Weak tracial approximation and a weak Rokhlin property for C*-dynamical systems

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Abstract

We extend results on tracial approximability of crossed products from the setting of topological dynamical systems to C*-dynamical systems. For actions of discrete amenable groups on unital C*-algebras, we introduce a weak Rokhlin property which leads to a weak tracial approximation of the crossed products by means of finite direct sums of matrix algebras over hereditary subalgebras of the original algebra. Examples are given.

Keywords: C*-dynamical system, Rokhlin property, weak tracial approximation.

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Chapter 1

Introduction and preliminaries

1.1 Introduction

The main motivation of this paper is to extend the correspondence between Rokhlin properties and weak tracial approximation from topological dynamical systems to C*-dynamical systems. In [47, 48], Rokhlin towers provide the local approximation structure through which the geometry of an action enters the study of its crossed product. To describe this starting point, let a countably infinite discrete amenable group Γ act freely and minimally on a compact metrizable space X. When the action has the uniform Rokhlin property (URP), one can choose finite families of disjoint open towers with sufficiently invariant shapes whose union has measure arbitrarily close to one, uniformly over all invariant probability measures. The tower levels give matrix coordinates in $\mathrm{C}(X) \rtimes \Gamma$. Controlling the boundaries of the tower shapes gives almost central positive contractions whose compressions of prescribed finite subsets are approximated by finite direct sums of possibly nonunital homogeneous C*-algebras [48, Theorem 3.9]. Uniform coverage by the towers gives a compression defect whose dimension functions are uniformly small on all tracial states of the crossed product. Together with Cuntz comparison of open sets (COS), this approximation connects the mean dimension of the action to comparison and Jiang–Su stability of its crossed product.

For an action on a general unital C*-algebra A, we seek an intrinsic formulation of the same mechanism. An open tower base $U \subseteq X$ is represented algebraically by the hereditary subalgebra $\mathrm{C}_0(U) \subseteq \mathrm{C}(X)$; this suggests using hereditary C*-subalgebras of A as the coefficients of the local matrix models. Positive contractions describe tower bases, orthogonality of their translates replaces disjointness of level sets, and invariant tracial states replace invariant probability measures in measuring the complement. In the noncommutative setting, the tower bases must also commute asymptotically with the coefficient algebra. This condition is automatic for commutative coefficients and is essential for the resulting matrix models to approximate compressions of general elements of the crossed product. Our formulation of the weak Rokhlin property (WRP) incorporates this centrality and uniform tracial control. Its principal structural consequence is weak tracial approximation of $A \rtimes_r \Gamma$ by finite direct sums of matrix algebras over hereditary

C*-subalgebras of A, with the compressing positive elements still belonging to A (Theorem 3.15). Allowing nonunital models and positive compressions is essential already in the topological setting, since open tower bases need not be compact. The resulting approximation structure provides a common setting for studying how comparison and regularity of the coefficient algebra influence the crossed product.

1.1.1 Weak tracial approximation

The preceding tower construction suggests comparing a finite subset of an ambient algebra with a model after compression by a positive contraction. The model need not contain the unit of the ambient algebra. We therefore use a positive contraction h in the model, require it to commute approximately with the prescribed elements, and approximate their compressions by elements of the model. The defect $1 - h$ is controlled either uniformly over tracial states or by comparison with a prescribed nonzero positive element. These two forms of control lead to the following approximation properties.

For a unital C*-algebra A, write $\mathrm{T}(A)$ for its tracial state space and put

$$\mathrm{d}_\tau(a) = \lim_{n\to\infty} \tau(a^{1/n}), \qquad a \in A_+.$$

We write $a \precsim b$ for Cuntz subequivalence.

