_n - A_n\|$ and $\delta_n = \|d_n - 1\|$, the two errors are bounded by

$$\|u_n'' \phi_n(e)(u_n'')^* - c_n e c_n^*\| \le \|u_n \phi_n(e) u_n^* - e\| + 2\delta_n \|e\|,$$
$$\|v_n' q - q_{1,n}(u_n'')^*\| \le \epsilon_n + \delta_n.$$

Stable rank one and zero corner K_1 turn these zero determinants into the full corner CU conditions.

Set $H_X = q^* H_Y q$ and $X = e^{2\pi i H_X}$. Reassociation writes the tag coefficient as

$$\mathcal{N} = (E *_G (G \otimes I_{H_Z})) *_{C_B} (C_B \otimes I_{H_X}).$$

On the faithful relative circle algebra

$$R^+ = (\mathcal{N} *_G (G \otimes \mathrm{C}(\mathbb{T}_q))) *_{\mathrm{r}} D$$

define

$$\Gamma|_{\mathcal{N}} = \mathrm{Ad}(Z), \qquad \Gamma(q) = qXZ^* = YqZ^*, \qquad \Gamma|_D = \mathrm{id}\,. \tag{9.12}$$

This is a relative Nielsen automorphism. In each normalized G-corner its multiplier has determinant $\Delta(X) - \Delta(Z) = [1/2] - [1/2] = 0$. Lemma 7.6 therefore gives actual automorphisms Γ_n of T^+ approaching this prescription on arbitrary finite packets in R^+, with full identity marking. One first extends $\mathrm{Ad}(Z)$ on the coefficient and then performs the right-multiplier move.

There is a source automorphism γ_S acting by $\mathrm{Ad}(Z)$ on N, by $\mathrm{Ad}(Y)$ on the whole B, and fixing D and the tag logarithms. Reassociate the two relative tag extensions, with the ordinary D external, as

$$N *_G (B \otimes I_{H_Y}) *_G (G \otimes I_{H_Z}).$$

The map is partial conjugation of N by the G-central Z; its inverse is the corresponding conjugation by Z^*. In this whole-finite-dimensional-leaf case Y commutes with B itself. The path $Z_t = e^{2\pi i t H_Z}$ gives trace-preserving automorphisms from the identity to γ_S. Homotopy retains total K-theory; trace preservation and $\mathrm{K}_1(S^+) = 0$ retain the full unitary map. The exact circle identity is

$$\Gamma \eta^+ = \eta^+ \gamma_S. \tag{9.13}$$

In particular $\eta^+(S^+) \subset R^+$.

For the phase correction let δ_{c_n} be the actual automorphism acting by $\mathrm{Ad}(c_n)$ on $\mathcal{N}$ and fixing q, D. It fixes both tag logarithms, since $c_n \in G \cap Z(C_B)$. The path $c_n(t) = \sum_\ell e^{2\pi i t r_{\ell,n}} b_\ell$ shows that its total K-map is the identity; trace preservation and $\mathrm{K}_1(T^+) = 0$ give identity full marking. Set $\Gamma_{c_n} = \delta_{c_n} \Gamma$ and take simply

$$\Gamma_{c_n, n} = \delta_{c_n} \Gamma_n.$$

Postcomposition is isometric, so the same finite-packet accuracy approximates Γ_{c_n}. Its circle formula in (9.12) is unchanged, because δ_{c_n} fixes X, Z, q.

Use Lemma 7.6 once more: let $\Theta_{Z,n}$ approximate $Z \mapsto Zu_n''$, and let $\Theta_{Y,n}$ approximate $Y \mapsto Yv_n'$. The exact corner conditions were checked above. Choose all coefficient packets after the moving core and circle images are known. The actual maps

$$F_n = \Theta_{Z,n}\phi_n\Gamma_n|_T, \qquad G_n = \Theta_{Y,n}\Gamma_{c_n,n}|_T$$

then have the following common images in the norm sequence algebra:

$$\begin{aligned} F(e) &= Zc\,e\,c^*Z^* = G(e) && (e \in E),\\ F(q) &= Yq_1u^*c^*Z^* = Yv'qZ^* = G(q),\\ F(d) &= d = G(d) && (d \in D). \end{aligned}$$

These equalities mean that the coordinate norm differences tend to zero; c_n is allowed to move. No equality on the added Y logarithm is needed, since the comparison domain is the original T.

The common relative circle is the faithful copy of

$$(E *_{\mathrm{r}} D) *_G (G \otimes \mathrm{C}(\mathbb{T})).$$

Only the two one-turn logarithms of its one interval differ. Theorem 9.3 applies: its coefficient has an external infinite reservoir, its source is simple by finite-tag absorption, and the native coefficient inclusion is a KK-equivalence. Thus the actual Cuntz-pair class vanishes. Here we use the theorem's first, comparison assertion, which requires a common faithful relative circle. Its separate descent assertion is not being used, so the common image need not be diagonal and the moving c_n is harmless. Lift the comparison unitary and diagonalize to get actual $V_n \in T^+$ with

$$\mathrm{Ad}(V_n)\Theta_{Z,n}\eta^+\beta_n^+\Gamma_n(x) - \Theta_{Y,n}\Gamma_{c_n,n}(x) \longrightarrow 0 \qquad (x \in T).$$

Put $H_n = \Theta_{Y,n}\Gamma_{c_n,n}$ and define the actual trace-preserving automorphism

$$a_n = H_n^{-1}\,\mathrm{Ad}(V_n)\Theta_{Z,n}.$$

Every factor has identity full marking. The preceding comparison gives

$$a_n\eta^+\beta_n^+\Gamma_n(x) \longrightarrow x \qquad (x \in T).$$

Now put $b_n = \gamma_S^{-1}\beta_n^+\Gamma_n$. Equations (9.9) and (9.13) give $b_n\eta^+ \to \mathrm{id}$ on S^+. For a fixed n and finite target packet $L \subset T$, choose a second actual lift Γ_m accurately on the already specified moving packet $\eta^+b_n(L) \subset R^+$. Then

$$\Gamma_m\eta^+b_n(x) \approx \Gamma\eta^+b_n(x) = \eta^+\gamma_S b_n(x) = \eta^+\beta_n^+\Gamma_n(x).$$

Consequently $\alpha_n = a_n\Gamma_m$ supplies target capture for that same source-good return. Every inverse is an actual coordinate inverse; none is passed through a limit.

The exactly natural finite-tag absorption of Lemma 7.4 removes the added Z, Y tags using common ordinary banks. Given finite source and target packets, first approximate them by polynomials using a common finite ordinary reservoir prefix D_{head}. Put

$$H_S = (N *_G B)*_{\mathrm{r}} D_{\mathrm{head}}, \qquad H_T = (E *_G (G \otimes I_h))*_{\mathrm{r}} D_{\mathrm{head}}.$$

Then $\eta(H_S) \subset H_T$, the labelled bases G, B map onto G, B_0, and η fixes the common unused scalar banks. In particular the original interval h belongs to H_T. Choose the same bank partitions and reindexings in the absorption lemma on both sides. This gives label-natural isomorphisms $J_S : S^+ \to S$, $J_T : T^+ \to T$ fixing H_S, H_T, respectively, and satisfying

$$J_T \eta^+ = \eta J_S.$$

Fix these isomorphisms, and apply the preceding construction to the protected polynomial packets to choose an enlarged return b^+ and enlarged correction α^+. Transport them by

$$b = J_S b^+ J_T^{-1}, \qquad \alpha = J_T \alpha^+ J_T^{-1}.$$

For protected polynomials $s \in H_S$ and $t \in H_T$, the commuting square gives

$$b\eta(s) = J_S b^+ \eta^+(s), \qquad \alpha\eta b(t) = J_T \alpha^+ \eta^+ b^+(t).$$

Hence the source and target errors are transported isometrically. Polynomial approximation now shows that the original S, T have, for every pair of finite packets, a common actual return b and actual target automorphism α with

$$b\eta \approx \mathrm{id}_S, \qquad \alpha\eta b \approx \mathrm{id}_T\,.$$

All maps have the required full markings: the returns have the inverse core marking; the Nielsen lifts have identity marking by their corner conditions; coefficient conjugacies and inner comparisons preserve it. The source gauges are homotopic to the identity. Native finite-tag inclusions are KK-equivalences, and zero K_1 and the selected trace retain the determinant compatibility.

To verify (CAAP) for $\eta(S) \subset T$, include $\eta(F)$ in the target packet whenever $F \subset S$ is the source packet. Then

$$\|\alpha\eta(s) - \eta(s)\| \le \|s - b\eta(s)\| + \|\alpha\eta b(\eta(s)) - \eta(s)\| \qquad (s \in F).$$

The same α places each target test element near $\alpha\eta(S)$. Lemma 4.3 therefore gives $S \cong T$ with the full marking of η. Finally finite-dimensional tag absorption on $E *_{\mathrm{r}} D$ gives $T \cong E *_{\mathrm{r}} D$, fixing E. □

9.6. **Zero-K_1 binary reduction and exact-unit compression.**

Proposition 9.7. *Let A_1, A_2 be separable unital nuclear UCT algebras with faithful selected traces and $\mathrm{K}_1(A_i) = 0$. Their free product, after adjoining a reservoir, is trace-preservingly isomorphic to a nuclear-core reservoir model with the aggregate selected pointed data.*

Proof. Write $u_i = [1_{A_i}]$ and

$$G_0 = \frac{\mathrm{K}_0(A_1) \oplus \mathrm{K}_0(A_2)}{\langle (u_1, -u_2) \rangle}, \qquad u = [(u_1, 0)] = [(0, u_2)].$$

The aggregate pairing sends $[(x_1, x_2)]$ to $\tau_{1*}(x_1) + \tau_{2*}(x_2)$ and sends u to 1. Choose the simple monotracial nuclear UCT range core E_0 with $\mathrm{K}_0(E_0) = G_0$, unit u, this pairing and $\mathrm{K}_1(E_0) = 0$, using its strict trace cone and [19, Theorem 2.8]. Theorem 3.16 supplies actual unital trace-preserving embeddings $a_i : A_i \to E_0$ with the canonical markings. Positivity uses the range core's strict trace cone; no identification of the original cores' ordered K_0-groups is required.

Apply the finite-base setup with

$$N = A_1 \oplus A_2, \qquad B = \mathrm{M}_2, \qquad G = \mathbb{C}^2, \qquad E = \mathrm{M}_2(E_0), \qquad a = \mathrm{diag}(a_1, a_2),$$

and the standard embedding $b : \mathrm{M}_2 \to \mathrm{M}_2(E_0)$. The trace on N is half-weighted and G is the common full diagonal. The map in the amalgamated K-theory sequence has columns

$$(u_1, 0, -h), \qquad (0, u_2, -h) \quad \text{in } \mathrm{K}_0(A_1) \oplus \mathrm{K}_0(A_2) \oplus \mathbb{Z}h,$$

where h is the minimal matrix class. It is injective: the selected trace on each A_i forces the corresponding integer coefficient to vanish. Its cokernel is G_0, with $h = u$. Thus $\mathrm{K}_1(S) = 0$, the amplified unit is $2u$, and the liberated embedding η is a K-isomorphism with the required selected trace. In the amplified diagram the pairing is half the aggregate pairing, so it sends $2u$ to 1 and u to $1/2$. The full-to-reduced KK equivalence and natural coefficient sequences give its total K-marking; zero K_1 and the selected trace give the unitary/determinant part of that marking. Its full selected marking is therefore an isomorphism.

The working source targets are admissible by the infinite external reservoir and the standard matrix identity. The target and its finite tag extensions are admissible by finite-dimensional tag absorption. Theorem 9.6 applies. Writing $P = A_1 *_{\mathrm{r}} A_2$, the exact identities in Proposition 9.4 give

$$\mathrm{M}_2(P *_{\mathrm{r}} D) \cong \mathrm{M}_2(E_0 *_{\mathrm{r}} D).$$

Under the constructed marking the distinguished projection e_{11} has class *exactly* u on both sides. Indeed $\eta(e_{11}) = e_{11}$, since $e_{11} \in G$ and q commutes with G. Its class is not inferred just by halving the amplified unit $2u$; that inference would lose possible two-torsion. Stable-rank-one cancellation in the target, for the projection and its complement, gives a unitary correcting its image to the standard e_{11}. Compressing this actual isomorphism gives

$$(A_1 *_{\mathrm{r}} A_2) *_{\mathrm{r}} D \cong E_0 *_{\mathrm{r}} D$$

with the selected unamplified unit and pairing. □

9.7. **Binary fusion with control on the old stabilized core.**

Proposition 9.8 (Tail-controlled binary fusion)**.** *Let R, A be nuclear UCT cores with faithful selected traces and $\mathrm{K}_1(R) = \mathrm{K}_1(A) = 0$. Let E be a simple monotracial nuclear UCT range core for their aggregate selected pointed data, and choose an actual trace-preserving unital embedding $\lambda : R \to E$ inducing the canonical old-core K_0-map. For an interval reservoir D, finite $\mathcal{Y} \subset R *_{\mathrm{r}} D$ and $\varepsilon > 0$, there is a trace-preserving isomorphism*

$$\psi : (R *_{\mathrm{r}} A) *_{\mathrm{r}} D \longrightarrow E *_{\mathrm{r}} D$$

inducing the canonical aggregate marking and satisfying

$$\|\psi(y) - (\lambda *_{\mathrm{r}} \mathrm{id}_D)(y)\| < \varepsilon \qquad (y \in \mathcal{Y}). \tag{9.14}$$

It may fix any specified finite external reservoir head pointwise.

Proof. Approximate the elements of $\mathcal{Y}$ by free-product polynomials using finitely many R-coefficients and finitely many reservoir intervals. Include those intervals and the prescribed protected head in a finite head H, and write $D = H *_{\mathrm{r}} D_{\mathrm{tail}}$ with an infinite complementary

tail. Run Proposition 9.7 only in that tail, obtaining

$$\chi : (R *_{\mathrm{r}} A) *_{\mathrm{r}} D_{\mathrm{tail}} \xrightarrow{\cong} E *_{\mathrm{r}} D_{\mathrm{tail}}$$

with the canonical aggregate marking, using marked core replacement (Theorem 4.7) to select the displayed core E.

The maps $\chi|_R$ and the native $\lambda : R \to E \subset E *_{\mathrm{r}} D_{\mathrm{tail}}$ have the same full selected marking. Here their ordinary K_0-maps and selected traces agree, while both source and target have zero K_1. Writing $U = E *_{\mathrm{r}} D_{\mathrm{tail}}$, the degree-zero UCT extension terms are

$$\mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_1(R), \mathrm{K}_0(U)) \oplus \mathrm{Ext}^1_{\mathbb{Z}}(\mathrm{K}_0(R), \mathrm{K}_1(U)) = 0.$$

Thus their KK classes agree, the coefficient maps are forced by the Bockstein sequences, and the compatible unitary map is determined by trace and K_0. There is no additional old-core rotation choice. Nuclear-source uniqueness on R gives a unitary $w \in E *_{\mathrm{r}} D_{\mathrm{tail}}$ such that $\mathrm{Ad}(w)\chi$ is as close to λ as required on those finitely many R-coefficients.

