ON THE SMALL BOUNDARY PROPERTY AND $\mathcal{Z}$-ABSORPTION, II

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ABSTRACT. Consider a minimal and free topological dynamical system $(X, \mathbb{Z}^d)$. It is shown that zero mean dimension of $(X, \mathbb{Z}^d)$ is characterized by $\mathcal{Z}$-absorption of the crossed product C*-algebra $A = \mathrm{C}(X) \rtimes \mathbb{Z}^d$, where $\mathcal{Z}$ is the Jiang-Su algebra. In fact, among other conditions, the following are shown to be equivalent:

(1) $(X, \mathbb{Z}^d)$ has the small boundary property.
(2) $A \cong A \otimes \mathcal{Z}$.
(3) A has uniform property Γ.
(4) $l^\infty(A)/J_{2,\omega,\mathrm{T}(A)}$ has real rank zero.

The same statement also holds for unital simple AH algebras with diagonal maps.

CONTENTS

1. INTRODUCTION

This is a continuation of our study [10] of the relation between the small boundary property of dynamical systems and the $\mathcal{Z}$-absorption of C*-algebras.

The Jiang-Su algebra $\mathcal{Z}$ is an infinite-dimensional unital simple separable amenable C*-algebra which has the same value of the Elliott invariant as $\mathbb{C}$. A C*-algebra A is said to be $\mathcal{Z}$-absorbing if $A \cong A \otimes \mathcal{Z}$, and the class of $\mathcal{Z}$-absorbing C*-algebras (which includes $\mathcal{Z}$ itself) is considered to be well behaved. In fact, the class of $\mathcal{Z}$-absorbing unital simple separable amenable C*-algebras

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which satisfy the Universal Coefficient Theorem of KK-theory (possibly redundant) is classified by the conventional Elliott invariant (see [12], [13], [8], [5], [4], [33], [2]).

On the other hand, the small boundary property of a topological dynamical system was introduced in [23] as a dynamical system analogue of the usual definition of zero-dimensional space. It is closely related to the mean (topological) dimension, which was introduced by Gromov ([14]), and then was developed and studied systematically by Lindenstrauss and Weiss ([23]), as a numerical invariant which measures the complexity of a dynamical system in terms of the dimension growth with respect to partial orbits. The small boundary property always implies the zero mean dimension ([23]), and the converse holds for $\mathbb{Z}^d$-actions with the marker property ([22], [16]).

It was shown in [9] that the small boundary property of $(X, \mathbb{Z})$ implies the $\mathcal{Z}$-absorption of the crossed product C*-algebra $\mathrm{C}(X) \rtimes \mathbb{Z}$. In this paper, we shall show that the converse also holds. Thus, these two regularity properties, one for C*-algebras and one for topological dynamical systems, are actually equivalent:

Theorem A. *Let $(X, \mathbb{Z}^d)$ be a free and minimal topological dynamical system, and let $A = \mathrm{C}(X) \rtimes \mathbb{Z}^d$. Then*

$$A \cong A \otimes \mathcal{Z} \quad \Longleftrightarrow \quad \mathrm{mdim}(X, \mathbb{Z}^d) = 0.$$

To prove this theorem, we shall introduce a new property of a C*-algebra, Property (S) (Definition 6.1), which states that every self-adjoint element can be approximated by self-adjoint elements with a neighbourhood of 0 uniformly small under all tracial spectral measures. This may perhaps be regarded as an analogue for C*-algebras of the small boundary property. All $\mathcal{Z}$-absorbing C*-algebras have Property (S), and more generally, all C*-algebras with uniform property Γ have Property (S).

It turns out that Property (S) of the C*-algebra A implies the (SBP) of $(X, \mathbb{Z}^d)$. In conjunction with other known results, this implies that $\mathcal{Z}$-absorption of A is characterized by the following list of equivalent properties:

Theorem B (Theorem 8.4)**.** *Let $(X, \mathbb{Z}^d)$ be a free and minimal dynamical system, and consider the crossed product C*-algebra $A = \mathrm{C}(X) \rtimes \mathbb{Z}^d$. Let $D = \mathrm{C}(X) \subseteq A$ be the canonical commutative subalgebra. Then the following conditions are equivalent:*

(1) *A has Property (S).*
(2) *$(D, \mathrm{T}(A))$ has the (SBP).*
(3) *$A \cong A \otimes \mathcal{Z}$.*
(4) *The strict order on $\mathrm{Cu}(A)$ is determined by traces.*
(5) $\mathrm{qRR}(l^\infty(A)/J_{2,\omega,\mathrm{T}(A)}) = 0$ *(Definition 6.2).*
(6) $\mathrm{RR}(l^\infty(A)/J_{2,\omega,\mathrm{T}(A)}) = 0$.
(7) $\mathrm{RR}(l^\infty(D)/J_{2,\omega,\mathrm{T}(A)}) = 0$.
(8) *A has uniform property Γ (Definition 2.5).*
(9) *(D, A) has strong uniform property Γ (Definition 2.15).*
(10) *$(D, \mathrm{T}(A))$ is approximately divisible (Definition 2.15).*
(11) $\mathrm{mdim}(X, \mathbb{Z}^d) = 0$.

This theorem also holds for an arbitrary free and minimal action of an amenable group with the (URP) and (COS) (Definition 2.12), and holds for a simple unital AH algebra with diagonal maps.

The main step is (1) $\Rightarrow$ (2). To prove this implication, we shall introduce the following two properties:

Definition (Definitions 4.1 and 5.1)**.** Let A be a unital C*-algebra and let D be a unital commutative subalgebra.

The pair (D, A) is said to have Property (C) if for any positive contractions $f, g, h \in D$ satisfying $f, g \in \overline{hDh}$, and

$$\mathrm{d}_\tau(f) < \mathrm{d}_\tau(g), \quad \tau \in \mathrm{T}(A),$$

and for any $\varepsilon > 0$, there is a contraction $u \in \overline{hAh} + \mathbb{C}1_A$ such that

$$ufu^* \in_{\varepsilon}^{\|\cdot\|_2} \overline{gAg},$$

$$\mathrm{dist}_{2,\mathrm{T}(A)}(udu^*, (D)_1) < \varepsilon, \quad \mathrm{dist}_{2,\mathrm{T}(A)}(u^*du, (D)_1) < \varepsilon, \quad d \in (D)_1,$$

and

$$\|uu^* - 1\|_{2,\mathrm{T}(A)}, \ \|u^*u - 1\|_{2,\mathrm{T}(A)} < \varepsilon.$$

The pair (D, A) is said to have Property (E) if for any positive contraction $a \in A$, any finite subset $\mathcal{F} \subseteq \mathrm{C}([0, 1])$, and any $\varepsilon > 0$, there is a positive contraction $b \in D$ such that

$$|\tau(f(a)) - \tau(f(b))| < \varepsilon, \quad f \in \mathcal{F}, \ \tau \in \mathrm{T}(A).$$

Property (C) can be regarded as a relative comparison property for D inside A, with respect to the uniform trace norm, but requires the comparison being implemented by almost normalizers. While property (E) is an existence property for affine functions of the trace simplex.

Properties (C) and (E) hold for the pair $(\mathrm{C}(X), \mathrm{C}(X) \rtimes \Gamma)$, where (X, Γ) is free and minimal with the (URP), and hold for the pair (D, A), where A is an AH algebra with diagonal maps and D is the standard diagonal subalgebra (Theorem 4.6 and Theorem 5.3).

For the implication (1) $\Rightarrow$ (2) of Theorem B, in order to show the (SBP) for the pair $(D, \mathrm{T}(A))$, by [10], it is enough to show that every self-adjoint element of D can be approximated by self-adjoint elements of D with a neighbourhood of 0 uniformly small under all tracial spectral measures (see Theorem 2.9). By Property (S), such approximating elements exist, but only in the ambient C*-algebra A. However, this can be fixed by using Properties (C) and (E): Upon using Property (E), one obtains a self-adjoint element in the subalgebra D which almost has the same trace spectral measure distributions as the self-adjoint element in A provided by Property (S), and then upon using Property (C), this element can be twisted inside D to approximate the given self-adjoint element and still have a neighbourhood of 0 uniformly small under all tracial spectral measures. This shows the (SBP) for $(D, \mathrm{T}(A))$.

Theorem B also has the following two immediate corollaries:

Corollary C (Corollary 8.7)**.** *Let $(X, \mathbb{Z}^d)$ be a free and minimal dynamical system. If the crossed product C*-algebra $A = \mathrm{C}(X) \rtimes \mathbb{Z}^d$ has real rank zero, then $\mathrm{mdim}(X, \mathbb{Z}^d) = 0$ and $A \cong A \otimes \mathcal{Z}$.*

Corollary D (Corollary 8.9). *Villadsen algebras of the first type (*[38]*) do not have uniform property* Γ.

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2. Preliminaries and notation

In this section, let us collect some notation and definitions concerning C*-algebras and dynamical systems.

2.1. **Comparison of positive elements and $\mathcal{Z}$-absorbing C*-algebras.** Let A be a C*-algebra, and let $a, b \in A$ be positive elements. One says that a is Cuntz subequivalent to b, denoted by $a \precsim b$, if there is a sequence (x_n) in A such that

$$\lim_{n\to\infty} x_n^* b x_n = a.$$

The following lemma will be frequently used.

Lemma 2.1 ([29]). *If $a, b \in A$ are positive elements such that $\|a - b\| < \varepsilon$, then $(a-\varepsilon)_+ \precsim b$, where $(a-\varepsilon)_+ = f(a)$ with $f(t) = \max\{t-\varepsilon, 0\}$, $t \in \mathbb{R}$.*

Let $\tau \in \mathrm{T}(A)$. For each positive element $a \in A$, define

$$\mathrm{d}_\tau(a) = \lim_{n\to\infty} \tau(a^{\frac{1}{n}}).$$

Then, if $a \precsim b$, one has

$$\mathrm{d}_\tau(a) \leq \mathrm{d}_\tau(b), \quad \tau \in \mathrm{T}(A).$$

The converse in general does not hold.

Definition 2.2 ([17]). The Jiang-Su algebra $\mathcal{Z}$ is the (unique) simple unital inductive limit of dimension drop C*-algebras such that

$$(\mathrm{K}_0(\mathcal{Z}), \mathrm{K}_0^+(\mathcal{Z}), [1_{\mathcal{Z}}]_0) \cong (\mathbb{Z}, \mathbb{Z}^+, 1), \quad \mathrm{K}_1(\mathcal{Z}) = \{0\}, \quad \text{and} \quad \mathrm{T}(\mathcal{Z}) = \{\mathrm{pt}\}.$$

A C*-algebra A is said to be $\mathcal{Z}$-absorbing if $A \cong A \otimes \mathcal{Z}$.

If A is simple and $\mathcal{Z}$-absorbing, then the strict order induced by the Cuntz subequivalence relation is determined by the rank functions; that is, for any positive elements $a, b \in A$,

$$\mathrm{d}_\tau(a) < \mathrm{d}_\tau(b), \quad \tau \in \mathrm{T}(A) \quad \Longrightarrow \quad a \precsim b.$$

The Toms-Winter conjecture asserts that strict comparison implies $\mathcal{Z}$-absorption for simple separable amenable C*-algebras (this was verified for C*-algebras with finitely many extreme traces in [24], and then it was generalized to C*-algebras with extreme traces being compact and finite dimensional; see [30], [19], [37]).

The class of simple separable amenable $\mathcal{Z}$-absorbing C*-algebras which satisfy the Universal Coefficient Theorem can be classified by the conventional Elliott Invariant, which in the unital case consists of the K-groups and the pairing with the trace simplex (the order on the K-group, not redundant in more general cases, is determined by the pairing) (see [12], [13], [8], [5], [4], [33], [2]):

Theorem 2.3. *Let A, B be unital simple separable amenable $\mathcal{Z}$-absorbing C*-algebras which satisfy the UCT. Then*

$$A \cong B \iff \mathrm{Ell}(A) \cong \mathrm{Ell}(B),$$

where $\mathrm{Ell}(\cdot)$ *denotes the Elliott invariant. Moreover, any isomorphism between the Elliott invariant can be lifted to an isomorphism between the C*-algebras.*

2.2. **Uniform trace norm.**

Definition 2.4. Let A be a unital C*-algebra, and let $\tau \in \mathrm{T}(A)$. Define

$$\|a\|_{2,\tau} = (\tau(a^*a))^{\frac{1}{2}}, \quad a \in A.$$

For any set $\Delta \subseteq \mathrm{T}(A)$, define the uniform trace norm

$$\|a\|_{2,\Delta} = \sup\{\|a\|_{2,\tau} : \tau \in \Delta\}, \quad a \in A.$$

The uniform trace norm satisfies

$$\|ab\|_{2,\Delta} \leq \min\{\|a\|\|b\|_{2,\Delta}, \|a\|_{2,\Delta}\|b\|\} \quad \text{and} \quad |\tau(a)| \leq \|a\|_{2,\Delta}, \quad a, b \in A,\ \tau \in \Delta.$$

We shall use $l^\infty(A)$ to denote the C*-algebra of bounded sequences of A, i.e.,

$$l^\infty(A) = \{(a_n) : a_n \in A,\ \sup\{\|a_n\| : n = 1, 2, ...\} < +\infty\}.$$

Let ω be a free ultrafilter; then the trace-kernel is the ideal

$$J_{2,\omega,\mathrm{T}(A)} := \{(a_n) \in l^\infty(A) : \lim_{n\to\omega} \|a_n\|_{2,\mathrm{T}(A)} = 0\}.$$

Definition 2.5 (Definition 2.1 of [1]). A C*-algebra A is said to have uniform property Γ if for each $n \in \mathbb{N}$, there is a partition of unity

$$p_1, p_2, ..., p_n \in (l^\infty(A)/J_{2,\omega,\Delta}) \cap A'$$

such that

$$\tau(p_i a p_i) = \frac{1}{n}\tau(a), \quad a \in A,\ \tau \in \mathrm{T}(A)_\omega,$$

where $\mathrm{T}(A)_\omega$ denotes the set of limit traces of $l^\infty(A)$, i.e., the traces of the form

$$\tau((a_i)) = \lim_{i\to\omega} \tau_i(a_i), \quad \tau_i \in \mathrm{T}(A),$$

and a is regarded as the constant sequence $(a) \in l^\infty(A)$.

All $\mathcal{Z}$-absorbing C*-algebras have uniform property Γ (see Theorem 5.6 of [1]). (Indeed, all unital simple amenable C*-algebras with unique trace have (uniform) property Γ, and if a C*-algebra U has uniform property Γ, then the tensor product C*-algebra $A \otimes U$ has uniform property Γ (an extreme trace on a tensor product is a product trace).)

2.3. **AH algebras with diagonal maps.**

Definition 2.6. An AH algebra with diagonal maps is the limit of an inductive sequence

$$A_1 \longrightarrow A_2 \longrightarrow \cdots \longrightarrow A = \varinjlim A_n,$$

where $A_i = \bigoplus_j \mathrm{M}_{n_{i,j}}(\mathrm{C}(X_{i,j}))$, and each connecting map preserves the diagonal subalgebras, i.e., it has the form

$$f \mapsto \mathrm{diag}\{f \circ \lambda_1, ..., f \circ \lambda_m\},$$

where the λs are continuous maps between the Xs.

All simple unital AH algebras with diagonal maps have stable rank one ([6]), but not all AH algebras with diagonal maps are $\mathcal{Z}$-absorbing. In the pioneering work [38], Villadsen constructed simple AH algebras with diagonal maps which have perforation in the ordered K_0-group. The construction was then used in [34] to obtain a simple AH algebra with diagonal maps which has the same value of the conventional Elliott invariant as an AI algebra, but is not isomorphic to this AI algebra. Although Villadsen algebras are not $\mathcal{Z}$-absorbing, a preliminary classification is obtained in [7].

2.4. **The small boundary property and the mean dimension.**

Definition 2.7. A topological dynamical system (X, Γ) is free if $x\gamma = x$ implies $\gamma = e$ where $x \in X$ and $\gamma \in \Gamma$. It is said to be minimal if the only closed invariant subspaces of X are $\emptyset$ and X.

A topological dynamical system induces an action of Γ on $\mathrm{C}(X)$ by

$$\gamma(f)(x) = f(x\gamma), \quad x \in X.$$

We shall assume Γ is discrete. The (universal) crossed product C*-algebra $\mathrm{C}(X) \rtimes \Gamma$ is the universal C*-algebra generated by $\mathrm{C}(X)$ and unitaries u_γ, $\gamma \in \Gamma$, with respect to the relations

$$u_\gamma^* f u_\gamma = \gamma(f) \quad \text{and} \quad u_{\gamma_1} u_{\gamma_2}^* = u_{\gamma_1\gamma_2^{-1}}, \quad f \in \mathrm{C}(X),\ \gamma, \gamma_1, \gamma_2 \in \Gamma.$$

If Γ is amenable and the dynamical system (X, Γ) is free and minimal, the C*-algebra $\mathrm{C}(X) \rtimes \Gamma$ is simple, unital, amenable, stably finite, and satisfies the UCT. However, the C*-algebra $\mathrm{C}(X) \rtimes \Gamma$ may fail to be $\mathcal{Z}$-absorbing, even for $\Gamma = \mathbb{Z}$ ([11]).

Let us consider the following property of dynamical systems.

Definition 2.8. A topological dynamical system (X, Γ) is said to have the small boundary property (SBP) if for any $x \in X$ and any open neighbourhood U of x, there is a neighbourhood V of x such that $V \subseteq U$ and $\mu(\partial V) = 0$ for all invariant measures μ. ([23])

More generally, consider a metrizable compact space X and a collection Δ of Borel probability measures on X. The pair (X, Δ) is said to have the (SBP) if for any $x \in X$ and any open neighbourhood U of x, there is a neighbourhood V of x such that $V \subseteq U$ and $\mu(\partial V) = 0$ for all $\mu \in \Delta$. ([10])

We have the following criterion for the (SBP):

Theorem 2.9 (Theorem 2.9 of [10])**.** *Let X be a metrizable compact space, and let Δ be a compact set of Borel probability measures on X. Then the pair (X, Δ) has the (SBP) if, and only if, for any continuous real-valued function $f : X \to \mathbb{R}$ and any $\varepsilon > 0$, there is a continuous real-valued function $g : X \to \mathbb{R}$ such that*

(1) $\|f - g\|_{2,\Delta} < \varepsilon$, *and*
(2) *there is $\delta > 0$ such that $\tau_\mu(\chi_\delta(g)) < \varepsilon$, $\mu \in \Delta$, where τ_μ is the tracial state of* $\mathrm{C}(X)$ *induced by μ, and*

$$\chi_\delta(t) = \begin{cases} 1, & |t| < \delta, \\ 2 - |t|/\delta, & |t| < 2\delta, \\ 0, & \text{otherwise.} \end{cases}$$

Mean topological dimension was introduced by Gromov ([14]), and then was developed and studied systematically by Lindenstrauss and Weiss ([23]):

Definition 2.10. Consider a topological dynamical system (X, Γ), where Γ is discrete and amenable. Its mean dimension is defined as

$$\mathrm{mdim}(X, \Gamma) := \sup_{\mathcal{U}} \lim_{n\to\infty} \frac{1}{|\Gamma_n|} \mathcal{D}\Big(\bigwedge_{\gamma\in\Gamma_n} \mathcal{U}\gamma^{-1}\Big),$$

where $\Gamma_1, \Gamma_2, ...$ is a Følner sequence of Γ, the supremum is taken over all finite open covers $\mathcal{U}$ of X, and $\mathcal{D}(\mathcal{U}) = \min\{\mathrm{ord}(\mathcal{V}) : \mathcal{V} \prec \mathcal{U}\}$ (ord is the maximal number of the mutually overlapping sets minus 1).

By [23], the small boundary property of (X, Γ) implies zero mean dimension. The converse was shown in [15] and [16] for $\Gamma = \mathbb{Z}^d$, and in [28] for actions with the (URP).

Zero mean dimension (or small boundary property) implies the $\mathcal{Z}$-absorption of the C*-algebra:

Theorem 2.11 ([9])**.** *Let $(X, \mathbb{Z})$ be a free and minimal dynamical system. If* $\mathrm{mdim}(X, \mathbb{Z}) = 0$, *then the C*-algebra* $\mathrm{C}(X) \rtimes \mathbb{Z}$ *is $\mathcal{Z}$-absorbing.*

The main motivation of this work is the converse of this theorem.

2.5. **Uniform Rokhlin property and Cuntz comparison of open sets.** The following two properties were introduced in [27].

Definition 2.12 (Definition 3.1 and Definition 4.1 of [27])**.** A topological dynamical system (X, Γ), where Γ is a discrete amenable group, is said to have the uniform Rokhlin property (URP) if for any $\varepsilon > 0$ and any finite set $K \subseteq \Gamma$, there exist closed sets $B_1, B_2, ..., B_S \subseteq X$ and (K, ε)-invariant sets $\Gamma_1, \Gamma_2, ..., \Gamma_S \subseteq \Gamma$ such that the transformed sets

$$B_s\gamma, \quad \gamma \in \Gamma_s, \quad s = 1, ..., S,$$

are mutually disjoint and

$$\mathrm{ocap}\Big(X \setminus \bigsqcup_{s=1}^{S} \bigsqcup_{\gamma\in\Gamma_s} B_s\gamma\Big) < \varepsilon,$$

where the abbreviation ocap stands for orbit capacity (see, for instance, Definition 5.1 of [23]).

The dynamical system (X, Γ) is said to have (λ, m)-Cuntz-comparison of open sets, where $\lambda \in (0, 1]$ and $m \in \mathbb{N}$, if for any open sets $E, F \subseteq X$ with

$$\mu(E) < \lambda\mu(F), \quad \mu \in \mathcal{M}_1(X, \Gamma),$$

where $\mathcal{M}_1(X, \Gamma)$ is the simplex of all invariant probability measures on X, it follows that

$$\varphi_E \precsim \underbrace{\varphi_F \oplus \cdots \oplus \varphi_F}_{m} \quad \text{in } \mathrm{C}(X) \rtimes \Gamma,$$

where φ_E and φ_F are continuous functions with open supports E and F respectively.

The dynamical system (X, Γ) is said to have Cuntz comparison of open sets (COS) if it has (λ, m)-Cuntz-comparison on open sets for some λ and m.

Theorem 2.13 ([27] and [28])**.** *Let (X, Γ) be a minimal and free dynamical system.*

- *If $\Gamma = \mathbb{Z}^d$, then (X, Γ) has the (URP) and (COS).*
- *If Γ is finitely generated and has sub-exponential growth, and if (X, Γ) has a Cantor factor, then (X, Γ) has the (URP) and (COS).*

The (UPR) implies that the C*-algebra $\mathrm{C}(X) \rtimes \Gamma$ can be weakly tracially approximated by the homogeneous C*-algebras generated by the Rokhlin towers. Together with the (COS), it has the following implications for the C*-algebra $\mathrm{C}(X) \rtimes \Gamma$:

Theorem 2.14 ([27], [26], [20])**.** *Let (X, Γ) be a minimal and free dynamical system with the (URP) and (COS), and let $A = \mathrm{C}(X) \rtimes \Gamma$. Then:*

- $\mathrm{rc}(A) \leq \frac{1}{2}\mathrm{mdim}(X, \Gamma)$, *where $\mathrm{rc}(A)$ is the radius of comparison of A;*
- *A has stable rank one, i.e., invertible elements are dense;*
- *$A \cong A \otimes \mathcal{Z}$ if, and only if, A has strict comparison for positive elements; in particular,*
- *if (X, Γ) has the (SBP), then $A \cong A \otimes \mathcal{Z}$.*

2.6. **Strong uniform property Γ and approximate divisibility.** In contrast to uniform property Γ, strong uniform property Γ and approximate divisibility were introduced in [10]:

Definition 2.15. Let A be a C*-algebra and let $D \subseteq A$ be a sub-C*-algebra. The pair (D, A) is said to have strong uniform property Γ if for each $n \in \mathbb{N}$, there is a partition of unity

$$p_1, p_2, ..., p_n \in (l^\infty(D)/J_{2,\omega,\Delta}) \cap A'$$

such that

$$\tau(p_i a p_i) = \frac{1}{n}\tau(a), \quad a \in A, \ \tau \in \mathrm{T}(A)_\omega,$$

where $\mathrm{T}(A)_\omega$ denotes the set of limit traces of $l^\infty(A)$, i.e., the traces of the form

$$\tau((a_i)) = \lim_{i \to \omega} \tau_i(a_i), \quad \tau_i \in \mathrm{T}(A),$$

and a is regarded as the constant sequence $(a) \in l^\infty(A)$.

Consider a commutative C*-algebra $D \cong \mathrm{C}(X)$, and consider a collection Δ of Borel probability measures on X. The pair (X, Δ) is said to be (tracially) approximately divisible if there is $K > 0$ such that for each $n \in \mathbb{N}$, there is a partition of unity

$$p_1, p_2, ..., p_n \in l^\infty(D)/J_{2,\omega,\Delta}$$

such that

$$\tau(p_i a p_i) \leq \frac{1}{n} K \tau(a), \quad a \in D^+,\ \tau \in \Delta_\omega,\ i = 1, ..., n.$$

It is clear that strong uniform property Γ for (D, A) implies approximate divisibility of $(D, \mathrm{T}(A)|_A)$.

Theorem 2.16 ([10])**.** *If (D, Δ) is approximately divisible, then (D, Δ) has the (SBP).*

3. Comparison in $\mathrm{M}_n(\mathrm{C}(X))$ using almost normalizers

Consider the homogeneous C*-algebra $A = \mathrm{M}_n(\mathrm{C}(X))$ and its diagonal subalgebra D, where X is a metrizable compact space. It is known that D always has relative comparison properties in A, regardless of the dimension of X (see, for instance, [27]). In this section, let us establish the technically important fact that the matrices implementing the (relative) comparison can be chosen to be close (with respect to the uniform trace norm) to permutation unitaries, so implemented by almost normalizers (Theorem 3.14).

3.1. **Well-supported elements.** Well-supported functions were introduced in [35] to study comparison of positive elements of $\mathrm{M}_n(\mathrm{C}(X))$ by means of their rank functions. We will also use the property of well-supportedness to study the comparison of diagonal elements of $\mathrm{M}_n(\mathrm{C}(X))$, but without conditions on the dimension of X (as in [35]).

Definition 3.1 (cf. [35])**.** Let

$$f = \mathrm{diag}\{f_1, f_2, ..., f_n\} \in D \subseteq \mathrm{M}_n(\mathrm{C}(X))$$

be a positive element. Assume $\mathrm{rank}(f)$ has values $n_1, ..., n_L$. and set

$$E_i = \{x \in X : \mathrm{rank}(f(x)) = n_i\}, \quad i = 1, ..., L.$$

The matrix-valued continuous function f is said to be well supported if

(1) for each $i = 0, 1, ..., n$, the range projection of $f|_{E_i}$ extends to a projection p_i over $\overline{E_i}$, and
(2) if $x \in \overline{E_i} \cap \overline{E_j}$ where $i < j$, then $p_i(x) \leq p_j(x)$.

Note that, since f is diagonal, its range projection is the rank n_i projection

$$p_i = \mathrm{diag}\{\mathbf{1}_{(0,\infty)}(f_1), ..., \mathbf{1}_{(0,\infty)}(f_n)\}$$

over E_i.

Lemma 3.2 (cf. Theorem 3.9 of [35])**.** *Let $A = \mathrm{M}_n(\mathrm{C}(X))$, where X is a finite simplicial complex, and let D be the diagonal subalgebra. Let $f \in D$ be a positive element. Then, for any $\varepsilon > 0$, there is a well-supported positive element $\tilde{f} \in D$ such that*

(1) $\tilde{f} \leq f$,

(2) $\|f - \tilde{f}\| < \varepsilon$, *and*

(3) *upon a refinement, the sets* $\overline{E_i}$ *of* $\tilde{f}$ *(of Definition 3.1) are finite subcomplexes of* X.