Definition A (Definition 2.1)**.** Let $\mathcal{S}$ be a class of C*-algebras, and let A be unital. We say that A is *weakly tracially approximated by* $\mathcal{S}$, and write $A \in \mathrm{TA}_0(\mathcal{S})$, if $\mathrm{T}(A) \neq \emptyset$ and, for every finite set $\mathcal{F} = \{x_1, \dots, x_n\} \subseteq A$ and every $\varepsilon > 0$, there are a C*-subalgebra $C \subseteq A$ with $C \in \mathcal{S}$, elements $x_1', \dots, x_n' \in A$, and a positive contraction $h \in C$ with $\|h\| = 1$ such that

(1) $\|x_i - x_i'\| < \varepsilon$, $1 \le i \le n$;

(2) $\|[h, x_i']\| < \varepsilon$, $1 \le i \le n$;

(3) $hx_i'h \in C$, $1 \le i \le n$;

(4) $\mathrm{d}_\tau(1 - h) < \varepsilon$ for every $\tau \in \mathrm{T}(A)$.

We say that A is *tracially approximated by* $\mathcal{S}$, and write $A \in \mathrm{TA}_1(\mathcal{S})$, if, for every such $\mathcal{F}$ and ε and every nonzero positive contraction $a_0 \in A$, the same data can be chosen to satisfy (1)–(3) and

$$(4') \qquad 1 - h \precsim a_0.$$

The definition of $\mathrm{TA}_1(\mathcal{S})$ does not require $\mathrm{T}(A) \neq \emptyset$.

Condition (4) measures the support of the defect. For example, if $B = \mathrm{C}(X) \rtimes \Gamma$, $h \in \mathrm{C}(X)$ is a positive contraction, and $\tau_\mu = \mu \circ \mathbb{E}$ is the trace induced by an invariant probability measure μ, then

$$\mathrm{d}_{\tau_\mu}(1 - h) = \mu\big(\{x \in X : h(x) < 1\}\big).$$

Thus the uniform measure estimate furnished by Rokhlin towers becomes exactly the defect estimate in weak tracial approximation. For a fixed h, a bound on $\mathrm{d}_\tau(1-h)$ is stronger than a bound on $\tau(1-h)$. At the level of the approximation property, however, the two formulations are equivalent after a functional-calculus cutoff and a change of tolerance (Remark 2.6).

The perturbed elements x_i' in Definition A ensure that the compressed elements belong to the model exactly. Equivalently, one may work directly with the prescribed finite set and require

$$\|[h,x]\| < \varepsilon, \qquad \mathrm{dist}(hxh, C) < \varepsilon, \qquad x \in \mathcal{F},$$

together with the corresponding defect condition; see Lemma 2.8. This is the form naturally supplied by the tower constructions.

The TA_1 condition expresses the smallness of the defect by Cuntz comparison and therefore does not require tracial states. When A is simple and has tracial states, $\mathrm{TA}_1(\mathcal{S})$ implies $\mathrm{TA}_0(\mathcal{S})$; strict comparison by tracial states on A_+ gives the converse (Remarks 2.5 and 2.4). For such A, Remark 2.4 also identifies the Cuntz-controlled version with nonunital generalized tracial approximation [22] and explains its relation to the formulation with unital models in [21]. For dynamical applications, the next step is to construct these positive compressions and local models from Rokhlin towers in the coefficient algebra.

1.1.2 Weak Rokhlin property

The classical Rokhlin lemma gives arbitrarily tall measurable towers for an aperiodic invertible measure-preserving transformation of a standard probability space, with complement of arbitrarily small measure. For topological actions, the uniform Rokhlin property replaces measurable bases by open sets and requires the complement to be small uniformly over invariant probability measures. For free minimal actions of countably infinite discrete amenable groups on compact metrizable spaces, URP together with COS bounds the radius of comparison of the crossed product by one half of the mean dimension [48], gives stable rank one [42], and gives Jiang–Su stability when the mean dimension is zero [47]. These results motivate a tower condition for C*-dynamical systems that retains uniform tracial coverage and produces the positive compressions used in weak tracial approximation.