Freely extend $\mathrm{Ad}(w)\chi$ by id_H and reassociate the scalar free products. This is the required actual isomorphism ψ. It fixes H exactly, and polynomial continuity gives (9.14), including the initial polynomial approximation errors. The inner correction is made before extending over H; it does not conjugate the protected reservoir coordinates. □

This is why finite-head protection suffices for countable coherence. Although uniqueness is applied only to the nuclear algebra R, it controls an arbitrary finite packet of $R *_{\mathrm{r}} D$: its finitely many reservoir coordinates are kept external to that uniqueness step.

9.8. **A two-sided finite-prefix construction.**

Proposition 9.9 (Countable zero-K_1 fusion)**.** *Let $((A_i, \tau_i))_{i\geq 1}$ be nuclear UCT cores with faithful selected traces and $\mathrm{K}_1(A_i) = 0$. For an interval reservoir D, there is a nuclear UCT core E_∞ with their aggregate selected pointed data and an actual trace-preserving isomorphism*

$$\left(\underset{i\geq 1}{*_{\mathrm{r}}} A_i\right) *_{\mathrm{r}} D \cong E_\infty *_{\mathrm{r}} D.$$

Proof. First use an interval reservoir. Put $P_n = A_1 *_{\mathrm{r}} \cdots *_{\mathrm{r}} A_n$. The free-product K-sequence gives

$$G_n = \mathrm{K}_0(P_n) = \frac{\bigoplus_{i\leq n} \mathrm{K}_0(A_i)}{\langle [1_{A_1}] - [1_{A_i}] : 2 \leq i \leq n \rangle}, \qquad \mathrm{K}_1(P_n) = 0.$$

The canonical maps $G_n \to G_{n+1}$ are injective. Indeed, in $(G_n \oplus \mathrm{K}_0(A_{n+1}))/\langle (u_n, -u_{n+1}) \rangle$, a class $(x, 0)$ can vanish only if $(x, 0) = k(u_n, -u_{n+1})$; applying the selected trace on the new factor gives $k = 0$.

Choose simple monotracial nuclear UCT $\mathcal{Z}$-stable range cores E_n for (G_n, u_n, ρ_n), with their strict trace cones and zero K_1. Theorem 3.16 supplies trace-preserving unital embeddings

$$\lambda_n : E_n \longrightarrow E_{n+1}$$

inducing the canonical maps of the selected pointed data. Their limit $E_\infty = \varinjlim(E_n, \lambda_n)$ is simple, monotracial, nuclear, UCT and $\mathcal{Z}$-stable. Continuity of ordinary K-theory gives the aggregate pointed groups and pairing; a class of positive trace is positive at a finite stage, so its order is the strict trace cone.

Set

$$S_n = P_n *_{\mathrm{r}} D, \qquad T_n = E_n *_{\mathrm{r}} D, \qquad i_n : S_n \hookrightarrow S_{n+1}, \qquad j_n = \lambda_n *_{\mathrm{r}} \mathrm{id}_D : T_n \hookrightarrow T_{n+1}.$$

These are actual trace-preserving embeddings. Start with a canonically marked isomorphism $\phi_1 : S_1 \to T_1$ given by nuclear-core replacement. Suppose ϕ_n has been constructed. Freely extending it by the identity on A_{n+1} gives an actual isomorphism

$$\phi_n *_{\mathrm{r}} \mathrm{id}_{A_{n+1}} : S_{n+1} \longrightarrow (E_n *_{\mathrm{r}} A_{n+1}) *_{\mathrm{r}} D.$$

For any prescribed finite $\mathcal{G}_n \subset T_n$, Proposition 9.8 gives an isomorphism

$$\psi_n : (E_n *_{\mathrm{r}} A_{n+1}) *_{\mathrm{r}} D \longrightarrow T_{n+1}$$

as close to j_n as required on $\mathcal{G}_n$. Define

$$\phi_{n+1} = \psi_n(\phi_n *_{\mathrm{r}} \mathrm{id}_{A_{n+1}}).$$

Thus every ϕ_n is an actual isomorphism onto its entire stage.

Here are explicit two-sided exhaustion conditions. Fix positive ε_n with $\sum_n \varepsilon_n < \infty$. After ϕ_n is known, choose finite $\mathcal{F}_n \subset S_n$ and put $\mathcal{G}_n = \phi_n(\mathcal{F}_n)$. Require

$$i_{n-1}(\mathcal{F}_{n-1}) \subset \mathcal{F}_n, \qquad j_{n-1}(\mathcal{G}_{n-1}) \subset \mathcal{G}_n. \tag{9.15}$$

Also include successively more points from a fixed countable dense subset of every earlier S_k, propagated by the i's, and of every earlier T_k, propagated by the j's. The second condition and all target dense points can be included simply by adding their ϕ_n-preimages to $\mathcal{F}_n$. Every stage has only finitely many requirements.

Choose ψ_n accurate within ε_n on $\mathcal{G}_n$. Then

$$\|\phi_{n+1} i_n(x) - j_n \phi_n(x)\| < \varepsilon_n \quad (x \in \mathcal{F}_n). \tag{9.16}$$

Applying the actual isometric inverse ϕ_{n+1}^{-1} gives the equally important reverse estimate

$$\|\phi_{n+1}^{-1} j_n(y) - i_n \phi_n^{-1}(y)\| < \varepsilon_n \quad (y \in \mathcal{G}_n). \tag{9.17}$$

The propagation conditions (9.15) and the summable estimates make the forward and inverse maps Cauchy on the two dense exhaustions. The coordinate maps are contractive homomorphisms, so convergence extends from these dense sets to pointwise convergence and preserves the algebraic operations. They extend to trace-preserving homomorphisms between the two limits. On a test point propagated to stage n, either composition differs from the identity by at most $2\sum_{k\ge n} \varepsilon_k$. Letting $n \to \infty$ proves that these maps are mutual inverses. This proves surjectivity, not just the existence of a limiting embedding.

Finally,

$$\varinjlim (S_n, i_n) = \left(*_{\mathrm{r}}\, A_i \right) *_{\mathrm{r}} D, \qquad \varinjlim (T_n, j_n) = E_\infty *_{\mathrm{r}} D.$$

These norm identifications follow directly from faithful reduced embeddings and density of finite free-product words. On the target side, the copies of E_n have dense union in E_∞; on the source side every word uses only finitely many factors. The constructed isomorphism has the canonical pointed groups and pairing, since all stage maps have those markings. □

9.9. **Arbitrary K_1 and the general fusion theorem.**

Theorem 9.10 (Fusion of finite and countable nuclear families)**.** *Let $((A_i, \tau_i))_{i\in J}$ be a nonempty finite or countable family of separable unital nuclear UCT algebras with faithful selected traces. Let F be a nuclear UCT core with faithful selected trace having the same selected pointed graded* K*-groups and pairing as $P = *_{\mathrm{r}i\in J} A_i$. For every reservoir D there is a trace-preserving isomorphism*

$$P*_{\mathrm{r}}D \cong F*_{\mathrm{r}}D.$$

Every prescribed compatible full marking can be realized, and a specified finite external reservoir head can be fixed pointwise.

Proof. For a finite zero-K_1 family, iterate Proposition 9.7 in independent banks. For a countable zero-K_1 family use Proposition 9.9. We now remove the K_1 restriction, writing $J = \mathbb{N}$ in the countable case; the same construction applies to a finite index set. Partition a working interval reservoir into mutually free infinite banks D_i and one spare infinite bank. For each i, choose nuclear range cores A_i^0, C_i with A_i^0 retaining the selected K_0, unit and pairing of A_i, $\mathrm{K}_1(A_i^0) = 0$, and $\mathrm{K}_0(C_i) = \mathbb{Z}[1_{C_i}]$, $\mathrm{K}_1(C_i) = \mathrm{K}_1(A_i)$. Theorem 8.1, applied to A_i^0, C_i with nuclear target A_i, supplies an actual marked isomorphism $(A_i^0 *_{\mathrm{r}} C_i) *_{\mathrm{r}} D_i \cong A_i *_{\mathrm{r}} D_i$. Invert it to obtain

$$A_i *_{\mathrm{r}} D_i \cong (A_i^0 *_{\mathrm{r}} C_i) *_{\mathrm{r}} D_i', \qquad \mathrm{K}_1(A_i^0) = 0, \quad \mathrm{K}_0(C_i) = \mathbb{Z}[1_{C_i}], \quad \mathrm{K}_1(C_i) = \mathrm{K}_1(A_i),$$

where A_i^0 retains the selected K_0, unit and pairing of A_i. Their countable reduced free product is an actual isomorphism: the trace-preserving isomorphisms give isometries on each finite free subproduct, and these finite subproducts have dense union. Regrouping the reservoir banks therefore gives

$$P*_{\mathrm{r}}D \cong \left(*_{\substack{\mathrm{r}\\ i\in J}} A_i^0\right) *_{\mathrm{r}} \left(*_{\substack{\mathrm{r}\\ i\in J}} C_i\right) *_{\mathrm{r}} D.$$

Fuse the A_i^0 in a spare bank, using Proposition 9.9 in the countable case and finite iteration of Proposition 9.7 otherwise. Extend the resulting actual isomorphism by the identity over every C_i. Theorem 8.1, including its countable assertion, now applies to the resulting one nuclear core and the family (C_i): only that one core may have nonsplit unit. This gives a nuclear-core reservoir model with the aggregate graded groups and pairing, and marked nuclear-core replacement gives $F*_{\mathrm{r}}D$.

For an arbitrary prescribed compatible global full marking, first take the actual fusion just constructed. Compose with the residual full-marking automorphism of $F*_{\mathrm{r}}D$, supplied by Theorem 4.7. We do not assume that a desired global marking splits factorwise, or that an arbitrary absolute KK representative is retained through the inductive limit.

Keep any originally protected finite reservoir head external before partitioning banks. All splittings, controlled binary steps, reservoir changes and final marking corrections are made in its complementary tail and freely extended by its identity. It is therefore fixed by the limiting isomorphism. Corollary 4.11 gives the stated reservoir class. For an original block head in a general reservoir, use the native tail-triple transport argument at the end of the proof of Theorem 8.1: isolate that head first, identify only its complementary reservoir tail

with the interval model, transport the desired marking through the invertible tail inclusions, and extend by the identity on the head. □

9.10. **Completion of the classification theorem.** We combine general fusion with marked nuclear-core replacement to complete the classification and morphism-realization assertions.

Theorem 9.11 (Nuclear factors: classification and morphism realization)**.** *Let D be a reservoir in the sense of Definition 2.1. Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty finite or countable families of separable unital nuclear UCT algebras equipped with faithful selected tracial states. Put*

$$A = P_{\mathcal{A}} *_{\mathrm{r}} D, \qquad B = P_{\mathcal{B}} *_{\mathrm{r}} D.$$

Each of A and B is isomorphic to $F *_{\mathrm{r}} D$ for a nuclear UCT core F with the same aggregate selected pointed graded K*-groups and pairing. Every unital morphism $\eta : \mathrm{Ell}(A) \to \mathrm{Ell}(B)$ is induced by a unital trace-preserving embedding $\Phi : A \to B$. If η is an isomorphism, Φ may be chosen to be a *-isomorphism. In particular,*

$$A \cong B \quad \Longleftrightarrow \quad \mathrm{Ell}(A) \cong \mathrm{Ell}(B).$$

If each family contains two distinct factors admitting Haar unitaries, the same classification and morphism-realization assertions hold for $P_{\mathcal{A}}$ and $P_{\mathcal{B}}$ without adjoining D.

Proof. Proposition 2.5 gives countable aggregate pointed groups (G_0, u, G_1) and a selected pairing $\rho : G_0 \to \mathbb{R}$ with $\rho(u) = 1$ for each family. Equip G_0 with the strict trace cone $\{0\} \cup \{g : \rho(g) > 0\}$. As in the proof of Lemma 4.1, this is a simple weakly unperforated scaled ordered group with unique normalized state ρ. The nuclear range theorem [19, Theorem 2.8] supplies a simple, infinite-dimensional, unital, monotracial, nuclear UCT, $\mathcal{Z}$-stable core with these data. For the exceptional data $(\mathbb{Z}, m, 0)$ one may use $\mathrm{M}_m(\mathcal{Z})$, as in that lemma. Theorem 9.10 therefore gives trace-preserving isomorphisms

$$\Theta_A : A \longrightarrow F_{\mathcal{A}} *_{\mathrm{r}} D, \qquad \Theta_B : B \longrightarrow F_{\mathcal{B}} *_{\mathrm{r}} D$$

with the chosen aggregate markings. This proves the core-model assertion.

Given η, transport it to the nuclear-core stabilizations:

$$\widetilde{\eta} = \mathrm{Ell}(\Theta_B) \circ \eta \circ \mathrm{Ell}(\Theta_A)^{-1}.$$

The canonical core inclusions identify the graded K-groups, units and selected pairings of these stabilizations with those of $F_{\mathcal{A}}$ and $F_{\mathcal{B}}$, by Proposition 2.5. Thus the graded part of $\widetilde{\eta}$ consists of unit- and selected-pairing-preserving homomorphisms between the two cores. Theorem 4.7 realizes these maps by a unital trace-preserving embedding

$$\Psi : F_{\mathcal{A}} *_{\mathrm{r}} D \longrightarrow F_{\mathcal{B}} *_{\mathrm{r}} D.$$

Its Elliott-invariant morphism is $\widetilde{\eta}$: the graded maps agree by construction, and the trace map between the one-point trace simplices is unique. Consequently

$$\Phi = \Theta_B^{-1} \circ \Psi \circ \Theta_A : A \longrightarrow B$$

induces η. If η is an isomorphism, both transported group maps are isomorphisms, so Theorem 4.7 allows Ψ, and hence Φ, to be chosen as a *-isomorphism. The converse implication in the classification assertion follows from functoriality of the Elliott invariant.

Finally, suppose each family has two distinct factors admitting Haar unitaries. Group those factors on opposite sides of the respective products. Semicircular absorption [45, Theorem 5.2], after the interval identification of Corollary 4.11, gives trace-preserving isomorphisms

$$P_{\mathcal{A}} \cong P_{\mathcal{A}} *_{\mathrm{r}} D, \qquad P_{\mathcal{B}} \cong P_{\mathcal{B}} *_{\mathrm{r}} D.$$

Transport a prescribed invariant morphism through these isomorphisms, apply the assertion just proved, and compose back. For an invariant isomorphism, choose the realizing map to be an isomorphism. This proves the unstabilized assertions. □

Corollary 9.12 (Closure under unital strong Morita equivalence)**.** *The class in Theorem 9.11 is closed, up to unital isomorphism, under strong Morita equivalence among unital* C*-*algebras. More precisely, let* $A = P_{\mathcal{A}} *_{\mathrm{r}} D$ *be in that class, let* $n \geq 1$*, and let* $p \in \mathrm{M}_n(A)$ *be a nonzero projection. There is a simple, infinite-dimensional, unital, monotracial, nuclear UCT,* $\mathcal{Z}$*-stable algebra* E *such that*

$$p\mathrm{M}_n(A)p \cong E *_{\mathrm{r}} D.$$

The isomorphism preserves the normalized corner trace. Under the Morita identifications of the ordered K*-groups, if* $g = [p] \in \mathrm{K}_0(A)$ *and* $\rho = (\tau_A)_*$*, the corner has unit class* g *and trace pairing* $\rho/\rho(g)$*.*

Proof. As in the proof of Theorem 9.11, choose a simple, infinite-dimensional, unital, monotracial, nuclear UCT, $\mathcal{Z}$-stable core N and a trace-preserving isomorphism $A \cong N *_{\mathrm{r}} D$. We use this identification throughout. The core inclusion $\iota : N \to A$ induces an isomorphism on K_0, preserving the trace pairing, by Proposition 2.5. Consequently

$$h = \iota_*^{-1}([p]) \in \mathrm{K}_0(N), \qquad (\tau_N)_*(h) = (\mathrm{Tr}_n \otimes \tau_A)(p) > 0.$$

The order on $\mathrm{K}_0(N)$ is the strict trace order. Indeed, strict comparison in N gives this assertion by the projection argument of Lemma 2.3. Thus h is represented by a projection $q \in \mathrm{M}_l(N)$ for some l.