Proof. The lemma follows from the proof of Theorem 3.9 of [35]. □

3.2. **Decomposition of X with respect to well-supported functions.** Consider well-supported positive diagonal matrix-valued functions

$$f = \mathrm{diag}\{f_1, ..., f_n\} \quad \text{and} \quad g = \mathrm{diag}\{g_1, ..., g_n\},$$

and assume both of them satisfy Condition (3) of Lemma 3.2.

Write

$$F_i = \{x \in X : \mathrm{rank}(f(x)) = m_i\} \quad \text{and} \quad G_j = \{x \in X : \mathrm{rank}(g(x)) = n_j\},$$

where $m_1 \leq \cdots \leq m_M$ and $n_1 \leq \cdots \leq n_N$. Assume that F_i and G_j are finite subcomplexes of X, upon a refinement.

Note that the sets

$$F_1 \sqcup \cdots \sqcup F_i \quad \text{and} \quad G_1 \sqcup \cdots \sqcup G_j, \quad i = 1, ..., M, \ j = 1, ..., N, \tag{3.1}$$

are closed in X.

For each $p = 1, ..., n$, denote the open supports of f_p and g_p respectively by

$$O_{f_p} = \{x \in X : f_p(x) > 0\} \quad \text{and} \quad O_{g_p} = \{x \in X : g_p(x) > 0\}. \tag{3.2}$$

Since f is well supported, the range projection of $f|_{E_i}$ extends to a projection p_i over $\overline{E_i}$, and therefore each set $\overline{F_i}$, $i = 1, ..., M$, has a (disjoint) decomposition

$$\overline{F_i} = \overline{F_{i,1}} \sqcup \cdots \sqcup \overline{F_{i,l_i}} \tag{3.3}$$

such that the restriction of (the extension of) the range projection p_i to each $\overline{F_{i,s}}$, $s = 1, ..., l_i$, is constant (diagonal), and the restrictions of p_i to different $\overline{E_{i,s}}$ have different values.

Write the induced decomposition of F_i as

$$F_i = F_{i,1} \sqcup \cdots \sqcup F_{i,l_i},$$

and denote the *shape* of f on $F_{i,s}$ by $\mathcal{F}_{i,s}$, i.e.,

$$\mathcal{F}_{i,s} := \{p = 1, ..., n : f_p(x) > 0, \ x \in F_{i,s}\}.$$

Note that, with the notation above, one has

$$F_{i,s} = \{x \in X : f_p(x) > 0, \ f_q(x) = 0, \ p \in \mathcal{F}_{i,s}, \ q \notin \mathcal{F}_{i,s}\}.$$

So, $F_{i,s} = F_{i,s'}$ if, and only if, $\mathcal{F}_{i,s} = \mathcal{F}_{i,s'}$. In other words, the sets $F_{i,s}$ are determined by their shapes.

Also note that, for any $i \leq i'$,

$$F_{i,s} \cap \overline{F_{i',s'}} = F_{i,s} \cap \overline{\Big(\bigcap_{p \in \mathcal{F}_{i',s'} \setminus \mathcal{F}_{i,s}} O_{f_p}\Big)}. \tag{3.4}$$

List the sets $F_{i,s}$, $s = 1, ..., l_i$, $i = 1, ..., M$, as

$$F_{1,1}, ..., F_{1,l_1}, F_{2,1}, ..., F_{2,l_2}, ..., F_{M,1}, ..., F_{M,l_M},$$

and re-index them as $F_1, ..., F_{M'}$ (with an abuse of notation). By Condition (2) of Definition 3.1, (3.3), and (3.1), this list of sets has the following properties:

(1) Each $F_1 \cup \cdots \cup F_i$, $i = 1, ..., M'$, is closed, and
(2) If $\overline{F_i} \cap \overline{F_j} \neq \emptyset$ and $i \leq j$, then $\mathcal{F}_i \subseteq \mathcal{F}_j$.

Similarly, for each G_j, $j = 1, ..., N$, there are decompositions

$$\overline{G_j} = \overline{G_{j,1}} \sqcup \cdots \sqcup \overline{G_{j,r_j}}$$

and

$$G_j = G_{j,1} \sqcup \cdots \sqcup G_{j,r_j}$$

such that the restriction of the range projection q_j to each $\overline{G_{j,t}}$, $t = 1, ..., r_j$, is constant; denote the *shape* of the function f on $G_{j,t}$ by

$$\mathcal{G}_{j,t} := \{p = 1, ..., n : g_p(x) > 0,\ x \in G_{j,t}\}.$$

List the sets $G_{j,t}$, $t = 1, ..., s_j$, $j = 1, ..., N$, as

$$G_{1,1}, ..., G_{1,s_1}, G_{2,1}, ..., G_{2,l_2}, ..., G_{M,1}, ..., G_{M,l_M},$$

and re-index them as $G_1, ..., G_{N'}$ (with an abuse of notation). This list of sets has the following properties:

(1) Each $G_1 \cup \cdots \cup G_i$, $i = 1, ..., M'$, is closed, and
(2) If $\overline{G_i} \cap \overline{G_j} \neq \emptyset$ and $i \leq j$, then $\mathcal{F}_i \subseteq \mathcal{F}_j$.

Let us set

$$Z_{i,j} = F_i \cap G_j,$$

(where F_i and G_j are the sets in the re-indexed sequence) so that we have the following decomposition of X:

$$X = \bigsqcup_{i,j} Z_{i,j}. \tag{3.5}$$

For each $Z_{i,j}$, define

$$\mathrm{dom}(Z_{i,j}) = \mathcal{F}_i \quad \text{and} \quad \mathrm{codom}(Z_{i,j}) = \mathcal{G}_j,$$

and the *shape* of $Z_{i,j}$ as

$$(\mathcal{F}_i, \mathcal{G}_j).$$

Note that, by the construction, each set $Z_{i,j}$ is determined by its shape.

Then list the sets $Z_{i,j}$, $i = 1, ..., M'$, $j = 1, ..., N'$, as

$$\underbrace{Z_{1,1}, ..., Z_{1,N'}}_{F_1}, \underbrace{Z_{2,1}, ..., Z_{2,N'}}_{F_2}, ..., \underbrace{Z_{M',1}, ..., Z_{M',N'}}_{F_{M'}}.$$

Definition 3.3 (Notation)**.** Re-index this list of the sets $\{Z_{i,j}\}$ as $Z_1, Z_2, ..., Z_L$, and also re-index the shape of Z_i as $(\mathcal{F}_i, \mathcal{G}_i)$. Then this list has the following properties:

(1) Each $Z_1 \sqcup \cdots \sqcup Z_k$ is closed in X.

(2) If $\overline{Z_i} \cap \overline{Z_j} \neq \emptyset$ and $i \leq j$, then $(\mathcal{F}_i, \mathcal{G}_i) \subseteq (\mathcal{F}_j, \mathcal{G}_j)$ (where $(\mathcal{F}_i, \mathcal{G}_i) \subseteq (\mathcal{F}_j, \mathcal{G}_j)$ denotes $\mathcal{F}_i \subseteq \mathcal{F}_j$ and $\mathcal{G}_i \subseteq \mathcal{G}_j$).
(3) For each $k = 1, ..., n-1$, the closed set $(Z_1 \cup \cdots \cup Z_k) \cap \overline{Z_{k+1}}$ has a neighbourhood retraction in $\overline{Z_{k+1}}$. (This follows from Condition (3) of Lemma 3.2.)

Note that

$$Z_i \cap \overline{Z_j} = Z_i \cap \overline{(\bigcap_{p \in \mathcal{F}_j \setminus \mathcal{F}_i} \bigcap_{q \in \mathcal{G}_j \setminus \mathcal{G}_i} O_{f_p} \cap O_{g_q})}, \quad \text{if } (\mathcal{F}_i, \mathcal{G}_i) \subseteq (\mathcal{F}_j, \mathcal{G}_j).$$

This implies that if

$$(\mathcal{F}_i, \mathcal{G}_i) \subseteq (\mathcal{F}_k, \mathcal{G}_k) \subseteq (\mathcal{F}_j, \mathcal{G}_j)$$

for some k, then

$$\begin{aligned} & Z_i \cap \overline{Z_j} \\ = \ & Z_i \cap \overline{(\bigcap_{p \in \mathcal{F}_j \setminus \mathcal{F}_i} \bigcap_{q \in \mathcal{G}_j \setminus \mathcal{G}_i} O_{f_p} \cap O_{g_q})} \\ \subseteq \ & Z_i \cap \overline{(\bigcap_{p \in \mathcal{F}_k \setminus \mathcal{F}_i} \bigcap_{q \in \mathcal{G}_k \setminus \mathcal{G}_i} O_{f_p} \cap O_{g_q})} \\ = \ & Z_i \cap \overline{Z_k}. \end{aligned} \tag{3.6}$$

3.3. **Almost normalizers and comparison.**

Definition 3.4. Let $\delta > 0$ and $n \in \mathbb{N}$. Consider a matrix algebra $A = \mathrm{M}_n(\mathbb{C})$ and its diagonal subalgebra D. Then a unitary $u \in A$ is said to be a δ-normalizer if there is a unitary $v \in A$ such that

(1) $vDv^* = D$, and
(2) $\|u - v\|_{2, \mathrm{T}(A)} < \delta$.

Denote by $P_n(\delta)$ the set of $n \times n$ unitaries which are also δ-normalizers.

Remark 3.5. If u is a δ-normalizer, then, for any contraction $d \in D$, one has

$$\mathrm{dist}_2(udu^*, D) \leq \mathrm{dist}_2(udu^*, vdv^*) < 2\delta.$$

Remark 3.6. Let σ be a permutation of $\{1, 2, ..., n\}$. Then the permutation unitary u_σ is a δ-normalizer for all $\delta > 0$.

Lemma 3.7. *Let σ_1, σ_2 be two permutations of $\{1, 2, ..., n\}$ and let $\mathcal{F}_1,\ \mathcal{F}_2,\ \mathcal{G} \subseteq \{1, 2, ..., n\}$ be such that*

$$\mathcal{F}_1 \subseteq \mathcal{F}_2,$$

$$\sigma_i(\mathcal{F}_1) \subseteq \mathcal{G} \quad \text{and} \quad \sigma_i|_{\{1,...,n\} \setminus (\mathcal{F}_2 \cup \mathcal{G})} = \mathrm{id}, \quad i = 1, 2.$$

Then there is a continuous path

$$u_t \in P_n(2/n), \quad t \in [0, 1], \tag{3.7}$$

such that

$$u_0 = u_{\sigma_1}, \quad u_1 = u_{\sigma_2},$$

and

$$u_t(e_i) \subseteq \mathrm{span}\{e_j : j \in \mathcal{G}\}, \quad i \in \mathcal{F}_1,\ t \in [0,1], \tag{3.8}$$

and

$$u_t(e_i) = e_i, \quad i \in \{1, ..., n\} \setminus (\mathcal{F}_2 \cup \mathcal{G}),\ t \in [0,1].$$

Proof. Write

$$\sigma_1 = \tau_1 \cdots \tau_k \sigma_2,$$

where τ_i, $i = 1, ..., k$, are transpositions of the elements of $\mathcal{G}$ (in particular, they are the identity on $\{1, ..., n\} \setminus (\mathcal{F}_2 \cup \mathcal{G})$). Then, for each each τ_i, $1 \leq i \leq k$, write $\tau_i = (i_1 i_2)$, and for $t \in [0,1]$, define the unitary matrix $v_t^{(i)}$ by

$$v_t^{(i)}(e_i) = e_i, \quad i \neq i_1, i_2,$$

$$v_t^{(i)}(e_{i_1}) = \frac{1}{2}((1 + e^{\pi\sqrt{-1}(1-t)})e_{i_1} + (1 - e^{\pi\sqrt{-1}(1-t)})e_{i_2})$$

and

$$v_t^{(i)}(e_{i_2}) = \frac{1}{2}((1 - e^{\pi\sqrt{-1}(1-t)})e_{i_1} + (1 + e^{\pi\sqrt{-1}(1-t)})e_{i_2}).$$

Note that

$$v_0^{(i)} = u_{\tau_i}, \quad v_1^{(i)} = I_n, \quad \text{and} \quad \|v_t^{(i)} - I_n\|_{2,\mathrm{tr}} \leq 2/n, \quad t \in [0,1].$$

Hence, defining

$$u_t^{(i)} = v_t^{(i)} u_{\tau_{i+1}} \cdots u_{\tau_k} u_{\sigma_2},$$

one has

$$u_0^{(i)} = u_{\tau_i} \cdots u_{\tau_k} u_{\sigma_2}, \quad u_1^{(i)} = u_{\tau_{i+1}} \cdots u_{\tau_k} u_{\sigma_2}, \quad \text{and} \quad \|u_t^{(i)} - u_{\tau_{i+1}} \cdots u_{\tau_k} u_{\sigma_2}\|_2 < 2/n, \quad t \in [0,1].$$

In particular,

$$u_t^{(i)} \in P_n(2/n), \quad t \in [0,1].$$

Then, connecting the paths

$$u_t^{(1)},\ u_t^{(2)}, ...,\ u_t^{(k)},$$

and renormalizing the parameter, one has the desired homotopy. □

Remark 3.8. It follows from (3.8) that

$$u_t f u_t^* \subseteq \mathrm{Her}(\mathcal{G}), \quad f \in \mathrm{Her}(\mathcal{F}_1),$$

where, for any set $\mathcal{F} \subseteq \{1, ..., n\}$, $\mathrm{Her}(\mathcal{F})$ denotes the hereditary subalgebra of $\mathrm{M}_n(\mathbb{C})$ generated by $\{e_p : p \in \mathcal{F}\}$.

Lemma 3.9. *Let $A = \mathrm{M}_n(\mathrm{C}(X))$, where X is a simplicial complex, and let $D \subseteq A$ be the diagonal subalgebra. Let $f, g, h \in D$ be positive elements such that f, g are well supported, $f, g \in \overline{hDh}$ and the sets $\overline{F_i}$ of f and $\overline{G_j}$ of g are subcomplexes of X (upon a refinement). If*

$$\mathrm{rank}(f(x)) \leq \mathrm{rank}(g(x)), \quad x \in X, \tag{3.9}$$

then, for any $\varepsilon > 0$, there is a unitary $u \in \overline{hAh} + \mathbb{C}1$ such that

$$u(x) \in P_n(2/n) \quad \text{and} \quad (ufu^*)(x) \in_\varepsilon \overline{(gAg)(x)}, \quad x \in X.$$

Proof. In the setting above, by (3.9), one has

$$|\mathcal{F}_i| \leq |\mathcal{G}_i|, \quad i = 1, 2, ..., L.$$

Then, for each Z_i, $i = 1, ..., L$, pick a permutation σ_i of $\{1, 2, ..., n\}$ such that

$$\sigma_i(\mathcal{F}_i) \subseteq \mathcal{G}_i \quad \text{and} \quad \sigma_i(p) = p, \quad p \in \{1, 2, ..., n\} \setminus (\mathcal{F}_i \cup \mathcal{G}_i).$$

Also note that, since $f, g \in \overline{hDh}$, one has

$$\mathcal{F}_i,\ \mathcal{G}_i \subseteq \{p = 1, ..., n : h_p(x) \neq 0\}, \quad x \in Z_i,\ i = 1, ..., n.$$

Therefore,

$$\sigma_i(p) = p, \quad \text{if } h_p(x) = 0 \text{ and } x \in Z_i. \tag{3.10}$$

Starting with Z_1, define the unitary u on Z_1 to be

$$u(x) = u_{\sigma_1}, \quad x \in Z_1.$$

It follows from (3.10) that

$$u(x) \in \overline{(hAh)(x)} + \mathbb{C}1, \quad x \in Z_1,$$

and

$$(ufu^*)(x) \in \overline{(gAg)(x)}, \quad x \in Z_1.$$

Moreover, for each $j \geq 2$ such that $Z_1 \cap \overline{Z_j} \neq \emptyset$, one has $(\mathcal{F}_1, \mathcal{G}_1) \subseteq (\mathcal{F}_j, \mathcal{G}_j)$. By Lemma 3.7 (with σ_1, σ_j, $\mathcal{F}_1$, $\mathcal{F}_j$, and $\mathcal{G}_j$ in place of σ_1, σ_2, $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{G}$, respectively), there is a continuous path

$$w_t \in P_n(2/n), \quad t \in [0, 1],$$

such that

$$w_0 = \sigma_1, \quad w_1 = \sigma_j,$$

$$(w_t f w_t^*)(x) \in \mathrm{Her}(\mathcal{G}_j), \quad x \in Z_1 \cap \overline{Z_j},$$

and

$$w_t(e_p) = e_p, \quad p \in \{1, ..., n\} \setminus (\mathcal{F}_j \cup \mathcal{G}_j).$$

Now, assume inductively that we have constructed a unitary $u \in \overline{hAh} + \mathbb{C}1$ defined on $Z_1 \cup \cdots \cup Z_k$ which satisfies

(A) $(ufu^*)(x) \in_{(k-1)\varepsilon/L} \overline{(gAg)(x)}, \quad x \in Z_1 \cup \cdots \cup Z_k$,

(H) for each Z_j, $j \geq k+1$, the restriction of u to $(Z_1 \cup \cdots \cup Z_k) \cap \overline{Z_j}$ is homotopic to u_{σ_j} through a path

$$w_t : (Z_1 \cup \cdots \cup Z_k) \cap \overline{Z_j} \to P_n(2/n), \quad t \in [0, 1],$$

such that

$$(w_t f w_t^*)(x) \in_{(k-1)\varepsilon/L} \mathrm{Her}(\mathcal{G}_j), \quad x \in (Z_1 \cup \cdots \cup Z_k) \cap \overline{Z_j},\ t \in [0, 1],$$

and

$$w_t(x)(e_p) = e_p, \quad p \in \{1, 2, ..., n\} \setminus (\mathcal{F}_j \cup \mathcal{G}_j).$$

Let us extend u to $(Z_1 \cup \cdots \cup Z_k) \cup Z_{k+1}$ so as still to have the properties (A) and (H) above (for $k+1$). Consider $(Z_1 \cup \cdots \cup Z_k) \cap \overline{Z_{k+1}}$. By the property (H) (for k), there is a path

$$w_t : (Z_1 \cup \cdots \cup Z_k) \cap \overline{Z_j} \to P_n(2/n), \quad t \in [0,1],$$

such that

$$w_0 = u|_{(Z_1 \cup \cdots \cup Z_k) \cap \overline{Z_j}}, \quad w_1 = u_{\sigma_{k+1}},$$

$$(w_t f w_t^*)(x) \in_{(k-1)\varepsilon} \mathrm{Her}(\mathcal{G}_{k+1}), \quad x \in (Z_1 \cup \cdots \cup Z_k) \cap \overline{Z_{k+1}},\ t \in [0,1], \tag{3.11}$$

and

$$w_t(x)(e_p) = e_p, \quad p \in \{1, 2, ..., n\} \setminus (\mathcal{F}_{k+1} \cup \mathcal{G}_{k+1}). \tag{3.12}$$

Inside $\overline{Z_{k+1}}$, pick a neighbourhood $U \supseteq (Z_1 \cup \cdots \cup Z_k) \cap \overline{Z_{k+1}}$ and a retraction

$$r : U \to (Z_1 \cup \cdots \cup Z_k) \cap \overline{Z_{k+1}}.$$

Without loss of generality, one may assume that U is sufficiently small that

$$\|f(r(x)) - f(x)\| < \varepsilon/L, \quad x \in U. \tag{3.13}$$

Choose a continuous function $s : \overline{Z_{k+1}} \to [0,1]$ such that

$$s|_{(Z_1 \cup \cdots \cup Z_k) \cap \overline{Z_{k+1}}} = 0 \quad \text{and} \quad s|_{\overline{Z_{k+1}} \setminus U} = 1.$$

Then extend u to $(Z_1 \cup \cdots \cup Z_k) \cup Z_{k+1}$ by

$$u(x) = \begin{cases} w_{s(x)}(r(x)), & x \in U, \\ u_{\sigma_{k+1}}, & x \in \overline{Z_{k+1}} \setminus U. \end{cases}$$

Note that, by (3.12), one has $u \in \overline{hAh} + \mathbb{C}1$.

Let us first verify that

$$(ufu^*)(x) \in_{k\varepsilon/L} \overline{(gAg)(x)}, \quad x \in (Z_1 \cup \cdots \cup Z_k) \cup Z_{k+1}. \tag{3.14}$$

One only needs to verify it over $\overline{Z_{k+1}}$. If $x \in \overline{Z}_{k+1} \setminus U$, then

$$(ufu^*)(x) = u_{\sigma_{k+1}} f(x) u^*_{\sigma_{k+1}} \in \overline{(gAg)(x)};$$

if $x \in U$, then, by (3.13) and (3.11),

$$\begin{array}{rcl} (ufu^*)(x) & = & w_{s(x)}(r(x)) f(x) w^*_{s(x)}(r(x)) \\ & \approx_{\varepsilon/L} & w_{s(x)}(r(x)) f(r(x)) w^*_{s(x)}(r(x)) \\ & \in_{(k-1)\varepsilon/L} & \mathrm{Her}(\mathcal{G}_{k+1}) \\ & = & \overline{(gAg)(x)}. \end{array}$$

This verifies (3.14). Thus u satisfies the inductive assumption (A) for $k+1$.

Let us now show that the unitary u also satisfies the inductive assumption (H) (for $k+1$). Consider the restriction of u to $\overline{Z_{k+1}}$, and note that the two-variable function

$$H_t(x) = \begin{cases} w_{(1-t)s(x)+t}(r(x)), & x \in U, \\ u_{\sigma_{k+1}}, & x \notin U, \end{cases} \quad t \in [0,1],$$

defines a homotopy between $u|_{\overline{Z_{k+1}}}$ and $u_{\sigma_{k+1}}$.

Then, for each $t \in [0,1]$, if $x \in \overline{Z_{k+1}} \setminus U$, then

$$(H_t f H_t^*)(x) = u_{\sigma_{k+1}} f(x) u_{\sigma_{k+1}}^* \in \overline{(gAg)(x)},$$

and if $x \in U$, then, by (3.13) and (3.11),

$$\begin{aligned} (H_t f H_t^*)(x) &= w_{(1-t)s(x)+t}(r(x)) f(x) w^*_{(1-t)s(x)+t}(r(x)) \\ &\approx_{\varepsilon/L} w_{(1-t)s(x)+t}(r(x)) f(r(x)) w^*_{(1-t)s(x)+t}(r(x)) \\ &\in_{(k-1)\varepsilon/L} \mathrm{Her}(\mathcal{G}_{k+1}). \end{aligned} \tag{3.15}$$

Also note that

$$H_t(x)(e_p) = e_p, \quad p \in \{1, ..., n\} \setminus (\mathcal{F}_{k+1} \cup \mathcal{G}_{k+1}),\ x \in \overline{Z_{k+1}}.$$

For each Z_j, $j \geq k+2$, consider the set

$$(Z_1 \cup \cdots \cup Z_{k+1}) \cap \overline{Z_j}.$$

If $Z_{k+1} \cap \overline{Z_j} = \varnothing$, then u has the homotopy property (H) by the inductive assumption. Thus, one may assume that

$$Z_{k+1} \cap \overline{Z_j} \neq \varnothing,$$

and hence that

$$(\mathcal{F}_{k+1}, \mathcal{G}_{k+1}) \subseteq (\mathcal{F}_j, \mathcal{G}_j).$$

List the special indices

$$\{i_1, ..., i_s\} = \{i = 1, ..., k : (\mathcal{F}_i, \mathcal{G}_i) \subseteq (\mathcal{F}_{k+1}, \mathcal{G}_{k+1})\}.$$

By (3.6),

$$Z_{i_1} \cap \overline{Z_j} \subseteq Z_{i_1} \cap \overline{Z_{k+1}}, \quad ..., \quad Z_{i_s} \cap \overline{Z_j} \subseteq Z_{i_s} \cap \overline{Z_{k+1}},$$

and therefore

$$(Z_{i_1} \cup \cdots \cup Z_{i_s}) \cap \overline{Z_j} \subseteq (Z_{i_1} \cup \cdots \cup Z_{i_s}) \cap \overline{Z_{k+1}} \subseteq \overline{Z_{k+1}}.$$

Note that, for any $i = 1, ..., k$ but $i \notin \{i_1, i_2, ..., i_s\}$, one has

$$\overline{Z}_i \cap \overline{Z_{k+1}} = \varnothing.$$

Indeed, if $\overline{Z}_i \cap \overline{Z_{k+1}} \neq \varnothing$, this would imply either $(\mathcal{F}_i, \mathcal{G}_i) \subseteq (\mathcal{F}_{k+1}, \mathcal{G}_{k+1})$ or $(\mathcal{F}_{k+1}, \mathcal{G}_{k+1}) \subseteq (\mathcal{F}_i, \mathcal{G}_i)$. Since $i \notin \{i_1, i_2, ..., i_s\}$, one must have $(\mathcal{F}_{k+1}, \mathcal{G}_{k+1}) \subseteq (\mathcal{F}_i, \mathcal{G}_i)$, which contradicts Z_{k+1} being a peak of $Z_1, ..., Z_k$ (see (2) of Definition 3.3).

Then

$$(Z_1 \cup \cdots \cup Z_{k+1}) \cap \overline{Z_j} \subseteq \overline{Z_{k+1}} \sqcup \Big(\bigcup_{\substack{i=1,...,k \\ i \notin \{i_1, ..., i_s\}}} \overline{Z}_i \Big).$$

That is, there is a decomposition

$$(Z_1 \cup \cdots \cup Z_{k+1}) \cap \overline{Z_j} = W_1 \sqcup W_2, \tag{3.16}$$

where

$$W_1 \subseteq \overline{Z_{k+1}} \quad \text{and} \quad W_2 \subseteq \bigcup_{\substack{i=1,...,k \\ i \notin \{i_1,...,i_s\}}} \overline{Z_i}.$$

Note that

$$\bigcup_{\substack{i=1,...,k \\ i \notin \{i_1,...,i_s\}}} \overline{Z_i} \subseteq Z_1 \cup \cdots \cup Z_k.$$

By the inductive assumption, the restriction of u to $(Z_1 \cup \cdots \cup Z_k) \cap \overline{Z_j}$ is homotopic to u_{σ_j} through a path

$$w_t : (Z_1 \cup \cdots \cup Z_k) \cap \overline{Z_j} \to P_n(2/n), \quad t \in [0,1],$$

such that

$$(w_t f w_t^*)(x) \in_{(k-1)\varepsilon/L} \mathrm{Her}(\mathcal{G}_j), \quad x \in (Z_1 \cup \cdots \cup Z_k) \cap \overline{Z_j},\ t \in [0,1],$$

and

$$w_t(x)(e_p) = e_p, \quad p \in \{1, ..., n\} \setminus (\mathcal{F}_j \cup \mathcal{G}_j),\ x \in (Z_1 \cup \cdots \cup Z_k) \cap \overline{Z_j},\ t \in [0,1].$$

Then the restriction of w_t to the closed subset W_2 provides a path, still denoted by w_t,

$$w_t : W_2 \to P_n(2/n), \quad t \in [0,1],$$

such that

$$(w_t f w_t^*)(x) \in_{(k-1)\varepsilon/L} \mathrm{Her}(\mathcal{G}_j), \quad x \in W_2,\ t \in [0,1],$$

and

$$w_t(x)(e_p) = e_p, \quad p \in \{1, ..., n\} \setminus (\mathcal{F}_j \cup \mathcal{G}_j),\ x \in W_2,\ t \in [0,1].$$

Let us now work on the closed set W_1, and it is enough to work on $\overline{Z_{k+1}}$. By (3.15), the restriction of u to $\overline{Z_{k+1}}$ is homotopic to $u_{\sigma_{k+1}}$ through a path (H_t), now denoted by w_t,

$$w_t : \overline{Z_{k+1}} \to P_n(2/n), \quad t \in [0,1],$$

such that

$$(w_t f w_t^*)(x) \in_{k\varepsilon/L} \mathrm{Her}(\mathcal{G}_{k+1}), \quad x \in \overline{Z_{k+1}},\ t \in [0,1],$$

and

$$w_t(x)(e_p) = e_p, \quad p \in \{1, ..., n\} \setminus (\mathcal{F}_{k+1} \cup \mathcal{G}_{k+1}),\ x \in \overline{Z_{k+1}}.$$

Since $\mathcal{G}_{k+1} \subseteq \mathcal{G}_j$, by Lemma 3.7 (with σ_{k+1}, σ_j, $\mathcal{F}_{k+1}$, $\mathcal{F}_j$, and $\mathcal{G}_j$ in place of σ_1, σ_2, $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{G}$ respectively), the unitary $u_{\sigma_{k+1}}$ can be connected further to u_{σ_j}, and this then provides the desired homotopy of $u|_{\overline{Z_{k+1}}}$. Thus, the unitary u constructed on $Z_1 \cup \cdots \cup Z_{k+1}$ satisfies the inductive condition (H).