Let A be a separable unital C*-algebra and let $\alpha\colon \Gamma \to \mathrm{Aut}(A)$ be an action of a countable discrete amenable group. Fix a free ultrafilter ω on $\mathbb{N}$, and write

$$A_\omega = \ell^\infty(\mathbb{N}, A)/\{(a_n) : \lim_{n\to\omega} \|a_n\| = 0\}.$$

We view A as the constant sequences in A_ω and extend the action coordinatewise. Denote by $\mathrm{T}^\alpha(A)$ the space of invariant tracial states, and put

$$J_{\mathrm{T}^\alpha(A)} = \left\{ [(a_n)] \in A_\omega : \lim_{n\to\omega} \sup_{\rho\in \mathrm{T}^\alpha(A)} \rho(a_n^* a_n) = 0 \right\}.$$

The supremum is taken before the ultralimit, so the estimate is uniform over the invariant traces. Recall that an exact tiling of Γ is a partition

$$\Gamma = \bigsqcup_{s=1}^{S} \bigsqcup_{t \in C_s} D_s t$$

with finitely many finite nonempty shapes $D_1, \ldots, D_S$ and center sets $C_s \subseteq \Gamma$. We make the following definition.

Definition B (Definition 3.4)**.** Let A be a unital separable C*-algebra with $\mathrm{T}(A) \neq \emptyset$, and let α be an action of a countable discrete amenable group Γ. The action has the *weak Rokhlin property* (WRP) if, for every exact tiling of Γ with shapes $D_1, \ldots, D_S$, there are positive contractions $c_s \in A_\omega \cap A'$, $1 \leq s \leq S$, such that

1. the levels $\alpha_g(c_s)$, $g \in D_s$, $1 \leq s \leq S$, are mutually orthogonal, and
2. $$1 - \sum_{s=1}^{S} \sum_{g \in D_s} \alpha_g(c_s) \in J_{\mathrm{T}^\alpha(A)}.$$

Amenability ensures that $\mathrm{T}^\alpha(A)$ is nonempty when $\mathrm{T}(A)$ is nonempty. It also provides exact tilings whose shapes have small boundary relative to any prescribed finite subset of Γ [19]. The quantifier over prescribed tilings lets us choose these shapes to fit the Fourier supports of the elements to be approximated.

For $A = \mathrm{C}(X)$, WRP agrees with the topological uniform Rokhlin property (URP): disjoint open towers cover all but a uniformly small amount of every invariant measure (Proposition 3.7). For general coefficient algebras, the finite-stage uniform Rokhlin condition in Definition 3.2 implies WRP (Proposition 3.6).

For comparison, the literature contains several Rokhlin properties with different tower and remainder requirements. For finite group actions, the strict Rokhlin property uses approximately central projection towers whose sum is the unit [54, Definition 1.1]. The tracial Rokhlin property retains projection towers but allows a complement which is Murray–von Neumann equivalent to a projection in the hereditary subalgebra generated by a prescribed nonzero positive element of A [54, Definition 1.2]. Weak tracial Rokhlin properties replace these projections by positive contractions while retaining Cuntz control of the complement in A. Such formulations are considered for amenable groups in [69, 52] and, including nonunital coefficient algebras, for finite groups in [29].

There are also positive-tower formulations based directly on tracial estimates, as in [59]. The weak Rohlin property of [45, Definition 2.5(3)] uses a central sequence and one sufficiently invariant tower shape, with the traces of the base converging uniformly over all of $\mathrm{T}(A)$ to the reciprocal of the tower size. The amenable-group formulation of [31, Definition 2.9] allows a finite family of shapes, requires the complement to vanish uniformly over all traces, and requires the levels within each tower to have one common positive tracial value, independent of the trace.

Thus even among positive-tower conditions, the trace space tested and the normalization of the levels matter.

A different relaxation is finite Rokhlin dimension [38], extended to residually finite groups in [61]. Here finitely many colours of positive towers cover the unit in norm, with orthogonality required within each colour. The version with commuting towers also requires approximate commutation between different colours. Our WRP requires all tower levels to be mutually orthogonal, but permits a complement invisible to invariant traces. These are different ways to relax projection Rokhlin towers. In particular, the permanence results for finite nuclear dimension and, with commuting towers, Jiang–Su stability in [38] arise from this coloured norm-covering structure.

Our formulation dispenses with projections, Cuntz control in A, and trace estimates outside $\mathrm{T}^{\alpha}(A)$. It is closest to the weak Rokhlin property with respect to invariant traces in [30, Definition 5.2]. Our WRP implies that condition, since sufficiently invariant exact tilings may be chosen. The additional requirement here is that towers exist for every prescribed exact tiling; in the cited formulation the sufficiently invariant shapes are chosen as part of the construction. This requirement on the shapes is useful for fitting the towers to the finite sets occurring in the approximation theorem below.