The images of p and q have the same $\mathrm{K}_0(A)$ class. Stable rank one of A gives cancellation of projections, so, after zero padding, they are Murray–von Neumann equivalent in $\mathrm{M}_{2^k}(A)$ for some k. In particular,

$$p\mathrm{M}_n(A)p \cong q\mathrm{M}_{2^k}(A)q.$$

Iterating Lemma 8.4 gives a trace-preserving isomorphism

$$\mathrm{M}_{2^k}(N *_{\mathrm{r}} D) \cong \mathrm{M}_{2^k}(N) *_{\mathrm{r}} D$$

fixing $\mathrm{M}_{2^k}(N)$ pointwise. The projection q is full in the simple algebra $\mathrm{M}_{2^k}(N)$. Lemma 8.5 therefore gives

$$q\mathrm{M}_{2^k}(A)q \cong q\Big(\mathrm{M}_{2^k}(N) *_{\mathrm{r}} D\Big)q \cong \Big(q\mathrm{M}_{2^k}(N)q\Big) *_{\mathrm{r}} D.$$

Take $E = q\mathrm{M}_{2^k}(N)q$. This full corner retains the listed properties of N, and its trace is the normalized restriction of $\mathrm{Tr}_{2^k} \otimes \tau_N$. All displayed isomorphisms preserve the normalized corner traces.

For completeness, every unital algebra strongly Morita equivalent to A is a full corner of a finite matrix algebra over A; compare [3, Theorem 1.2]. Indeed, its imprimitivity module is finitely generated projective as a right A-module, since its algebra of compact operators

is unital. The preceding argument therefore proves the asserted closure. Finally, the Morita identification sends the corner unit to $[p]$, and its normalized trace is

$$\tau_p(x) = \frac{(\mathrm{Tr}_n \otimes \tau_A)(x)}{(\mathrm{Tr}_n \otimes \tau_A)(p)}.$$

This gives the stated pairing $\rho/\rho(g)$, with the positive cone and both K-groups unchanged under those identifications. □

Corollary 9.13 (Commutative factors: classification and morphism realization)**.** *Let D be a reservoir. Let $\mathcal{X}$ and $\mathcal{Y}$ be nonempty finite or countable families of nonempty compact metrizable spaces equipped with full-support Borel probability measures. Put $A = P_{\mathcal{X}} *_{\mathrm{r}} D$ and $B = P_{\mathcal{Y}} *_{\mathrm{r}} D$. Every unital morphism $\mathrm{Ell}(A) \to \mathrm{Ell}(B)$ is induced by a unital trace-preserving embedding $A \to B$. If the invariant morphism is an isomorphism, the realizing map may be chosen to be a *-isomorphism. In particular,*

$$A \cong B \quad \Longleftrightarrow \quad \mathrm{Ell}(A) \cong \mathrm{Ell}(B).$$

If each family has two distinct factors admitting Haar unitaries, these assertions also hold without adjoining D. In particular, this applies when each family contains at least two nonatomic measures.

Proof. For a nonempty compact metrizable space X, the algebra $\mathrm{C}(X)$ is separable, unital, nuclear and UCT. A full-support probability measure defines a faithful tracial state. The classification and morphism-realization assertions therefore follow from Theorem 9.11, including the assertion without D under the Haar-unitary hypothesis. If each family has two nonatomic measures, Lemma 2.6 supplies Haar unitaries in the corresponding two distinct factors of each product. □

Remark 9.14 (Commutative factors and concrete models). Theorem 9.10 includes arbitrary finite or countable mixtures of faithfully tracial nuclear UCT factors. For commutative factors, evaluation at a point splits the unit; their sharper invariant range and standard models are described in Section 10.

The AF and real-rank-zero constructions in Subsection 10.3 give concrete models for selected pointed data. An AF product may have torsion even though each individual AF factor has torsion-free K_0. These model descriptions are useful in examples, while general fusion permits arbitrary unit classes and both K-groups. The matrix enlargement in its proof retains the exact minimal projection class before compression, including its torsion part. Section 10 gives standard free-product models for the whole nuclear-factor class by adding homogeneous Moore-space corners to the commutative list.

Remark 9.15 (Arbitrary inductive limits need not be classified). The classification theorem does not extend to arbitrary inductive limits of the free products considered here, even when the connecting maps are unital, injective and trace preserving and the limits absorb D. Take D in its interval presentation. Let $U = \mathrm{M}_{2^\infty}$, with its unique trace, and use the minimal tensor product to define

$$X = (D \otimes U) *_{\mathrm{r}} D, \qquad Y = U *_{\mathrm{r}} D.$$

For $n \geq 1$, put

$$E_n = \mathrm{M}_{2^n}(D) *_{\mathrm{r}} D, \qquad F_n = \mathrm{M}_{2^n}(\mathbb{C}) *_{\mathrm{r}} D.$$

Let $\alpha_n : \mathrm{M}_{2^n}(D) \to \mathrm{M}_{2^{n+1}}(D)$ be $x \mapsto \mathrm{diag}(x, x)$, and let μ_n be the same embedding on scalar matrices. Taking their free products with id_D gives

$$X \cong \varinjlim(E_n, \alpha_n *_{\mathrm{r}} \mathrm{id}_D), \qquad Y \cong \varinjlim(F_n, \mu_n *_{\mathrm{r}} \mathrm{id}_D).$$

Indeed, the nested core algebras have dense unions in $D \otimes U$ and U, respectively, and finite reduced words give the asserted continuity. Equation (9.5) and reindexing $D *_{\mathrm{r}} D \cong D$ give trace-preserving isomorphisms

$$E_n \cong \mathrm{M}_{2^n}(D *_{\mathrm{r}} D) \cong \mathrm{M}_{2^n}(D) \cong F_n.$$

In particular, all stages belong to the class of Theorem 9.11. Both invariant diagrams have $\mathrm{K}_0 = \mathbb{Z}$, positive cone $\mathbb{Z}_{\geq 0}$, unit 2^n, pairing $m \mapsto m/2^n$, zero K_1, and connecting K_0-maps $m \mapsto 2m$. The stages, and hence their limits, are simple and monotracial. Continuity of ordered K-theory therefore gives

$$\mathrm{Ell}(X) \cong \mathrm{Ell}(Y) \cong \Big(\mathbb{Z}[1/2], \mathbb{Z}[1/2]_{\geq 0}, 1, 0, \{\tau\}, \rho\Big), \qquad \rho(q) = q.$$

Moreover, associativity gives $X *_{\mathrm{r}} D \cong X$ and $Y *_{\mathrm{r}} D \cong Y$.

Nevertheless, $X \ncong Y$. In the respective tracial GNS representations, write $M = D'' \cong L(\mathbb{F}_\infty)$ and $R = U''$, the hyperfinite II_1 factor. Their von Neumann closures are

$$X'' \cong (M \mathbin{\overline{\otimes}} R) * M, \qquad Y'' \cong R * M \cong L(\mathbb{F}_\infty),$$

where the last identification is [13, Theorem 4.1]. Recall that a finite von Neumann algebra is *solid* if the relative commutant of every diffuse von Neumann subalgebra is amenable. The factor Y'' is solid by [35, Theorem 1 and Proposition 2]; here $\mathbb{F}_\infty$ is a subgroup of $\mathbb{F}_2$. In X'', however, the diffuse algebra $1 \otimes R$ in the first free factor has relative commutant containing the nonamenable algebra $M \otimes 1$. Every von Neumann subalgebra of a finite von Neumann algebra admits a trace-preserving conditional expectation, so this relative commutant cannot be amenable. Thus X'' is not solid. A C*-isomorphism $X \cong Y$ would preserve the unique traces and extend to an isomorphism of these von Neumann closures, a contradiction.

Thus isomorphic stages and isomorphic Elliott-invariant diagrams do not determine the limit, even with free D-absorption. By Corollary 10.12, the stages may even be replaced by finite standard free products, transporting the connecting maps through the chosen isomorphisms. The connecting maps in Theorem 10.11 are chosen more specifically: after marked identifications $C_n \cong N_n *_{\mathrm{r}} D$, they have the form $\nu_n *_{\mathrm{r}} \mathrm{id}_D$ with nuclear core maps $\nu_n : N_n \to N_{n+1}$. This identifies that limit as $(\varinjlim N_n) *_{\mathrm{r}} D$ and permits application of the nuclear-core theorem.

10. Range and models

We first give the full range for nuclear cores and the products of Theorem 9.11, then determine the sharper commutative range. We then describe models built from AF factors and construct standard free-product models for the whole class, retaining commutative models for the commutative subclass.

10.1. **The general nuclear range.**

Theorem 10.1 (Range for nuclear cores)**.** *The Elliott invariants of the D-stabilized products in Theorem 9.11 are precisely*

$$\big(G_0, G_0^+, u, G_1, \{\tau\}, \rho\big), \qquad G_0^+ = \{0\} \cup \{g : \rho(g) > 0\},$$

where G_0, G_1 are countable abelian groups, $u \in G_0$, and $\rho : G_0 \to \mathbb{R}$ is a homomorphism with $\rho(u) = 1$. Every such invariant is already attained, without adjoining D, by a reduced free product of two simple infinite-dimensional monotracial nuclear UCT $\mathcal{Z}$-stable algebras, one of which is $\mathcal{Z}$. The unit of the other algebra need not split.

Proof. Necessity follows from separability and Proposition 2.5. Conversely, the displayed strict trace cone makes G_0 a simple weakly unperforated directed ordered group, with order unit u. Its unique normalized state is ρ: rational upper and lower comparisons with u determine the value of every element. The range theorem [19, Theorem 2.8] provides a simple infinite-dimensional monotracial nuclear UCT $\mathcal{Z}$-stable algebra F with these pointed groups and pairing. The exceptional data $(G_0, u, G_1) = (\mathbb{Z}, m, 0)$, $m \geq 1$, can be realized explicitly by $\mathrm{M}_m(\mathcal{Z})$.

The free-product K-sequence identifies the pointed groups and pairing of $F *_{\mathrm{r}} \mathcal{Z}$ with those of F. Both factors contain Haar unitaries by [47, Corollary B], since they are simple, unital and infinite dimensional. Semicircular absorption therefore gives $F *_{\mathrm{r}} \mathcal{Z} \cong (F *_{\mathrm{r}} \mathcal{Z}) *_{\mathrm{r}} D$. The latter algebra has the displayed Elliott invariant by Proposition 2.5. The representatives $F *_{\mathrm{r}} \mathcal{Z}$ belong to the class of Theorem 9.11 and realize every listed invariant. This proves its range assertion as well. □

10.2. **The commutative range.** For commutative factors, rank imposes a restriction on the selected pairing beyond splitting the unit.

Lemma 10.2 (Rank decomposition)**.** *Let X be a nonempty compact metrizable space. There are a finite or countable set J_X and a countable abelian group H_X such that*

$$\mathrm{K}_0(\mathrm{C}(X)) \cong \mathbb{Z}[1] \oplus \mathbb{Z}^{(J_X)} \oplus H_X, \qquad H_X = \ker(\mathrm{rk}),$$

where $\mathrm{rk} : \mathrm{K}_0(\mathrm{C}(X)) \to \mathrm{C}(X, \mathbb{Z})$ is the pointwise rank map. For every Borel probability measure μ on X, its pairing vanishes on H_X and is integration on the first two summands, identified with $\mathrm{C}(X, \mathbb{Z})$.

Proof. The rank map has the canonical group section

$$f \longmapsto \sum_{k \in f(X)} k[1_{f^{-1}(k)}].$$

Here $f(X)$ is finite, and the characteristic functions are projections in $\mathrm{C}(X)$. Thus

$$\mathrm{K}_0(\mathrm{C}(X)) \cong \mathrm{C}(X, \mathbb{Z}) \oplus \ker(\mathrm{rk}).$$

There are only countably many clopen subsets of X: every such subset is compact and therefore is a finite union of elements of a fixed countable base. Enumerating them gives nested finite clopen partitions $\mathcal{P}_n$, starting with $\mathcal{P}_0 = \{X\}$, whose locally constant integer-valued functions exhaust $\mathrm{C}(X, \mathbb{Z})$. A refinement replaces the characteristic function of each old atom by the sum of those of its nonempty subatoms. This is a split inclusion between

finitely generated free groups, with free cokernel: in each refinement block the diagonal vector $(1,\dots,1)$ extends to an integral basis. Choosing a complement at every step therefore gives

$$\mathrm{C}(X,\mathbb{Z}) = \mathbb{Z}1_X \oplus \mathbb{Z}^{(J_X)}.$$

Finally, for a projection p in a matrix algebra over $\mathrm{C}(X)$,

$$(\tau_\mu \otimes \mathrm{Tr})(p) = \int_X \mathrm{rank}(p(x))\, d\mu(x).$$

This proves the pairing assertion. All groups are countable because $\mathrm{C}(X)$ is separable. □

Lemma 10.3 (Connected pointed range)**.** *Let H_0, H_1 be countable abelian groups. There is a connected compact metrizable space Y of dimension at most four, a point $y_0 \in Y$, and isomorphisms*

$$\mathrm{K}_0(\mathrm{C}(Y)) \cong \mathbb{Z}[1] \oplus H_0, \qquad \mathrm{K}_1(\mathrm{C}(Y)) \cong H_1,$$

under which evaluation at y_0 is $(k,h) \mapsto k$. Consequently every probability measure on Y pairs by this same rank map. The space can be chosen nontrivial and hence admits a full-support nonatomic Borel probability measure. If $H_0 = 0$, the dimension may be taken at most three.

Proof. We record the pointed construction to specify the unit and rank splitting. The existence of finite-dimensional connected spaces with arbitrary prescribed countable K-groups is also used at the beginning of [11, Section 6].

For a finitely generated abelian group $L = \mathbb{Z}^r \oplus \bigoplus_a \mathbb{Z}/n_a\mathbb{Z}$, let

$$W(L) = \bigvee_{j=1}^{r} S^3 \;\vee\; \bigvee_a M(\mathbb{Z}/n_a\mathbb{Z}, 2), \qquad M(\mathbb{Z}/n\mathbb{Z}, 2) = S^2 \cup_{\deg n} D^3.$$

Take $W(0)$ to be a point. We use complex topological K-theory in this proof. These finite complexes satisfy $\widetilde{\mathrm{K}}_0(W(L)) = 0$ and $\mathrm{K}_1(W(L)) \cong L$. Every homomorphism $\alpha : L \to L'$ is induced contravariantly by a pointed continuous map $W(L') \to W(L)$. Here is an explicit verification. On a spherical summand of the domain, degree maps and the pinch map realize all the required free coordinates. On a summand $M(\mathbb{Z}/m\mathbb{Z}, 2)$, collapsing its two-skeleton gives the reduction map $\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ from a target S^3-summand. A homomorphism $\mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/m\mathbb{Z}$ is multiplication by an integer b with $nb = ma$ for some integer a. Degree a on the lower two-sphere and degree b on the attaching two-sphere give a homotopy-commutative square of the degree-m and degree-n maps. The resulting map of their mapping cones induces the required homomorphism. Finally, every domain summand is a suspension, so its suspension pinch map combines the maps to the separate target summands; on reduced K-theory the resulting homomorphisms add.