By induction, there is a unitary $u \in \overline{hAh} + \mathbb{C}1$, defined on X, such that

$$u(x) \in P_n(2/n) \quad \text{and} \quad (ufu^*)(x) \in_\varepsilon \overline{(gAg)(x)}, \quad x \in X,$$

as desired. □

The following lemma asserts that if an element is pointwisely close to a hereditary subalgebra, then it is close to the hereditary subalgebra.

Lemma 3.10. *Let $A = \mathrm{M}_n(\mathrm{C}(X))$, where X is a metrizable compact space, and let D be the diagonal subalgebra. Let $f \in A$ and $h \in D$ be positive elements. If*

$$f(x) \in_\varepsilon \overline{(hAh)(x)}, \quad x \in X,$$

for some $\varepsilon > 0$, then

$$f \in_\varepsilon \overline{hAh}.$$

Proof. For each $x \in X$, by the assumption, there is an $n \times n$ matrix $g_x \in \overline{(hAh)(x)}$ such that

$$\|f(x) - g_x\| < \varepsilon.$$

By the continuity of f and h, there is an open set $U \ni x$ such that

$$\|f(y) - g_x\| < \varepsilon \quad \text{and} \quad g_x \in \overline{(hAh)(y)}, \quad y \in U.$$

Since X is compact, there are a finite open cover $U_1, U_2, ..., U_n$, points $x_1 \in U_1, ..., x_n \in U_n$, and $n \times n$ matrices $g_{x_1}, ..., g_{x_n}$ such that

$$\|f(y) - g_{x_i}\| < \varepsilon \quad \text{and} \quad g_{x_i} \in \overline{(hAh)(y)}, \quad y \in U_i,\ i = 1, ..., n.$$

Choose a partition of unity $\{\phi_1, ..., \phi_n : X \to [0, 1]\}$ subordinate to $U_1, ..., U_n$, and define

$$g = \phi_1 g_{x_1} + \cdots + \phi_n g_{x_n} \in A.$$

Since $g_{x_i} \in \overline{(hAh)(y)}$, $y \in U_i$, $i = 1, ..., n$, one has $g \in \overline{hAh}$. Moreover, a straightforward calculation shows that

$$\|f(x) - g(x)\| < \varepsilon, \quad x \in X,$$

which implies that $\|f - g\| < \varepsilon$, and hence $f \in_\varepsilon \overline{hAh}$, as desired. ☐

Proposition 3.11. *Let $A = \mathrm{M}_n(\mathrm{C}(X))$, where X is a finite simplicial complex, and let D be the diagonal subalgebra. Let $f, g, h \in D$ be positive elements such that $f, g \in \overline{hDh}$ and*

$$\mathrm{rank}(f(x)) \leq \mathrm{rank}(g(x)), \quad x \in X.$$

Then, for any $\varepsilon > 0$, there is a unitary $u \in \overline{hAh} + \mathbb{C}1$ such that

$$u(x) \in P_n(2/n), \quad x \in X,$$

and

$$ufu^* \in_\varepsilon \overline{gAg}.$$

Proof. With the given f, g and ε, by Lemma 2.8 of [36], there is $\delta > 0$ such that

$$\mathrm{rank}((f - \varepsilon/2)_+(x)) \leq \mathrm{rank}((g - \delta)_+(x)), \quad x \in X.$$

By Lemma 3.2, there are well-supported elements $\tilde{f}, \tilde{g} \in D$ such that

$$\tilde{f} \leq (f - \varepsilon/2)_+ \quad \text{and} \quad \tilde{g} \leq (g - \delta/2)_+,$$

and

$$\|\tilde{f} - (f - \varepsilon/2)_+\| < \varepsilon/4 \quad \text{and} \quad \|\tilde{g} - (g - \delta/2)_+\| < \delta/4.$$

Applying Lemma 5.1 of [25], one has

$$(g - \delta)_+ \precsim (\tilde{g} - \delta/4)_+.$$

Then, for each $x \in X$,

$$\mathrm{rank}(\tilde{f}(x)) \leq \mathrm{rank}((f-\varepsilon/2)_+(x)) \leq \mathrm{rank}((g-\delta)_+(x)) \leq \mathrm{rank}((\tilde{g}-\delta/4)(x)) \leq \mathrm{rank}(\tilde{g}(x)).$$

Since $\tilde{f}$ and $\tilde{g}$ are well supported, by Lemma 3.9, there is a unitary $u \in \overline{hAh} + \mathbb{C}1$ such that

$$u(x) \in P_n(2/n) \quad \text{and} \quad (u\tilde{f}u^*)(x) \in_{\varepsilon/4} \overline{(\tilde{g}A\tilde{g})(x)}, \quad x \in X.$$

By Lemma 3.10, the second equation implies that

$$u\tilde{f}u^* \in_{\varepsilon/4} \overline{\tilde{g}A\tilde{g}} \subseteq \overline{gAg}.$$

Since

$$\|\tilde{f} - f\| < \varepsilon/4 + \varepsilon/2 = 3\varepsilon/4,$$

one has

$$ufu^* \in_{\varepsilon} \overline{gAg},$$

as desired. ☐

The following lemma asserts that if an element is pointwisely close to the diagonal subalgebra, then it is close to the diagonal subalgebra.

Lemma 3.12. *Let $A = \mathrm{M}_n(\mathrm{C}(X))$, where X is a metrizable compact space, and let D be the diagonal subalgebra. Let $f \in A$ be such that*

$$\mathrm{dist}_{2,\mathrm{tr}_n}(f(x), D(x)) < \delta, \quad x \in X,$$

for some $\delta > 0$. Then there is $g \in D$ such that

$$\|f(x) - g(x)\|_{2,\mathrm{tr}_n} < 2\delta, \quad x \in X.$$

In other words,

$$\mathrm{dist}_{2,\mathrm{T}(A)}(f, D) < 2\delta.$$

Proof. Choose an open cover $U_1, U_2, ..., U_n$ of X such that

$$\|f(x) - f(y)\| < \delta, \quad x, y \in U_i,\ 1 \leq i \leq n.$$

Pick $x_i \in U_i$, $1 \leq i \leq n$, and choose a partition of unity $\{\phi_i,\ 1 \leq i \leq n\}$, subordinate to $U_1, ..., U_n$. For each x_i, $1 \leq i \leq n$, choose a diagonal matrix g_{x_i} such that

$$\|f(x_i) - g_{x_i}\|_{2,\mathrm{tr}_n} < \delta, \quad 1 \leq i \leq n,$$

and therefore

$$\|f(x) - g_{x_i}\|_{2,\mathrm{tr}_n} < 2\delta, \quad x \in U_i,\ 1 \leq i \leq n.$$

Define

$$g(x) = \phi_1(x)g_{x_1} + \cdots + \phi_n(x)g_{x_n}, \quad x \in X.$$

Then, for each $x \in X$, one has

$$\begin{aligned}
\|f(x) - g(x)\|_{2,\mathrm{tr}_n} &= \|(\phi_1(x)f(x) + \cdots + \phi_n(x)f(x)) - (\phi_1(x)g_1 + \cdots + \phi_n(x)g_n)\|_{2,\mathrm{tr}_n} \\
&= \|\phi_1(x)(f(x) - g_1) + \cdots + \phi_n(x)(f(x) - g_n)\|_{2,\mathrm{tr}_n} \\
&< 2\delta,
\end{aligned}$$

as desired. □

The following lemma will be used in the proofs of Theorem 3.14 and Proposition 4.3.

Lemma 3.13. *Let X be a metrizable compact space, and let C_i, $i = 1, 2, ...$, be a family of unital subalgebras of $C := \mathrm{C}(X)$ with dense union. Let $f, g, h \in C$ be positive elements with norm 1 such that*

$$fh = f \quad \text{and} \quad gh = h.$$

Then, for any $\varepsilon > 0$, there are positive elements $\tilde{f}, \tilde{g}, \tilde{h} \in C_i$, where i is sufficiently large, such that

$$\|f - \tilde{f}\| < \varepsilon, \quad \|g - \tilde{g}\| < \varepsilon,$$
$$\tilde{g} \in \overline{gCg}, \quad \tilde{h} \in \overline{hCh},$$

and

$$\tilde{f},\ \tilde{g} \in \overline{\tilde{h}C_i\tilde{h}}.$$

Proof. Set $\varepsilon' = \varepsilon/5$. With sufficiently large i, pick $f', g', h' \in D_i$ such that

$$\|f - f'\| < \varepsilon', \quad \|g - g'\| < \varepsilon', \quad \text{and} \quad \|h - h'\| < \varepsilon'.$$

Then, noting that C is commutative, one has

$$(g' - \varepsilon')_+ \in \overline{gCg} \quad \text{and} \quad (h' - \varepsilon')_+ \in \overline{hCh}. \tag{3.17}$$

Consider

$$\tilde{f} := (h' - \varepsilon')_+ f', \quad \tilde{g} = (h' - \varepsilon')_+(g' - \varepsilon')_+, \quad \text{and} \quad \tilde{h} = (h' - \varepsilon')_+.$$

Since C_i and C are commutative, by (3.17), one has

$$(h' - \varepsilon')_+ f',\ (h' - \varepsilon')_+(g' - \varepsilon')_+ \in \overline{(h' - \varepsilon')_+ C_i (h' - \varepsilon')_+},$$
$$(h' - \varepsilon')_+(g' - \varepsilon')_+ \in \overline{gCg},$$

and

$$\|\tilde{h} - h\| < 2\varepsilon',$$
$$\|\tilde{f} - f\| = \|(h' - \varepsilon')_+ f' - f\| \approx_{3\varepsilon'} \|hf - f\| = 0,$$
$$\|\tilde{g} - g\| = \|(h' - \varepsilon')_+(g' - \varepsilon')_+ - g\| \approx_{5\varepsilon'} \|hg - g\| = 0,$$

as desired. □

We are now ready to prove the main theorem of this section.

Theorem 3.14. *Let $A = \mathrm{M}_n(\mathrm{C}(X))$, where X is a metrizable compact space, and let D be the diagonal subalgebra. Let $f, g, h \in D$ be positive elements such that $f, g \in \overline{hDh}$ and*

$$\mathrm{rank}(f(x)) \leq \mathrm{rank}(g(x)), \quad x \in X.$$

Then, for any $\varepsilon > 0$, there is a unitary $u \in \overline{hAh} + \mathbb{C}1$ such that

$$ufu^* \in_\varepsilon \overline{gAg},$$

and

$$u(x) \in P_n(2/n), \quad x \in X.$$

In particular,

$$\mathrm{dist}_{2,\mathrm{T}(A)}(udu^*, (D)_1) < 4/n, \quad d \in (D)_1.$$

Proof. Let $f, g, h \in D$ and ε be given. By Lemma 2.8 of [36], there is $\delta > 0$ such that

$$\mathrm{rank}((f-\varepsilon/2)_+(x)) \leq \mathrm{rank}((g-\delta)_+(x)), \quad x \in X.$$

Without loss of generality, one may assume that

$$h(f-\varepsilon/2)_+ = (f-\varepsilon/2)_+ \quad \text{and} \quad h(g-\delta/2)_+ = (g-\delta/2)_+.$$

By Lemma 3.13 (applying to $(f-\varepsilon/2)_+$, $(g-\delta/2)_+$, and h), with a sufficiently fine simplicial complex approximation of X, there is a unital homomorphism $\phi : \mathrm{M}_n(\mathrm{C}(W)) \to \mathrm{M}_n(\mathrm{C}(X))$, where W is a simplicial complex, such that there are functions

$$f', g', h' \in \phi(\mathrm{M}_n(\mathrm{C}(W)))$$

with the properties

$$\|f' - (f-\varepsilon/2)_+\| < \varepsilon/4 \quad \text{and} \quad \|g' - (g-\delta/2)_+\| < \delta/4, \tag{3.18}$$

$$g' \in \overline{(g-\delta/2)_+ D (g-\delta/2)_+}, \quad h' \in \overline{hDh}, \tag{3.19}$$

and

$$f', g' \in \overline{h'\phi(\mathrm{M}_n(\mathrm{C}(W)))h'}. \tag{3.20}$$

By Lemma 5.1 of [25] and (3.18),

$$(f-\varepsilon)_+ \precsim (f'-\varepsilon/4)_+ \precsim (f-\varepsilon/2)_+$$

and

$$(g-\delta)_+ \precsim (g'-\delta/4)_+ \precsim g.$$

Therefore,

$$\mathrm{rank}((f'-\varepsilon/4)_+(x)) \leq \mathrm{rank}((f-\varepsilon/2)_+(x)) \leq \mathrm{rank}((g-\delta)_+(x)) \leq \mathrm{rank}((g'-\delta/4)(x)), \quad x \in X,$$

and hence

$$\mathrm{rank}((f'-\varepsilon/4)_+(x)) \leq \mathrm{rank}(g'(x)), \quad x \in X.$$

Lift f', g', h' to positive contractions $\tilde{f}, \tilde{g}, \tilde{h}$ of $\mathrm{M}_n(\mathrm{C}(W))$, respectively, such that $\tilde{f}, \tilde{g} \in \overline{\tilde{h}\mathrm{M}_n(\mathrm{C}(X))\tilde{h}}$. Then

$$\mathrm{rank}((\tilde{f}-\varepsilon/4)_+(x)) \leq \mathrm{rank}(\tilde{g}(x)), \quad x \in W_0,$$

where $W_0 \subseteq W$ is the closed set which induces the homomorphism ϕ. Pick a continuous function $\theta : W \to [0,1]$ such that $\theta(x) \neq 0$, $x \in W \setminus W_0$, and $\theta|_W = 0$, and consider the function $\theta\tilde{h}$. Then

$$\mathrm{rank}((\tilde{f}-\varepsilon/4)_+) \leq \mathrm{rank}((\tilde{g}+\theta\tilde{h})(x)), \quad x \in W.$$

By Proposition 3.11, there is a unitary $u \in \overline{\tilde{h}\mathrm{M}_n(\mathrm{C}(W))\tilde{h}} + \mathbb{C}1$ such that

$$u(x) \in P_n(2/n), \quad x \in W,$$

and

$$u((\tilde{f}-\varepsilon/4)_+)u^* \in_\varepsilon \overline{(\tilde{g}+\theta\tilde{h})A(\tilde{g}+\theta\tilde{h})}.$$

On passing to the image of ϕ, by (3.20) and (3.19), we obtain a unitary, still denoted by u,

$$u \in \overline{h'\mathrm{M}_n(\mathrm{C}(W))h'} + \mathbb{C}1 \subseteq \overline{hAh} + \mathbb{C}1$$

such that

$$u(x) \in P_n(2/n), \quad x \in W_0,$$

and

$$u(f' - \varepsilon/4)_+u^* \in_\varepsilon \overline{g'Ag'} \subseteq \overline{gAg},$$

which implies

$$ufu^* \in_{2\varepsilon} \overline{gAg}.$$

Regarding u as a function on X, one has

$$u(x) \in P_n(2/n), \quad x \in X,$$

and hence

$$(udu^*)(x) \in^{\|\cdot\|_2}_{4/n} (D(x))_1, \quad x \in X,\ d \in (D)_1.$$

By Lemma 3.12, it follows that

$$udu^* \in^{\|\cdot\|_2}_{4/n} (D)_1,$$

as desired. □

4. Property (C)

Let us introduce the following relative comparison property of a commutative C*-algebra inside an ambient C*-algebra:

Definition 4.1. Let A be a unital C*-algebra and let D be a unital commutative subalgebra. Then the pair (D, A) is said to have Property (C) if for any positive contractions $f, g, h \in D$ satisfying $f, g \in \overline{hDh}$, and

$$\mathrm{d}_\tau(f) < \mathrm{d}_\tau(g), \quad \tau \in \mathrm{T}(A),$$

and for any $\varepsilon > 0$, there is a contraction $u \in \overline{hAh} + \mathbb{C}1_A$ such that

$$ufu^* \in^{\|\cdot\|_2}_\varepsilon \overline{gAg},$$

$$\mathrm{dist}_{2,\mathrm{T}(A)}(udu^*, (D)_1) < \varepsilon, \quad \mathrm{dist}_{2,\mathrm{T}(A)}(u^*du, (D)_1) < \varepsilon, \quad d \in (D)_1,$$

and

$$\|uu^* - 1\|_{2,\mathrm{T}(A)},\ \|u^*u - 1\|_{2,\mathrm{T}(A)} < \varepsilon.$$

Remark 4.2. Comparing to Property (COS), the approximations in Property (C) are with respect to the uniform trace norm, but on the other hand, the comparison in Property (C) is implemented by an almost unitary which is also an almost normalizer (with respect to the uniform trace norm).

Using Theorem 3.14, let us show that AH algebras with diagonal maps and the C*-algebras $\mathrm{C}(X) \rtimes \Gamma$ for which (X, Γ) has the (URP) have Property (C). (Theorem 4.6.)

Let us first work on the AH algebras. They actually have the following property which is slightly stronger than Property (C):

Proposition 4.3. *Let A be a simple AH algebra with diagonal maps, and let D be the canonical diagonal subalgebra of A. Then, for any positive contractions $f, g, h \in D$ satisfying $f, g \in \overline{hDh}$, and*

$$\mathrm{d}_\tau(f) < \mathrm{d}_\tau(g), \quad \tau \in \mathrm{T}(A),$$

and for any $\varepsilon > 0$, there is a unitary $u \in \overline{hAh} + \mathbb{C}1_A$ such that

$$ufu^* \in_\varepsilon \overline{gAg}$$

and

$$\mathrm{dist}_{2,\mathrm{T}(A)}(udu^*, (D)_1) < \varepsilon, \quad \mathrm{dist}_{2,\mathrm{T}(A)}(u^*du, (D)_1) < \varepsilon, \quad d \in (D)_1.$$

Proof. Write $A = \varinjlim A_i$ and the induced decomposition $D = \varinjlim D_i$, where $A_i = \bigoplus_j \mathrm{M}_{n_{i,j}}(\mathrm{C}(X_{i,j}))$. For the given ε, by the compactness of $\mathrm{T}(A)$, there is $\delta > 0$ such that

$$\mathrm{d}_\tau((f-\varepsilon)_+) < \mathrm{d}_\tau((g-\delta)_+), \quad \tau \in \mathrm{T}(A).$$

Without loss of generality, one may assume that

$$h \cdot (f-\varepsilon)_+ = (f-\varepsilon)_+ \quad \text{and} \quad h \cdot (g-\delta/2)_+ = (g-\delta/2)_+.$$

Applying Lemma 3.13 to $(f-\varepsilon)_+$, $(g-\delta)_+$, h, we obtain $\tilde{f}$, $\tilde{g}$, and $\tilde{h} \in D_{i_0}$ where i_0 is sufficiently large that

$$\|(f-\varepsilon)_+ - \tilde{f}\| < \varepsilon/2, \quad \|(g-\delta/2)_+ - \tilde{g}\| < \delta/4,$$
$$\tilde{g} \in \overline{gDg}, \quad \tilde{h} \in \overline{hDh},$$

and

$$\tilde{f},\ \tilde{g} \in \overline{\tilde{h}D_{i_0}\tilde{h}}.$$

Then

$$(\tilde{f}-\varepsilon) \precsim (f-\varepsilon)_+ \quad \text{and} \quad (g-\delta)_+ \precsim (\tilde{g}-\delta/2)_+,$$

and hence

$$\mathrm{d}_\tau((\tilde{f}-\varepsilon)) \leq \mathrm{d}_\tau((f-\varepsilon)_+) < \mathrm{d}_\tau((g-\delta)_+) \leq \mathrm{d}_\tau((\tilde{g}-\delta/2)_+), \quad \tau \in \mathrm{T}(A).$$

Since A is simple, by (the proof of) Proposition 3.2 of [29], there is $m \in \mathbb{N}$ such that

$$\bigoplus_{m+1}(\tilde{f}-\varepsilon)_+ \precsim \bigoplus_{m}(\tilde{g}-\delta/2)_+$$

in A. Therefore, with $i_1 > i_0$ sufficiently large, there are $(x_{i,j}) \in \mathrm{M}_\infty(A_{i_1})$ such that $2/n_{i_1} < \varepsilon/2$ and

$$\|\bigoplus_{m+1}(\tilde{f}-\varepsilon)_+ - (x_{i,j})(\bigoplus_{m}(\tilde{g}-\delta/2)_+)(x_{i,j})^*\| < \varepsilon$$

in A_{i_1}. This implies

$$\bigoplus_{m+1}(\tilde{f}-\varepsilon-\varepsilon)_+ \precsim \bigoplus_{m}(\tilde{g}-\delta)_+$$

in A_{i_1}, and hence

$$\mathrm{rank}(\tilde{f}-2\varepsilon)_+(x) < \mathrm{rank}(\tilde{g}-\delta/2)_+(x), \quad x \in \bigsqcup_j X_{i_1,j}.$$

Note that
$$(\tilde{f}-2\varepsilon)_+,\ (\tilde{g}-\delta/2)_+ \in \overline{\tilde{h}A_{i_1}\tilde{h}}.$$
Then, by Theorem 3.14, there is a unitary $u \in \overline{\tilde{h}A_{i_1}\tilde{h}}+\mathbb{C}1 \subseteq \overline{hAh}+\mathbb{C}1$ such that
$$u(\tilde{f}-2\varepsilon)_+u^* \in_\varepsilon \overline{(\tilde{g}-\delta/2)_+A_{i_1}(\tilde{g}-\delta/2)_+}$$
and
$$u(x) \in P_{n_{i_1}}(2/n_{i_1}),\quad x \in X_{n_{i_1}}.$$
Then
$$u \in \overline{\tilde{h}A_{i_1}\tilde{h}}+\mathbb{C}1 \subseteq \overline{hAh}+\mathbb{C}1$$
and
$$ufu^* \approx_{3\varepsilon} u(\tilde{f}-2\varepsilon)_+u^* \in_\varepsilon \overline{(\tilde{g}-\delta/2)_+A_{i_1}(\tilde{g}-\delta/2)_+} \subseteq \overline{gAg}.$$
Note that, since all connecting maps of $\varinjlim A_i$ are diagonal maps, inside each A_i, $i>i_1$, the image of u, regarded as a function on X_i, takes values in $P_{n_i}(\varepsilon')$, where $\varepsilon' \leq 2/n_{i_1} < \varepsilon/2$, and therefore
$$(udu^*)(x) \in^{\|\cdot\|_2}_{2\varepsilon'} (D_x)_1,\quad d\in (D_x)_1,\ x\in X_i,\ i>i_1.$$
By Lemma 3.12,
$$udu^* \in_{2\varepsilon'} (D)_1,\quad d \in (D)_1,$$
as desired. □

Let us now consider the crossed product C*-algebras $A=\mathrm{C}(X)\rtimes\Gamma$. We first need the following lemma, which states that A can be weakly tracially approximated by homogeneous C*-algebras induced by Rokhlin towers:

Lemma 4.4 (c.f. Theorem 3.9 of [27]). *Let (X,Γ) be a free and minimal dynamical system with the (URP). Then, for any finite set $\mathcal{F}\subseteq \mathrm{C}(X)$ and any $\varepsilon>0$, there exist a positive element $p\in\mathrm{C}(X)$ with $\|p\|=1$ and a sub-C*-algebra $C\subseteq \mathrm{C}(X)\rtimes\Gamma$ such that*

(1) $\|[p,f]\|<\varepsilon$, $f\in\mathcal{F}$,
(2) $pfp \in_\varepsilon C$, $f\in\mathcal{F}$,
(3) $pdp\in C$, $d\in\mathrm{C}(X)$,
(4) $C\cong\bigoplus_{i=1}^S \mathrm{M}_{n_i}(\mathrm{C}_0(Z_i))$, *where $Z_i\subseteq X$, $i=1,...,S$, are mutually disjoint, and under this isomorphism, the elements pdp are diagonal elements of C for all $d\in\mathrm{C}(X)$,*
(5) *under the isomorphism above, all diagonal elements of $\bigoplus_{i=1}^S \mathrm{M}_{n_i}(\mathrm{C}_0(Z_i))$ are in $C\cap\mathrm{C}(X)$,*
(6) $\mathrm{d}_\tau(1-p)<\varepsilon$, $\tau\in\mathrm{T}(A)$,
(7) *there is a closed subset $[Z_i]\subseteq Z_i$ for each $i=1,...,S$ such that if $a\in C$, $\|a\|\leq 1$, and a is supported in $\bigsqcup_{i=1}^S(Z_i\setminus[Z_i])$ under the isomorphism $C\cong\bigoplus_{i=1}^S\mathrm{M}_{n_i}(\mathrm{C}_0(Z_i))$, then*
$$\tau(a)<\varepsilon,\quad \tau\in\mathrm{T}(A),$$
(8) *for any $d\in\mathrm{C}(X)$, there are $d_1,d_2\in\mathrm{C}(X)$ such that*
$$d=d_1+d_2,\quad \|d_1\|_{2,\mathrm{T}(A)}<\varepsilon,\quad d_2\in C\cap\mathrm{C}(X),$$
and the support of d_2 is inside $\bigsqcup_{i=1}^S[Z_i]$ under the isomorphism $C\cong\bigoplus_{i=1}^S\mathrm{M}_{n_i}(\mathrm{C}_0(Z_i))$.

Proof. This follows from the same (actually a simpler) argument as Theorem 3.9 of [27]. $\square$

Proposition 4.5. *Let (X, Γ) be a free and minimal dynamical system with the (URP). Then the pair $(\mathrm{C}(X), \mathrm{C}(X) \rtimes \Gamma)$ has Property (C).*

Proof. Let $f, g, h \in \mathrm{C}(X)$ be such that $f, g \in \overline{hDh}$, and

$$\mathrm{d}_\tau(f) < \mathrm{d}_\tau(g), \quad \tau \in \mathrm{T}(A),$$

and let $\varepsilon > 0$ be arbitrary.