The link with weak tracial approximation is the following structure theorem. It approximates the crossed product by matrix algebras over hereditary subalgebras of the coefficient algebra, with the compression itself still in A.

Theorem C (Theorem 3.15)**.** *Let A be a unital separable C^*-algebra with $\mathrm{T}(A) \neq \emptyset$, and let α be an action of a countable discrete amenable group Γ with WRP. Put $B = A \rtimes_{\alpha,r} \Gamma$. For every finite $\mathcal{F} \subseteq B$ and every $\varepsilon > 0$, there are a subalgebra*

$$C \cong \bigoplus_{s=1}^{S} \mathrm{M}_{n_s}(H_s) \subseteq B, \qquad H_s \subseteq A \text{ hereditary},$$

and a norm-one positive contraction $h \in A \cap C$ such that

$$\|[h, x]\| < \varepsilon, \qquad \operatorname{dist}(hxh, C) < \varepsilon \quad (x \in \mathcal{F}), \qquad \mathrm{d}_{\tau}(1-h) < \varepsilon \quad (\tau \in \mathrm{T}(B)).$$

When $x \in \mathcal{F} \cap A$, its compressed approximation may be chosen in $A \cap C$. In particular, $B \in \mathrm{TA}_0(\mathcal{S})$, where $\mathcal{S}$ consists of finite direct sums of matrix algebras over hereditary subalgebras of A.

The matrix coordinates in these models record the levels of a tower; the hereditary algebra records its base. In the commutative case, $H_s = \mathrm{C}_0(U_s)$ for an open subset $U_s \subseteq X$, recovering the models supplied by topological towers. In the general case, the proof constructs matrix cones from central sequence towers and lifts them while keeping their diagonals in A. Enlarging the cones gives the hereditary matrix models, and a spectral cutoff changes the uniform tracial 2-norm estimate into the rank estimate on $1-h$. The conclusion requires neither simplicity nor nuclearity of A. WRP also identifies the traces of B with the invariant traces of A; when the action is minimal, it makes B simple (Proposition 3.8 and Corollary 3.32).

The location of the defect is important here: $1 - h \in A$, but WRP alone gives its smallness through traces. When A is simple and infinite-dimensional, B is simple, and the action also has COS, the same approximation can be chosen so that, for any prescribed $0 \neq z \in B_+$,

$$1 - h \precsim_B z$$

(Proposition 4.9). This upgrades the conclusion to $B \in \mathrm{TA}_1(\mathcal{S})$. The Cuntz comparison takes place in the crossed product B, even though the compressing element and its defect belong to A. This is the form of remainder control used below to transfer comparison and rank properties through the tower models.

1.1.3 Sources of weak tracial approximation and the weak Rokhlin property

Chapter 7 collects the examples of weak tracial approximation and WRP discussed here, together with their proofs and constructions. We begin with interval approximation for classifiable C*-algebras, which does not require an action, and then describe two sources of WRP: central topological towers and towers obtained from invariant traces. By Theorem C, each of these WRP examples also gives weak tracial approximation of the corresponding crossed product.

The first result shows that the Cuntz-controlled approximation in Definition A holds throughout the unital simple infinite-dimensional stably finite classifiable setting. The local models can be chosen to be matrix algebras over half-open or open intervals.

Theorem D (Theorem 7.1)**.** *Let A be a unital, separable, simple, infinite-dimensional, nuclear, stably finite, $\mathcal{Z}$-stable* C*-*algebra satisfying the UCT. Let $\mathcal{I}_{\mathrm{h}}$ and $\mathcal{I}_{\mathrm{o}}$ be the classes of finite direct sums*

$$\bigoplus_{j=1}^{r} \mathrm{M}_{n_j}(\mathrm{C}_0([0,1))) \quad \textit{and} \quad \bigoplus_{j=1}^{r} \mathrm{M}_{n_j}(\mathrm{C}_0((0,1))), \qquad r, n_1, \ldots, n_r \in \mathbb{N},$$

respectively. Then

$$A \in \mathrm{TA}_1(\mathcal{I}_{\mathrm{h}}) \cap \mathrm{TA}_1(\mathcal{I}_{\mathrm{o}}).$$

In particular, A can be weakly tracially approximated by either class.