Choose increasing finitely generated subgroups $H_{i,n}$ with union H_i, for $i = 0, 1$. Realize their inclusions by pointed maps $W(H_{i,n+1}) \to W(H_{i,n})$ as above, and form the inverse system

$$Y_n = \Sigma W(H_{0,n}) \vee W(H_{1,n}) \vee [0,1],$$

with the suspended maps on the first summand and the identity on the last. Here Σ is reduced suspension and 0 is the basepoint of the interval. Each Y_n is connected, is at most four-dimensional, and has pointed groups $\mathrm{K}_0(Y_n) = \mathbb{Z}[1] \oplus H_{0,n}$ and $\mathrm{K}_1(Y_n) = H_{1,n}$. All

bonding maps preserve the basepoint and fix the interval. Their inverse limit Y is therefore nontrivial, connected, compact, metrizable and at most four-dimensional; surjectivity of the bonding maps is not needed. Indeed, in $\prod_{n\geq 1} Y_n$ the set satisfying only the first m bonding relations is homeomorphic to $Y_{m+1} \times \prod_{k>m+1} Y_k$ and is compact and connected. Their nested intersection is Y, so it is connected; the fixed interval makes it nontrivial. Finite covers refine to pullbacks from a finite coordinate image, giving the dimension bound. When $H_0 = 0$, the suspended summand is absent and each Y_n, and hence Y, has dimension at most three. The canonical map from the C*-inductive limit of the pullback system $\mathrm{C}(Y_n)$ to $\mathrm{C}(Y)$ is an isomorphism even when the bonding maps are not surjective: the coordinate functions are dense by Stone–Weierstrass, and the decreasing compact coordinate images give equality of norms. Continuity of K-theory therefore gives the displayed pointed groups. Since Y is connected, the rank of a projection is constant, proving the pairing assertion. Finally, Y has no isolated points. For a countable basis $(U_j)_{j\geq 1}$ of nonempty open sets, choose a Cantor subset of each U_j and an atomless probability measure μ_j supported on it. Then $\sum_{j\geq 1} 2^{-j}\mu_j$ is a full-support nonatomic probability measure on Y. □

Theorem 10.4 (Range of the Elliott invariant)**.** *The Elliott invariants of the algebras $P_{\mathcal{X}} *_{\mathrm{r}} D$ in Corollary 9.13 are exactly the following invariants, up to isomorphism:*

$$\mathrm{K}_0 = \mathbb{Z}u \oplus \mathbb{Z}^{(J)} \oplus H_0, \qquad \mathrm{K}_1 = H_1, \qquad [1] = u, \qquad \mathrm{T} = \{\tau\}, \tag{10.1}$$

$$\rho\Big(mu + (n_j), h\Big) = m + \sum_{j\in J} a_j n_j, \qquad \mathrm{K}_0^+ = \{0\} \cup \{g : \rho(g) > 0\}. \tag{10.2}$$

Here J is finite or countable, H_0, H_1 are arbitrary countable abelian groups, and the a_j are arbitrary real numbers. Equivalently, K_0 has a decomposition $F \oplus H_0$ with F free abelian, the unit belonging to a basis of F, and $\rho(H_0) = 0$; apart from $\rho(u) = 1$, the homomorphism on F is unrestricted.

Every invariant in this list is realized by a finite or countable reduced free product containing two Lebesgue-interval factors, with all commutative spectra of dimension at most four. Thus the range is the same for the D-stabilized class and the Haar-unitary class in Corollary 9.13.

Proof. For necessity, apply Lemma 10.2 to each commutative factor. Identifying their unit classes in Proposition 2.5 gives

$$\mathrm{K}_0(P_{\mathcal{X}}) = \mathbb{Z}u \oplus \bigoplus_i \mathbb{Z}^{(J_{X_i})} \oplus \bigoplus_i H_{X_i}.$$

The middle summand is countable free abelian and the pairing vanishes on the last. The degree-one group is countable, and adjoining D preserves the groups, unit and pairing. Regularity and Lemma 2.3 give the positive cone and the one-point trace simplex.

Conversely, fix the displayed data. By Lemma 10.3, choose a connected compact metrizable Y with pointed groups $(\mathbb{Z}[1] \oplus H_0, H_1)$ and rank pairing, and give it a full-support nonatomic probability measure. Its unit is marked by u, its rank-zero summand by H_0, and its degree-one group by H_1. For the standard generator e_j of $\mathbb{Z}^{(J)}$, put

$$q_j = \lfloor a_j \rfloor, \qquad s_j = a_j - q_j \in [0, 1).$$

If $s_j > 0$, take the space $[0, 1] \sqcup [0, 1]$ with component masses $s_j, 1 - s_j$ and Lebesgue measure within each component. Mark the first-component projection by $e_j - q_j u$. If $s_j = 0$, take

S^2 with any full-support nonatomic measure and mark its rank-zero Bott class by $e_j - q_j u$. These choices realize exactly the required trace values. The classes u and $e_j - q_j u$, together with H_0, still give the prescribed direct-sum decomposition.

Form P from the factor $\mathrm{C}(Y)$, these finite or countably many additional factors, and two Lebesgue-interval factors. Proposition 2.5 computes the data of $P *_{\mathrm{r}} D$ as (10.1)–(10.2). The two interval factors contain Haar unitaries, so semicircular absorption gives $P \cong P *_{\mathrm{r}} D$. This proves realization in both classes. All spectra used have dimension at most four. □

Example 10.5 (A split unit is not sufficient)**.** Consider $\mathbb{Z}u \oplus \mathbb{Z}[1/2]$ with unit u and normalized pairing

$$\rho(mu, q) = m + q.$$

This pointed group and pairing cannot occur in Corollary 9.13. Indeed, every homomorphism from $\mathbb{Z}[1/2]$ to a free abelian group is zero, since its image lies in $\bigcap_{n\geq 1} 2^n F = \{0\}$. Consequently the pairing in Theorem 10.4 must vanish on this subgroup, whereas the displayed pairing does not. Theorem 10.1 realizes these same pointed data in the larger class with nuclear factors.

10.3. **Models built from AF factors and split-unit fusion.** We say that (A, τ_A) *admits a model built from AF factors* if its pointed graded K-groups and selected trace pairing agree with those of a nonempty finite or countable reduced free product of faithfully tracial separable unital AF algebras. The original order and other traces are not prescribed. An AF-product model may be nonnuclear. The replacement statements below start from nuclear UCT factors.

Corollary 10.6 (AF factors and an arbitrary nuclear root)**.** *Let R be a nuclear UCT core with faithful selected trace, and let $((B_i, \tau_i))$ be a finite or countable family of faithfully tracial unital AF algebras. If F is a nuclear UCT core carrying the aggregate selected pointed data, then*

$$(R *_{\mathrm{r}} (*_{\mathrm{r}i} B_i)) *_{\mathrm{r}} D \cong F *_{\mathrm{r}} D.$$

Every prescribed compatible full marking is realized, and a finite external reservoir head may be fixed pointwise.

Proof. Apply Theorem 9.10. The empty AF family is Theorem 4.7. □

Proposition 10.7 (Replacement by AF products)**.** *Let $\mathcal{A} = ((A_i, \tau_i))_{i\in S}$ be a nonempty finite or countable family of nuclear UCT cores with faithful selected traces. Suppose each A_i admits a model built from AF factors, and choose such a model*

$$R_i = *_{\mathrm{r}j\in J_i}(F_{ij}, \sigma_{ij})$$

with a specified unit- and selected-pairing-preserving isomorphism of graded K*-groups from A_i to R_i. For every reservoir D and every faithfully tracial separable unital algebra Q there is a trace-preserving isomorphism*

$$(P_{\mathcal{A}} *_{\mathrm{r}} Q) *_{\mathrm{r}} D \cong ((*_{\mathrm{r}i\in S} R_i) *_{\mathrm{r}} Q) *_{\mathrm{r}} D. \tag{10.3}$$

It fixes Q pointwise and induces the specified maps on the canonical factor K*-classes. A prescribed finite reservoir prefix may also be fixed pointwise. The flattened family $(F_{ij})_{i\in S,\, j\in J_i}$ is finite or countable.*

Proof. First use the interval reservoir and partition its coordinates into disjoint infinite banks so that

$$D_I = D_* *_{\mathrm{r}} (*_{\mathrm{r}\, i\in S} D_i)\,.$$

Keep the prescribed finite head among the coordinates of D_*. For each i, Corollary 10.6 supplies a nuclear UCT range core E_i and a marked trace-preserving isomorphism $R_i *_{\mathrm{r}} D_i \cong E_i *_{\mathrm{r}} D_i$. The selected data of A_i and E_i agree. Theorem 4.7 therefore gives a marked trace-preserving isomorphism $A_i *_{\mathrm{r}} D_i \cong E_i *_{\mathrm{r}} D_i$. Choose the markings so that their composition gives

$$\theta_i : A_i *_{\mathrm{r}} D_i \longrightarrow R_i *_{\mathrm{r}} D_i$$

with the specified graded map on A_i.

Take the reduced free product of all the actual isomorphisms θ_i, together with the identities on Q and D_*. Associativity identifies its source and target as

$$(*_{\mathrm{r}\, i\in S}(A_i *_{\mathrm{r}} D_i)) *_{\mathrm{r}} Q *_{\mathrm{r}} D_* \cong (P_{\mathcal{A}} *_{\mathrm{r}} Q) *_{\mathrm{r}} D_I,$$
$$(*_{\mathrm{r}\, i\in S}(R_i *_{\mathrm{r}} D_i)) *_{\mathrm{r}} Q *_{\mathrm{r}} D_* \cong ((*_{\mathrm{r}\, i\in S} R_i) *_{\mathrm{r}} Q) *_{\mathrm{r}} D_I.$$

Trace preservation identifies the reduced completions, and the free product of the inverse maps is the inverse. In the countable case this is a single countable free product of actual isomorphisms; no infinite composition or limiting choice of corrections is used. The free-product K-sequence gives the stated factor markings. The algebra Q and the head in D_* are fixed exactly.

For a general reservoir, keep its protected finite prefix external and use Corollary 4.11 to identify the complementary tail with D_I. Apply the preceding construction there and extend by the identity on the prefix. This proves all the assertions. □

Proposition 10.8 (Models with real rank zero and torsion-free K_0)**.** *Let (A,τ) be a separable unital nuclear UCT algebra with faithful selected trace. Suppose its selected pointed data agree with those of a separable unital nuclear UCT algebra (B,τ_B) with faithful trace, real rank zero and torsion-free $\mathrm{K}_0(B)$. There are a faithfully tracial unital AF algebra F and a nuclear UCT core C with faithful selected trace such that*

$$(\mathrm{K}_0(F),[1_F],\tau_{F*}) \cong (\mathrm{K}_0(A),[1_A],\tau_*), \qquad \mathrm{K}_1(F)=0,$$
$$\mathrm{K}_0(C)=\mathbb{Z}[1_C], \qquad \mathrm{K}_1(C)=\mathrm{K}_1(A).$$

For every reservoir D there is a trace-preserving isomorphism

$$A *_{\mathrm{r}} D \cong (F *_{\mathrm{r}} C) *_{\mathrm{r}} D \tag{10.4}$$

with the corresponding selected-data marking. Every compatible full marking may be realized, and a finite reservoir prefix may be fixed pointwise.

Proof. Identify the selected data of B with those of A. For finite-dimensional B, the selected data give $\mathrm{K}_1(A)=0$. Take $F=B$ and $C=\mathbb{C}$; Theorem 4.7 gives the assertion. Assume now that B is infinite dimensional. Real rank zero gives, for every N, mutually orthogonal nonzero projections $p_1,\dots,p_N$ in B: take orthogonal nonzero positive elements and a nonzero projection in each of their hereditary subalgebras. Faithfulness gives $\tau_B(p_j)>0$, while $\sum_j \tau_B(p_j)\le 1$. Thus the common selected trace range has arbitrarily small positive elements and is dense in $\mathbb{R}$.

Put $G = \mathrm{K}_0(A)$, $u = [1_A]$ and $\rho = \tau_*$, and give G the cone

$$G^+ = \{0\} \cup \{g : \rho(g) > 0\}.$$

This is a directed unperforated ordered group with order unit u; unperforation uses torsion-freeness also when $ng = 0$. For interpolation, suppose $a_i \le b_j$ for $i, j = 1, 2$. If $\max_i \rho(a_i) < \min_j \rho(b_j)$, density supplies an element with trace strictly between the two bounds. If the bounds agree, a maximizing a_i and a minimizing b_j have equal trace and are comparable, hence are equal; this common element interpolates. The countable AF range theorem [17] therefore gives a unital AF algebra F with this ordered pointed group and a faithful trace inducing ρ. Indeed the displayed order is simple and every normalized state equals ρ, by rational comparisons with u, so F may be chosen simple and monotracial. The original order and other traces of A are not part of the selected data.

The nuclear range theorem [19, Theorem 2.8] supplies C with the displayed groups and pairing $\tau_{C*}(k[1_C]) = k$. The scalar unit relation gives

$$\mathrm{K}_0(F *_{\mathrm{r}} C) = \frac{G \oplus \mathbb{Z}[1_C]}{\langle u - [1_C] \rangle} \cong G, \qquad \mathrm{K}_1(F *_{\mathrm{r}} C) = \mathrm{K}_1(A),$$

with unit u and selected pairing ρ. Since C has split unit, Theorem 8.1 gives $(F *_{\mathrm{r}} C) *_{\mathrm{r}} D \cong E *_{\mathrm{r}} D$ for a nuclear range core E with these data. Theorem 4.7 compares $A *_{\mathrm{r}} D$ with $E *_{\mathrm{r}} D$, proving (10.4) with the specified markings and finite-prefix control. □

10.4. **Commutative standard blocks.** We first describe the commutative blocks and their finite realization property. For general nuclear factors we will add homogeneous corners over finite wedges of Moore spaces. The spaces needed at each finite stage come from fixed types, although the connecting homomorphisms need not come from maps between the spaces. Put

$$T_{\mathrm{II},n} = S^1 \cup_{z \mapsto z^n} D^2, \qquad T_{\mathrm{III},n} = S^2 \cup_{\deg n} D^3 \quad (n \ge 2).$$

These are the usual two- and three-dimensional Moore spaces, introduced as standard model spaces for the classification of AH algebras in [18, Section 4.2]. In particular, $T_{\mathrm{III},n}$ is not a three-dimensional real projective space. Our list consists of

$$[0,1], \quad [0,1] \sqcup [0,1], \quad S^1, \quad S^2, \quad T_{\mathrm{II},n}, \quad T_{\mathrm{III},n} \quad (n \ge 2). \tag{10.5}$$

For the connected spaces, use the complement to the unit given by the kernel of evaluation at a point. For the disconnected interval space, use the complement generated by the characteristic function e of its first component. The last column records the pairing on these chosen complements, using the displayed generator lifts:

X	$\mathrm{K}_0/\mathbb{Z}[1]$	K_1	ρ on the complement
$[0,1]$	0	0	0
$[0,1] \sqcup [0,1]$	$\mathbb{Z}[e]$	0	$[e] \mapsto t$
S^1	0	$\mathbb{Z}$	0
S^2	$\mathbb{Z}$	0	0
$T_{\mathrm{II},n}$	$\mathbb{Z}/n\mathbb{Z}$	0	0
$T_{\mathrm{III},n}$	0	$\mathbb{Z}/n\mathbb{Z}$	0

Here $t \in (0,1)$ is the mass of the first interval. Every space has a full-support nonatomic probability measure; on a disconnected interval space use normalized Lebesgue measure on each component with the indicated masses. For connected spaces the pairing is the rank map and is independent of this choice.