For the given ε, by the compactness of $\mathrm{T}(A)$, there is $\delta > 0$ such that

$$\mathrm{d}_\tau((f-\varepsilon)_+) < \mathrm{d}_\tau((g-\delta)_+), \quad \tau \in \mathrm{T}(A).$$

Since A is simple, by (the proof of) Proposition 3.2 of [29], there is $m \in \mathbb{N}$ such that

$$\bigoplus_{m+1}(f-\varepsilon)_+ \precsim \bigoplus_{m}(g-\delta)_+$$

in A. Since (X, Γ) has the (URP), by Lemma 4.4(3)(1)(2), there are $p \in \mathrm{C}(X)$ and $C \subseteq A$ with $C \cong \bigoplus_{i=1}^S \mathrm{M}_{n_i}(\mathrm{C}_0(Z_i))$ such that

$$p(f-\varepsilon)_+p, \quad p(g-\delta)_+p, \quad php \in C,$$

and

$$\bigoplus_{m+1}(p(f-\varepsilon)_+p-\varepsilon)_+ \precsim \bigoplus_{m}p(g-\delta)_+p$$

in C. Thus

$$\text{(4.1)} \qquad \mathrm{rank}((p(f-\varepsilon)_+p-\varepsilon)_+(z)) < \mathrm{rank}((p(g-\delta)_+p)(z)), \quad z \in \bigsqcup_{i=1}^S Z_i.$$

Note that

$$\text{(4.2)} \qquad ((p(f-\varepsilon)_+p)-\varepsilon)_+,\ p(g-\delta)_+p \in \overline{(php)A(php)}.$$

On passing to the restriction to $[Z_i]$, $i = 1, ..., S$, by (4.1), (4.2), and Theorem 3.14, we obtain a unitary

$$u \in \overline{(php)|_{[Z]}(\bigoplus_{i=1}^S \mathrm{M}_{n_i}(\mathrm{C}([Z_i])))(php)|_{[Z]}} + \mathbb{C}1$$

such that

$$\text{(4.3)} \qquad u\pi_{[Z]}((p(f-\varepsilon)_+p-\varepsilon)_+)u^* \in_\varepsilon \pi_{[Z]}(\overline{(p(g-\delta)_+p)C(p(g-\delta)_+p)}),$$

and

$$\text{(4.4)} \qquad udu^*,\ u^*du \in_\varepsilon^{\|\cdot\|_2} (D([Z]))_1, \quad d \in (D([Z]))_1,$$

where $[Z] = \bigsqcup_{i=1}^S [Z_i]$, and $D([Z])$ is the diagonal subalgebra of $\pi_{[Z]}(C) \cong \bigoplus_{i=1}^S \mathrm{M}_{n_i}(\mathrm{C}([Z_i]))$.

Extend u to a contraction in $\overline{(php)(\bigoplus_{i=1}^S \mathrm{M}_{n_i}(\mathrm{C}_0(Z_i)))(php)} + \mathbb{C}1$, and still denote it by u. Then uu^* and u^*u, as functions on Z, are 1_{n_i} on $[Z_i]$, and hence

$$\|uu^*-1\|_{2,\mathrm{T}(A)},\ \|u^*u-1\|_{2,\mathrm{T}(A)} < \varepsilon.$$

Moreover, for any $d \in (D)_1$, where $D = \mathrm{C}(X)$, by Lemma 4.4(8), it can be written as

$$d = d_1 + d_2,$$

where $d_1, d_2 \in D$, $\|d_1\|_{2,\mathrm{T}(A)} < \varepsilon$, and the support of d_2 is inside $[Z]$. In particular, by (4.4),

$$ud_2u^* \in_{\varepsilon}^{\|\cdot\|_2} (D)_1,$$

and hence,

$$udu^* = ud_1u^* + ud_2u^* \approx_{\varepsilon}^{\|\cdot\|_2} ud_2u^* \in_{\varepsilon}^{\|\cdot\|_2} (D_1).$$

This shows

$$\mathrm{dist}_{2,\mathrm{T}(A)}(udu^*, (D)_1) < 2\varepsilon.$$

The same argument shows

$$\mathrm{dist}_{2,\mathrm{T}(A)}(u^*du, (D)_1) < 2\varepsilon.$$

By (4.3), there is a contraction $c \in \pi_{Z_0}(\overline{(p(g-\delta)_+p)C(p(g-\delta)_+p)})$ such that

$$\|u\pi_{Z_0}((p(f-\varepsilon)_+p-\varepsilon)_+)u^* - c\| < \varepsilon.$$

Extend c to a contraction of $\overline{(p(g-\delta)_+p)C(p(g-\delta)_+p)}$, and still denote it by c. Then, the element $c - u(p(f-\varepsilon)_+ - \varepsilon)_+u^*$, regarded as a (matrix-valued) function on Z, has norm at most on ε on Z_0. Therefore,

$$\|c - u(p(f-\varepsilon)_+ - \varepsilon)_+u^*\|_{2,\mathrm{T}(A)} < 2\varepsilon$$

and

$$\begin{array}{rcl} u(f-\varepsilon)_+u^* & = & up(f-\varepsilon)_+u^* + u(1-p)(f-\varepsilon)_+u^* \\ & \approx_{\varepsilon} & u(p(f-\varepsilon)_+ - \varepsilon)_+u^* + u(1-p)(f-\varepsilon)_+u^* \\ & \approx_{2\varepsilon}^{\|\cdot\|_2} & c + u(1-p)(f-\varepsilon)_+u^* \\ & \approx_{\varepsilon}^{\|\cdot\|_2} & c. \end{array}$$

That is,

$$ufu^* \in_{4\varepsilon}^{\|\cdot\|_2} \overline{(p(g-\delta)_+p)C(p(g-\delta)_+p)} \subseteq \overline{gAg}.$$

Since ε is arbitrary, this shows that (D, A) has Property (C). $\square$

Combining Proposition 4.3 and Proposition 4.5, one has the following theorem:

Theorem 4.6. *Let A be a simple AH algebra with diagonal maps, or let $A = \mathrm{C}(X) \rtimes \Gamma$, where (X, Γ) is free and minimal with the (URP). Let D be the canonical Cartan subalgebra of A. Then the pair (D, A) has Property (C).*

5. Property (E)

Definition 5.1. Let A be a unital C*-algebra and let D be a unital commutative subalgebra.

The pair (D, A) is said to have Property (E) if for any positive contraction $a \in A$, any finite subset $\mathcal{F} \subseteq \mathrm{C}([0,1])$, and any $\varepsilon > 0$, there is a positive contraction $b \in D$ such that

$$|\tau(f(a)) - \tau(f(b))| < \varepsilon, \quad f \in \mathcal{F},\ \tau \in \mathrm{T}(A).$$

Recall the following theorem which is due to Thomsen and Li ([32] and [21]).

Theorem 5.2. *Suppose that X is a path connected metrizable compact space. For any finite subset $\mathcal{F} \subseteq \mathrm{Aff}\mathrm{T}(\mathrm{C}(X))$ and $\varepsilon > 0$, there is $N > 0$ with the following property:*

For any unital positive linear map $\xi : \mathrm{Aff}\mathrm{T}(\mathrm{C}(X)) \to \mathrm{Aff}\mathrm{T}(\mathrm{C}(Y))$, where Y is an arbitrary metrizable compact space, and any $n \geq N$, there are homomorphisms

$$\phi_1, ..., \phi_n : \mathrm{C}(X) \to \mathrm{C}(Y)$$

such that

$$|\xi(f)(\tau) - \frac{1}{n}\sum_{i=1}^{n} \tau(\phi_i(f))| < \varepsilon, \quad f \in \mathcal{F},\ \tau \in \mathrm{T}(\mathrm{C}(Y)).$$

Theorem 5.3. *Let A be a simple AH algebra with diagonal maps, or let $A = \mathrm{C}(X) \rtimes \Gamma$, where (X, Γ) is a free and minimal dynamical system with the (URP). Then the pair (D, A) has Property (E).*

Proof. Let $(\mathcal{F}, \varepsilon)$ be given. Without loss of generality, one may assume that each element of $\mathcal{F}$ has norm 1 and is real valued, so $\mathcal{F}$ can be regarded as a subset of $\mathrm{Aff}(\mathrm{T}(\mathrm{C}([0,1])))$. Applying the Thomsen-Li Theorem above to $(\mathcal{F}, \varepsilon)$ (where $X = [0,1]$), one obtains N.

Let us first consider the case of AH algebras. Let $a \in A$ be a positive element with norm 1; to prove the theorem, without loss of generality, one may assume that $a \in \mathrm{M}_n(\mathrm{C}(X)) \subseteq A$ for some $n \geq N$. Consider the homomorphism $\phi : \mathrm{C}[0,1] \to \mathrm{M}_n(\mathrm{C}(X))$ induced by a, and consider the induced unital positive linear map $\phi^* : \mathrm{Aff}(\mathrm{T}(\mathrm{C}[0,1])) \to \mathrm{Aff}(\mathrm{T}(\mathrm{C}(X)))$. By Thomsen-Li Theorem, there are continuous maps $\lambda_1, ..., \lambda_n : X \to [0,1]$ such that

$$|\phi^*(f)(\tau) - \frac{1}{n}(\tau(f \circ \lambda_1) + \cdots + \tau(f \circ \lambda_n))| < \varepsilon, \quad f \in \mathcal{F},\ \tau \in \mathrm{T}(\mathrm{C}(X)).$$

Then, with

$$b = \mathrm{diag}\{(\lambda_1)_*(\mathrm{id}), ..., (\lambda_n)_*(\mathrm{id})\} \in D_n,$$

(and since $\phi(f) = f(a)$,) one has that

$$|\tau(f(a)) - \tau(f(b))| < \varepsilon, \quad f \in \mathcal{F},\ \tau \in \mathrm{T}(\mathrm{M}_n(\mathrm{C}(X))).$$

Let us now consider the case that $A = \mathrm{C}(X) \rtimes \Gamma$, where (X, Γ) is free and minimal, and has the (URP).

Let $a \in A$ be a positive element with norm 1. Choose $\delta > 0$ such that if $\|x - y\|_{2,\mathrm{T}(A)} < \delta$, where x, y are positive contractions, then $\|f(x) - f(y)\|_{2,\mathrm{T}(A)} < \varepsilon$ for all $f \in \mathcal{F}$.

By Lemma 4.4, there exist a sub-C*-algebra $C \subseteq A$ and a positive contraction $p \in C$ such that

(1) $C \cong \bigoplus_{i=1}^S \mathrm{M}_{n_i}(\mathrm{C}_0(Z_i))$, where $n_i > N$, $i = 1, ..., S$,
(2) $pap \in_{\delta/2} C$,
(3) $\mathrm{d}_\tau(1-p) < \delta/2$, $\tau \in \mathrm{T}(A)$,
(4) there is a closed subset $[Z_i] \subseteq Z_i$ for each $i = 1, ..., S$ such that if $a \in C$, $\|a\| \leq 1$, and a is supported in $\bigsqcup_{i=1}^S (Z_i \setminus [Z_i])$ under the isomorphism $C \cong \bigoplus_{i=1}^S \mathrm{M}_{n_i}(\mathrm{C}_0(Z_i))$, then

$$\tau(a) < \varepsilon, \quad \tau \in \mathrm{T}(A),$$

Choose $\tilde{a} \in C$ such that $\|\tilde{a} - pap\| < \delta/2$; hence

$$\|a - \tilde{a}\|_{2,\mathrm{T}(A)} < \delta.$$

and so, by the choice of δ,

$$\|f(a) - f(\tilde{a})\|_{2,\mathrm{T}(A)} < \varepsilon, \quad f \in \mathcal{F}. \tag{5.1}$$

Consider the homomorphism

$$\phi : \mathrm{C}([0,1]) \ni f \mapsto \pi_{[Z]}(f(\tilde{a})) \in \pi_{[Z]}(C) \cong \bigoplus_{i=1}^S \mathrm{M}_{n_i}(\mathrm{C}([Z_i])),$$

where $[Z] = \bigsqcup_{i=1}^S [Z_i]$, and consider the unital positive linear map $\phi^* : \mathrm{Aff}(\mathrm{T}(\mathrm{C}[0,1])) \to \mathrm{Aff}(\mathrm{T}(\mathrm{C}([Z])))$. Then, by Theorem 5.2, there are continuous maps $\lambda_{i,1}, ..., \lambda_{i,n_i} : [Z_i] \to [0,1]$, $i = 1, ..., S$, such that

$$|\phi^*(f)(\tau) - \frac{1}{n_i}(\tau(f \circ \lambda_{i,1}) + \cdots + \tau(f \circ \lambda_{i,n_i}))| < \varepsilon, \quad f \in \mathcal{F},\ \tau \in \mathrm{T}(\mathrm{C}([Z_i])).$$

Define

$$b_i = \mathrm{diag}\{\mathrm{id} \circ \lambda_{i,1}, ..., \mathrm{id} \circ \lambda_{i,n}\} \in \mathrm{M}_{n_i}(\mathrm{C}([Z_i])),$$

where id is the identity function on $[0,1]$, and define

$$b = b_1 \oplus \cdots \oplus b_{n_S}.$$

Then

$$|\tau(\pi_{Z_0}(f(\tilde{a}))) - \tau(f(b))| < \varepsilon, \quad f \in \mathcal{F},\ \tau \in \mathrm{T}(\bigoplus_{i=1}^S \mathrm{M}_{n_i}(\mathrm{C}([Z_i]))). \tag{5.2}$$

Note that b is a diagonal element. Extend b to a positive diagonal contraction $\hat{b} \in C$. Note that, by Lemma 4.4(5), the element $\hat{b}$ is also in D. By (4), a calculation shows

$$|\tau(f(\hat{b})) - \tau|_{[Z]}(f(b))| < 4\varepsilon, \quad f \in \mathcal{F},\ \tau \in \mathrm{T}(A), \tag{5.3}$$

where $\tau|_{[Z]}$ is the tracial state of $\bigoplus_{i=1}^S \mathrm{M}_{n_i}(\mathrm{C}([Z_i]))$ induced by $\tau|_C$:

$$\tau|_{[Z]}(x) = \lim_{n\to\infty} \frac{1}{\tau(e_n^2)} \tau(e_n \tilde{x} e_n),$$

where $x \in \bigoplus_{i=1}^{S} \mathrm{M}_{n_i}(\mathrm{C}([Z_i]))$, $\tilde{x} \in C \cong \bigoplus_{i=1}^{S} \mathrm{M}_{n_i}(\mathrm{C}_0(Z_i))$ is an extension of x, and (e_n) is a decreasing sequence of positive contractions of C which converges (pointwisely) to $\mathbf{1}_Z$. Also note that, by (4),

$$|\tau(f(\tilde{a})) - \tau|_{[Z]}(\pi_{[Z]}(f(\tilde{a})))| < 2\varepsilon, \quad f \in \mathcal{F},\ \tau \in \mathrm{T}(A). \tag{5.4}$$

Then, for all $f \in \mathcal{F}$ and $\tau \in \mathrm{T}(A)$, by (5.1), (5.4), (5.2), and (5.3),

$$\tau(f(a)) \approx_{\varepsilon} \tau(f(\tilde{a})) \approx_{2\varepsilon} \tau|_{[Z]}(\pi_{Z_0}(f(\tilde{a}))) \approx_{\varepsilon} \tau|_{[Z]}(f(b)) \approx_{4\varepsilon} \tau(f(\hat{b})),$$

as desired. □

6. Property (S)

Definition 6.1. Let A be a unital C*-algebra, and let $\Delta \subseteq \mathrm{T}(A)$ be a closed set of tracial states. The pair (A, Δ) will be said to have Property (S) if for any self-adjoint element f and any $\varepsilon > 0$, there is a self-adjoint $g \in A$ such that

(1) $\|f - g\|_{2,\Delta} < \varepsilon$, and
(2) there is $\delta > 0$ such that $\tau(\chi_\delta(g)) < \varepsilon$, $\tau \in \Delta$, where

$$\chi_\delta(t) = \begin{cases} 1, & |t| < \delta, \\ 2 - |t|/\delta, & |t| < 2\delta, \\ 0, & \text{otherwise.} \end{cases} \tag{6.1}$$

In the case that $\Delta = \mathrm{T}(A)$, we shall just say that A has Property (S) if $(A, \mathrm{T}(A))$ has Property (S).

Compared to Theorem 2.9, Property (S) can be regarded as a weaker version of the small boundary property, without referring to a commutative subalgebra D. Eventually, it will be shown (Proposition 8.3) that Property (S) for A implies the small boundary property of the pair (D, A), provided that (D, A) has Properties (C) and (E).

Definition 6.2. Let A be a C*-algebra, and let $\Delta \subseteq \mathrm{T}(A)$. Let us say that $\mathrm{qRR}(l^\infty(A)/J_{2,\omega,\Delta}) = 0$ if the class of the constant sequence of any self-adjoint element of A can be approximated by invertible self-adjoint elements of $l^\infty(A)/J_{2,\omega,\Delta}$ with respect to the uniform limit trace norm $\|\cdot\|_{2,\omega,\Delta}$. It is clear that if $\mathrm{RR}(A) = 0$ or if $\mathrm{RR}(l^\infty(A)/J_{2,\omega,\Delta}) = 0$, then $\mathrm{qRR}(l^\infty(A)/J_{2,\omega,\Delta}) = 0$.

Property (S) can be characterized in terms of the real rank of the sequence algebra:

Proposition 6.3. *Let A be a unital C*-algebra, and let $\Delta \subseteq \mathrm{T}(A)$ be closed. The following conditions are equivalent:*

(1) $\mathrm{qRR}(l^\infty(A)/J_{2,\omega,\Delta}) = 0$*;*
(2) (A, Δ) *has Property (S);*
(3) $\mathrm{RR}(l^\infty(A)/J_{2,\omega,\Delta}) = 0$.

Proof. (1) ⇒ (2): Assume $\mathrm{qRR}(l^\infty(A)/J_{2,\omega,\Delta}) = 0$. Let (f, ε) be given, where $f \in A$ is a self-adjoint contraction and $\varepsilon > 0$. Consider the constant sequence (f), and then there is a sequence $(g_k) \in l^\infty(A)$ such that

$$\|(f - g_k)\|_{2,\omega,\Delta} = \lim_{k\to\omega} \|f - g_k\|_{2,\Delta} < \varepsilon$$

and $\overline{(g_k)}$ is invertible in $l^\infty(A)/J_{2,\omega,\Delta}$. Then there is $\delta > 0$ such that

$$\overline{(\chi_\delta(g_k))} = \chi_\delta(\overline{(g_k)}) = 0.$$

In other words,

$$(\chi_\delta(g_k)) \in J_{2,\omega,\Delta}.$$

Then, with some sufficiently large k, one has

- $\|f - g_k\|_{2,\Delta} < \varepsilon$, and
- $\tau(\chi_\delta(g_k)) < \varepsilon$, $\tau \in \Delta$,

as desired.

(2) $\Rightarrow$ (3): Assume that (A, Δ) has Property (S), and let us show that $l^\infty(A)/J_{2,\omega,\Delta}$ has real rank zero. Let $\overline{(a_1, a_2, ...)} \in l^\infty(A)/J_{2,\omega,\Delta}$ be a self-adjoint element, and let $\varepsilon > 0$ be arbitrary. By Property (S), for each $n = 1, 2, ...$, there is a self-adjoint element b_n such that

- $\tau((a_n - b_n)^2) < 1/n$ for all $\tau \in \Delta$, and
- there is $\delta_n > 0$ such that $\tau(\chi_{\delta_n}(b_n)) < 1/n$ for all $\tau \in \Delta$.

Without loss of generality, one may assume that $\delta_n < \varepsilon$. Note that

$$\overline{(a_1, a_2, ...)} = \overline{(b_1, b_2, ...)}.$$

Consider $b'_n = g_n(b_n)$ and $c_n = h_n(b_n)$, where

$$g_n(t) = \begin{cases} \varepsilon t/\delta_n, & |t| < \delta_n, \\ t + \mathrm{sign}(t)(\varepsilon - \delta_n), & \text{otherwise} \end{cases}$$

and

$$h_n(t) = \begin{cases} t/\varepsilon\delta_n, & |t| < \delta_n, \\ 1/(t + \mathrm{sign}(t)(\varepsilon - \delta_n)), & \text{otherwise.} \end{cases}$$

Then

$$\|b'_n - b_n\| < \varepsilon, \quad \|c_n\| < 1/\varepsilon, \quad \text{and} \quad \tau((b'_n c_n - 1)^2) < \tau(\chi_{\delta_n}(b_n)) < 1/n, \quad \tau \in \Delta.$$

In particular, the element $\overline{(b'_1, b'_2, ...)}$ is invertible in $l^\infty(A)/J_{\omega,\Delta}$ with the inverse $\overline{(c_1, c_2, ...)}$. Moreover,

$$\|\overline{(a_1, a_2, ...)} - \overline{(b'_1, b'_2, ...)}\| = \|\overline{(b_1, b_2, ...)} - \overline{(b'_1, b'_2, ...)}\| < \varepsilon.$$

This shows that $l^\infty(A)/J_{2,\omega,\Delta}$ has real rank zero.

(3) $\Rightarrow$ (1): Trivial. ◻

Uniform property Γ implies Property (S):

Proposition 6.4. *If a unital C*-algebra A has uniform property Γ, then A has Property (S).*

Proof. The proof is similar to the proof of Theorem 3.5 of [10]:

Let (f, ε) be given, where $f \in A$ is a self-adjoint contraction and $\varepsilon > 0$. By Corollary 3.9 of [10], for the given ε, there exist $n \in \mathbb{N}$ and self-adjoint elements

$$f_1, f_2, ..., f_n \in A$$

such that

$$\|f - f_i\| < \varepsilon, \quad i = 1, 2, ..., n, \tag{6.2}$$

and there is $\delta > 0$ such that

$$\frac{1}{n}(\tau((f_1)_\delta) + \cdots + \tau((f_n)_\delta)) < \varepsilon, \quad \tau \in \mathrm{T}(A),$$

where $(f)_\delta = \chi_\delta(f)$ (see (6.1)). In particular, regarding $(f_1)_\delta, ..., (f_n)_\delta$ as constant sequences in A, we have

$$\frac{1}{n}(\tau((f_1)_\delta) + \cdots + \tau((f_n)_\delta)) < \varepsilon, \quad \tau \in \mathrm{T}(A)_\omega. \tag{6.3}$$

Since A has uniform property Γ, there is a partition of unity

$$p_1, p_2, ..., p_n \in (l^\infty(A)/J_{2,\omega,\Delta}) \cap A'$$

such that

$$\tau(p_i a p_i) = \frac{1}{n}\tau(a), \quad a \in A,\ \tau \in \mathrm{T}(A)_\omega. \tag{6.4}$$

Consider the element

$$g := p_1\overline{(f_1)}p_1 + \cdots + p_n\overline{(f_n)}p_n \in l^\infty(A)/J_{2,\omega,\Delta}.$$

By (6.2),

$$\|f - g\|_{2,\omega,\mathrm{T}(A)} = \|p_1\overline{(f - f_1)}p_1 + \cdots + p_n\overline{(f - f_n)}p_n\|_{2,\omega,\mathrm{T}(A)} < \varepsilon. \tag{6.5}$$

Note that, for each $\tau \in \mathrm{T}(A)_\omega$, by (6.4),

$$\tau(p_i\overline{((f_i)_\delta)}p_i) = \frac{1}{n}\tau((f_i)_\delta), \quad i = 1, ..., n,\ \tau \in \mathrm{T}(A)_\omega,$$

and hence, together with (6.3),

$$\begin{aligned} \tau((g)_\delta) &= \tau(p_1\overline{((f_1)_\delta)}p_1) + \cdots + \tau(p_n\overline{((f_n)_\delta)}p_n) \\ &= \frac{1}{n}(\tau((f_1)_\delta) + \cdots + \tau((f_n)_\delta)) \\ &< \varepsilon. \end{aligned} \tag{6.6}$$

Pick a representative sequence $g = \overline{(g_k)}$ with g_k, $k = 1, 2, ...$, self-adjoint elements of A. By (6.5) and (6.6), with some sufficiently large k, the function g_k satisfies

(1) $\|f - g_k\|_{2,\mathrm{T}(A)} < \varepsilon$, and
(2) $\tau((g_k)_\delta) < \varepsilon$, $\tau \in \mathrm{T}(A)$,

as desired. $\square$

7. Some approximation lemmas

In the section, let us prepare some approximation lemmas for the next section.

Lemma 7.1. *Let X be a metrizable compact space, and let Δ be a compact set of probability Borel measures of X. Then, for any $\varepsilon \in (0,1)$, there is $\delta > 0$ such that if $a_0, a_1, d : X \to [0,1]$ are continuous functions satisfying*

(1) $a_0 a_1 = a_1$,
(2) $\|a_0 d - d\|_{2,\Delta} < \delta$, *and*
(3) $\|d a_1 - a_1\|_{2,\Delta} < \delta$,

then there is a continuous function $\tilde{d} : X \to [0,1]$ such that

(1) $\|d - \tilde{d}\|_{2,\Delta} < \varepsilon$,
(2) $\|a_0\tilde{d} - \tilde{d}\| < \varepsilon$, *and*
(3) $\|\tilde{d} a_1 - a_1\| < \varepsilon$.

Proof. With the given ε, choose $\varepsilon' \in (0,1)$ such that $\varepsilon' < \varepsilon$ and $2\sqrt{\varepsilon'} < \varepsilon$. Then

$$\delta = \varepsilon' / \sqrt{2/(\varepsilon')^3}$$

has the property of the lemma.

Indeed, let a_0, a_1, d satisfy the conditions of the statement. Define sets

$$A_{0,\leq 1-\varepsilon'} = a_0^{-1}([0, 1-\varepsilon']) \quad \text{and} \quad A_{1,\geq\varepsilon'} = a_1^{-1}([\varepsilon', 1]).$$

Note that $A_{0,\leq 1-\varepsilon'} \cap A_{1,\geq\varepsilon'} = \emptyset$.

On $A_{0,\leq 1-\varepsilon'}$, consider the set

$$W_0 = \{x \in A_{0,\leq 1-\varepsilon'} : |d(x)|^2 \geq \varepsilon'\}.$$

Then, for any $\mu \in \Delta$,

$$(\varepsilon')^2 \varepsilon \mu(W_0) \leq (\varepsilon')^2 \int_{A_{0,\leq 1-\varepsilon'}} d^2(x) \mathrm{d}\mu(x) \leq \int_{A_{0,\leq 1-\varepsilon'}} (a_0(x) - 1)^2 d^2(x) \mathrm{d}\mu(x) < \delta^2,$$

and hence

$$\mu(W_0) < \delta^2/(\varepsilon')^3.$$

On $A_{1,\geq\varepsilon'}$, consider the set

$$W_1 = \{x \in A_{1,\geq\varepsilon'} : |d(x) - 1|^2 \geq \varepsilon'\}.$$

Then, for any $\mu \in \Delta$,

$$(\varepsilon')^2 (\varepsilon' \mu(W_1)) \leq (\varepsilon')^2 \int_{A_{1,\geq\varepsilon'}} |d(x) - 1|^2 \mathrm{d}\mu(x) \leq \int_{A_{1,\geq\varepsilon'}} |d(x) - 1|^2 a_1^2(x) \mathrm{d}\mu(x) < \delta^2,$$

and hence

$$\mu(W_1) < \delta^2/(\varepsilon')^3.$$

Choose disjoint open sets $U_0 \supseteq W_0$ and $U_1 \supseteq$. Since Δ is compact, U_0 and U_1 can be chosen such that

$$\mu(U_0) < \delta^2/(\varepsilon')^3 \quad \text{and} \quad \mu(U_1) < \delta^2/(\varepsilon')^3, \quad \mu \in \Delta.$$

Choose continuous functions $r_0, r_1 : X \to [0,1]$ such that

$$r_i|_{W_i} = 1 \quad \text{and} \quad r_i|_{X\setminus U_i} = 0, \quad i = 0, 1.$$

Define

$$\tilde{d}(x) = \begin{cases} 0, & x \in W_0, \\ (1 - r_0(x))d(x), & x \in U_0, \\ d(x), & x \in X \setminus (U_0 \cup U_1), \\ (1 - r_1(x))d(x) + r_1(x), & x \in U_1, \\ 1, & x \in W_1. \end{cases}$$

Then

$$\int_X (d(x) - \tilde{d}(x))^2 \mathrm{d}\mu(x) \leq \mu(U_0) + \mu(U_1) < 2\delta^2/(\varepsilon')^3, \quad \mu \in \Delta.$$

That is,

$$\|d - \tilde{d}\|_{2,\Delta} < \delta\sqrt{2/(\varepsilon')^3} = \varepsilon' < \varepsilon.$$

Note that

$$\tilde{d}(x) < \sqrt{\varepsilon'}, \quad x \in A_{0,\leq 1-\varepsilon'}.$$

So

$$\|a_0\tilde{d} - \tilde{d}\| < \max\{\varepsilon', 2\sqrt{\varepsilon'}\} < 2\sqrt{\varepsilon'} < \varepsilon.$$

Also note that

$$1 - \tilde{d}(x) < \sqrt{\varepsilon'}, \quad x \in A_{1,\geq \varepsilon'}.$$

So

$$\|\tilde{d}a_1 - a_1\| < \max\{\sqrt{\varepsilon'}, 2\varepsilon'\} < 2\sqrt{\varepsilon'} < \varepsilon,$$

as desired. ☐

The following lemma certainly is well known:

Lemma 7.2. *Let A be a C^*-algebra. Let $N \in \mathbb{N}$ and $\varepsilon > 0$. Then there is $\delta > 0$ such that if $c_1, ..., c_N$ are self-adjoint contractions such that*

$$\|c_ic_j\| < \delta, \quad i, j = 1, ..., N, \ i \neq j,$$

then

$$\|c_1 + \cdots + c_N\| < \max\{\|c_i\| : i = 1, ..., N\} + \varepsilon.$$

Proof. For the given (N, ε), there is $\delta > 0$ such that if $c_1, ..., c_N$ are self-adjoint contractions such that

$$\|c_ic_j\| < \delta, \quad i, j = 1, ..., N, \ i \neq j,$$

then there are self-adjoint elements $\tilde{c}_1, ..., \tilde{c}_N \in A$ such that

$$\|\tilde{c}_i - c_i\| < \varepsilon/2N, \quad \text{and} \quad \tilde{c}_i \perp \tilde{c}_j, \quad i, j = 1, ..., N, \ i \neq j.$$

Then, this δ has the property of the lemma, as

$$\begin{aligned} \|c_1 + \cdots + c_N\| &\approx_{\varepsilon/2} \|\tilde{c}_1 + \cdots + \tilde{c}_N\| \\ &= \max\{\|\tilde{c}_i\| : i = 1, ..., N\} \\ &\approx_{\varepsilon/2} \max\{\|c_i\| : i = 1, ..., N\}, \end{aligned}$$

as desired. □

Lemma 7.3 (cf. [3]). *Let A be a C*-algebra. For any $\varepsilon > 0$, there are $N \in \mathbb{N}$ and $\delta > 0$ with the following property:*

Let $a \in A$ be a positive element with norm at most 1. Define

$$a_i = \chi_i(a), \quad i = 1, ..., N,$$

where

$$\chi_i(t) = \begin{cases} 0, & t \leq \frac{i-1}{N}, \\ \text{linear}, & t \in [\frac{i-1}{N}, \frac{i}{N}], \\ 1, & t \geq \frac{i}{N}. \end{cases}$$

Assume there are positive elements $d_1, ..., d_N \in A$ with norm at most 1 such that

(1) $[a, d_i] = 0$, $i = 1, ..., N$,
(2) $\|a_i d_{i+1} - d_{i+1}\| < \delta$, $i = 1, ..., N-1$,
(3) $\|d_i a_{i+1} - a_{i+1}\| < \delta$, $i = 1, ..., N-1$.