These interval models are projectionless. They approximate finite subsets after compression by a positive contraction, as in Definition A, so the ambient algebra may still have nontrivial projections and K-theory. The proof starts from the classification models of [32, 33] and constructs half-open interval subalgebras with uniformly small tracial defect. Strict comparison gives the required Cuntz control, and $\mathcal{Z}$-absorption gives the open interval version.

For actions, a first source of WRP is a central topological subsystem. Suppose that $\mathrm{C}(X) \subseteq Z(A)$ is invariant under the action. Tower functions in $\mathrm{C}(X)$ then commute with all of A, and invariant traces on A restrict to invariant probability measures on X. Thus open towers on X supply algebraic towers with the centrality and uniform coverage required in the preceding subsection. This gives the following criterion.

Theorem E (Theorem 7.2)**.** *Let A be a unital separable C^*-algebra with $\mathrm{T}(A) \neq \emptyset$, and let α be an action of a countable discrete amenable group Γ on A. Suppose that A contains an α-invariant unital subalgebra $\mathrm{C}(X) \subseteq Z(A)$, where X is compact Hausdorff. If the induced action on X has topological URP, then (A, Γ) has the URP of Definition 3.2 and hence WRP.*

In particular, this applies to a unital separable C^*-algebra that is a continuous field over X and has a nonempty tracial state space, provided that the action preserves its canonical central copy of $\mathrm{C}(X)$ and the induced action on X has topological URP.

A second source comes from the von Neumann algebras associated with invariant traces. Each $\rho \in \mathrm{T}^\alpha(A)$ gives a GNS von Neumann algebra $M_\rho = \pi_\rho(A)''$ with an induced action. The action on A is called W^*-*centrally free* when all these induced actions are centrally free (Definition 7.11). The construction begins with Rokhlin towers in these von Neumann algebras and lifts them to positive towers in the norm central sequence algebra of A.

When $\mathrm{T}^\alpha(A)$ has only finitely many extreme points, W^*-central freeness suffices for WRP without a nuclearity assumption on A (Theorem 7.19). This includes the case of a unique invariant trace (Theorem 7.20). For a larger family of invariant traces, nuclearity allows the construction to be made uniform over a compact finite-dimensional extreme boundary. The next theorem states this criterion; recall that a Bauer simplex is a Choquet simplex whose extreme boundary is compact.

Theorem F (Theorem 7.48)**.** *Let A be a unital separable nuclear C^*-algebra with $\mathrm{T}(A) \neq \emptyset$, and let $\alpha\colon \Gamma \to \mathrm{Aut}(A)$ be an action of a countable discrete amenable group. Suppose that α is W^*-centrally free and that $S = \mathrm{T}^\alpha(A)$ is a Bauer simplex with $\dim(\partial_e S) < \infty$. If Γ is infinite, then α has WRP. The same conclusion holds for finite Γ when A has no finite-dimensional quotients.*

The dimension assumption concerns the extreme boundary of the *invariant* trace simplex, which need not be a face of $\mathrm{T}(A)$. For infinite groups, finite-dimensional quotients of A are allowed. This is important for commutative coefficients: a free action of a countably infinite discrete amenable group on a compact metrizable space has WRP, and hence topological URP, whenever its invariant-measure simplex is Bauer with finite-dimensional extreme boundary (Corollary 7.50). The theorem therefore also supplies topological towers from information about invariant measures.

1.1.4 Comparison and Jiang–Su stability

The preceding constructions give local models on large positive compressions. To use these models to study the whole algebra, one must also control the part left outside the compression. Comparison provides this control by absorbing the defect into a prescribed positive element. Chapter 4 develops the comparison results, and Chapter 5 uses them to obtain Jiang–Su stability.