Lemma 10.9 (Finite standard realization)**.** *Let H_0, H_1 be finitely generated abelian groups, let $H_0 = \mathbb{Z}u \oplus H_0'$, and let $\rho : H_0 \to \mathbb{R}$ satisfy $\rho(u) = 1$. There is a finite reduced free product C of measured commutative algebras over the spaces in* (10.5)*, including two Lebesgue-interval factors, and isomorphisms*

$$\eta_i : \mathrm{K}_i(C) \longrightarrow H_i \quad (i = 0, 1)$$

such that $\eta_0([1_C]) = u$ and $\tau_{C*} = \rho\eta_0$. Its positive cone corresponds to $\{0\} \cup \{h \in H_0 : \rho(h) > 0\}$.

Proof. Write $H_0' = \bigoplus_{s=1}^{r} \mathbb{Z}h_s \oplus \bigoplus_a \mathbb{Z}/n_a\mathbb{Z}$ and $H_1 = \mathbb{Z}^q \oplus \bigoplus_b \mathbb{Z}/m_b\mathbb{Z}$. Set $k_s = \lfloor \rho(h_s) \rfloor$ and $t_s = \rho(h_s) - k_s$. If $t_s > 0$, use a disconnected interval factor of first-component mass t_s and mark its class $[e]$ by $h_s - k_s u$. If $t_s = 0$, use S^2 and mark its Bott class by $h_s - k_s u$. Use T_{II,n_a} for the cyclic summands of H_0', and q circle factors and the spaces T_{III,m_b} for H_1. Torsion has zero pairing with $\mathbb{R}$. Finally add two Lebesgue-interval factors. They add no reduced K-theory and ensure the Haar-unitary hypothesis. The free-product calculation in Proposition 2.5 gives exactly the stated markings and pairing. Haar absorption identifies C with $C *_{\mathrm{r}} D$, so regularity and the strict trace cone follow from Lemmas 2.4 and 2.3. □

10.5. **Homogeneous corners and arbitrary unit classes.** In addition to the commutative blocks in (10.5), allow the homogeneous blocks

$$B(X, V) = \Gamma(X, \operatorname{End} V) \cong p\, \mathrm{M}_N(\mathrm{C}(X))p, \qquad V = \mathbf{1}^{\,d-1} \oplus L, \quad d \geq 1. \tag{10.6}$$

Here X is a finite wedge of two-dimensional Moore spaces $T_{\mathrm{II},n}$, with a point allowed for the empty wedge, and L is a complex line bundle on X. The full projection p represents V and has constant rank d. Equip this block with the faithful selected trace

$$\tau_{X,V}(b) = \int_X \frac{1}{d} \operatorname{Tr}(b(x))\, d\mu(x),$$

where μ is any full-support Borel probability measure. For X a point these blocks are the full matrix algebras. Let $\mathscr{B}$ denote this enlarged collection of faithfully tracial blocks, together with the commutative blocks above.

Lemma 10.10 (Finite type I realization with an arbitrary unit)**.** *Let H_0 be a finitely generated abelian group of free rank r, let H_1 be any countable abelian group, let $u \in H_0$, and let $\rho : H_0 \to \mathbb{R}$ satisfy $\rho(u) = 1$. There is a reduced free product C of at most* $\max\{2, r\}$ *separable unital type I algebras with faithful selected traces, and isomorphisms*

$$\eta_i : \mathrm{K}_i(C) \longrightarrow H_i \quad (i = 0, 1), \qquad \eta_0([1_C]) = u, \qquad \tau_{C*} = \rho\eta_0.$$

Its positive cone corresponds to $\{0\} \cup \{h \in H_0 : \rho(h) > 0\}$. All factors have central Haar unitaries. At most one factor is noncommutative; it is a homogeneous corner over a connected compact metrizable space of dimension at most three. The other factors are commutative, with spectra of dimension at most two. If the unit splits off a copy of $\mathbb{Z}$, all factors may be chosen commutative.

If H_1 is also finitely generated, one may alternatively use a finite product of blocks in $\mathscr{B}$, containing two Lebesgue-interval factors and at most one corner from (10.6)*. In this alternative the split-unit case uses only the commutative blocks in* (10.5)*.*

Proof. Put $T_0 = \operatorname{Tor} H_0$. The unit has infinite order, so $r \geq 1$. An integral change of basis in H_0/T_0 gives

$$H_0 = \mathbb{Z}v \oplus \bigoplus_{s=1}^{r-1} \mathbb{Z}h_s \oplus T_0, \qquad u = dv + t_0, \quad d \geq 1, \quad t_0 \in T_0.$$

Thus $\rho(v) = 1/d$, since ρ vanishes on T_0. Write $T_0 = \bigoplus_a \mathbb{Z}/n_a\mathbb{Z}$ and take $X_0 = \bigvee_a T_{\mathrm{II},n_a}$, using a point when $T_0 = 0$. Rank and first Chern class give $\mathrm{K}_0(\mathrm{C}(X_0)) = \mathbb{Z}[1] \oplus T_0$ and $\mathrm{K}_1(\mathrm{C}(X_0)) = 0$. Choose a complex line bundle L_0 with $[L_0] - [1] = t_0$; compare [30, Lemmas 3.1–3.2 and the construction following Lemma 3.7].

Apply Lemma 10.3 with $H_0 = 0$ to obtain a pointed connected compact metrizable space Y of dimension at most three with

$$\mathrm{K}_0(\mathrm{C}(Y)) = \mathbb{Z}[1], \qquad \mathrm{K}_1(\mathrm{C}(Y)) = H_1.$$

Set $X = X_0 \vee Y \vee [0,1]$ and pull L_0 back under the retraction $X \to X_0$, obtaining L. The split evaluation sequences for the wedge give

$$\mathrm{K}_0(\mathrm{C}(X)) = \mathbb{Z}[1] \oplus T_0, \qquad \mathrm{K}_1(\mathrm{C}(X)) = H_1.$$

Take $V = \mathbf{1}^{d-1} \oplus L$ and let $B = \Gamma(X, \operatorname{End} V)$. This is a unital homogeneous type I algebra, and Morita equivalence gives

$$\mathrm{K}_0(B) = \mathbb{Z}v \oplus T_0, \qquad [1_B] \longmapsto dv + t_0, \qquad \mathrm{K}_1(B) = H_1.$$

Give X a full-support nonatomic probability measure and use the integrated normalized fibre trace on B. Since X is connected, its pairing is rank divided by d, so it is $1/d$ on v and zero on T_0. Lemma 2.6 supplies a Haar unitary in the central copy of $\mathrm{C}(X)$.

For each remaining generator h_s, put $k_s = \lfloor \rho(h_s) \rfloor$ and $\lambda_s = \rho(h_s) - k_s \in [0,1)$. If $\lambda_s > 0$, use the disconnected interval block with first-component mass λ_s, marking its first-component class by $h_s - k_s u$. If $\lambda_s = 0$, use S^2 with a full-support nonatomic measure, marking its rank-zero Bott class by $h_s - k_s u$. These commutative factors have Haar unitaries by Lemma 2.6. Replacing h_s by $h_s - k_s u$ is an invertible triangular change of generators fixing $\mathbb{Z}v \oplus T_0$, even when $t_0 \neq 0$. Proposition 2.5 therefore gives exactly the required pointed groups and pairing.

There are r factors when $r \geq 2$. When $r = 1$, add one Lebesgue-interval factor, giving two factors. Thus there are at most $\max\{2, r\}$ factors and two distinct factors with Haar unitaries. Haar absorption gives $C \cong C *_{\mathrm{r}} D$. The factors are nuclear and UCT, so Lemmas 2.4 and 2.3 give a unique trace and the asserted positive cone. If the unit splits, choose $v = u$, $d = 1$ and $t_0 = 0$. Then $B = \mathrm{C}(X)$, giving the commutative assertion.

For the final assertion, when H_1 is finitely generated use $B(X_0, \mathbf{1}^{d-1} \oplus L_0)$ in place of B. Realize H_1 by finitely many circle and $T_{\mathrm{III},m}$ factors, and add two Lebesgue-interval factors. The same group and pairing calculation proves the assertion. When the unit splits, Lemma 10.9 gives the stated standard blocks. □

10.6. **Type I free-product models for the general class.**

Theorem 10.11 (Type I models and standard-block refinements)**.** *Let $\mathcal{A}$ satisfy Theorem 9.11, and put $A = P_{\mathcal{A}} *_{\mathrm{r}} D$. Choose increasing finitely generated subgroups $G_{0,n} \subset \mathrm{K}_0(A)$ containing $[1_A]$ and exhausting $\mathrm{K}_0(A)$, and put $r_n = \operatorname{rank} G_{0,n}$. There is an inductive system*

$$C_1 \xrightarrow{\varphi_1} C_2 \xrightarrow{\varphi_2} \cdots$$

with unital trace-preserving injective connecting homomorphisms, such that each C_n is a reduced free product of at most $\max\{2, r_n\}$ separable unital type I algebras with faithful traces, and

$$A \cong \varinjlim (C_n, \varphi_n).$$

The marked stage groups are $\mathrm{K}_0(C_n) = G_{0,n}$ and $\mathrm{K}_1(C_n) = \mathrm{K}_1(A)$; the connecting maps induce the inclusions in degree zero and the identity in degree one. At most one factor at each stage is noncommutative, of the homogeneous form in Lemma 10.10. All factor spectra have dimension at most three. If $[1_A]$ splits off a copy of $\mathbb{Z}$ in $\mathrm{K}_0(A)$, the subgroups and models may be chosen so that all factors are commutative. The limiting isomorphism may realize any prescribed Elliott-invariant identification.

There is also a standard-block refinement: one may instead choose every C_n as a finite reduced free product of blocks in $\mathscr{B}$, including two Lebesgue-interval factors and at most one homogeneous Moore-space corner. In this refinement both graded groups are exhausted by finitely generated stage groups. If all original factors are commutative, only the commutative blocks in (10.5) *are needed. If two distinct original factors admit Haar unitaries, the same conclusions hold for $P_{\mathcal{A}}$ without adjoining D.*

Proof. Apply Lemma 10.10 to $G_{0,n}, [1_A], \mathrm{K}_1(A)$ and the restricted selected pairing. This gives the primary stage models with the asserted factor bound. If the unit splits, write $\mathrm{K}_0(A) = \mathbb{Z}[1_A] \oplus H$ and choose $G_{0,n} = \mathbb{Z}[1_A] \oplus H_n$ for increasing finitely generated subgroups H_n exhausting H. The commutative assertion follows from the same lemma.

The stage products absorb D, and their factors are nuclear and UCT. Choose simple monotracial nuclear UCT $\mathcal{Z}$-stable cores N_n by the strict-cone range construction in the proof of Lemma 4.1. Theorems 9.10 and 4.7 give marked isomorphisms

$$\psi_n : C_n \longrightarrow N_n *_{\mathrm{r}} D,$$

with the indicated stage data. Theorem 9.11 allows the absorption isomorphism to be chosen with the identity marking on the identified pointed groups and pairing. The inclusions $G_{0,n} \subset G_{0,n+1}$ preserve the unit and the strict trace cones. Together with the identity in degree one and the unique trace maps, they define Elliott-invariant morphisms carrying the target trace to the faithful source trace. Nuclear embedding existence, with compatible full-invariant extensions [4, Theorem B and Theorem 3.9], supplies unital embeddings $\nu_n : N_n \to N_{n+1}$ inducing the inclusions on K_0 and the identity on K_1. Their unique traces agree. Define

$$\varphi_n = \psi_{n+1}^{-1} \circ (\nu_n *_{\mathrm{r}} \mathrm{id}_D) \circ \psi_n.$$

These maps are injective and trace preserving. Identifying the N_n with nested subalgebras of $N = \varinjlim (N_n, \nu_n)$, finite reduced words give

$$\varinjlim (C_n, \varphi_n) \cong N *_{\mathrm{r}} D.$$

The algebra N is simple, separable, unital, nuclear and UCT, with faithful unique trace. Its groups, unit and pairing are those of A by continuity; its positive cone is strict because each positive-trace class is positive at a stage. Theorem 9.11 also identifies A with $F_\infty *_{\mathrm{r}} D$ for a nuclear core with these data. Theorem 4.7 compares $N *_{\mathrm{r}} D$ and $F_\infty *_{\mathrm{r}} D$ with any prescribed Elliott marking. This proves the primary model assertion.

For the standard-block refinement, choose in addition increasing finitely generated subgroups $G_{1,n}$ exhausting $\mathrm{K}_1(A)$. Use the final assertion of Lemma 10.10 and the same construction, now inducing subgroup inclusions in both degrees. For the commutative subclass, evaluation on the original factors splits the common unit; use $G_{0,n} = \mathbb{Z}[1_A] \oplus H_n$ and Lemma 10.9. Haar absorption proves the last assertion. □

Corollary 10.12 (A bound on the number of type I factors)**.** *If* $\mathrm{K}_0(A)$ *in Theorem 10.11 is finitely generated with free rank* r*, then* A *is isomorphic to a reduced free product of at most* $\max\{2, r\}$ *faithfully tracial separable unital type I factors. No finite-generation assumption on* $\mathrm{K}_1(A)$ *is needed. At most one factor is noncommutative, of the homogeneous form in Lemma 10.10. If* $[1_A]$ *splits off a copy of* $\mathbb{Z}$*, all factors may be chosen commutative; in particular this holds for commutative original factors. The isomorphism may realize any prescribed Elliott-invariant identification.*

If both groups are finitely generated, there is also a finite standard-product model using $\mathscr{B}$*, with at most one homogeneous Moore-space corner and two Lebesgue-interval factors. For commutative original factors this alternative uses only the blocks in* (10.5)*. Under the Haar-unitary hypothesis the same conclusions hold for* $P_{\mathcal{A}}$*.*

Proof. Apply Lemma 10.10 to the full invariant. The resulting product C absorbs D and has the same Elliott invariant as A. Theorem 9.11 supplies the desired marked isomorphism. The standard-block refinement follows from the final assertion of the lemma, and Haar absorption gives the unstabilized conclusion. □

Remark 10.13 (The unit class and the corner). The projection in (10.6) retains the entire unit class, including its torsion component. For example, if L is the nontrivial complex line bundle on $\mathbb{RP}^2 = T_{\mathrm{II},2}$, the block $\mathrm{End}(\mathbf{1} \oplus L)$ has $\mathrm{K}_0 = \mathbb{Z}v \oplus \mathbb{Z}/2\mathbb{Z}\,t$ and unit $2v + t$. This unit is not divisible by two, whereas the unit of $\mathrm{M}_2(\mathrm{C}(\mathbb{RP}^2))$ is. The corner is used here as an individual nuclear free factor; no absorption identity for an arbitrary projection corner of a free product is needed. Elliott–Thomsen point–line factors may also be allowed, but the realizations above already give the stated type I and standard-block models.