Then

$$\|a - \frac{1}{N}(d_1 + \cdots + d_N)\| < \varepsilon.$$

Proof. Choose $N > 32/\varepsilon$. Applying Lemma 7.2 to N and $\varepsilon/16$, one obtains δ_1. Choose $\delta > 0$ such that

$$16\delta < \delta_1 \quad \text{and} \quad \frac{\varepsilon}{8} + \delta + 4((4\delta + \frac{\varepsilon}{8}) + \frac{\varepsilon}{16}) < \varepsilon.$$

Then the pair (N, δ) has the property of the lemma.

Since $a_i a_j = a_j$, $i < j$, by (2),

$$a_i d_j \approx_\delta a_i a_{j-1} d_j = a_{j-1} d_j \approx_\delta d_j, \quad i < j.$$

Hence, together with (2),

$$\|a_i d_j - d_j\| < 2\delta, \quad i < j. \tag{7.1}$$

A similar argument applied to (3) shows

$$\|d_i a_j - a_j\| < 2\delta, \quad i < j. \tag{7.2}$$

Also note that, if $i < j - 1$, then

$$d_i d_j \approx_\delta d_i a_{j-1} d_j \approx_{2\delta} a_{j-1} d_j \approx_\delta d_j,$$

and therefore, a straightforward calculation shows that

$$\|(d_i - d_{i+1})(d_j - d_{j+1})\| < 16\delta < \delta_1, \quad i < j - 2. \tag{7.3}$$

Note that

$$a \approx_{\frac{2}{N}} aa_2,$$

and then, together with (3), one has

$$a \approx_{\frac{2}{N}} aa_2 \approx_\delta aa_2 d_1 \approx_{\frac{2}{N}} ad_1. \tag{7.4}$$

Noting that

$$a = \frac{1}{N}(a_1 + \cdots + a_N),$$

together with (7.1) and (7.2), one also has that, for $i = 1, ..., N-1$,

$$\begin{aligned}
& a(d_i - d_{i+1}) \\
= \quad & \frac{1}{N}(a_1 + \cdots + a_N)(d_i - d_{i+1}) \\
= \quad & \frac{1}{N}((a_1d_i + \cdots + a_Nd_i) - (a_1d_{i+1} + \cdots + a_Nd_{i+1})) \\
\approx_{4\delta} \quad & \frac{1}{N}((\underbrace{d_i + \cdots + d_i}_{i-1} + a_id_i + a_{i+1} + \cdots + a_N) \\
& -(\underbrace{d_{i+1} + \cdots + d_{i+1}}_{i} + a_{i+1}d_{i+1} + a_{i+2} + \cdots + a_N)) \\
= \quad & \frac{i}{N}(d_i - d_{i+1}) + \frac{1}{N}(a_{i+1} - d_i + a_id_i - a_{i+1}d_{i+1}) \\
\approx_{\frac{4}{N}} \quad & \frac{i}{N}(d_i - d_{i+1}).
\end{aligned}$$

That is,

$$\|a(d_i - d_{i+1}) - \frac{i}{N}(d_i - d_{i+1})\| < 4\delta + \frac{4}{N} < 4\delta + \frac{\varepsilon}{8}, \quad i = 1, 2, ..., N-1. \tag{7.5}$$

A similar argument also shows

$$\|ad_N - d_N\| < 2\delta + \frac{2}{N} < 4\delta + \frac{\varepsilon}{8}. \tag{7.6}$$

Then, on applying (7.4), (7.5), (7.6), (7.3), and Lemma 7.2 (note that $a, d_1, ..., d_N$ commute),

$$\begin{aligned}
a \quad \approx_{\frac{4}{N}+\delta} \quad & ad_1 \\
= \quad & a(d_1 - d_2 + d_2 - d_3 + \cdots + d_{N-1} - d_N + d_N) \\
= \quad & a(d_1 - d_2) + a(d_2 - d_3) + \cdots + a(d_{N-1} - d_N) + ad_N \\
\approx_{4((4\delta+\frac{\varepsilon}{8})+\frac{\varepsilon}{16})} \quad & \frac{1}{N}(d_1 - d_2) + \frac{2}{N}(d_2 - d_3) + \cdots + \frac{N-1}{N}(d_{N-1} - d_N) + d_N \\
= \quad & \frac{1}{N}(d_1 + d_2 + \cdots + d_N),
\end{aligned}$$

as desired. $\square$

Lemma 7.4. *Let (D, A) be a pair of unital C*-algebras, where D is commutative. For any $\varepsilon > 0$, there are $\delta > 0$ and $N \in \mathbb{N}$ with the following property:*

Let $a \in D \subseteq A$ be a positive element with norm at most 1, and set

$$a_i = \chi_i(a), \quad i = 1, ..., N,$$

where

$$\chi_i(t)=\begin{cases} 0, & t\leq \frac{i-1}{N},\\ \text{linear}, & t\in[\frac{i-1}{N},\frac{i}{N}],\\ 1, & t\geq\frac{i}{N}.\end{cases}$$

Let $d_1, ..., d_N \in A$ be positive elements with norm at most 1 such that

(1) $\mathrm{dist}_{2,\mathrm{T}(A)}(d_i, (D)_1^+) < \delta$, $i = 1, 2, ..., N$,
(2) $\|a_i d_{i+1} - d_{i+1}\|_{2,\mathrm{T}(A)} < \delta$, $i = 1, ..., N-1$, *and*
(3) $\|d_i a_{i+1} - a_{i+1}\|_{2,\mathrm{T}(A)} < \delta$, $i = 1, ..., N-1$.

Then

$$\|a - \frac{1}{N}(d_1 + \cdots + d_N)\|_{2,\mathrm{T}(A)} < \varepsilon.$$

Proof. Applying Lemma 7.3 to $\varepsilon/2$, one obtains, say, the pair (N, δ_2). Applying Lemma 7.1 to $\min\{\delta_2, \varepsilon/4\}$, one obtains δ_1. Set $\delta = \min\{\delta_1/4, \varepsilon/4\}$. Then (N, δ) has the property of the lemma.

Indeed, let $d_1, ..., d_N$ be given. Since $\mathrm{dist}_{2,\mathrm{T}(A)}(d_i, (D)_1^+) < \delta$, $i = 1, 2, ..., N$, there are positive contractions $\tilde{d}_1, ...\tilde{d}_N \in D$ such that

$$\|d_i - \tilde{d}_i\|_{2,\mathrm{T}(A)} < \min\{\delta_1/4, \varepsilon/4\}, \quad i = 1, ..., N. \tag{7.7}$$

Then, it follows from (2) and (3) that

$$\|a_i\tilde{d}_{i+1} - \tilde{d}_{i+1}\|_{2,\mathrm{T}(A)} < \delta_1 \quad \text{and} \quad \|\tilde{d}_i a_{i+1} - a_{i+1}\|_{2,\mathrm{T}(A)} < \delta_1, \quad i = 1, 2, ..., N-1.$$

Applying Lemma 7.1 to each element $\tilde{d}_i$, $i = 1, ..., N$, one obtains a positive contraction $\tilde{\tilde{d}}_i \in D$ such that

$$\|\tilde{d}_i - \tilde{\tilde{d}}_i\|_{2,\mathrm{T}(A)} < \varepsilon/4, \quad i = 1, ..., N \tag{7.8}$$

and

$$\|a_i\tilde{\tilde{d}}_{i+1} - \tilde{\tilde{d}}_{i+1}\| < \delta_2 \quad \text{and} \quad \|\tilde{\tilde{d}}_i a_{i+1} - a_{i+1}\| < \delta_2, \quad i = 1, 2, ..., N-1.$$

Then, by Lemma 7.3, one has

$$\|a - \frac{1}{N}(\tilde{\tilde{d}}_1 + \cdots + \tilde{\tilde{d}}_N)\| < \frac{\varepsilon}{2},$$

and hence, together with (7.7) and (7.8), one has

$$\|a - \frac{1}{N}(d_1 + \cdots + d_N)\|_{2,\mathrm{T}(A)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

as desired. ∎

Lemma 7.5. *Let A be a unital C*-algebra. For any $\varepsilon > 0$, any $N \in \mathbb{N}$, and any $\chi \in \mathrm{C}([0,1])^+$, there are $\delta > 0$ and $M \in \mathbb{N}$ such that if $b_1, b_2, ..., b_N \in A$ and $d_1, d_2, ..., d_N \in A$ are positive elements with norm at most 1 such that*

(1) $\|d_i d_{i+1} - d_{i+1}\|_{2,\mathrm{T}(A)} < \delta$, $i = 1, ..., N-1$,
(2) $b_i b_{i+1} = b_{i+1}$, $i = 1, 2, ..., N$, *and*
(3) $|\tau(b_i^j) - \tau(d_i^j)| < \delta$, $i = 1, 2, ..., N$, $j = 1, ..., M$, $\tau \in \mathrm{T}(A)$,

then

$$|\tau(\chi(\frac{1}{N}(b_1+\cdots+b_N)))-\tau(\chi(\frac{1}{N}(d_1+\cdots+d_N)))|<\varepsilon,\quad \tau\in\mathrm{T}(A).$$

Proof. It is enough to prove the statement for χ a monomial, i.e., $\chi(t)=t^n$. Note that there are positive numbers $\alpha_{i,j}$, $i=1,...,N$, $j=1,...,n$, such that

$$\begin{aligned} b^n &= (\frac{1}{N}(b_1+\cdots+b_N))^n \\ &= \frac{1}{N^n}\sum_{i_1+\cdots+i_N=n} b_1^{i_1}\cdots b_N^{i_N} \\ &= \sum_{i=1}^{N}\sum_{j=1}^{n}\alpha_{i,j}b_i^j. \end{aligned}$$

Hence,

$$\tau(b^n)=\sum_{i=1}^{N}\sum_{j=1}^{n}\alpha_{i,j}\tau(b_i^j).$$

Then there is $\delta>0$ such that if

$$\|d_id_{i+1}-d_{i+1}\|_{2,\mathrm{T}(A)}<\delta,\quad i=1,...,N-1,$$

then

$$\|(\frac{1}{N}(d_1+\cdots+d_N))^n-\sum_{i=1}^{N}\sum_{j=1}^{n}\alpha_{i,j}d_i^j\|_{2,\mathrm{T}(A)}<\varepsilon/2.$$

In particular,

$$\tau((\frac{1}{N}(d_1+\cdots+d_N))^n)\approx_{\varepsilon/2}\tau(\sum_{i=1}^{N}\sum_{j=1}^{n}\alpha_{i,j}d_i^j)=\sum_{i=1}^{N}\sum_{j=1}^{n}\alpha_{i,j}\tau(d_i^j).$$

Moreover, one may assume that $\delta>0$ is sufficiently small such that if

$$|\tau(b_i^j)-\tau(d_i^j)|<\delta,\quad i=1,2,...,N,\ j=1,...,n,$$

then

$$|\sum_{i=1}^{N}\sum_{j=1}^{n}\alpha_{i,j}\tau(b_i^j)-\sum_{i=1}^{N}\sum_{j=1}^{n}\alpha_{i,j}\tau(d_i^j)|<\varepsilon/2.$$

Then this δ and $M:=n$ have the desired property. ∎

8. The small boundary property

In this section, let us show that Property (S) for the ambient C*-algebra A indeed implies the (SBP) for the subalgebra D when Properties (C) and (E) are present (Theorem 8.3).

For each $\varepsilon>0$, define

$$\eta_\varepsilon(t)=\begin{cases}0, & t\leq 1-\varepsilon,\\ \text{linear}, & t\in[1-\varepsilon,1-\varepsilon/2],\\ 1, & t\geq 1-\varepsilon/2.\end{cases}\tag{8.1}$$

In the proof of the following lemma, we use $O(\varepsilon)$ to denote a quantity which converges to 0 when ε approaches 0.

Lemma 8.1. *Let A be a unital C*-algebra, and let $D \subseteq A$ be a unital commutative subalgebra such that the pair (D, A) has Property (C). Let $\phi_1, \phi_2, \phi_3, \psi_1, \psi_2 \in (D)_1^+$ and $\varepsilon_0 > 0$ have the following properties:*

(1) $\phi_2, \phi_3 \in \overline{\phi_1 D \phi_1}$,
(2) $\psi_1\psi_2 = \psi_2$, $\phi_2\phi_3 = \phi_3$,
(3) $\mathrm{d}_\tau(\psi_1) < \mathrm{d}_\tau(\phi_1)$ *for all* $\tau \in \mathrm{T}(A)$,
(4) $\inf\{\tau(\psi_2) - \mathrm{d}_\tau(\phi_3) : \tau \in \mathrm{T}(A)\} > 0$, *and*
(5) $\inf\{\mathrm{d}_\tau(\phi_2) - \tau(\eta_{\varepsilon_0}(\psi_1)) : \tau \in \mathrm{T}(A)\} > 0$.

Then, for any $\varepsilon > 0$, there is a contraction $u \in A$ such that

$$u^*\psi_1 u \in_{\varepsilon}^{\|\cdot\|_2} \overline{\phi_1 A \phi_1}, \quad u^*\eta_{\varepsilon_0/2}(\psi_1)u \in_{\varepsilon}^{\|\cdot\|_2} \overline{\phi_2 A \phi_2}, \quad \eta_\varepsilon(u^*\psi_1 u)\phi_3 \approx_{\varepsilon}^{\|\cdot\|_2} \phi_3,$$

and

$$\mathrm{dist}_{2,\mathrm{T}(A)}(udu^*, (D)_1) < \varepsilon, \quad \mathrm{dist}_{2,\mathrm{T}(A)}(u^*du, (D)_1) < \varepsilon, \quad d \in (D)_1,$$

and

$$\|uu^* - 1\|_{2,\mathrm{T}(A)}, \quad \|u^*u - 1\|_{2,\mathrm{T}(A)} < \varepsilon.$$

Proof. Let $\varepsilon > 0$ be given. Choose $\delta \in (0, \min\{\varepsilon_0, \varepsilon\}/4)$ such that if x, y are positive contractions of a unital C*-algebra A such that $\|x - y\|_{2,\mathrm{T}(A)} < \delta$, then

$$\|\eta_\varepsilon(x) - \eta_\varepsilon(y)\|_{2,\mathrm{T}(A)} < \varepsilon/2. \tag{8.2}$$

Define

$$\delta_1 := \inf\{\tau(\psi_2) - \mathrm{d}_\tau(\phi_3) : \tau \in \mathrm{T}(A)\} > 0$$

and

$$\delta_2 := \inf\{\mathrm{d}_\tau(\phi_2) - \tau(\eta_{\varepsilon_0}(\psi_1)) : \tau \in \mathrm{T}(A)\} > 0.$$

By Property (C) and Assumption (3), for any $\varepsilon' > 0$ (to be determined later), there is a contraction $u_1 \in A$ such that

$$u_1^*\psi_1 u_1 \in_{\varepsilon'}^{\|\cdot\|_2} \overline{\phi_1 A \phi_1}, \tag{8.3}$$

$$\mathrm{dist}_{2,\mathrm{T}(A)}(u_1 d u_1^*, (D)_1) < \varepsilon', \quad \mathrm{dist}_{2,\mathrm{T}(A)}(u_1^* d u_1, (D)_1) < \varepsilon', \quad d \in (D)_1, \tag{8.4}$$

and

$$\|u_1u_1^* - 1\|_{2,\mathrm{T}(A)}, \quad \|u_1^*u_1 - 1\|_{2,\mathrm{T}(A)} < \varepsilon'. \tag{8.5}$$

By (8.3), there is n large enough that

$$\|(\phi_1^{\frac{1}{n}})(u_1^*\psi_1 u_1) - u_1^*\psi_1 u_1\|_{2,\mathrm{T}(A)} < \varepsilon'.$$

By (8.4), there is a positive contraction $d_1 \in D$ such that

$$\|d_1 - u_1^*\psi_1 u_1\|_{2,\mathrm{T}(A)} < \varepsilon',$$

and then
$$\|\phi_1^{\frac{1}{n}}d_1 - u_1^*\psi_1u_1\|_{2,\mathrm{T}(A)} < 2\varepsilon'.$$
Consider $\phi_1^{\frac{1}{n}}d_1$, and still denote this element by d_1. One has
$$d_1 \in \overline{\phi_1 D\phi_1}$$
and
$$\|d_1 - u_1^*\psi_1u_1\|_{2,\mathrm{T}(A)} < 2\varepsilon'. \tag{8.6}$$

Consider the contraction
$$\eta_{\varepsilon_0}(u_1^*\psi_1u_1),$$
and note that (by (8.6), (8.5) and Condition (5))
$$\mathrm{d}_\tau(\eta_{\varepsilon_0/2}(d_1)) \leq \tau(\eta_{\varepsilon_0}(d_1)) \approx^{\|\cdot\|_2}_{O(\varepsilon')} \tau(\eta_{\varepsilon_0}(u_1^*\psi_1u_1)) \approx^{\|\cdot\|_2}_{O(\varepsilon')} \tau(\eta_{\varepsilon_0}(\psi_1)) < \mathrm{d}_\tau(\phi_2) - \delta_2/2, \quad \tau \in \mathrm{T}(A).$$
With ε' sufficiently small, one has
$$\mathrm{d}_\tau(\eta_{\varepsilon_0/2}(d_1)) < \mathrm{d}_\tau(\phi_2), \quad \tau \in \mathrm{T}(A).$$
By Property (C), for any $\varepsilon'' > 0$ (to be determined later), there is $u_2 \in \overline{\phi_1 A\phi_1} + \mathbb{C}1$ such that
$$u_2^*\eta_{\varepsilon_0/2}(d_1)u_2 \in^{\|\cdot\|_2}_{\varepsilon''} \overline{\phi_2 A\phi_2}, \tag{8.7}$$
$$\mathrm{dist}_{2,\mathrm{T}(A)}(u_2du_2^*, D_1) < \varepsilon'', \quad \mathrm{dist}_{2,\mathrm{T}(A)}(u_2^*du_2, D_1) < \varepsilon'', \quad d \in D_1, \tag{8.8}$$
and
$$\|u_2u_2^* - 1\|_{2,\mathrm{T}(A)}, \quad \|u_2^*u_2 - 1\|_{2,\mathrm{T}(A)} < \varepsilon''. \tag{8.9}$$

Since (as $\delta < \varepsilon_0/4$)
$$\eta_{\varepsilon_0/2}(d_1)\eta_\delta(d_1) = \eta_\delta(d_1),$$
it follows from (8.7) that
$$u_2^*\eta_\delta(d_1)u_2 \in^{\|\cdot\|_2}_{2\varepsilon''} \overline{\phi_2 A\phi_2}. \tag{8.10}$$

By (8.8), there are positive contractions
$$d_{1,1},\ d_{1,\delta} \in D$$
such that
$$\|d_{1,1} - u_2^*d_1u_2\|_{2,\mathrm{T}(A)} < \varepsilon'', \quad \|d_{1,\delta} - u_2^*\eta_\delta(d_1)u_2\|_{2,\mathrm{T}(A)} < \varepsilon''. \tag{8.11}$$
By (8.10),
$$d_{1,\delta} \in^{\|\cdot\|_2}_{3\varepsilon''} \in \overline{\phi_2 A\phi_2}.$$
With the same argument as above for d_1, there is a positive contraction, still denoted by $d_{1,\delta}$, such that
$$d_{1,\delta} \in \overline{\phi_2 D\phi_2} \quad \text{and} \quad \|d_{1,\delta} - u_2^*\eta_\delta(d_1)u_2\|_{2,\mathrm{T}(A)} < 5\varepsilon''. \tag{8.12}$$
Also note that, by (8.9),
$$\|\eta_\delta(d_{1,1}) - d_{1,\delta}\|_{2,\mathrm{T}(A)} = O(\varepsilon'').$$

Define

$$\tilde{d}_{1,1} = f_\delta(d_{1,1}) \quad \text{and} \quad \tilde{d}_{1,\delta} = \eta_\delta(d_{1,1}),$$

where

$$f_\delta(t) = \begin{cases} 0, & t \leq 0, \\ \text{linear}, & t \in [0, 1-\delta], \\ 1, & t \geq 1-\delta. \end{cases}$$

Then

$$\tilde{d}_{1,1}\tilde{d}_{1,\delta} = \tilde{d}_{1,\delta} \quad \text{and} \quad \|\tilde{d}_{1,1} - d_{1,1}\| < \delta, \tag{8.13}$$

and by (8.11),

$$\|\tilde{d}_{1,\delta} - d_{1,\delta}\|_{2,\mathrm{T}(A)} \approx_{O(\varepsilon'')} \|\eta_\delta(u_2^* d_1 u_2) - u_2^*\eta_\delta(d_1)u_2\| = O(\varepsilon'').$$

Then, with ε'' sufficiently small, there is $n \in \mathbb{N}$ such that

$$\|(\tilde{d}_{1,\delta})^{\frac{1}{n}} d_{1,\delta} - d_{1,\delta}\|_{2,\mathrm{T}(A)} < \delta_1/4,$$

and hence, for all $\tau \in \mathrm{T}(A)$,

$$\begin{array}{rcl} \mathrm{d}_\tau((\tilde{d}_{1,\delta})^{\frac{1}{n}} d_{1,\delta}) + \delta_1/4 & \geq & \tau((\tilde{d}_{1,\delta})^{\frac{1}{n}} d_{1,\delta}) + \delta_1/4 \\ & > & \tau(d_{1,\delta}) \approx_{5\varepsilon''} \tau(u_2^*\eta_\delta(d_1)u_2) \qquad ((8.12)) \\ & \approx_{O(\varepsilon')} & \tau(u_2^*\eta_\delta(u_1^*\psi_1 u_1)u_2) \qquad ((8.6)) \\ & \approx_{O(\varepsilon'+\varepsilon'')} & \tau(\eta_\delta(\psi_1)) > \tau(\psi_2) \\ & > & \mathrm{d}_\tau(\phi_3) + \delta_1/2. \end{array}$$

With ε' and ε'' sufficiently small, one has

$$\mathrm{d}_\tau((\tilde{d}_{1,\delta})^{\frac{1}{n}} d_{1,\delta}) > \mathrm{d}_\tau(\phi_3), \quad \tau \in \mathrm{T}(A).$$

Note that $(\tilde{d}_{1,\delta})^{\frac{1}{n}} d_{1,\delta} \in \overline{\phi_2 D \phi_2}$ (both $(\tilde{d}_{1,\delta})^{\frac{1}{n}}$ and $d_{1,\delta}$ belong to D, so they commute). By Property (C), for any $\varepsilon''' > 0$ (to be determined later), there is $u_3 \in \overline{\phi_2 A \phi_2} + \mathbb{C}1$ such that

$$u_3\phi_3 u_3^* \in_{\varepsilon'''}^{\|\cdot\|_2} \overline{(\tilde{d}_{1,\delta})^{\frac{1}{n}} d_{1,\delta} A (\tilde{d}_{1,\delta})^{\frac{1}{n}} d_{1,\delta}}, \tag{8.14}$$

$$\mathrm{dist}_{2,\mathrm{T}(A)}(u_3 d u_3^*, D_1) < \varepsilon''', \quad \mathrm{dist}_{2,\mathrm{T}(A)}(u_3^* d u_3, D_1) < \varepsilon''', \quad d \in D_1,$$

and

$$\|u_3 u_3^* - 1\|_{2,\mathrm{T}(A)}, \quad \|u_3^* u_3 - 1\|_{2,\mathrm{T}(A)} < \varepsilon'''. \tag{8.15}$$

Note that, by (8.13),

$$\tilde{d}_{1,1} c = c, \quad c \in \overline{(\tilde{d}_{1,\delta})^{\frac{1}{n}} d_{1,\delta} A (\tilde{d}_{1,\delta})^{\frac{1}{n}} d_{1,\delta}}.$$

By (8.14), there is $c \in \overline{(\tilde{d}_{1,\delta})^{\frac{1}{n}} d_{1,\delta} D (\tilde{d}_{1,\delta})^{\frac{1}{n}} d_{1,\delta}}$ such that $\|c\| \leq 1$ and

$$\|c - u_3\phi_3 u_3^*\|_{2,\mathrm{T}(A)} < \varepsilon'''.$$

Then

$$
\begin{aligned}
(8.16)\qquad (u_2^*\eta_\varepsilon(u_1^*\psi_1u_1)u_2)(u_3\phi_3u_3^*) &\approx^{\|\cdot\|_2}_{O(\varepsilon')} (u_2^*\eta_\varepsilon(d_1)u_2)(u_3\phi_3u_3^*) \qquad ((8.6))\\
&\approx^{\|\cdot\|_2}_{O(\varepsilon'')} \eta_\varepsilon(u_2^*d_1u_2)(u_3\phi_3u_3^*) \qquad ((8.9))\\
&\approx^{\|\cdot\|_2}_{O(\varepsilon'')} \eta_\varepsilon(d_{1,1})(u_3\phi_3u_3^*) \qquad ((8.11))\\
&\approx^{\|\cdot\|_2}_{\varepsilon/2+O(\varepsilon'')} \eta_\varepsilon(\tilde{d}_{1,1})(u_3\phi_3u_3^*) \qquad ((8.13)(8.2))\\
&\approx^{\|\cdot\|_2}_{\varepsilon'''} \eta_\varepsilon(\tilde{d}_{1,1})c\\
&= c \approx^{\|\cdot\|_2}_{\varepsilon'''} u_3\phi_3u_3^*.
\end{aligned}
$$

Then, consider the contraction

$$u = u_1u_2u_3.$$

Since

$$u_2 \in \overline{\phi_1A\phi_1} + \mathbb{C}1 \quad \text{and} \quad u_3 \in \overline{\phi_2A\phi_2} + \mathbb{C}1,$$

one has

$$(u_1u_2u_3)^*\psi_1(u_1u_2u_3) \in \overline{\phi_1A\phi_1}.$$

Moreover,

$$
\begin{aligned}
(u_1u_2u_3)^*\eta_{\varepsilon_0/2}(\psi_1)(u_1u_2u_3) &\approx^{\|\cdot\|_2}_{O(\varepsilon')} u_3^*u_2^*\eta_{\varepsilon_0/2}(u_1^*\psi_1u_1)u_2u_3 \qquad ((8.5))\\
&\approx^{\|\cdot\|_2}_{O(\varepsilon')} u_3^*(u_2^*\eta_{\varepsilon_0/2}(d_1)u_2)u_3 \qquad ((8.6))\\
&\in^{\|\cdot\|_2}_{\varepsilon''} \overline{\phi_2A\phi_2}, \qquad ((8.7))
\end{aligned}
$$

and

$$
\begin{aligned}
\eta_\varepsilon((u_1u_2u_3)^*\psi_1(u_1u_2u_3))\phi_3 &\approx^{\|\cdot\|_2}_{O(\varepsilon'+\varepsilon'')} u_3^*u_2^*\eta_\varepsilon(u_1^*\psi_1u_1)u_2u_3\phi_3 \qquad ((8.5),(8.9))\\
&\approx^{\|\cdot\|_2}_{\varepsilon'''} u_3^*(u_2^*\eta_\varepsilon(u_1^*\psi_1u_1)u_2)(u_3\phi_3u_3^*)u_3 \qquad ((8.15))\\
&\approx_{\varepsilon+O(\varepsilon'+\varepsilon''+\varepsilon''')} u_3^*(u_3\phi_3u_3^*)u_3 = \phi_3. \qquad ((8.16))
\end{aligned}
$$

With $\varepsilon', \varepsilon'', \varepsilon'''$ sufficiently small, the contraction u has the desired property. $\square$

For technical reasons, we also need the following lemma which, very roughly, asserts that, after a perturbation with respect to the uniform trace norm, the spectrum of a positive element of the subalgebra D is, in a strong sense, dense.

Lemma 8.2. *Let A be a unital C*-algebra and let $D = \mathrm{C}(X) \subseteq A$ be a unital commutative sub-C*-algebra. Assume A is simple, X has no isolated points, and the pair (D, A) has Property (E). Then, for any positive contraction $g \subseteq D$, any finite set $\{x_1, ..., x_n\} \subseteq (0, 1]$, and any $\varepsilon > 0$, there is a positive contraction $\tilde{g} \in D$ such that*

1. *$\|\tilde{g} - g\|_{2,\mathrm{T}(A)} < \varepsilon$, and*
2. *each point x_i, $i = 1, ..., n$, is in $\mathrm{sp}(\tilde{g})$, and is not isolated from the left inside $\mathrm{sp}(\tilde{g})$ (i.e., $(s, x_i) \cap \mathrm{sp}(\tilde{g}) \neq \emptyset$ for all $s < x_i$).*

Proof. Since A is simple (and non-elementary), there are mutually orthogonal positive elements $a_1, ..., a_n \in A$ such that

$$\|a_i\| = 1 \quad \text{and} \quad \mathrm{d}_\tau(a_i) < \varepsilon/n^2, \quad i = 1, ..., n, \ \tau \in \mathrm{T}(A).$$

(See, for instance, Lemma 4.7 of [27].)

Consider the contraction

$$a = \frac{1}{n}a_1 + \cdots + \frac{n}{n}a_n,$$

and note that it has the property

$$0 < \tau(h_i(a)) < \varepsilon/n, \quad i = 1, ..., n, \ \tau \in \mathrm{T}(A),$$

where $h_i : [0, 1] \to [0, 1]$ is the continuous function taking value 1 at $[(4i-1)/4n, (4i+1)/4n]$, 0 on $[0, (2i-1)/2n]$ and $[(2i+1)/2n, 1]$, and linear between. By Property (E), there is a positive contraction $d \in D$ such that

$$0 < \tau(h_i(d)) < \varepsilon/n, \quad i = 1, ..., n, \ \tau \in \mathrm{T}(A),$$

and there are mutually orthogonal positive elements $b_1, ..., b_n \in D$ such that

$$\|b_i\| = 1 \quad \text{and} \quad \mathrm{d}_\tau(b_i) < \varepsilon/n, \quad i = 1, ..., n, \ \tau \in \mathrm{T}(A).$$

Consider the sets

$$U_i = b_i^{-1}((0, 1]) \quad \text{and} \quad V_i = b_i^{-1}((1/2, 1]), \quad i = 1, ..., n.$$

Then

$$\overline{V_i} \subseteq U_i \quad \text{and} \quad \mu_\tau(U_i) < \varepsilon/n, \quad i = 1, ..., n, \ \tau \in \mathrm{T}(A).$$

For each V_i, $i = 1, ..., n$, since X has no isolated points, there is a continuous function $g_i : X \to [0, 1]$ such that $g_i|_{V_i^c} = 0$ and 1 is not isolated from the left in $g_i(X)$ (i.e., $(s, 1) \cap g_i(X) \neq \emptyset$ for all $s < 1$). Also pick a continuous function $r : X \to [0, 1]$ such that

$$r|_{X \setminus (\bigcup_{i=1}^n U_i)} = 1 \quad \text{and} \quad r|_{\bigcup_{i=1}^n \overline{V_i}} = 0.$$

Then the function

$$\tilde{g} := gr + (x_1 g_1 + x_2 g_2 + \cdots + x_n g_n)$$

has the desired property. $\square$

We are now ready to prove the main theorem of the paper, which states that Property (S) of A implies the (SBP) of $(D, \mathrm{T}(A))$ if Properties (C) and (E) are present.

Theorem 8.3. *Let A be a unital simple C*-algebra, and let $D = \mathrm{C}(X) \subseteq A$ be a unital commutative sub-C*-algebra such that X has no isolated points. Assume that the pair (D, A) has Properties (C) and (E). Then, if the C*-algebra A has Property (S), the pair $(D, \mathrm{T}(A))$ has the (SBP).*

Proof. By Theorem 2.9, it is enough to show that for any self-adjoint contraction $f \in D$ and any $\varepsilon > 0$, there is a self-adjoint element $g \in D$ such that

(1) $\|f - g\|_{2,\mathrm{T}(A)} < \varepsilon$, and
(2) there is $\delta > 0$ such that $\tau(\chi_\delta(g)) < \varepsilon$, $\tau \in \mathrm{T}(A)$.

To show this statement, it is enough to prove it for f such that $\mathrm{sp}(f) = [-1, 1]$. Indeed, set

$$t_- = \sup\{t < 0 : t \notin \mathrm{sp}(h)\} \quad \text{and} \quad t_+ = \inf\{t > 0 : t \notin \mathrm{sp}(h)\}.$$

If $t_- = 0$ or $t_+ = 0$, then it is straightforward to perturb f to produce g (with $\chi_\delta(g) = 0$). Assume neither of t_- and t_+ is zero. Choose $s_-,\ s_+ \notin \mathrm{sp}(f)$ such that

$$0 \leq t_- - s_- < \min\{\varepsilon, -t_-\} \quad \text{and} \quad 0 \leq s_+ - t_+ < \min\{\varepsilon, t_+\}$$

and consider the self-adjoint element $h(f)$ where

$$h = \begin{cases} 0, & t < t_-, \\ t_- - s_-, & t \in [t_-, s_-], \\ t, & t \in [t_-, t + -], \\ s_+ - s_-, & t \in [t_+, s_+], \\ 0 & t > s_+. \end{cases}$$

Then $\mathrm{sp}(h(f)) = [t_-, t_+]$, and

$$\|f - (f^-_{s_-} + h(f) + f^+_{s_+})\| < \varepsilon,$$

where $f^-_{s_-}(t) = t$ if $t < s_-$ and $f^-_{s_-}(t) = 0$ otherwise, and $f^+_{s_+}$ is defined similarly. Then, applying the statement to the self-adjoint element $h(f)$, one obtains the desired approximation g.

Now, let us assume that $\mathrm{sp}(f) = [-1, 1]$. Identifying $[-1, 1]$ with $[0, 1]$, let us show the following (equivalent) statement:

Let $f \in D$ be a positive contraction with $\mathrm{sp}(f) = [0, 1]$, and let $\varepsilon > 0$. Then there is a positive contraction $g \in D$ such that

$$\|f - g\|_{2,\mathrm{T}(A)} < \varepsilon, \tag{8.17}$$

and there is $\delta > 0$ such that

$$\tau(\chi_{\frac{1}{2},\delta}(g)) < \varepsilon, \quad \tau \in \mathrm{T}(A), \tag{8.18}$$

where

$$\chi_{\frac{1}{2},\delta}(t) = \begin{cases} 0, & t < \frac{1}{2} - \delta, \\ \text{linear}, & t \in [\frac{1}{2} - \delta, \frac{1}{2} - \frac{\delta}{2}], \\ 1, & t \in [\frac{1}{2} - \frac{\delta}{2}, \frac{1}{2} + \frac{\delta}{2}], \\ \text{linear}, & t \in [\frac{1}{2} + \frac{\delta}{2}, \frac{1}{2} + \delta], \\ 0, & t > \frac{1}{2} + \delta. \end{cases}$$

Let (f, ε) be given. Applying Lemma 7.4 to $\varepsilon/2$ (in place of ε), we obtain N and ε_0 (in place of δ). Choose $\varepsilon_1 > 0$ such that

$$3\varepsilon_1 < 1/N, \quad 8N\varepsilon_1 < \varepsilon_0 \quad \text{and} \quad \varepsilon_1 < \varepsilon/4.$$

For each $i = 1, 2, ..., N$, consider the following functions

$$\chi_i(t) = \begin{cases} 0, & t \leq \frac{i-1}{N}, \\ \text{linear}, & t \in [\frac{i-1}{N}, \frac{i}{N}], \\ 1, & t \geq \frac{i}{N}, \end{cases} \qquad \chi_{i,\varepsilon_1}(t) = \begin{cases} 0, & t \leq \frac{i-1}{N} + \varepsilon_1, \\ \text{linear}, & t \in [\frac{i-1}{N} + \varepsilon_1, \frac{i}{N} + \varepsilon_1], \\ 1, & t \geq \frac{i}{N} + \varepsilon_1, \end{cases}$$

$$\kappa_{i,\varepsilon_1}(t)=\begin{cases}0, & t\le \frac{i-1}{N}+\frac{\varepsilon_1}{2},\\ \text{linear}, & t\in[\frac{i-1}{N}+\frac{\varepsilon_1}{2}, \frac{i-1}{N}+\varepsilon_1],\\ 1, & t\ge \frac{i-1}{N}+\varepsilon_1,\end{cases}\qquad \theta_{i,\varepsilon_1}=\begin{cases}0, & t\le \frac{i}{N}+\varepsilon_1,\\ \text{linear}, & t\in[\frac{i}{N}+\varepsilon_1, \frac{i}{N}+2\varepsilon_1],\\ 1, & t\ge \frac{i}{N}+2\varepsilon_1,\end{cases}$$

$$\xi^+_{i,\varepsilon_1}=\begin{cases}0, & t\le \frac{i}{N},\\ \text{linear}, & t\in[\frac{i}{N}, \frac{i}{N}+\frac{\varepsilon_1}{2}],\\ 1, & t\ge \frac{i}{N}+\frac{\varepsilon_1}{2},\end{cases}\quad\text{and}\quad \xi^-_{i,\varepsilon_1}=\begin{cases}0, & t\le \frac{i}{N}+2\varepsilon_1,\\ \text{linear}, & t\in[\frac{i}{N}+2\varepsilon_1, \frac{i}{N}+3\varepsilon_1],\\ 1, & t\ge \frac{i}{N}+3\varepsilon_1.\end{cases}$$

Consider the finite set of functions

$$\mathcal{H}=\{\chi_i,\ \chi_{i,\varepsilon_1},\ \kappa_{i,\varepsilon_1},\ \theta_{i,\varepsilon_1},\ \eta_{\varepsilon_1/2}\circ\chi_{i,\varepsilon_1}:\ i=1,2,...,N\}. \tag{8.19}$$

Note that

$$\eta_{\varepsilon_1/2}(\chi_{i,\varepsilon_1}(t))=0,\quad t\le i/N+\varepsilon_1/2,\ i=1,...,N,$$

where, recall ((8.1)),

$$\eta_\varepsilon(t)=\begin{cases}0, & t\le 1-\varepsilon,\\ \text{linear}, & t\in[1-\varepsilon, 1-\varepsilon/2],\\ 1, & t\ge 1-\varepsilon/2.\end{cases}$$

Since A is simple and $\mathrm{sp}(f)=[0,1]$, there is $\gamma>0$ such that

$$\begin{aligned}\gamma\ <\ \frac{1}{4}\min\{&d_\tau(\chi_i(f))-\tau(\kappa_{i,\varepsilon_1}(f)),\ \tau(\theta_{i,\varepsilon_1}(f))-\mathrm{d}_\tau(\xi^-_{i,\varepsilon_1}(f)),\\ &\tau(\xi^+_{i,\varepsilon_1}(f))-\tau(\eta_{\varepsilon_1/2}(\chi_{i,\varepsilon_1}(f))): i=1,...,N,\ \tau\in\mathrm{T}(A)\}.\end{aligned} \tag{8.20}$$

Without loss of generality, one may assume that $\gamma<\varepsilon/4$.

Since A has Property (S), there is a positive contraction $\tilde{g}\in A$ such that $\|f-\tilde{g}\|_{2,\mathrm{T}(A)}$ is sufficiently small that

$$|\tau(\chi(f)-\tau(\chi(\tilde{g}))|<\gamma,\quad \chi\in\mathcal{H}\subseteq\mathrm{C}([0,1]),\ \tau\in\mathrm{T}(A), \tag{8.21}$$

and there is $\delta>0$ such that

$$\tau(\chi_{\frac{1}{2},\delta}(\tilde{g}))<\varepsilon/4,\quad \tau\in\mathrm{T}(A). \tag{8.22}$$

(Note that the choice of δ depends on $\tilde{g}$.)

Applying Lemma 7.5 to $\varepsilon/4$, N, and $\chi_{\frac{1}{2}+\varepsilon_1,\delta}$ (in place of ε, N, and χ, respectively), one obtains δ_0 (in place of δ) and M which have the property specified in Lemma 7.5 with respect to $\varepsilon/2$, N, and $\chi_{\frac{1}{2}+\varepsilon_1,\delta}$. Choose $\delta_1>0$ such that

$$4\delta_1<\varepsilon_0,\quad 6\delta_1<\delta_0,\quad\text{and}\quad 3^N\delta_1<\min\{\varepsilon_1,\gamma/2\}.$$

Also choose $\delta_2>0$ such that if a,b are positive contractions of a C*-algebra A, then

$$\|a-b\|_{2,\mathrm{T}(A)}<\delta_2\quad\Longrightarrow\quad \|\chi_{\frac{1}{2}+\varepsilon_1,\delta}(a))-\chi_{\frac{1}{2}+\varepsilon_1,\delta}(b)\|_{2,\mathrm{T}(A)}<\varepsilon/4. \tag{8.23}$$

Since (D,A) has Property (E), there is a positive contraction $\tilde{\tilde{g}}\in D$ such that

$$|\tau(\chi(\tilde{g}))-\tau(\chi(\tilde{\tilde{g}}))|<\gamma,\quad \chi\in\mathcal{H}\cup\{\chi_{\frac{1}{2},\delta}\},\ \tau\in\mathrm{T}(A). \tag{8.24}$$

In particular, together with (8.21) and (8.22), one has

$$|\tau(\chi(f)) - \tau(\chi(\tilde{\tilde{g}}))| < 2\gamma, \quad \chi \in \mathcal{H},\ \tau \in \mathrm{T}(A), \tag{8.25}$$

and

$$\tau(\chi_{\frac{1}{2},\delta}(\tilde{\tilde{g}})) < \varepsilon/4 + \gamma < \varepsilon/2. \tag{8.26}$$

So

$$\tau(\chi_{\frac{1}{2}+\varepsilon_1,\delta}((\tilde{\tilde{g}} - \varepsilon_1)_+)) < \varepsilon/2. \tag{8.27}$$

By Lemma 8.2, after a small perturbation (with respect to $\|\cdot\|_{2,\mathrm{T}(A)}$), without loss of generality, one may assume that the numbers

$$i/N + \varepsilon_1, \quad i = 1, 2, ..., N-1,$$

are in $\mathrm{sp}(\tilde{\tilde{g}})$, and are not isolated from the left.

Now, let us consider the elements

$$\chi_1(f),\ \chi_2(f),\ ...,\ \chi_N \in D$$

and

$$\chi_{1,\varepsilon_1}(\tilde{\tilde{g}}),\ \chi_{1,\varepsilon_1}(\tilde{\tilde{g}}),\ ...,\ \chi_{1,\varepsilon_1}(\tilde{\tilde{g}}) \in D.$$

By (8.24), one has

$$\mathrm{d}_\tau(\chi_{1,\varepsilon_1}(\tilde{\tilde{g}})) < \tau(\kappa_{1,\varepsilon_1}(\tilde{\tilde{g}})) \approx_{2\gamma} \tau(\kappa_{1,\varepsilon_1}(f)) < \mathrm{d}_\tau(\chi_1(f)), \quad \tau \in \mathrm{T}(A). \tag{8.28}$$

By the choice of γ ((8.20)), one has

$$\mathrm{d}_\tau(\chi_{1,\varepsilon_1}(\tilde{\tilde{g}})) < \mathrm{d}_\tau(\chi_1(f)), \quad \tau \in \mathrm{T}(A). \tag{8.29}$$

Note that, by the construction of ξ^+_{1,ε_1}, ξ^-_{1,ε_1}, and θ_{1,ε_1}, we have

$$\xi^+_{1,\varepsilon_1}(f),\ \xi^-_{1,\varepsilon_1}(f) \in \overline{\chi_1(f) D \chi_1(f)},$$

$$\chi_{1,\varepsilon_1}(\tilde{\tilde{g}})\theta_{1,\varepsilon_1}(\tilde{\tilde{g}}) = \theta_{1,\varepsilon_1}(\tilde{\tilde{g}}), \quad \xi^+_{1,\varepsilon_1}(f)\xi^-_{1,\varepsilon_1}(f) = \xi^-_{1,\varepsilon_1}(f),$$

$$\tau(\theta_{1,\varepsilon_1}(\tilde{\tilde{g}})) \approx_{2\gamma} \tau(\theta_{1,\varepsilon_1}(f)) > \mathrm{d}_\tau(\xi^-_{1,\varepsilon_1}(f)), \quad \tau \in \mathrm{T}(A),$$

and

$$\tau(\eta_{\varepsilon_1/2}(\chi_{1,\varepsilon_1}(\tilde{\tilde{g}}))) \approx_{2\gamma} \tau(\eta_{\varepsilon_1/2}(\chi_{1,\varepsilon_1}(f))) < \tau(\xi^+_{1,\varepsilon_1}(f)) < \mathrm{d}_\tau(\xi^+_{1,\varepsilon_1}(f)), \quad \tau \in \mathrm{T}(A).$$

By Lemma 8.1, for any $\delta'' > 0$ (to be fixed later), there is a contraction $u_1 \in A$ such that

$$u_1^* \chi_{1,\varepsilon_1}(\tilde{\tilde{g}}) u_1 \in^{\|\cdot\|_2}_{\delta''} \overline{\chi_1(f) A \chi_1(f)}, \tag{8.30}$$

$$\eta_{\varepsilon_1/4}(u_1^* \chi_{1,\varepsilon_1}(\tilde{\tilde{g}}) u_1) \in^{\|\cdot\|_2}_{\delta''} \overline{\xi^+_{1,\varepsilon_1}(f) A \xi^+_{1,\varepsilon_1}(f)}, \tag{8.31}$$

$$\eta_{\delta''}(u_1^* \chi_{1,\varepsilon_1}(\tilde{\tilde{g}}) u_1)\xi^-_{1,\varepsilon_1}(f) \approx^{\|\cdot\|_2}_{\delta''} \xi^-_{1,\varepsilon_1}(f), \tag{8.32}$$

$$\mathrm{dist}_{2,\mathrm{T}(A)}(u_1 d u_1^*, (D)_1) < \delta'', \quad \mathrm{dist}_{2,\mathrm{T}(A)}(u_1^* d u_1, (D)_1) < \delta'', \quad d \in (D)_1, \tag{8.33}$$

and

$$\|u_1u_1^* - 1\|_{2,\mathrm{T}(A)},\ \|u_1^*u_1 - 1\|_{2,\mathrm{T}(A)} < \delta''. \tag{8.34}$$

With δ'' sufficiently small, one has

$$|\tau((u_1^*xu_1)^j) - \tau(x^j)| < \delta_1, \quad j = 1, ..., M,\ x \in (A)_1,\ \tau \in \mathrm{T}(A), \tag{8.35}$$

and (by (8.33) and (8.34))

$$\mathrm{dist}_{2,\mathrm{T}(A)}(u_1^*\chi_{1,\varepsilon_1}(\tilde{\tilde{g}})u_1, (D)_1^+) < 3\delta'' < \min\{\varepsilon_0, \delta_2, \varepsilon/4\}. \tag{8.36}$$

Set

$$\rho_1 := \min\{\tau(\rho_{1,\delta_1}(\tilde{\tilde{g}})) : \tau \in \mathrm{T}(A)\},$$

where

$$\rho_{1,\delta_1} = \begin{cases} 0, & t \leq \frac{1}{N} + \varepsilon_1 - \delta_1, \\ \text{linear}, & \frac{1}{N} + \varepsilon_1 - \delta_1 \leq t \leq \frac{1}{N} + \varepsilon_1 - \delta_1/2, \\ 1, & t = \frac{1}{N} + \varepsilon_1 - \delta_1/2, \\ \text{linear}, & \frac{1}{N} + \varepsilon_1 - \delta_1/2 \leq t \leq \frac{1}{N} + \varepsilon_1, \\ 0, & t \geq \frac{1}{N} + \varepsilon_1. \end{cases}$$

Since $1/N + \varepsilon_1$ is not isolated from the left in $\mathrm{sp}(\tilde{\tilde{g}})$ and A is simple, we have that $\rho_1 > 0$.

By (8.36), there is a positive contraction

$$[u_1^*\chi_{1,\varepsilon_1}(\tilde{\tilde{g}})u_1] \in D$$

such that

$$\|[u_1^*\chi_{1,\varepsilon_1}(\tilde{\tilde{g}})u_1] - u_1^*\chi_{1,\varepsilon_1}(\tilde{\tilde{g}})u_1\|_{2,\mathrm{T}(A)} < 3\delta'' < \delta_1. \tag{8.37}$$

With δ'' sufficiently small, one has

$$\|\eta_{\delta_1}([u_1^*\chi_{1,\varepsilon_1}(\tilde{\tilde{g}})u_1]) - u_1^*\eta_{\delta_1}(\chi_{1,\varepsilon_1}(\tilde{\tilde{g}}))u_1\|_{2,\mathrm{T}(A)} < \min\{\rho_1/8, \delta_1\}.$$

Define

$$[u_1^*\eta_{\delta_1}(\chi_{1,\varepsilon_1}(\tilde{\tilde{g}}))u_1] := \eta_{\delta_1}([u_1^*\chi_{1,\varepsilon_1}(\tilde{\tilde{g}})u_1]) \in D. \tag{8.38}$$

Then

$$\|[u_1^*\eta_{\delta_1}(\chi_{1,\varepsilon_1}(\tilde{\tilde{g}}))u_1] - u_1^*\eta_{\delta_1}(\chi_{1,\varepsilon_1}(\tilde{\tilde{g}}))u_1\|_{2,\mathrm{T}(A)} < \min\{\rho_1/8, \delta_1\}. \tag{8.39}$$

Moreover (by (8.38)),

$$\eta_{2\delta_1}([u_1^*\chi_{1,\varepsilon_1}(\tilde{\tilde{g}})u_1])[u_1^*\eta_{\delta_1}(\chi_{1,\varepsilon_1}(\tilde{\tilde{g}}))u_1] = [u_1^*\eta_{\delta_1}(\chi_{1,\varepsilon_1}(\tilde{\tilde{g}}))u_1]. \tag{8.40}$$

One should assume δ'' is sufficiently small that

$$\|\eta_{\varepsilon_1/4}([u_1^*\chi_{1,\varepsilon_1}(\tilde{\tilde{g}})u_1]) - \eta_{\varepsilon_1/4}(u_1^*\chi_{1,\varepsilon_1}(\tilde{\tilde{g}})u_1)\|_{2,\mathrm{T}(A)} < \delta_1, \tag{8.41}$$

and

$$\|\eta_{2\delta_1}([u_1^*\chi_{1,\varepsilon_1}(\tilde{\tilde{g}})u_1]) - \eta_{2\delta_1}(u_1^*\chi_{1,\varepsilon_1}(\tilde{\tilde{g}})u_1)\|_{2,\mathrm{T}(A)} < \delta_1. \tag{8.42}$$

Note that (use (8.32) in the third and fifth steps),

$$\begin{array}{rcl} (u_1^*\chi_{1,\varepsilon_1}(\tilde{\tilde{g}})u_1)\chi_2(f) & \approx_{3N\varepsilon_1} & (u_1^*\chi_{1,\varepsilon_1}(\tilde{\tilde{g}})u_1)\chi_{2,3\varepsilon_1}(f) \\ & = & (u_1^*\chi_{1,\varepsilon_1}(\tilde{\tilde{g}})u_1)\xi^-_{1,\varepsilon_1}(f)\chi_{2,3\varepsilon_1}(f) \\ & \approx^{\|\cdot\|_2}_{\delta''} & (u_1^*\chi_{1,\varepsilon_1}(\tilde{\tilde{g}})u_1)\eta_{\delta''}(u_1^*\chi_{1,\varepsilon_1}(\tilde{\tilde{g}})u_1)\xi^-_{1,\varepsilon_1}(f)\chi_{2,3\varepsilon_1}(f) \\ & \approx_{\delta''} & \eta_{\delta''}(u_1^*\chi_{1,\varepsilon_1}(\tilde{\tilde{g}})u_1)\xi^-_{1,\varepsilon_1}(f)\chi_{2,3\varepsilon_1}(f) \\ & \approx^{\|\cdot\|_2}_{\delta''} & \xi^-_{1,\varepsilon_1}(f)\chi_{2,3\varepsilon_1}(f) \\ & = & \chi_{2,3\varepsilon_1}(f) \approx_{3N\varepsilon_1} \chi_2(f). \end{array}$$

With δ'' sufficiently small, one has

$$[u_1^*\chi_{1,\varepsilon_1}(\tilde{\tilde{g}})u_1]\chi_2(f) \approx^{\|\cdot\|_2}_{7N\varepsilon_1} \chi_2(f). \tag{8.43}$$

Note that, by (8.39),

$$|\tau([u_1^*\eta_{\delta_1}(\chi_{1,\varepsilon_1}(\tilde{\tilde{g}}))u_1]) - \tau(u_1^*\eta_{\delta_1}(\chi_{1,\varepsilon_1}(\tilde{\tilde{g}}))u_1)| < \rho_1/8, \quad \tau \in \mathrm{T}(A). \tag{8.44}$$

With δ'' sufficiently small, one has

$$\tau(\eta_{\delta_1}(\chi_{1,\varepsilon_1}(\tilde{\tilde{g}}))) \approx_{\rho_1/8} \tau(u_1^*\eta_{\delta_1}(\chi_{1,\varepsilon_1}(\tilde{\tilde{g}}))u_1), \quad \tau \in \mathrm{T}(A), \tag{8.45}$$

and

$$\|\eta_{\delta_1}(u_1^*\chi_{1,\varepsilon_1}(\tilde{\tilde{g}})u_1) - u_1^*\eta_{\delta_1}(\chi_{1,\varepsilon_1}(\tilde{\tilde{g}}))u_1\|_2 < \delta_1/4. \tag{8.46}$$

Hence, by (8.44) and (8.45),

$$\begin{array}{rcl} \mathrm{d}_\tau(\chi_{2,\varepsilon_1}(\tilde{\tilde{g}})) & \leq & \tau(\eta_{\delta_1}(\chi_{1,\varepsilon_1}(\tilde{\tilde{g}}))) - \rho_1/2 \\ & \approx_{\rho_1/8} & \tau(u_1^*\eta_{\delta_1}(\chi_{1,\varepsilon_1}(\tilde{\tilde{g}}))u_1) - \rho_1/2 \\ & \approx_{\rho_1/8} & \tau([u_1^*\eta_{\delta_1}(\chi_{1,\varepsilon_1}(\tilde{\tilde{g}}))u_1]) - \rho_1/2 \\ & < & \mathrm{d}_\tau([u_1^*\eta_{\delta_1}(\chi_{1,\varepsilon_1}(\tilde{\tilde{g}}))u_1]) - \rho_1/2, \end{array}$$

for all $\tau \in \mathrm{T}(A)$, and therefore

$$\mathrm{d}_\tau(\chi_{2,\varepsilon_1}(\tilde{\tilde{g}})) < \mathrm{d}_\tau([u_1^*\eta_{\delta_1}(\chi_{1,\varepsilon_1}(\tilde{\tilde{g}}))u_1]), \quad \tau \in \mathrm{T}(A). \tag{8.47}$$

By (8.32) (and (8.39), (8.46)), (note that $3\varepsilon_1 < 1/N$ and $\delta'' \ll \delta_1$)

$$\begin{array}{rcl} [u_1^*\eta_{\delta_1}(\chi_{1,\varepsilon_1}(\tilde{\tilde{g}}))u_1](\xi^+_{2,\varepsilon_1}(f)) & = & [u_1^*\eta_{\delta_1}(\chi_{1,\varepsilon_1}(\tilde{\tilde{g}}))u_1](\xi^-_{1,\varepsilon_1}(f))(\xi^+_{2,\varepsilon_1}(f)) \\ & \approx^{\|\cdot\|_2}_{\delta_1} & (u_1^*\eta_{\delta_1}(\chi_{1,\varepsilon_1}(\tilde{\tilde{g}}))u_1)(\xi^-_{1,\varepsilon_1}(f))(\xi^+_{2,\varepsilon_1}(f)) \\ & \approx^{\|\cdot\|_2}_{\delta_1} & \eta_{\delta_1}((u_1^*(\chi_{1,\varepsilon_1}(\tilde{\tilde{g}}))u_1))(\xi^-_{1,\varepsilon_1}(f))(\xi^+_{2,\varepsilon_1}(f)) \\ & \approx^{\|\cdot\|_2}_{\delta''} & (\xi^-_{1,\varepsilon_1}(f))(\xi^+_{2,\varepsilon_1}(f)) \\ & = & \xi^+_{2,\varepsilon_1}(f), \end{array} \tag{8.48}$$

and therefore, one also has

$$[u_1^*\eta_{\delta_1}(\chi_{1,\varepsilon_1}(\tilde{\tilde{g}}))u_1](\xi^-_{2,\varepsilon_1}(f)) \approx^{\|\cdot\|_2}_{3\delta_1} \xi^-_{2,\varepsilon_1}(f). \tag{8.49}$$

Now, fixing δ'', we have the contraction $u_1 \in A$.