Remark 10.14 (Type I factors: unrestricted K_1 and a pointed K_0 obstruction). Arbitrary unital type I factors impose no restriction on the countable K_1-group. For any countable abelian group G, Lemma 10.3 gives a connected compact metrizable space Y with $\mathrm{K}_0(\mathrm{C}(Y)) = \mathbb{Z}[1]$ and $\mathrm{K}_1(\mathrm{C}(Y)) = G$. Choose a full-support probability measure on Y and take its reduced free product with two Lebesgue-interval algebras. All three factors are commutative and faithfully tracial, and Proposition 2.5 gives $\mathrm{K}_1 = G$.

For the full invariant, however, no finite or countable number of unital type I factors with faithful traces suffices in general. In fact, the pointed group

$$(\mathrm{K}_0, [1]) = (\mathbb{Z}[1/2], 1)$$

cannot occur for such a free product, even though it occurs for $\mathrm{M}_{2^\infty} *_{\mathrm{r}} D$, whose degree-one group is zero and whose selected pairing is the inclusion $\mathbb{Z}[1/2] \subset \mathbb{R}$.

To prove the obstruction, suppose $P = *_{\mathrm{r}i}(A_i, \tau_i)$ has this pointed group, and write $G_i = \mathrm{K}_0(A_i)$ and $u_i = [1_{A_i}]$. The faithful traces make every u_i have infinite order. The free-product formula gives

$$\mathrm{K}_0(P) = \left(\bigoplus_i G_i\right) / \langle u_i - u_j \rangle, \qquad \mathrm{K}_0(P)/\mathbb{Z}[1_P] \cong \bigoplus_i G_i/\mathbb{Z}u_i.$$

Each G_i injects into $\mathrm{K}_0(P)$. Indeed, fix i and present the relations as $u_j - u_i$ for $j \neq i$. If an element of G_i is a finite sum of these relations, its j-coordinate is an integer multiple of u_j and must vanish for every $j \neq i$; infinite order forces all coefficients to be zero. Thus each G_i is a subgroup of $\mathbb{Z}[1/2]$ containing 1.

Every nonzero unital type I algebra has a finite-dimensional irreducible quotient: quotient by a maximal proper closed ideal; the resulting simple unital type I algebra is an algebra of compact operators and hence a full matrix algebra. Consequently there is a homomorphism $G_i \to \mathbb{Z}$ sending u_i to an integer $n_i > 0$. A subgroup of $\mathbb{Z}[1/2]$ containing 1 is either $2^{-k_i}\mathbb{Z}$ for some $k_i \geq 0$, or all of $\mathbb{Z}[1/2]$. The latter is impossible, since every homomorphism $\mathbb{Z}[1/2] \to \mathbb{Z}$ is zero. Hence every $G_i/\mathbb{Z}u_i$ is a finite cyclic 2-group. Their direct sum is not 2-divisible unless it is zero. But $\mathbb{Z}[1/2]/\mathbb{Z}$ is a nonzero divisible group, a contradiction.

Thus the inductive-limit formulation in Theorem 10.11 remains necessary for the full general range, even after allowing arbitrary type I factors. Corollary 10.12 gives a finite bound when K_0 is finitely generated; the bound is not asserted to be optimal. For the fixed standard list $\mathscr{B}$, there is also the simpler degree-one obstruction: a single countable free product of those individual blocks has K_1 a direct sum of cyclic groups, so it cannot have $\mathbb{Z}[1/2]$ as its degree-one group.

The range of the actual commutative free products is exactly that of Theorem 10.4; arbitrary inductive systems of standard products need not have invariants in this range. The connecting maps constructed above need not be pullbacks of continuous maps between the spectra.

11. Exponential length and exponential rank

We compute the exponential rank and exponential length using only the trace range in the classification data. We first obtain the required upper estimates directly from Theorem 3.16; no additional classification theorem for nonnuclear algebras is needed.

11.1. **Definitions and interval lifts.** For a unital algebra B and $u \in \mathrm{U}_0(B)$, put

$$\mathrm{cel}_B(u) = \inf\left\{\sum_{j=1}^{r} \|h_j\| : u = \prod_{j=1}^{r} e^{ih_j},\ h_j = h_j^* \in B\right\}, \qquad \mathrm{cel}(B) = \sup_{u \in \mathrm{U}_0(B)} \mathrm{cel}_B(u).$$

This agrees with the infimum of lengths of rectifiable unitary paths from 1 to u [39]. Here

$$\mathrm{CU}(B) = \overline{[\mathrm{U}_0(B), \mathrm{U}_0(B)]}, \qquad \mathrm{cel}_{\mathrm{CU}}(B) = \sup_{u \in \mathrm{CU}(B)} \mathrm{cel}_B(u).$$

The paths defining $\mathrm{cel}_{\mathrm{CU}}$ may run through all of $\mathrm{U}_0(B)$. The assertion $\mathrm{cer}(B) \leq 1+\varepsilon$ means that single exponentials are norm dense in $\mathrm{U}_0(B)$; equality means additionally that some connected unitary has no self-adjoint logarithm. The symbol ε in this rank notation is formal.

We use two elementary facts. If $\|u - e^{ih}\| < \delta < 2$, the residual unitary has a logarithm of norm at most $2\arcsin(\delta/2)$. Thus uniform approximation with $\|h\| \leq L$ implies $\mathrm{cel}_B(u) \leq L$. Also, if $h = h^*$ vanishes on every tracial state, then $e^{ih} \in \mathrm{CU}(B)$, by the trace-commutator result of [7] and the Trotter product formula.

Proposition 11.1 (Bounded approximate logarithms)**.** *Let T be an admissible target, with unique trace τ, and put $H = \tau_*(\mathrm{K}_0(T))$. Every $u \in \mathrm{U}_0(T)$ is a norm limit of exponentials e^{ih} with $h = h^*$ and $\|h\| \leq 2\pi$. Consequently*

$$\mathrm{cer}(T) \leq 1+\varepsilon, \qquad \mathrm{cel}(T) \leq 2\pi.$$

Proof. If the spectrum of u omits a point of the circle, a branch of the argument gives a continuous logarithm of norm at most 2π. We may therefore suppose that $\mathrm{sp}(u) = \mathbb{T}$. Let μ be its spectral distribution for τ; it has full support. Let ν_0 be the lift of μ by a Borel argument in $[0, 2\pi)$, regarded as a measure on $[0, 2\pi]$, and put

$$a = \frac{1}{2\pi} \int s \, d\nu_0(s).$$

Every open interval at either endpoint contains the lift of an open arc, so ν_0 has full support on $[0, 2\pi]$. The Thomsen sequence identifies the connected unitary class of u with a determinant $\Delta_T(u) \in \mathbb{R}/\overline{H}$. Choose a representative $d \in \mathbb{R}$. Choose $t \in [0, 1)$ such that $a - t - d \in \mathbb{Z}$, and let

$$\nu_t = (1-t)\nu_0 + t(s \mapsto s - 2\pi)_*\nu_0.$$

For $t = 0$ use $Y = [0, 2\pi]$; otherwise use $Y = [-2\pi, 2\pi]$. Then ν_t is faithful on $\mathrm{C}(Y)$, its exponential pushforward is μ, and its normalized mean is $a - t$.

Send the unit of $\mathrm{K}_0(\mathrm{C}(Y)) = \mathbb{Z}$ to $[1_T]$, and use ν_t for the trace component. These data extend to a compatible full-invariant morphism [4, Theorem 3.9]. Theorem 3.16 gives a unital embedding $\phi : \mathrm{C}(Y) \to T$ with $\tau\phi = \nu_t$. For $h = \phi(s \mapsto s)$ and $v = e^{ih}$ we have

$$\|h\| \leq 2\pi, \qquad \tau(f(v)) = \tau(f(u)) \quad (f \in \mathrm{C}(\mathbb{T})), \qquad \Delta_T(v) = a - t = d \pmod{\overline{H}}.$$

Here $\mathbb{Z} \subseteq H$, since the trace is normalized.

The two circle embeddings defined by u and v have the same full invariant. Indeed, their K_0-maps are unital and their K_1-maps are zero. The free K-groups of $\mathrm{C}(\mathbb{T})$ imply equality on all coefficient groups. Their trace components agree by the displayed spectral-distribution identity. Finally, the Thomsen kernel and the class of the coordinate unitary z generate $\mathrm{U}_\infty(\mathrm{C}(\mathbb{T}))/\mathrm{CU}_\infty(\mathrm{C}(\mathbb{T}))$, because its K_1-quotient is $\mathbb{Z}[z]$. The trace action determines the map on that kernel, and the matched determinants give equality on $[z]$. Thus Theorem 3.16 makes u and v approximately unitarily conjugate. Conjugation preserves the norm of h.

The small-residual estimate above gives the length bounds. □

11.2. **Sharpness for a discrete range.**

Lemma 11.2. *A rectifiable unitary loop based at 1 of length less than 2π is null-homotopic. Its trace determinant integral is zero.*

Proof. Every point on the loop is connected to 1 by one of its two subarcs of length less than π. Its spectrum therefore omits -1. The continuous principal logarithm contracts the whole loop to 1. □

Proposition 11.3. *Let B be unital with a unique faithful trace τ, and suppose it contains a Haar unitary. If $H = \tau_*(\mathrm{K}_0(B))$ is discrete, then $\mathrm{cel}_{\mathrm{CU}}(B) \geq 2\pi$.*

Proof. For a piecewise smooth unitary path γ, use the normalized determinant integral

$$D_\tau(\gamma) = \frac{1}{2\pi i} \int_0^1 \tau(\gamma'(s)\gamma(s)^*)\, ds.$$

It is invariant under endpoint-preserving homotopies, has absolute value at most $\mathrm{Length}(\gamma)/(2\pi)$, and has periods in H on loops [12].

Let b be a Haar unitary and choose $0 < t < 1/2$ with $t \notin H$. Choose an arc of Haar measure t and its Fejér approximants f_n, so $0 \leq f_n \leq 1$, $\int f_n = t$, and f_n tends in L^2 to the indicator of the arc. Put

$$h_n = 2\pi(t1 - f_n(b)), \qquad u_n = e^{ih_n}, \qquad \lambda = e^{2\pi i t}.$$

Then $\tau(h_n) = 0$, so $u_n \in \mathrm{CU}(B)$, and

$$\mathrm{cel}_B(u_n) \leq 2\pi(1-t), \qquad \|u_n - \lambda 1\|_{2,\tau} \longrightarrow 0.$$

We show that the first upper bound is asymptotically sharp. Otherwise a subsequence admits paths γ_n from 1 to u_n with lengths at most some $\ell < 2\pi(1-t)$. Comparing with the trace-zero exponential path gives $D_\tau(\gamma_n) \in H$. These numbers are bounded. Discreteness of H permits a subsequence on which they are a fixed $c \in H$.

Choose $\eta > 0$ with $\ell + \eta + 2\pi t < 2\pi$. Fix $0 < \delta < 1$ so small that

$$2\arcsin(r/2) \leq \left(1 + \frac{\eta}{\ell+1}\right) r \qquad (0 \leq r \leq \delta).$$

Parametrize each γ_n by arclength and subdivide it into subpaths of length at most δ. The number of subpaths is at most $N = \lceil \ell/\delta \rceil + 1$, independently of n. After left translation by its initial unitary, each subpath lies in the norm-δ neighborhood of 1, where the principal logarithm is continuous. Interpolating its self-adjoint logarithm with the linear path to the endpoint logarithm gives a homotopy relative to the endpoints to the principal-logarithm path. Thus this replacement preserves the determinant integral. If the original subpath has length $l_{j,n}$ and endpoint increment $e^{ik_{j,n}}$, with $k_{j,n}$ its principal self-adjoint logarithm, then

$$\|k_{j,n}\| \leq 2\arcsin(l_{j,n}/2) \leq \left(1 + \frac{\eta}{\ell+1}\right) l_{j,n}.$$

Consequently the replacements have total length at most $\ell + \eta$. Pad the list of logarithms by zeros to obtain exactly N terms. This fixed number of uniformly bounded logarithms allows the paths and their determinant integrals to pass to the tracial ultrapower. Their images $k_j = (k_{j,n})_\omega$ in the tracial ultrapower of $\pi_\tau(B)''$ satisfy

$$\sum_j \|k_j\| \leq \lim_\omega \sum_j \|k_{j,n}\| \leq \ell + \eta, \qquad \frac{1}{2\pi} \sum_j \tau_\omega(k_j) = \lim_\omega c = c.$$

They therefore define a genuine piecewise exponential path from 1 to $\lambda 1$, with determinant integral c and length at most $\ell + \eta$. Appending the reverse scalar path of length $2\pi t$ gives a

loop of length less than 2π with integral $c - t \neq 0$. This contradicts Lemma 11.2. Hence $\mathrm{cel}_B(u_n) \to 2\pi(1-t)$ along the original sequence. Letting $t \downarrow 0$ outside H proves the proposition. □

11.3. **A unitary with no ambient logarithm.**

Lemma 11.4. *Let P be faithfully tracial and let $A = P *_{\mathrm{r}} D$. There is a Haar unitary $w \in \mathrm{CU}(A)$ such that*

$$A \cap \{w\}' = \mathrm{C}^*(w).$$

In particular w has no self-adjoint logarithm in A.

Proof. Let x_1, x_2 be the first two interval coordinates, put $a_j = e^{2\pi i x_j}$, and set $w = a_1 a_2$. The two-interval subalgebra has a unique trace by the Avitzour condition [2]. Since $x_j - 1/2$ is traceless there,

$$w = e^{2\pi i(x_1 - 1/2)} e^{2\pi i(x_2 - 1/2)} \in \mathrm{CU}(A).$$

Every nonzero power of w is a centered reduced word, so w is Haar.

In the tracial von Neumann closure M of A, each a_j generates the corresponding interval von Neumann algebra by measurable functional calculus. These are free Haar unitaries, free from the von Neumann closure of P. The free-group substitution $(a_1, a_2, a_3, \ldots) \mapsto (w, a_2, a_3, \ldots)$ gives $M = \mathrm{W}^*(w) * R$ for a finite tracial von Neumann algebra R. The reduced-word decomposition identifies $L^2(M) \ominus L^2(\mathrm{W}^*(w))$ with

$$\ell^2(\mathbb{Z}) \otimes \mathcal{H} \otimes \ell^2(\mathbb{Z})$$

where $\mathcal{H}$ is the Hilbert space of reduced words beginning and ending with centered R-letters. On this space conjugation by w shifts the two integer indices by $(m, n) \mapsto (m+1, n-1)$. Every orbit is infinite, so there are no nonzero fixed vectors. Consequently $M \cap \{w\}' = \mathrm{W}^*(w)$.