Let us inductively assume that contractions $u_1, ..., u_k$, where $k \leq N-1$, have been constructed such that (note that (8.51) and (8.53) are void if $k=1$)

$$\mathrm{dist}_{2,\mathrm{T}(A)}(u_i^*\chi_{i,\varepsilon_1}(\tilde{\tilde{g}})u_i, (D)_1^+) < \min\{\varepsilon_0, \delta_2, \varepsilon/4\}, \quad i=1,...,k, \qquad ((8.36) \text{ when } k=1) \tag{8.50}$$

$$\|\chi_{i-1}(f)(u_i^*\chi_{i,\varepsilon_1}(\tilde{\tilde{g}})u_i) - (u_i^*\chi_{i,\varepsilon_1}(\tilde{\tilde{g}})u_i)\|_{2,\mathrm{T}(A)} < 4\delta_1 < \varepsilon_1, \quad i=1,...,k, \tag{8.51}$$

$$\|(u_i^*\chi_{i,\varepsilon_1}(\tilde{\tilde{g}})u_i)\chi_{i+1}(f) - \chi_{i+1}(f)\|_{2,\mathrm{T}(A)} < 7N\varepsilon_1 + 3^i\delta_1, \quad i=1,...,k, \qquad ((8.43) \text{ when } k=1) \tag{8.52}$$

and

$$\|(u_{i-1}^*\chi_{i,\varepsilon_1}(\tilde{\tilde{g}})u_{i-1})(u_i^*\chi_{i,\varepsilon_1}(\tilde{\tilde{g}})u_i) - (u_i^*\chi_{i,\varepsilon_1}(\tilde{\tilde{g}})u_i)\|_{2,\mathrm{T}(A)} < 6\delta_1, \quad i=1,...,k, \tag{8.53}$$

$$|\tau((u_i^*xu_i)^j) - \tau(x^j)| < \delta_1, \quad i=1,...,k,\ j=1,...,M,\ x\in(A)_1,\ \tau\in\mathrm{T}(A). \qquad ((8.35) \text{ when } k=1) \tag{8.54}$$

Moreover, the contraction u_k satisfies

$$\eta_{\varepsilon_1/4}(u_k^*\chi_{k,\varepsilon_1}(\tilde{\tilde{g}})u_k) \in_{\delta_1}^{\|\cdot\|_2} \overline{\xi_{k,\varepsilon_1}^+(f)A\xi_{k,\varepsilon_1}^+(f)}, \qquad ((8.31) \text{ when } k=1;\text{ note that } \delta''<\delta) \tag{8.55}$$

and there are positive contractions $[u_k^*\chi_{k,\varepsilon_1}(\tilde{\tilde{g}})u_k], [u_k^*\eta_{\delta_1}(\chi_{k,\varepsilon_1}(\tilde{\tilde{g}}))u_k] \in D$ such that

$$\eta_{2\delta_1}([u_k^*\chi_{k,\varepsilon_1}(\tilde{\tilde{g}})u_k])[u_k^*\eta_{\delta_1}(\chi_{k,\varepsilon_1}(\tilde{\tilde{g}}))u_k] = [u_k^*\eta_{\delta_1}(\chi_{k,\varepsilon_1}(\tilde{\tilde{g}}))u_k], \qquad ((8.40) \text{ when } k=1) \tag{8.56}$$

$$\|\eta_{\varepsilon_1/4}([u_k^*\chi_{k,\varepsilon_1}(\tilde{\tilde{g}})u_k]) - \eta_{\varepsilon_1/4}(u_k^*\chi_{k,\varepsilon_1}(\tilde{\tilde{g}})u_k)\|_{2,\mathrm{T}(A)} < \delta_1, \qquad ((8.41) \text{ when } k=1) \tag{8.57}$$

$$\|\eta_{2\delta_1}([u_k^*\chi_{k,\varepsilon_1}(\tilde{\tilde{g}})u_k]) - \eta_{2\delta_1}(u_k^*\chi_{k,\varepsilon_1}(\tilde{\tilde{g}})u_k)\|_{2,\mathrm{T}(A)} < \delta_1, \qquad ((8.42) \text{ when } k=1) \tag{8.58}$$

$$\mathrm{d}_\tau(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}})) < \mathrm{d}_\tau([u_k^*\eta_{\delta_1}(\chi_{k,\varepsilon_1}(\tilde{\tilde{g}}))u_k]), \quad \tau\in\mathrm{T}(A), \qquad ((8.47) \text{ when } k=1) \tag{8.59}$$

$$[u_k^*\eta_{\delta_1}(\chi_{k,\varepsilon_1}(\tilde{\tilde{g}}))u_k](\xi_{k+1,\varepsilon_1}^+(f)) \approx_{3^k\delta_1}^{\|\cdot\|_2} \xi_{k+1,\varepsilon_1}^+(f), \qquad ((8.48) \text{ when } k=1) \tag{8.60}$$

and

$$[u_k^*\eta_{\delta_1}(\chi_{k,\varepsilon_1}(\tilde{\tilde{g}}))u_k](\xi_{k+1,\varepsilon_1}^-(f)) \approx_{3^k\delta_1}^{\|\cdot\|_2} \xi_{k+1,\varepsilon_1}^-(f). \qquad ((8.49) \text{ when } k=1) \tag{8.61}$$

Let us construct u_{k+1}. Define

$$\left\langle \xi_{k+1,\varepsilon_1}^-(f)\right\rangle := [u_k^*\eta_{\delta_1}(\chi_{k,\varepsilon_1}(\tilde{\tilde{g}}))u_k](\xi_{k+1,\varepsilon_1}^-(f)) \in \mathrm{Her}([u_k^*\eta_{\delta_1}(\chi_{k,\varepsilon_1}(\tilde{\tilde{g}}))u_k]) \cap D.$$

It follows from (8.61) that

$$\|\left\langle \xi_{k+1,\varepsilon_1}^-(f)\right\rangle - \xi_{k+1,\varepsilon_1}^-(f)\|_{2,\mathrm{T}(A)} < 3^k\delta_1. \tag{8.62}$$

Then, for all $\tau\in\mathrm{T}(A)$,

$$\tau(\theta_{k+1,\varepsilon_1}(f)) > d_\tau(\xi_{k+1,\varepsilon_1}^-(f)) + 2\gamma \geq d_\tau([u_k^*\eta_{\delta_1}(\chi_{k,\varepsilon_1}(\tilde{\tilde{g}}))u_k]\xi_{k+1,\varepsilon_1}^-(f)) + 2\gamma = \mathrm{d}_\tau(\left\langle \xi_{k+1,\varepsilon_1}^-(f)\right\rangle) + 2\gamma,$$

and hence

$$\tau(\theta_{k+1,\varepsilon_1}(\tilde{\tilde{g}})) \approx_\gamma \tau(\theta_{k+1,\varepsilon_1}(f)) > \mathrm{d}_\tau(\left\langle \xi_{k+1,\varepsilon_1}^-(f)\right\rangle) + 2\gamma.$$

In particular,

$$\tau(\theta_{k+1,\varepsilon_1}(\tilde{\tilde{g}})) > \mathrm{d}_\tau(\left\langle \xi_{k+1,\varepsilon_1}^-(f)\right\rangle), \quad \tau\in\mathrm{T}(A). \tag{8.63}$$

Also define

$$\left\langle \xi^+_{k+1,\varepsilon_1}(f)\right\rangle := [u_k^*(\eta_{\delta_1}(\chi_{k,\varepsilon_1}(\tilde{\tilde{g}}))u_k](\xi^+_{k+1,\varepsilon_1}(f)) \in \mathrm{Her}([u_k^*(\eta_{\delta_1}((\chi_{k,\varepsilon_1}(\tilde{\tilde{g}})))u_k]) \cap D. \tag{8.64}$$

Then,

$$\begin{array}{ll} & \tau(\eta_{\varepsilon_1/2}(\chi_{k+1,\varepsilon_1}(f))) \\ < & \tau(\xi^+_{k+1,\varepsilon_1}(f)) - 3\gamma \qquad ((8.20)) \\ \approx_{3^k\delta_1} & \tau([u_k^*(\eta_{\delta_1}(\chi_{k,\varepsilon_1}(\tilde{\tilde{g}}))u_k](\xi^+_{k+1,\varepsilon_1}(f))) - 3\gamma \qquad ((8.60)) \\ = & \tau(\left\langle \xi^+_{k+1,\varepsilon_1}(f)\right\rangle) - 3\gamma \\ < & \mathrm{d}_\tau(\left\langle \xi^+_{k+1,\varepsilon_1}(f)\right\rangle) - 3\gamma, \end{array}$$

and therefore

$$\tau(\eta_{\varepsilon_1/2}(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}))) \approx_{2\gamma} \tau(\eta_{\varepsilon_1/2}(\chi_{k+1,\varepsilon_1}(f))) < d_\tau([\xi^+_{k+1,\varepsilon_1}(f)]) - 3\gamma + 3^k\delta_1.$$

In particular (note that $3^N\delta_1 < \gamma/2$),

$$\tau(\eta_{\varepsilon_1/2}(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}))) < d_\tau([\xi^+_{k+1,\varepsilon_1}(f)]), \quad \tau \in \mathrm{T}(A). \tag{8.65}$$

With (8.59), (8.63), and (8.65), by Lemma 8.1, for any $\delta'' > 0$ (to be fixed later), there is a contraction $u_{k+1} \in A$ such that

$$u_{k+1}^*\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}})u_{k+1} \in^{\|\cdot\|_2}_{\delta''} \overline{[u_k^*\eta_{\delta'}(\chi_{k,\varepsilon_1}(\tilde{\tilde{g}}))u_k]A[u_k^*\eta_{\delta'}(\chi_{k,\varepsilon_1}(\tilde{\tilde{g}}))u_k]}, \tag{8.66}$$

$$\eta_{\varepsilon_1/4}(u_{k+1}^*\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}})u_{k+1}) \in^{\|\cdot\|_2}_{\delta''} \overline{\left\langle \xi^+_{k+1,\varepsilon_1}(f)\right\rangle A \left\langle \xi^+_{k+1,\varepsilon_1}(f)\right\rangle}, \tag{8.67}$$

$$\eta_{\delta''}(u_{k+1}^*\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}})u_{k+1}) \left\langle \xi^-_{k+1,\varepsilon_1}(f)\right\rangle \approx^{\|\cdot\|_2}_{\delta''} \left\langle \xi^-_{k+1,\varepsilon_1}(f)\right\rangle, \tag{8.68}$$

$$\mathrm{dist}_{2,\mathrm{T}(A)}(u_{k+1}du_{k+1}^*, D_1) < \delta'', \quad \mathrm{dist}_{2,\mathrm{T}(A)}(u_{k+1}^*du_{k+1}, D_1) < \delta'', \quad d \in D_1, \tag{8.69}$$

and

$$\|u_{k+1}u_{k+1}^* - 1\|_{2,\mathrm{T}(A)},\ \|u_{k+1}^*u_{k+1} - 1\|_{2,\mathrm{T}(A)} < \delta''. \tag{8.70}$$

By (8.69) and (8.70), with δ'' sufficiently small,

$$\mathrm{dist}_{2,\mathrm{T}(A)}(u_{k+1}^*\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}})u_{k+1}, (D)_1^+) < 3\delta'' < \min\{\varepsilon_0, \delta_2, \varepsilon/2\}. \tag{8.71}$$

This verifies Assumption (8.50) for $k+1$.

Also note

$$\begin{array}{ll} & u_{k+1}^*(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}))u_{k+1} \\ \approx^{\|\cdot\|_2}_{\delta''} & (u_{k+1}^*(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}))u_{k+1})(\eta_{2\delta_1}([u_k^*\chi_{k,\varepsilon_1}(\tilde{\tilde{g}})u_k])) \qquad ((8.66), (8.56)) \\ \approx^{\|\cdot\|_2}_{\delta_1} & (u_{k+1}^*(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}))u_{k+1})(\eta_{2\delta_1}(u_k^*\chi_{k,\varepsilon_1}(\tilde{\tilde{g}})u_k)) \qquad ((8.58)) \\ \approx^{\|\cdot\|_2}_{2\delta_1} & (u_{k+1}^*(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}))u_{k+1})(\eta_{2\delta_1}(u_k^*\chi_{k,\varepsilon_1}(\tilde{\tilde{g}})u_k))(u_k\chi_{k,\varepsilon_1}(\tilde{\tilde{g}})u_k) \\ \approx_{\delta_1+\delta''} & (u_{k+1}^*(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}))u_{k+1})(u_k^*\chi_{k,\varepsilon_1}(\tilde{\tilde{g}})u_k). \qquad ((8.66), (8.56), (8.58)) \end{array}$$

In other words,

$$(u_k^*\chi_{k,\varepsilon_1}(\tilde{\tilde{g}})u_k)(u_{k+1}^*(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}))u_{k+1}) \approx_{6\delta_1}^{\|\cdot\|_2} u_{k+1}^*(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}))u_{k+1}.$$

This verifies Assumption (8.53) for $k+1$.

By (8.55) and (8.66), (8.56), (8.57),

$$\begin{aligned}
\chi_k(f)(u_{k+1}^*(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}))u_{k+1}) &\approx_{\delta''} \chi_k(f)\eta_{2\delta_1}([u_k^*\chi_{k,\varepsilon_1}(\tilde{\tilde{g}})u_k])(u_{k+1}^*(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}))u_{k+1}) && ((8.56))\\
&= \chi_k(f)\eta_{\varepsilon_1/4}([u_k^*\chi_{k,\varepsilon_1}(\tilde{\tilde{g}})u_k])(u_{k+1}^*(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}))u_{k+1})\\
&\approx_{\delta_1}^{\|\cdot\|_2} (\chi_k(f)\eta_{\varepsilon_1/4}(u_k^*\chi_{k,\varepsilon_1}(\tilde{\tilde{g}})u_k))(u_{k+1}^*(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}))u_{k+1}) && ((8.57))\\
&\approx_{\delta_1}^{\|\cdot\|_2} \eta_{\varepsilon_1/4}(u_k^*\chi_{k,\varepsilon_1}(\tilde{\tilde{g}})u_k)(u_{k+1}^*(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}))u_{k+1}) && ((8.55))\\
&\approx_{\delta_1}^{\|\cdot\|_2} \eta_{\varepsilon_1/4}([u_k^*\chi_{k,\varepsilon_1}(\tilde{\tilde{g}})u_k])(u_{k+1}^*(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}))u_{k+1}) && ((8.57))\\
&= u_{k+1}^*(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}))u_{k+1}.
\end{aligned}$$

So,

$$\chi_k(f)(u_{k+1}^*(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}))u_{k+1}) \approx_{4\delta_1}^{\|\cdot\|_2} u_{k+1}^*(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}))u_{k+1}.$$

This verifies Assumption (8.51) for $k+1$.

If $k+1 \leq N-1$, with the same argument as for (8.43),

$$\begin{aligned}
& (u_{k+1}^*\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}})u_{k+1})\chi_{k+2}(f)\\
\approx_{3N\varepsilon_1}\ & (u_{k+1}^*\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}})u_{k+1})\chi_{k+2,3\varepsilon_1}(f)\\
=\ & (u_{k+1}^*\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}})u_{k+1})\xi_{k+1,\varepsilon_1}^-(f)\chi_{k+2,3\varepsilon_1}(f)\\
\approx_{3^k\delta_1}^{\|\cdot\|_2}\ & (u_{k+1}^*\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}})u_{k+1})\left\langle\xi_{k+1,\varepsilon_1}^-(f)\right\rangle\chi_{k+2,3\varepsilon_1}(f) \qquad ((8.62))\\
\approx_{\delta''}^{\|\cdot\|_2}\ & (u_{k+1}^*\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}})u_{k+1})\eta_{\delta''}(u_{k+1}^*\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}})u_{k+1})\left\langle\xi_{k+1,\varepsilon_1}^-(f)\right\rangle\chi_{k+2,3\varepsilon_1}(f) \qquad ((8.68))\\
\approx_{\delta''}\ & \eta_{\delta''}(u_{k+1}^*\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}})u_{k+1})\left\langle\xi_{k+1,\varepsilon_1}^-(f)\right\rangle\chi_{k+2,3\varepsilon_1}(f)\\
\approx_{\delta''}^{\|\cdot\|_2}\ & \left\langle\xi_{k+1,\varepsilon_1}^-(f)\right\rangle\chi_{k+2,3\varepsilon_1}(f) \qquad ((8.68))\\
\approx_{3^k\delta_1}^{\|\cdot\|_2}\ & \chi_{k+2,3\varepsilon_1}(f) \approx_{3N\varepsilon_1} \chi_{k+2}(f). \qquad ((8.62))
\end{aligned}$$

Thus,

$$(u_{k+1}^*(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}))u_{k+1})\chi_{k+2}(f) \approx_{7N\varepsilon_1+3^{k+1}\delta_1}^{\|\cdot\|_2} \chi_{k+2}(f).$$

This verifies Assumption (8.52) for $k+1$ (which is void if $k+1=N$).

With δ'' sufficiently small, one has

$$|\tau((u_{k+1}^*xu_{k+1})^j)-\tau(x^j)| < \delta_1, \quad j=1,...,M,\ x\in(A)_1,\ \tau\in\mathrm{T}(A).$$

This verifies (8.54) for $k+1$.

If $k+1\leq N-1$, let us verify that (with δ'' sufficiently small), the contraction u_{k+1} satisfies the inductive assumptions (8.55), (8.56), (8.57), (8.58), (8.59), (8.60), and (8.61) for $k+1$.

By (8.64) and noting that $[u_k^*(\eta_{\delta_1}(\chi_{k,\varepsilon_1}(\tilde{\tilde{g}}))u_k]$ and $(\xi_{k+1,\varepsilon_1}^+(f))$ commute (both are in D), one has

$$\overline{\left\langle\xi_{k+1,\varepsilon_1}^+(f)\right\rangle A\left\langle\xi_{k+1,\varepsilon_1}^+(f)\right\rangle} \subseteq \overline{\xi_{k+1,\varepsilon_1}^+(f)A\xi_{k+1,\varepsilon_1}^+(f)}.$$

Thus, Assumption (8.55) for $k+1$ follows from (8.67).

For the other assumptions, let us repeat the argument of u_1: Set

$$\rho_{k+1} := \min\{\tau(\rho_{k+1,\delta_1}(\tilde{\tilde{g}})) : \tau \in \mathrm{T}(A)\},$$

where

$$\rho_{k+1,\delta_1} = \begin{cases} 0, & t \leq \frac{k+1}{N} + \varepsilon_1 - \delta_1, \\ \text{linear}, & \frac{k+1}{N} + \varepsilon_1 - \delta_1 \leq t \leq \frac{k+1}{N} + \varepsilon_1 - \delta_1/2, \\ 1, & t = \frac{k+1}{N} + \varepsilon_1 - \delta_1/2, \\ \text{linear}, & \frac{k+1}{N} + \varepsilon_1 - \delta_1/2 \leq t \leq \frac{k+1}{N} + \varepsilon_1, \\ 0, & t \geq \frac{k+1}{N} + \varepsilon_1. \end{cases}$$

Since $(k+1)/N + \varepsilon_1$ is not isolated from the left in $\mathrm{sp}(\tilde{\tilde{g}})$ and A is simple, we have that $\rho_1 > 0$.

By (8.71), with a sufficiently small δ'', there is a positive contraction

$$[u_{k+1}^* \chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}) u_{k+1}] \in D$$

such that

$$\|[u_{k+1}^* \chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}) u_{k+1}] - u_{k+1}^* \chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}) u_{k+1}\|_{2,\mathrm{T}(A)} < 3\delta'' < \delta_1.$$

With δ'' sufficiently small, one has

$$\|\eta_{\delta_1}([u_{k+1}^* \chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}) u_{k+1}]) - u_{k+1}^* \eta_{\delta_1}(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}})) u_{k+1}\|_{2,\mathrm{T}(A)} < \min\{\rho_{k+1}/8, \delta_1\}.$$

Define

$$[u_{k+1}^* \eta_{\delta_1}(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}})) u_{k+1}] := \eta_{\delta_1}([u_{k+1}^* \chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}) u_{k+1}]) \in D. \tag{8.72}$$

Then

$$\|[u_{k+1}^* \eta_{\delta_1}(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}})) u_{k+1}] - u_{k+1}^* \eta_{\delta_1}(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}})) u_{k+1}\|_{2,\mathrm{T}(A)} < \min\{\rho_{k+1}/8, \delta_1\}. \tag{8.73}$$

Moreover (by (8.72)),

$$\eta_{2\delta_1}([u_{k+1}^* \chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}) u_{k+1}])[u_{k+1}^* \eta_{\delta_1}(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}})) u_{k+1}] = [u_{k+1}^* \eta_{\delta_1}(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}})) u_{k+1}]. \tag{8.74}$$

This verifies (8.56) for $k+1$.

One should assume δ'' is sufficiently small that

$$\|\eta_{\varepsilon_1/4}([u_{k+1}^* \chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}) u_{k+1}]) - \eta_{\varepsilon_1/4}(u_{k+1}^* \chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}) u_{k+1})\|_{2,\mathrm{T}(A)} < \delta_1, \tag{8.75}$$

and

$$\|\eta_{2\delta_1}([u_{k+1}^* \chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}) u_{k+1}]) - \eta_{2\delta_1}(u_{k+1}^* \chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}) u_{k+1})\|_{2,\mathrm{T}(A)} < \delta_1. \tag{8.76}$$

This verifies (8.57) and (8.58) for $k+1$.

Note that, by (8.73),

$$|\tau([u_{k+1}^* \eta_{\delta_1}(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}})) u_{k+1}]) - \tau(u_{k+1}^* \eta_{\delta_1}(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}})) u_{k+1})| < \rho_{k+1}/8, \quad \tau \in \mathrm{T}(A). \tag{8.77}$$

With δ'' sufficiently small, one has

$$\tau(\eta_{\delta_1}(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}))) \approx_{\rho_{k+1}/8} \tau(u_{k+1}^* \eta_{\delta_1}(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}})) u_{k+1}), \quad \tau \in \mathrm{T}(A), \tag{8.78}$$

and

$$\|\eta_{\delta_1}(u_{k+1}^* \chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}) u_{k+1}) - u_{k+1}^* \eta_{\delta_1}(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}})) u_{k+1}\|_{2,\mathrm{T}(A)} < \delta_1/4. \tag{8.79}$$

Hence, by (8.77) and (8.78),

$$\begin{array}{rcl} \mathrm{d}_\tau(\chi_{k+2,\varepsilon_1}(\tilde{\tilde{g}})) & \leq & \tau(\eta_{\delta_1}(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}))) - \rho_{k+1}/2 \\ & \approx_{\rho_{k+1}/8} & \tau(u_{k+1}^*\eta_{\delta_1}(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}))u_{k+1}) - \rho_{k+1}/2 \\ & \approx_{\rho_{k+1}/8} & \tau([u_{k+1}^*\eta_{\delta_1}(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}))u_{k+1}]) - \rho_{k+1}/2 \\ & < & \mathrm{d}_\tau([u_{k+1}^*\eta_{\delta_1}(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}))u_{k+1}]) - \rho_{k+1}/2, \end{array}$$

for all $\tau \in \mathrm{T}(A)$, and therefore

$$\mathrm{d}_\tau(\chi_{k+2,\varepsilon_1}(\tilde{\tilde{g}})) < \mathrm{d}_\tau([u_{k+1}^*\eta_{\delta_1}(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}))u_{k+1}]). \tag{8.80}$$

This verifies (8.59) for $k+1$.