Let $\mathbb{E}_w : M \to \mathrm{W}^*(w)$ be the trace-preserving expectation. For centered functions f_j in the first interval and g_j in the second, reduced-word orthogonality gives

$$\mathbb{E}_w(f_1 g_1 \cdots f_r g_r) = \left(\prod_{j=1}^{r} \tau(a_1^* f_j) \tau(a_2^* g_j) \right) w^r.$$

The reverse pattern has expectation a scalar multiple of w^{-r}. Every other centered reduced-word pattern has expectation zero, including a word with a letter from P or a later interval. Thus $\mathbb{E}_w$ maps the algebraic free product into Laurent polynomials in w. Norm density and contractivity give $\mathbb{E}_w(A) \subseteq \mathrm{C}^*(w)$, and hence $A \cap \{w\}' = \mathrm{C}^*(w)$.

If $w = e^{ih}$ with $h = h^* \in A$, then h commutes with w and therefore equals $f(w)$ for a continuous real function f on the circle. But $e^{if(z)} = z$ is impossible by winding number. □

11.4. **The trace-pairing dichotomy.**

Theorem 11.5. *Let (P, τ_P) be a separable unital exact faithfully tracial algebra, put $A = P *_{\mathrm{r}} D$, and let $H = \tau_{P*}(\mathrm{K}_0(P))$. Then*

$$\mathrm{cel}(A) = \mathrm{cel}_{\mathrm{CU}}(A) = \begin{cases} 2\pi, & H \text{ is discrete,} \\ \pi, & H \text{ is dense in } \mathbb{R}, \end{cases} \qquad \mathrm{cer}(A) = 1 + \varepsilon.$$

All three values are unchanged on passing to $\mathrm{M}_d(A)$, $d \geq 1$.

Proof. Lemma 2.4 makes A an admissible target. The free-product K-sequence identifies $\mathrm{K}_*(P) \to \mathrm{K}_*(A)$ as an isomorphism preserving unit and trace [20]. Thus $\tau_*(\mathrm{K}_0(A)) = H$. A subgroup of $\mathbb{R}$ containing 1 is either discrete or dense.

For discrete H, Proposition 11.1 and Proposition 11.3 give both lengths equal to 2π. For dense H, strict comparison makes $\mathrm{W}(A)$ almost unperforated. Admissibility supplies exactness, simplicity and stable rank one, and the unique trace identifies $\mathrm{Aff}(\mathrm{T}(A))$ with $\mathbb{R}$. Thus [44, Proposition 7.1] gives real rank zero directly. The real-rank-zero approximation theorem of [31] approximates each unitary in $\mathrm{U}_0(A)$ by single exponentials e^{ih}. Real rank zero then approximates h by a finite-spectrum self-adjoint element. Reducing its finitely many eigenvalues modulo 2π into $[-\pi, \pi]$ leaves its exponential unchanged and gives an approximate logarithm of norm at most π. The resulting short correction paths give $\mathrm{cel}(A) \leq \pi$. The unitary $v = e^{2\pi i(x_1 - 1/2)}$ belongs to $\mathrm{CU}(A)$ and has -1 in its spectrum. Any path from 1 to v has length at least π, so $\mathrm{cel}_{\mathrm{CU}}(A) \geq \pi$. The approximate logarithms prove $\mathrm{cer}(A) \leq 1 + \varepsilon$; Lemma 11.4 excludes rank at most 1.

Matrix algebras over A remain admissible targets, and their normalized trace range is $d^{-1}H$. Real rank zero passes to matrix algebras, so the same upper bounds apply. For the discrete lower bound use the Haar unitary $\mathrm{diag}(b, \ldots, b)$, where b is an interval Haar unitary. For the dense lower bound use $v \oplus 1_{d-1} \in \mathrm{CU}(\mathrm{M}_d(A))$. Finally, $w \oplus 1_{d-1}$ has no logarithm. A self-adjoint logarithm would commute with this unitary. Since w is Haar, $w - 1$ has zero kernel in the faithful tracial von Neumann representation; hence the off-diagonal entries between the first coordinate and its complement must vanish. The first diagonal entry would be a logarithm of w, contradicting Lemma 11.4. □

Remark 11.6 (Abstract hypotheses and free-product structure). The length dichotomy in Theorem 11.5 holds for every admissible target T. More precisely, if $H = \tau_*(\mathrm{K}_0(T))$, then

$$\mathrm{cel}(T) = \mathrm{cel}_{\mathrm{CU}}(T) = \begin{cases} 2\pi, & H \text{ is discrete,} \\ \pi, & H \text{ is dense in } \mathbb{R}, \end{cases} \qquad \mathrm{cer}(T) \leq 1 + \varepsilon.$$

Proposition 11.1 already gives the abstract upper bounds. For discrete H, every admissible target contains a Haar unitary by [47, Corollary B], so Proposition 11.3 gives sharpness. For dense H, strict comparison makes $\mathrm{W}(T)$ almost unperforated, and [44, Proposition 7.1] gives real rank zero; that criterion requires neither nuclearity nor $\mathcal{Z}$-stability. The finite-spectrum approximation argument above, using [31], gives the upper bound π. A trace-preserving embedding of the uniform interval into T, supplied by Theorem 3.16, gives a trace-zero exponential with -1 in its spectrum, proving the reverse bound as above. These length formulas also hold for all matrix amplifications, whose normalized trace ranges are $d^{-1}H$.

The exact rank equality $\mathrm{cer}(P *_{\mathrm{r}} D) = 1 + \varepsilon$ in our proof additionally uses the concrete free-product structure: Lemma 11.4 produces a connected unitary with no self-adjoint logarithm in the ambient algebra, by computing its commutant using reduced words. The abstract argument above establishes the upper bound on exponential rank. In the free-product applications, the concrete structure also verifies admissibility and identifies H from the trace pairings of the factors.

Proof of Theorem E. Apply Theorem 11.5 to $P = P_{\mathcal{A}}$. Proposition 2.5 identifies its trace range with $H = \tau_{A*}(\mathrm{K}_0(A))$. Under the Haar-unitary hypothesis, $P_{\mathcal{A}} \cong P_{\mathcal{A}} *_{\mathrm{r}} D$, so all the

conclusions, including those for matrix amplifications, transfer to the original free product. In the commutative subclass, if every X_i is connected, each projection has constant integral rank, so $H = \mathbb{Z}$. □

12. Examples and consequences

We begin with the circle–annulus comparison of Hirshberg and Phillips, then illustrate the roles of reduced classes, torsion, infinite generation and the trace pairing.

Example 12.1 (Circles and annuli)**.** Let Y be a compact planar annulus with normalized area measure μ, and give S^1 its normalized Haar measure λ_{S^1}. For a faithful tracial state σ on $\mathbb{C} \oplus \mathbb{C}$, put

$$B_\sigma = (\mathbb{C} \oplus \mathbb{C}, \sigma)^{*_{\mathrm{r}}\infty}.$$

Then, for $R = D$ or $R = B_\sigma$, there is a trace-preserving isomorphism

$$(\mathrm{C}(S^1), \tau_{\lambda_{S^1}}) *_{\mathrm{r}} R \cong (\mathrm{C}(Y), \tau_\mu) *_{\mathrm{r}} R.$$

For $R = D$ and, respectively, $R = B_\sigma$ with $\sigma(1,0)$ irrational, these are the comparisons asked for by Hirshberg and Phillips in [26, Question 4.9(2)(b) and (1)(b)]. The conclusion for B_σ also holds when $\sigma(1,0)$ is rational. Moreover, for $2 \le n \le \infty$,

$$(\mathrm{C}(Y), \tau_\mu)^{*_{\mathrm{r}} n} \cong \mathrm{C}^*_{\mathrm{r}}(\mathbb{F}_n).$$

Proof. The annulus deformation retracts onto a circle. Thus both root factors have $\mathrm{K}_0 = \mathbb{Z}[1]$ and $\mathrm{K}_1 = \mathbb{Z}$, and their degree-zero trace pairings are rank. Appending the same tail R gives identical aggregate pointed groups and selected pairings. Corollary 9.13 therefore identifies the two products after adjoining D.

Both tails freely absorb D. This is immediate for $R = D$ by reindexing. For $R = B_\sigma$, put $p = \sigma(1,0) \in (0,1)$ and $I = (\mathrm{C}([0,1]), \tau_\lambda)$. The atom parameter of each two-point factor is $\max(p, 1-p) < 1$, so [26, Theorem 3.1] gives $I *_{\mathrm{r}} B_\sigma \cong B_\sigma$. Since $B_\sigma^{*_{\mathrm{r}}\infty} \cong B_\sigma$ by reindexing, grouping countably many copies gives

$$D *_{\mathrm{r}} B_\sigma \cong (I *_{\mathrm{r}} B_\sigma)^{*_{\mathrm{r}}\infty} \cong B_\sigma.$$

These isomorphisms preserve the unique free-product traces, so they remove the adjoined reservoir and prove the first assertion. For the free powers, apply the two-nonatomic-factor clause of Corollary 9.13 to the annulus factors and the circle factors. Their aggregate invariants agree, and the circle free power is $\mathrm{C}^*_{\mathrm{r}}(\mathbb{F}_n)$. □

Example 12.2 (Two spheres)**.** For faithful nonatomic sphere and wedge measures,

$$(\mathrm{C}(S^2), \mu_1) *_{\mathrm{r}} (\mathrm{C}(S^2), \mu_2) *_{\mathrm{r}} D \cong (\mathrm{C}(S^2 \vee S^2), \nu) *_{\mathrm{r}} D. \tag{12.1}$$

The two sphere Bott classes map to the two wedge Bott classes.

Example 12.3 (A torsion example)**.** For faithful nonatomic measures on the real projective plane and its wedge,

$$(\mathrm{C}(\mathbb{RP}^2), \mu_1) *_{\mathrm{r}} (\mathrm{C}(\mathbb{RP}^2), \mu_2) *_{\mathrm{r}} D \cong (\mathrm{C}(\mathbb{RP}^2 \vee \mathbb{RP}^2), \nu) *_{\mathrm{r}} D.$$

Both sides have pointed groups $\mathrm{K}_0 = \mathbb{Z}[1] \oplus (\mathbb{Z}/2)^2$, $\mathrm{K}_1 = 0$; the trace pairing is 1 on the unit and zero on the two torsion summands.

Example 12.4 (Non-AF factors with matrix models)**.** Let X be a nonempty compact metrizable contractible space, let $n \geq 1$, and give $A = \mathrm{M}_n(\mathrm{C}(X))$ any faithful tracial state. Under Morita identification, its selected data are

$$\mathrm{K}_0(A) = \mathbb{Z}g, \qquad [1_A] = ng, \qquad \tau_{A*}(g) = 1/n, \qquad \mathrm{K}_1(A) = 0.$$

These are the selected data of M_n with normalized trace. The same data are carried by $\mathrm{M}_n(\mathcal{Z})$. Proposition 10.7 permits simultaneous replacement of any finite or countable family of these factors by the corresponding matrix factors after adjoining D. Theorem 9.10 allows arbitrary additional nuclear UCT factors with faithful selected traces. For example, it applies to arbitrarily many factors $\mathrm{M}_{n_i}(\mathrm{C}([0,1]))$, although these interval algebras are not AF and their unit classes do not split when $n_i > 1$.

Example 12.5 (Two matrix factors and the projective plane)**.** Give each M_2 its normalized trace, and give $E = \mathrm{M}_2(\mathrm{C}(\mathbb{RP}^2))$ the trace

$$\tau_E(f) = \int_{\mathbb{RP}^2} \mathrm{tr}_2(f(x))\, d\mu(x),$$

where μ is any full-support Borel probability measure. Then

$$(\mathrm{M}_2 *_{\mathrm{r}} \mathrm{M}_2) *_{\mathrm{r}} D \cong \mathrm{M}_2(\mathrm{C}(\mathbb{RP}^2)) *_{\mathrm{r}} D.$$

Indeed, if p_i is a minimal projection in the ith matrix factor, the free-product calculation gives

$$\begin{aligned} \mathrm{K}_0(\mathrm{M}_2 *_{\mathrm{r}} \mathrm{M}_2) = \frac{\mathbb{Z}[p_1] \oplus \mathbb{Z}[p_2]}{\langle 2[p_1] - 2[p_2] \rangle} &= \mathbb{Z}a \oplus (\mathbb{Z}/2)t, \\ a = [p_1], \qquad t = [p_2] - [p_1], \qquad [1] &= 2a, \\ \rho(a) = \tfrac{1}{2}, \qquad \rho(t) = 0, \qquad \mathrm{K}_1 &= 0. \end{aligned}$$

Under the standard Morita identification for E, let a be the trivial line class and let $t = [L] - [\mathbf{1}]$, where L is the nontrivial complex line bundle over $\mathbb{RP}^2$. These give the same pointed groups and pairing. The trace is independent of μ on K_0, since the rank of each projection is constant. Corollary 10.6 and marked nuclear-core replacement give the displayed isomorphism with these prescribed classes.

The two matrix markings also have a geometric realization in E. The trivial decomposition $\mathbf{1} \oplus \mathbf{1}$ gives a unital matrix embedding with minimal class a. Since $L \oplus L$ is trivial, a trivialization gives another unital matrix embedding with minimal class $a + t$. Their difference retains the nonzero torsion class. The reservoir remains in the displayed fusion statement; the two finite-dimensional factors do not satisfy the Haar-unitary hypothesis.

The algebra E admits a model built from AF factors, namely the two matrix algebras above. Proposition 10.7 therefore permits simultaneous replacement of any finite or countable family of these E factors by their paired matrix models after adjoining D, while fixing the remaining factors. Theorem 9.10 applies without any model restriction on the other nuclear UCT factors. This use of E as an input factor does not require E itself to be AF.

Example 12.6 (A unit-twisted Moore corner and a matrix factor)**.** Let L be the nontrivial complex line bundle over $\mathbb{RP}^2$, and put

$$A = \Gamma\Big(\mathrm{End}(\mathbf{1} \oplus L)\Big), \qquad P_4 = S^1 \cup_4 D^2.$$

Equip A with the normalized fibre trace integrated against any full-support probability measure. Under Morita identification,

$$\mathrm{K}_0(A) = \mathbb{Z}a \oplus (\mathbb{Z}/2)t, \qquad [1_A] = 2a + t, \qquad \tau_*(a) = \tfrac{1}{2}, \quad \tau_*(t) = 0, \qquad \mathrm{K}_1(A) = 0.$$

Corollary 10.6 applies with A as the arbitrary root and M_2 as its AF leaf. If b is the minimal matrix class and $d = b - a$, the unit relation gives

$$t = 2d, \qquad 4d = 0, \qquad \mathrm{K}_0(A *_{\mathrm{r}} \mathrm{M}_2) = \mathbb{Z}b \oplus (\mathbb{Z}/4)d, \quad [1] = 2b, \quad \mathrm{K}_1 = 0.$$

The selected pairing sends b to $1/2$ and d to zero. These are exactly the selected pointed data of $\mathrm{M}_2(\mathrm{C}(P_4))$, for any full-support probability measure on P_4. Marked nuclear-core replacement therefore gives

$$(A *_{\mathrm{r}} \mathrm{M}_2) *_{\mathrm{r}} D \cong \mathrm{M}_2(\mathrm{C}(P_4)) *_{\mathrm{r}} D.$$

The markings may send the original classes a, t, b to $b - d, 2d, b$, respectively. The original order-two class becomes twice the new order-four class; replacing A by the trivial bundle corner would lose its unit marking. The proof uses no freeness assertion inferred from approximate conjugacy of the whole root.

Example 12.7 (Two unit-twisted Moore corners)**.** Let L be the nontrivial complex line bundle over $\mathbb{RP}^2$ and put

$$A = \Gamma\Big(\mathrm{End}(\mathbf{1} \oplus L)\Big), \qquad P_4 = S^1 \cup_{z \mapsto z^4} D^2, \qquad Y = \mathbb{RP}^2 \vee P_4.$$

Give each copy of A its normalized fibre trace integrated against a full-support probability measure. Write its two pointed groups as

$$\mathrm{K}_0(A_1) = \mathbb{Z}a \oplus (\mathbb{Z}/2)t, \quad [1_{A_1}] = 2a + t, \qquad \mathrm{K}_0(A_2) = \mathbb{Z}b \oplus (\mathbb{Z}/2)s, \quad [1_{A_2}] = 2b + s.$$

Both degree-one groups vanish. Setting $d = b - a$ in the free-product unit relation gives $s = t - 2d$ and $4d = 0$. Thus $s = t + 2d$ as well, and

$$\mathrm{K}_0(A_1 *_{\mathrm{r}} A_2) = \mathbb{Z}a \oplus (\mathbb{Z}/4)d \oplus (\mathbb{Z}/2)t, \qquad [1] = 2a + t, \quad \rho(a) = \tfrac{1}{2}, \quad \rho(d) = \rho(t) = 0.$$

Let L_t be the pullback of L through the collapse $Y \to \mathbb{RP}^2$, and set $F = \Gamma(\mathrm{End}(\mathbf{1} \oplus L_t))$. Rank and first Chern class on the two-dimensional complex Y give the same pointed groups and pairing for F, with $\mathrm{K}_1(F) = 0$. Theorem 9.10 gives

$$(A_1 *_{\mathrm{r}} A_2) *_{\mathrm{r}} D \cong F *_{\mathrm{r}} D$$

with factor markings $a \mapsto a$, $t \mapsto t$, $b \mapsto a + d$ and $s \mapsto t + 2d$. The exact unit $2a + t$ and the order-four class are both retained. The two factors have nonsplit units and torsion in degree zero.

Example 12.8 (Solenoids)**.** Let $\Sigma_p = \varprojlim(S^1, z \mapsto z^p)$ be the p-adic solenoid, with its Haar probability measure. Then, for any faithful selected probability ν on the wedge,

$$(\mathrm{C}(\Sigma_2), \mu_2) *_{\mathrm{r}} (\mathrm{C}(\Sigma_3), \mu_3) *_{\mathrm{r}} D \cong (\mathrm{C}(\Sigma_2 \vee \Sigma_3), \nu) *_{\mathrm{r}} D.$$

Here $\mathrm{K}_0 = \mathbb{Z}[1]$ and $\mathrm{K}_1 = \mathbb{Z}[1/2] \oplus \mathbb{Z}[1/3]$. Neither summand in degree one is finitely generated or free abelian. The left-hand side may also be written without D.

Proof. Continuity for $\mathrm{C}(\Sigma_p) = \varinjlim(\mathrm{C}(S^1), f(z) \mapsto f(z^p))$ gives $\mathrm{K}_0 = \mathbb{Z}[1]$ and $\mathrm{K}_1 = \mathbb{Z}[1/p]$. Apply Theorem 8.9. The Haar measures are nonatomic, so [45, Theorem 5.2] absorbs D on the left. □

Example 12.9 (The K-theory of a point)**.** Let $P = *_{\mathrm{r}\, j\in S}(E_j, \tau_{\mu_j})$, with $E_j = \mathrm{C}(X_j)$, be as in Corollary 9.13. Suppose each factor satisfies $\mathrm{K}_0(E_j) = \mathbb{Z}[1]$ and $\mathrm{K}_1(E_j) = 0$. Then $P *_{\mathrm{r}} D \cong D$. If two factors contain Haar unitaries, then $P \cong D$.

Proof. The aggregate data are $(\mathbb{Z}, 1, 0, \mathrm{id})$, which are the data of D. Apply Corollary 9.13 and the reindexing $D *_{\mathrm{r}} D \cong D$. □

Example 12.10 (Odd spheres)**.** Let $d \geq 1$ and let μ be a nonatomic full-support probability measure on S^{2d+1}. For every $n \in \{2, 3, \ldots, \infty\}$,

$$(\mathrm{C}(S^{2d+1}), \tau_\mu)^{*_{\mathrm{r}} n} \cong \mathrm{C}^*_{\mathrm{r}}(\mathbb{F}_n).$$

Proof. The pointed graded groups of $\mathrm{C}(S^{2d+1})$ and $\mathrm{C}(S^1)$ are $(\mathbb{Z}[1], \mathbb{Z})$, and their selected trace pairings in degree zero are rank. Their corresponding free powers therefore have the same aggregate invariant. Use Corollary 9.13 with Haar measure on S^1; the free power on the circle side is $\mathrm{C}^*_{\mathrm{r}}(\mathbb{F}_n)$. □

Example 12.11 (The pairing for disconnected spaces)**.** Let $X = [0,1] \sqcup [0,1]$, let $0 < t < 1$, and give X the measure $\mu_t = t\lambda \sqcup (1-t)\lambda$. Put $P_t = (\mathrm{C}(X), \tau_{\mu_t}) *_{\mathrm{r}} (\mathrm{C}(X), \tau_{\mu_t})$. All these measures are faithful and nonatomic. The pointed groups are $\mathrm{K}_0(P_t) = \mathbb{Z}[1] \oplus \mathbb{Z}^2$ and $\mathrm{K}_1(P_t) = 0$. With the two reduced generators represented by the first-component projections in the two free factors, the trace pairing is

$$\tau_{t*}(m[1] + ng_1 + kg_2) = m + t(n+k).$$

Its image is $\mathbb{Z} + t\mathbb{Z}$. For $t = 1/2$ this is the discrete group $\frac{1}{2}\mathbb{Z}$; for irrational t it is dense in $\mathbb{R}$. Consequently these two types of free products are nonisomorphic, although their pointed graded groups agree. Their traces are unique, so any isomorphism would preserve the displayed trace range. Theorem E also gives

$$\mathrm{cel}(P_t) = \mathrm{cel}_{\mathrm{CU}}(P_t) = \begin{cases} 2\pi, & t \in \mathbb{Q}, \\ \pi, & t \notin \mathbb{Q}, \end{cases} \qquad \mathrm{cer}(P_t) = 1 + \varepsilon.$$

Thus the trace pairing detects an exact change in exponential length even though the pointed graded groups are independent of t.

Example 12.12 (Finite matrix Moore-space models)**.** For a finite family $A_i = \mathrm{M}_{n_i}$, put

$$L = \mathrm{lcm}(n_1, \ldots, n_r), \qquad G = \frac{\bigoplus_{i=1}^r \mathbb{Z} a_i}{\langle n_i a_i - n_j a_j \rangle}.$$

There is a connected finite complex Y of dimension at most two, a finite wedge of Moore complexes, such that

$$H^1(Y; \mathbb{Z}) = 0, \qquad H^2(Y; \mathbb{Z}) = \mathrm{Tor}\, G, \qquad \Big(\mathop{*_{\mathrm{r}}}_{i=1}^{r} \mathrm{M}_{n_i}\Big) *_{\mathrm{r}} D \cong \mathrm{M}_L(\mathrm{C}(Y)) *_{\mathrm{r}} D$$

with the indicated selected marking. If $\mathrm{Tor}\, G = 0$, take Y to be a point. In particular, for two factors one may take $Y = Y_d = S^1 \cup_d e^2$, where $d = \gcd(m,n)$ and $L = \mathrm{lcm}(m,n)$; for $d = 1$ take a point and for $d = 2$ take $\mathbb{RP}^2$.

Proof. Put $r_i = L/n_i$. Since $\gcd(r_1, \dots, r_r) = 1$, choose integers k_i with $\sum_i k_i r_i = 1$ and set $g = \sum_i k_i a_i$. Writing $u = n_i a_i$ for the common unit gives

$$Lg = u, \qquad \rho(g) = 1/L, \qquad t_i = a_i - r_i g, \qquad n_i t_i = 0.$$

The t_i generate the torsion subgroup and $G = \mathbb{Z}g \oplus \operatorname{Tor} G$. Choose a Moore-space wedge with the displayed cohomology. The two-dimensional K-calculation gives

$$\mathrm{K}_0(\mathrm{M}_L(\mathrm{C}(Y))) = \mathbb{Z}g \oplus \operatorname{Tor} G, \quad [1] = Lg, \quad \rho(g) = 1/L, \quad \rho(\operatorname{Tor} G) = 0, \quad \mathrm{K}_1 = 0.$$

Every faithful selected measure on Y has this pairing. Corollary 10.6 applies.

The geometric frames can also be retained. Let L_{t_i} be the line bundle with first Chern class t_i, and set $V_i = \mathbf{1}^{r_i - 1} \oplus L_{t_i}$. The bundle $V_i^{\oplus n_i}$ has rank L and zero first Chern class, hence is trivial over the two-complex. Its decomposition gives a unital matrix frame $\mathrm{M}_{n_i} \subset \mathrm{M}_L(\mathrm{C}(Y))$ with minimal class $r_i g + t_i$. This identifies the asserted marking. For two factors the unit relation has torsion $\mathbb{Z}/d$, giving the stated Y_d. □

12.1. **Free products without adjoining a reservoir.** The following cases extend the observation in Theorem 9.11 by grouping the original factors before applying semicircular absorption.

Theorem 12.13 (Classification without adjoining a reservoir)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be families as in Theorem 9.11. Suppose that each family has two disjoint nonempty subfamilies whose reduced free products contain Haar unitaries for their selected free-product traces. Then $P_{\mathcal{A}}$ and $P_{\mathcal{B}}$ are simple and monotracial, and*

$$P_{\mathcal{A}} \cong P_{\mathcal{B}} \quad \Longleftrightarrow \quad \mathrm{Ell}(P_{\mathcal{A}}) \cong \mathrm{Ell}(P_{\mathcal{B}}).$$

Every unital morphism of these invariants is induced by a unital embedding. If the invariant morphism is an isomorphism, the realizing map may be chosen to be a *-isomorphism. The invariant range is exactly that of Theorem C; every invariant in this range is realized by $F *_{\mathrm{r}} \mathcal{Z}$ with F simple, infinite-dimensional, unital, monotracial, nuclear, UCT and $\mathcal{Z}$-stable.

For each family, the grouping hypothesis holds under any one of the following sufficient conditions:

(i) *Two distinct factors have diffuse selected traces in the sense of* [47, Theorem A]*; in particular, two factors are simple and infinite dimensional.*

(ii) *Four distinct factors contain unitaries z_i with $\tau_i(z_i) = 0$; in particular, four factors are full matrix algebras M_{n_i} with $n_i \geq 2$.*

(iii) *The family is countably infinite and*

$$\sum_i \big(1 - \mathrm{At}(A_i, \tau_i)\big) = \infty,$$

with the analogous condition for $\mathcal{B}$.

The two families may satisfy different sufficient conditions.

Proof. First consider either family and write its product as P. Group the first distinguished subfamily into B_1 and the second, together with all remaining factors, into B_2. Associativity gives $P = B_1 *_{\mathrm{r}} B_2$, with faithful selected traces, and each B_j contains a Haar unitary. By Corollary 4.11, the reservoir D has a countable semicircular presentation. Shlyakhtenko's

absorption theorem [45, Theorem 5.2] therefore gives a trace-preserving isomorphism $P \cong P*_{\mathrm{r}}D$. Nuclearity and the UCT are not required for this absorption step; the grouped algebras may be nonnuclear.

Apply this to both families, obtaining isomorphisms $\alpha_{\mathcal{A}} : P_{\mathcal{A}} \to P_{\mathcal{A}}*_{\mathrm{r}}D$ and $\alpha_{\mathcal{B}} : P_{\mathcal{B}} \to P_{\mathcal{B}}*_{\mathrm{r}}D$. Simplicity and uniqueness of trace follow from Lemma 2.4. For a prescribed unital Elliott-invariant morphism χ, apply Theorem 9.11 to

$$\mathrm{Ell}(\alpha_{\mathcal{B}}) \circ \chi \circ \mathrm{Ell}(\alpha_{\mathcal{A}})^{-1}.$$

Composing its realizing embedding with $\alpha_{\mathcal{A}}$ and $\alpha_{\mathcal{B}}^{-1}$ realizes χ between the original products. If χ is invertible, choose the realizing isomorphism in Theorem 9.11. The converse classification implication is functoriality. Theorem 10.1 gives the claimed range through $F*_{\mathrm{r}}\mathcal{Z}$, whose two factors are simple and infinite dimensional.

It remains to verify the concrete sufficient conditions. A diffuse tracial state admits a Haar unitary [47, Theorem A]; this applies to every tracial state on a simple unital infinite-dimensional algebra [47, Corollary B]. This proves (i). For (ii), the unitaries z_1z_2 and z_3z_4 are Haar in the two disjoint paired products: every nonzero power is a centered alternating word. This is also [45, Remark 5.5]. Each M_n, $n \geq 2$, has a trace-zero unitary, for example the diagonal matrix of all nth roots of unity.

For (iii), enumerate the factors by $\mathbb{N}$ and choose $a_i = a_i^* \in A_i$ so that, writing μ_i for its selected spectral distribution and $\beta_i = \sup_t \mu_i(\{t\})$, one has $\beta_i < \mathrm{At}(A_i, \tau_i) + 2^{-i}$. Then $\sum_i (1 - \beta_i) = \infty$. Choose two disjoint finite sets S_1, S_2 with $\sum_{i \in S_j}(1 - \beta_i) > 1$ for $j = 1, 2$. In the corresponding free block, $b_j = \sum_{i \in S_j} a_i$ has distribution $\boxplus_{i \in S_j} \mu_i$. The atom formula for free additive convolution makes this distribution atomless: an atom would require atoms t_i with

$$\sum_{i \in S_j} \mu_i(\{t_i\}) > |S_j| - 1, \qquad \text{whereas} \qquad \sum_{i \in S_j} \mu_i(\{t_i\}) \leq \sum_{i \in S_j} \beta_i < |S_j| - 1.$$

This is the grouping argument in the proof of [26, Lemma 2.6]. The continuous distribution function H_j of b_j gives the Haar unitary $\exp(2\pi i H_j(b_j))$. Thus the two disjoint blocks satisfy the required hypothesis. Finally, for a nonscalar faithfully tracial algebra, any nonscalar selfadjoint has a non-point-mass spectral distribution, so $\mathrm{At}(A, \tau) < 1$. Repeated copies therefore satisfy the divergence condition. □

The additional factors in the first two cases are arbitrary subject to the standing nuclear UCT and faithful-trace hypotheses. The last case includes countably many copies of any fixed nonscalar separable unital nuclear UCT algebra with a faithful selected trace.

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Department of Mathematics, University of Toronto, Toronto, Ontario M5S 2E4, Canada
Email address: elliott@math.toronto.edu

School of Mathematics and Statistics, Northeast Normal University, Changchun 130024, P. R. China
Email address: licg864@nenu.edu.cn

Department of Mathematics and Statistics, University of Wyoming, Laramie, Wyoming 82071, USA
Email address: zhuangniu@icloud.com