By (8.68) and (8.62) (and (8.73), (8.79)), (note that $3\varepsilon_1 < 1/N$ and $\delta'' \ll \delta_1$)

$$\begin{array}{rll} & [u_{k+1}^*\eta_{\delta_1}(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}))u_{k+1}](\xi_{k+2,\varepsilon_1}^+(f)) & \\ = & [u_{k+1}^*\eta_{\delta_1}(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}))u_{k+1}](\xi_{k+1,\varepsilon_1}^-(f))(\xi_{k+2,\varepsilon_1}^+(f)) & \\ \approx_{\delta_1}^{\|\cdot\|_2} & (u_{k+1}^*\eta_{\delta_1}(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}))u_{k+1})(\xi_{k+1,\varepsilon_1}^-(f))(\xi_{k+2,\varepsilon_1}^+(f)) & ((8.73)) \\ \approx_{\delta_1}^{\|\cdot\|_2} & \eta_{\delta_1}((u_{k+1}^*(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}))u_{k+1}))(\xi_{k+1,\varepsilon_1}^-(f))(\xi_{k+2,\varepsilon_1}^+(f)) & ((8.79)) \\ \approx_{3^k\delta_1}^{\|\cdot\|_2} & \eta_{\delta_1}((u_{k+1}^*(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}))u_{k+1}))\left\langle \xi_{k+1,\varepsilon_1}^-(f)\right\rangle(\xi_{k+2,\varepsilon_1}^+(f)) & ((8.62)) \\ \approx_{\delta''}^{\|\cdot\|_2} & \left\langle \xi_{k+1,\varepsilon_1}^-(f)\right\rangle(\xi_{k+2,\varepsilon_1}^+(f)) & ((8.68)) \\ \approx_{3^k\delta_1}^{\|\cdot\|_2} & (\xi_{k+1,\varepsilon_1}^-(f))(\xi_{k+2,\varepsilon_1}^+(f)) & ((8.62)) \\ = & \xi_{k+2,\varepsilon_1}^+(f), & \end{array} \tag{8.81}$$

and therefore, one also has

$$[u_{k+1}^*\eta_{\delta_1}(\chi_{k+1,\varepsilon_1}(\tilde{\tilde{g}}))u_{k+1}](\xi_{k+2,\varepsilon_1}^-(f)) \approx_{3^{k+1}\delta_1}^{\|\cdot\|_2} \xi_{k+2,\varepsilon_1}^-(f). \tag{8.82}$$

This verifies (8.60) and (8.61) for $k+1$. Fix δ'', and we obtain the desired u_{k+1}.

By induction, there are contractions $u_1, u_2, ..., u_N \in A$ such that (note that $3^N\delta_1 < \varepsilon_1$)

$$\mathrm{dist}_{2,\mathrm{T}(A)}(u_i^*\chi_{i,\varepsilon_1}(\tilde{\tilde{g}})u_i, (D)_1^+) < \min\{\varepsilon_0, \delta_2, \varepsilon/4\} \leq \varepsilon_0, \quad i = 1, ..., N, \tag{8.83}$$

$$\|\chi_{i-1}(f)(u_i^*\chi_{i,\varepsilon_1}(\tilde{\tilde{g}})u_i) - (u_i^*\chi_{i,\varepsilon_1}(\tilde{\tilde{g}})u_i)\|_{2,\mathrm{T}(A)} < 4\delta_1 < \varepsilon_0, \quad i = 2, ..., N, \tag{8.84}$$

$$\|(u_i^*\chi_{i,\varepsilon_1}(\tilde{\tilde{g}})u_i)\chi_{i+1}(f) - \chi_{i+1}(f)\|_{2,\mathrm{T}(A)} < 7N\varepsilon_1 + 3^i\delta_1 < 8N\varepsilon_1 < \varepsilon_0, \quad i = 1, ..., N-1, \tag{8.85}$$

$$\|(u_{i-1}^*\chi_{i,\varepsilon_1}(\tilde{\tilde{g}})u_{i-1})(u_i^*\chi_{i,\varepsilon_1}(\tilde{\tilde{g}})u_i) - (u_i^*\chi_{i,\varepsilon_1}(\tilde{\tilde{g}})u_i)\|_{2,\mathrm{T}(A)} < 6\delta_1 < \delta_0, \quad i = 1, ..., N, \tag{8.86}$$

and

$$|\tau((u_i^*xu_i)^j) - \tau(x^j)| < \delta_1 < \delta_0, \quad i = 1, ..., N, \ j = 1, ..., M, \ x \in (A)_1, \ \tau \in \mathrm{T}(A). \tag{8.87}$$

Define

$$g := \frac{1}{N}(u_1^*\chi_{1,\varepsilon_1}(\tilde{\tilde{g}})u_1 + \cdots + u_N^*\chi_{N,\varepsilon_1}(\tilde{\tilde{g}})u_N).$$

Then, by (8.83), (8.84) and (8.85), it follows from Lemma 7.4 that

$$\|f-g\|_{2,\mathrm{T}(A)} < \varepsilon/2. \tag{8.88}$$

Note that

$$(\tilde{\tilde{g}}-\varepsilon_1)_+ = \frac{1}{N}(\chi_{1,\varepsilon_1}(\tilde{\tilde{g}}) + \cdots + \chi_{N,\varepsilon_1}(\tilde{\tilde{g}})).$$

By (8.86) and (8.87), it follows from Lemma 7.5 that

$$|\tau(\chi_{\frac{1}{2}+\varepsilon_1,\delta}((\tilde{\tilde{g}}-\varepsilon_1))) - \tau(\chi_{\frac{1}{2}+\varepsilon_1,\delta}(g))| < \varepsilon/4, \quad \tau\in\mathrm{T}(A). \tag{8.89}$$

By 8.83 again, one has

$$\mathrm{dist}_{2,\mathrm{T}(A)}(g,(D)_1^+) < \min\{\delta_2, \varepsilon/4\};$$

together with (8.88), (8.89), and the choice of δ_2 ((8.23)), there is a positive contraction in D, still denoted by g, such that

$$\|f-g\|_{2,\mathrm{T}(A)} < \varepsilon/2+\varepsilon/4 = 3\varepsilon/4$$

and,

$$|\tau(\chi_{\frac{1}{2}+\varepsilon_1,\delta}((\tilde{\tilde{g}}-\varepsilon_1))) - \tau(\chi_{\frac{1}{2}+\varepsilon_1,\delta}(g))| < \varepsilon/4+\varepsilon/4 = \varepsilon/2, \quad \tau\in\mathrm{T}(A)$$

By (8.27), one has

$$\tau(\chi_{\frac{1}{2}+\varepsilon_1,\delta}(g)) < \varepsilon/2+\varepsilon/2 = \varepsilon.$$

Stretching g to move $\frac{1}{2}+\varepsilon_1$ to $\frac{1}{2}$ (and note that $\varepsilon_1 < \varepsilon/2$), it satisfies the desired approximations (8.17) and (8.18). □

As a consequence of Theorem 8.3, one has the following characterizations of $\mathcal{Z}$-absorption of AH algebras with diagonal maps or the crossed product C*-algebras $\mathrm{C}(X)\rtimes\Gamma$:

Theorem 8.4. *Let A be a simple AH algebra with diagonal maps, or let $A=\mathrm{C}(X)\rtimes\Gamma$, where (X,Γ) is free, minimal, and has the (URP) and (COS). Let $D\subseteq A$ be the canonical commutative subalgebra. Then the following conditions are equivalent:*

(1) *A has Property (S).*
(2) *$(D,\mathrm{T}(A))$ has the (SBP).*
(3) *$A\cong A\otimes\mathcal{Z}$.*
(4) *The strict order on $\mathrm{Cu}(A)$ is determined by traces.*
(5) $\mathrm{qRR}(l^\infty(A)/J_{2,\omega,\mathrm{T}(A)})=0$ *(Definition 6.2).*
(6) $\mathrm{RR}(l^\infty(A)/J_{2,\omega,\mathrm{T}(A)})=0$.
(7) $\mathrm{RR}(l^\infty(D)/J_{2,\omega,\mathrm{T}(A)})=0$.
(8) *A has uniform property Γ (Definition 2.5).*
(9) *(D,A) has strong uniform property Γ (Definition 2.15).*
(10) *$(D,\mathrm{T}(A))$ is approximately divisible (Definition 2.15).*

In the case that $A=\mathrm{C}(X)\rtimes\Gamma$, each statement above is also equivalent to

(11) $\mathrm{mdim}(X,\Gamma)=0$.

Proof. (1) $\Rightarrow$ (2): By Theorem 4.6 and Theorem 5.3 respectively, the C*-algebra pair (D, A) has Properties (C) and (E). Since A has Property (S), by Theorem 8.3, $(D, \mathrm{T}(A))$ has the (SBP).

(2) $\Rightarrow$ (3): For the crossed product C*-algebras, this follows from Theorem 4.7 of [9] in the case $\Gamma = \mathbb{Z}$ and follows in general from Theorem 5.4 of [23] and Theorem 4.8 of [26]. For the AH algebras with diagonal maps, this implication follows from Proposition 4.10 of [10].

(3) $\Rightarrow$ (8): Theorem 5.6 of [1].

(8) $\Rightarrow$ (1): Proposition 6.4.

(2) $\Leftrightarrow$ (7): Theorem 2.12 of [10].

(10) $\Rightarrow$ (2): Theorem 3.5 of [10].

(1) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6): Proposition 6.3.

(2) $\Rightarrow$ (9): In the case of $\mathrm{C}(X) \rtimes \Gamma$, this follows from the proof of Theorem 9.4 of [18]. For AH algebras with diagonal maps, this follows from Theorem A.1 of the appendix.

(9) $\Rightarrow$ (10): Trivial.

(3) $\Leftrightarrow$ (4): In the case of $\mathrm{C}(X) \rtimes \Gamma$, this follows from Corollary 7.14 of [20]. In the case of AH algebras with diagonal maps, this follows from Theorem 4.1 of [6] and Theorem 9.5 of [31]

This shows the equivalence of Conditions (1)–(10).

(11) $\Rightarrow$ (2): Theorem 5.1 of [26] ([15] and [16] for $\mathbb{Z}^d$-actions).

(2) $\Rightarrow$ (11): Theorem 5.4 of [23]. $\square$

Remark 8.5. The relative comparison property (COS) only plays a role in (2) $\Rightarrow$ (3). So, all conditions except (3) and (4) are equivalent without assuming the (COS). Indeed, a Villadsen algebra of the second type ([39]), which has unique trace but is not $\mathcal{Z}$-absorbing, satisfies all conditions of the theorem above except (3) and (4).

Since free and minimal $\mathbb{Z}^d$-actions always have the (URP) and (COS) ([28]), a special case of Theorem 8.4 is the following corollary:

Corollary 8.6. *Let $(X, \mathbb{Z}^d)$ be a free and minimal dynamical system, and let $A = \mathrm{C}(X) \rtimes \mathbb{Z}^d$. Then the conditions (1)–(11) of Theorem 8.4 (with $D = \mathrm{C}(X)$) are equivalent.*

Since real rank zero of A implies Condition (5) of Theorem 8.4, then it actually implies the zero mean dimension of the dynamical system, and hence the $\mathcal{Z}$-absorption of A. (Note that for AH algebras with diagonal maps, this implication was already shown in [25].)

Corollary 8.7. *Let (X, Γ) be a free and minimal dynamical system with the (URP), and let $A = \mathrm{C}(X) \rtimes \Gamma$. If $\mathrm{RR}(A) = 0$, then $\mathrm{mdim}(X, \Gamma) = 0$. If (X, Γ) also has the (COS), then $A \otimes \mathcal{Z} \cong A$.*

Proof. It is clear that $\mathrm{RR}(A) = 0$ implies $\mathrm{qRR}(l^\infty(A)/J_{2,\omega,\mathrm{T}(A)}) = 0$. Then the corollary follows from Theorem 8.4 (and Remark 8.5). $\square$

Remark 8.8. In fact, a slightly stronger statement was shown in [25]: if a simple unital AH algebra with diagonal maps has the property that its projections separate traces, then it is $\mathcal{Z}$-absorbing. It would be interesting to ask if this statement also holds for the crossed product C*-algebra $\mathrm{C}(X) \rtimes \Gamma$.

Since Villadsen algebras of the first type ([38]) are AH algebras with diagonal maps, and they are not $\mathcal{Z}$-absorbing, the following corollary is straightforward.

Corollary 8.9. *Villadsen algebras of the first type do not have uniform property* Γ*.*

Remark 8.10. Although Villadsen algebras of the first type are not $\mathcal{Z}$-absorbing, if the seed spaces are finite products of a given contractible finite CW-complex, they are classified by the Cuntz semigroup (indeed, by the K_0-group together with the radius of comparison)([7]).

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Appendix A. Strong uniform property Γ for AH algebras

Theorem A.1. *Let A be a simple unital AH algebra with diagonal maps, and let D be the standard diagonal subalgebra. If $(D, \mathrm{T}(A))$ has the (SBP), then (D, A) has strong uniform property Γ.*

Proof. Let $\mathcal{F} \subseteq A$, $\varepsilon > 0$ and $n \in \mathbb{N}$, and let us construct mutually orthogonal positive contractions $p_1, ..., p_n \in D$ such that

$$\|1 - (p_1 + \cdots + p_n)\|_{2,\mathrm{T}(A)} < \varepsilon, \tag{A.1}$$

$$\|p_i f - f p_i\| < \varepsilon, \quad f \in \mathcal{F},\ i = 1, ..., n, \tag{A.2}$$

and

$$|\tau(p_i f p_i) - \frac{1}{n}\tau(f)| < \varepsilon, \quad i = 1, ..., n,\ \tau \in \mathrm{T}(A). \tag{A.3}$$

Then, strong uniform property Γ follows.

To simplify the notation, let us only prove the theorem for the case that $A_i = \mathrm{M}_{n_1}(\mathrm{C}(X_i))$. The general case follows from the same argument.

Let us first construct a new AH decomposition of A with diagonal maps such that each building block contains D as its diagonal subalgebra.

Note that the AH algebra A is generated by the diagonal subalgebra D and the standard AF subalgebra F. Start with $A_1 = \mathrm{M}_{n_1}(\mathrm{C}(X_1))$. Consider the rank-one diagonal projections $p_1^{(1)}, ..., p_{n_1}^{(1)} \in \mathrm{M}_{n_1}(\mathbb{C}) \subseteq A$ and the partial isometry

$$v^{(1)} = \begin{pmatrix} 0 & 1 & & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix} \in \mathrm{M}_{n_1}(\mathbb{C}) \subseteq A,$$

and consider the C*-algebra

$$C_1 := \mathrm{C}^*\{p_i^{(1)} D p_i^{(1)}, v^{(1)} : i = 1, ..., n_1\} \subseteq A.$$

Note that

$$C_1 \cong \mathrm{M}_{n_1}(p_1^{(1)} D p_1^{(1)}).$$

Moreover, since $p_1^{(1)}, ..., p_{n_1}^{(1)} \subseteq D$ and $p_1^{(1)} + \cdots + p_{n_1}^{(1)} = 1$, one has that $D \subseteq C_1$ as the diagonal subalgebra, and there is a canonical homomorphism $\Lambda_1^* : A_1 \to C_1$ induced by the inductive limit $D_1 \to D_2 \to \cdots \to D$, where D_i is the diagonal subalgebra of A_i, $i = 1, 2, ...$

Similarly, consider the rank-one diagonal projections $p_1^{(2)}, ..., p_{n_2}^{(2)} \in \mathrm{M}_{n_2}(\mathbb{C}) \subseteq A$ and the partial isometry

$$v^{(2)} = \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \\ & & & 0 \end{pmatrix} \in \mathrm{M}_{n_2}(\mathbb{C}) \subseteq A,$$

and consider the C*-algebra

$$C_2 := \mathrm{C}^*\{p_i^{(2)} D p_i^{(2)}, v^{(2)} : i = 1, ..., n_2\} \subseteq A.$$

Note that

$$C_2 \cong \mathrm{M}_{n_2}(p_1^{(2)} D p_1^{(2)}).$$

Since the partition of unity $\{p_1^{(2)}, ..., p_{n_2}^{(2)}\}$ refines $\{p_1^{(1)}, ..., p_{n_1}^{(1)}\}$, one has that $C_1 \subseteq C_2$, and the following diagram commutes:

$$\begin{tikzcd} \mathrm{M}_{n_1}(p_1^{(1)} D p_1^{(1)}) \arrow[r, hook] & \mathrm{M}_{n_2}(p_1^{(2)} D p_1^{(2)}) \arrow[r, hook] & A \\ \mathrm{M}_{n_1}(\mathrm{C}(X_1)) \arrow[r] \arrow[u, "\Lambda_1^*"] & \mathrm{M}_{n_2}(\mathrm{C}(X_2)) \arrow[r] \arrow[u, "\Lambda_2^*"] & A. \arrow[u, equal] \end{tikzcd}$$

Repeating this process, one has the following commutative diagram:

$$\begin{tikzcd} \mathrm{M}_{n_1}(p_1^{(1)} D p_1^{(1)}) \arrow[r, hook] & \mathrm{M}_{n_2}(p_1^{(2)} D p_1^{(2)}) \arrow[r, hook] & \cdots \arrow[r, hook] & A \\ \mathrm{M}_{n_1}(\mathrm{C}(X_1)) \arrow[r] \arrow[u, "\Lambda_1^*"] & \mathrm{M}_{n_2}(\mathrm{C}(X_2)) \arrow[r] \arrow[u, "\Lambda_2^*"] & \cdots \arrow[r] & A, \arrow[u, equal] \end{tikzcd}$$

and therefore, we have the following AH decomposition of A:

$$\mathrm{M}_{n_1}(p_1^{(1)} D p_1^{(1)}) \hookrightarrow \mathrm{M}_{n_2}(p_1^{(2)} D p_1^{(2)}) \hookrightarrow \cdots \hookrightarrow A. \tag{A.4}$$

For each $i = 1, 2, ...$, also write

$$p_1^{(i)} D p_1^{(i)} = \mathrm{C}(\tilde{X}_i),$$

where $\tilde{X}_i$ is metrizable and compact.

Without loss of generality, one may assume that $\mathcal{F} \subseteq \mathrm{M}_{n_1}(\mathrm{C}(\tilde{X}_1))$. Choose a finite open cover $\mathcal{U}$ of $\tilde{X}_{n_1}$ such that

$$\|f(x) - f(y)\| < \varepsilon, \quad x, y \in U,\ U \in \mathcal{U},\ f \in \mathcal{F}. \tag{A.5}$$

Since $(D, \mathrm{T}(A))$ has the (SBP), by Lemma 2.4 of [10], there is an open set $\hat{U} \subseteq U$ for each $U \in \mathcal{U}$ such that the subsets

$$\hat{U}, \quad U \in \mathcal{U},$$

are mutually disjoint, and

$$\mu_\tau(\tilde{X}_1 \setminus \bigcup_{U \in \mathcal{U}} \hat{U}) < \frac{\varepsilon}{n_1}, \quad \tau \in \mathrm{T}(A). \tag{A.6}$$

Choose $x_U \in \hat{U}$ for each $U \in \mathcal{U}$.

Choose $m_0 \in \mathbb{N}$ such that if $m_0 = nk + r$, where $0 \leq r < n$, then $r/m_0 < \varepsilon$.

Move to the next stage sufficiently far out, say C_2, such that by the simplicity of A,

$$|\{i = 1, ..., m : \lambda_i(x) \in \hat{U}\}| \geq m_0, \quad U \in \mathcal{U},\ x \in \tilde{X}_2,$$

and by (A.6),

$$\frac{1}{m}|\{i = 1, ..., m : \lambda_i(x) \in \bigcup_{U \in \mathcal{U}} \hat{U}\}| > 1 - \varepsilon, \quad x \in \tilde{X}_2,$$

where $\lambda_1, ..., \lambda_m : \tilde{X}_2 \to \tilde{X}_1$ are the eigenvalue maps of $C_1 \to C_2$.

By the compactness of $\tilde{X}_{n_2}$, there is an open cover $\mathcal{V}$ of $\tilde{X}_2$ such that for each $V \in \mathcal{V}$ and each $U \in \mathcal{U}$, there is $\alpha_{U,V} \subseteq \{1, 2, ..., m\}$ such that

$$\lambda_i(V) \subseteq \hat{U}, \quad i \in \alpha_{U,V}, \tag{A.7}$$

$$\frac{1}{m}\sum_{U \in \mathcal{U}} |\alpha_{U,V}| > 1 - \varepsilon \quad \text{and} \quad |\alpha_{U,V}| > m_0, \quad U \in \mathcal{U},\ V \in \mathcal{V}. \tag{A.8}$$

Since $(D, \mathrm{T}(A))$ has the (SBP), by Lemma 2.4 of [10], there is an open set $\hat{V} \subseteq V$ for each $V \in \mathcal{V}$ such that the open sets $\hat{V}$, $V \in \mathcal{V}$, are mutually disjoint, and

$$\mu_\tau(\tilde{X}_2 \setminus \bigcup_{V \in \mathcal{V}} \hat{V}) < \frac{\varepsilon}{n_2}, \quad \tau \in \mathrm{T}(A). \tag{A.9}$$

Then there are continuous functions $h_V : \tilde{X}_2 \to [0, 1]$, $V \in \mathcal{V}$, such that $h_V|_{\hat{V}^c} = 0$ and

$$\|1 - \sum_{V \in \mathcal{V}} h_V\|_{2,\mathrm{T}(A)} < \varepsilon. \tag{A.10}$$

Inside each $\alpha_{U,V}$, by the choice of m_0, there are mutually disjoint sets

$$\beta_{U,V}^{(1)}, ..., \beta_{U,V}^{(n)} \subseteq \alpha_{U,V}$$

such that

$$|\beta_{U,V}^{(1)}| = \cdots = |\beta_{U,V}^{(n)}| \tag{A.11}$$

and

$$\frac{1}{|\alpha_{U,V}|}(|\beta_{U,V}^{(1)}| + \cdots + |\beta_{U,V}^{(n)}|) > 1 - \varepsilon. \tag{A.12}$$

Then, define

$$p_1 = \sum_{V \in \mathcal{V}} \sum_{U \in \mathcal{U}} \sum_{i \in \beta_{U,V}^{(1)}} (h_V)e_i, \quad ..., \quad p_n = \sum_{V \in \mathcal{V}} \sum_{U \in \mathcal{U}} \sum_{i \in \beta_{U,V}^{(n)}} (h_V)e_i,$$

where, for each $i = 1, ..., m$, e_i denotes the diagonal projection of C_2 corresponding to the eigenvalue map λ_i. Then the contractions $p_1, ..., p_n$ have the desired property.

Indeed, it is clear that $p_1, ..., p_n \in D$ and are mutually orthogonal. It follows from (A.12), (A.8), and (A.10) that

$$\begin{aligned}
\|1-(p_1+\cdots+p_n)\|_{2,\mathrm{T}(A)} &= \|1-\sum_{V\in\mathcal{V}}\sum_{U\in\mathcal{U}}(\sum_{i\in\beta_{U,V}^{(1)}}(h_V)e_i+\cdots+\sum_{i\in\beta_{U,V}^{(n)}}(h_V)e_i)\|_{2,\mathrm{T}(A)} \\
&= \|1-\sum_{V\in\mathcal{V}}\sum_{U\in\mathcal{U}}(\sum_{j=1}^{n}\sum_{i\in\beta_{U,V}^{(j)}}e_i)h_V\|_{2,\mathrm{T}(A)} \\
&\approx_\varepsilon \|1-\sum_{V\in\mathcal{V}}\sum_{U\in\mathcal{U}}(\sum_{i\in\alpha_{U,V}}e_i)h_V\|_{2,\mathrm{T}(A)} \\
&\approx_\varepsilon \|1-\sum_{V\in\mathcal{V}}h_V\|_{2,\mathrm{T}(A)} \\
&\approx_\varepsilon 0.
\end{aligned}$$

This verifies (A.1).

Let us verify (A.2). For each $f \in \mathcal{F}$, its image in C_2 is

$$\sum_{j=1}^{m}(f\circ\lambda_j)e_j.$$

Therefore, for each $i = 1, ..., n$, noting that h_V, $V \in \mathcal{V}$, are in the center of C_2, we have

$$\begin{aligned}
(\sum_{j=1}^{m}(f\circ\lambda_j)e_j)p_i &= (\sum_{j=1}^{m}(f\circ\lambda_j)e_j)(\sum_{V\in\mathcal{V}}\sum_{U\in\mathcal{U}}\sum_{j\in\beta_{U,V}^{(i)}}(h_V)e_j) \\
&= \sum_{V\in\mathcal{V}}\sum_{U\in\mathcal{U}}\sum_{j\in\beta_{U,V}^{(i)}}(h_V)(f\circ\lambda_j)e_j \\
&= (\sum_{V\in\mathcal{V}}\sum_{U\in\mathcal{U}}\sum_{j\in\beta_{U,V}^{(i)}}(h_V)e_j)(\sum_{j}(f\circ\lambda_j)e_j) \\
&= p_i(\sum_{j=1}^{m}(f\circ\lambda_i)e_j).
\end{aligned}$$

This verifies (A.2).

Let us verify (A.3). For each $f \in \mathcal{F}$, its image in C_2 is

$$\sum_{j=1}^{m}(f\circ\lambda_j)e_j$$

and then, for each $i = 1, ..., n$ and $\tau \in \mathrm{T}(A)$, by (A.7) and (A.5),

$$\begin{aligned}\tau(p_i(\sum_{j=1}^{m}(f\circ\lambda_j)e_j)) &= \tau((\sum_{V\in\mathcal{V}}\sum_{U\in\mathcal{U}}\sum_{j\in\beta^{(i)}_{U,V}}(h_V)e_j)(\sum_{j=1}^{m}(f\circ\lambda_j)e_j))\\ &= \tau(\sum_{V\in\mathcal{V}}\sum_{U\in\mathcal{U}}\sum_{j\in\beta^{(i)}_{U,V}}(h_V)(f\circ\lambda_j)e_j)\\ &\approx_\varepsilon \tau(\sum_{V\in\mathcal{V}}\sum_{U\in\mathcal{U}}\sum_{j\in\beta^{(i)}_{U,V}}(h_V)(f(x_U))e_j).\end{aligned}$$

By (A.11) and (A.12), it follows that

$$\sum_{V\in\mathcal{V}}\sum_{U\in\mathcal{U}}\tau(f(x_U)\sum_{j\in\beta^{(i)}_{U,V}}h_Ve_j)\approx_\varepsilon\sum_{V\in\mathcal{V}}\sum_{U\in\mathcal{U}}\frac{1}{n}\tau(f(x_U)\sum_{j\in\alpha_{U,V}}h_Ve_j),\quad U\in\mathcal{U},\ V\in\mathcal{V},\ \tau\in\mathrm{T}(A),$$

and therefore, by (A.7), (A.5), (A.6), and (A.10),

$$\begin{aligned}\tau(\sum_{V\in\mathcal{V}}\sum_{U\in\mathcal{U}}\phi(x_U)\sum_{j\in\beta^{(i)}_{U,V}}h_Ve_j) &\approx_\varepsilon \frac{1}{n}\tau(\sum_{V\in\mathcal{V}}\sum_{U\in\mathcal{U}}f(x_U)\sum_{j=1}^{m}h_Ve_j)\\ &= \frac{1}{n}\tau(\sum_{V\in\mathcal{V}}\sum_{U\in\mathcal{U}}\sum_{j=1}^{m}(h_V)(f(x_U))e_j)\\ &\approx_\varepsilon \frac{1}{n}\tau(\sum_{V\in\mathcal{V}}\sum_{U\in\mathcal{U}}\sum_{j=1}^{m}(h_V)(\phi\circ\lambda_j)e_j)\\ &= \frac{1}{n}\tau((\sum_{V\in\mathcal{V}}\sum_{U\in\mathcal{U}}h_Ve_j)(\sum_{j=1}^{m}(f\circ\lambda_j)e_j))\\ &\approx_{2\varepsilon} \frac{1}{n}\tau(\sum_{j=1}^{m}(f\circ\lambda_j)e_j).\end{aligned}$$

Thus,

$$\tau(p_ifp_i)\approx_{2\varepsilon}\tau(p_if)\approx_{5\varepsilon}\frac{1}{n}\tau(f),\quad i=1,...,n,\ \tau\in\mathrm{T}(A),$$

as desired. ☐

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