For dynamical applications, the first comparison condition concerns positive elements of the coefficient algebra. It extends the Cuntz comparison of open sets used for topological actions and retains the name COS. Let α be an action of a discrete group Γ on a unital C^*-algebra A, suppose

that $\mathrm{T}^{\alpha}(A) \neq \emptyset$, and put $B = A \rtimes_{\alpha,r} \Gamma$. For $\lambda \in (0,1]$ and $m \in \mathbb{N}$, the condition (λ, m)-COS requires

$$\mathrm{d}_{\rho}(a) < \lambda \mathrm{d}_{\rho}(b) \quad (\rho \in \mathrm{T}^{\alpha}(A)) \quad \Longrightarrow \quad a \precsim_B b^{\oplus m}, \qquad a, b \in A_+.$$

The inequalities are tested on invariant traces of A, and the comparison takes place in B. The constants allow a fixed loss in the trace inequality and a fixed number of copies of the target. For a simple infinite-dimensional coefficient algebra with simple crossed product, WRP and COS turn the tracially small defect of Theorem C into a Cuntz-small defect in B. Thus B satisfies the TA_1 version of the tower approximation described in the preceding subsections (Proposition 4.9).

COS can also follow from regularity of the coefficient algebra. If A is simple and unital, has strict comparison by tracial states, and has almost divisible Cuntz semigroup, then every discrete group action admitting an invariant tracial state has $(1,1)$-COS (Proposition 4.12). In particular, this applies when A is simple, unital, exact, stably finite, and $\mathcal{Z}$-stable (Corollary 4.16). The amenable-group case of this criterion was established in [52, Proposition 5.4]. The conclusion compares coefficient elements, including their finite matrix amplifications; comparison of arbitrary elements of B uses the approximation argument as well.

The abstract form of that argument is Theorem 4.5. Let A be simple, separable, exact, unital, and infinite-dimensional, with $\mathrm{T}(A) \neq \emptyset$ and $A \in \mathrm{TA}_1(\mathcal{S})$. If $\mathrm{Cu}(C)$ is almost unperforated for every $C \in \mathcal{S}$, then A has strict comparison. The theorem also transfers quantitative comparison bounds. For nonunital models, the admissible gap is expressed by a finite-valued continuous affine function on the cone of densely finite traces, including zero, where the function vanishes. Such bounds pass to A when they are compatible with the chosen compressions. This compatibility is part of the hypothesis: traces on a model need not extend to traces on A.

For Jiang–Su stability, comparison is accompanied by a second requirement: sufficiently many positive elements must be available to realize prescribed rank functions. A positive element $x \in (A \otimes \mathcal{K})_+$ determines the function $\tau \mapsto \mathrm{d}_{\tau}(x)$ on $\mathrm{T}(A)$. Write $\mathrm{LAff}_{++}(\mathrm{T}(A))$ for the strictly positive lower semicontinuous affine functions on $\mathrm{T}(A)$, allowing the value ∞, and $\mathrm{V}(A)$ for the semigroup of projection classes. The next theorem shows that, under the stated local approximation hypothesis, strict comparison and realization of all these functions are exactly what is needed for $\mathcal{Z}$-absorption.

Theorem G (Theorem 5.10)**.** *Let $\mathcal{S}$ be a class such that every $C \in \mathcal{S}$ can be locally approximated by C^*-subalgebras of finite nuclear dimension. Let A be a separable, simple, unital, infinite-dimensional, nuclear C^*-algebra with $A \in \mathrm{TA}_0(\mathcal{S})$. Then $A \cong A \otimes \mathcal{Z}$ if and only if*

1. *A has strict comparison of positive elements; and*
2. *for every $f \in \mathrm{LAff}_{++}(\mathrm{T}(A))$ there is $x \in (A \otimes \mathcal{K})_+$ such that $\mathrm{d}_{\tau}(x) = f(\tau)$ for every $\tau \in \mathrm{T}(A)$.*

Consequently, within this class, $\mathcal{Z}$-stability is also equivalent to the canonical ordered-semigroup description

$$\mathrm{Cu}(A) \cong \mathrm{V}(A) \sqcup \mathrm{LAff}_{++}(\mathrm{T}(A)).$$

In the displayed description of $\mathrm{Cu}(A)$, $\mathrm{V}(A)$ records projection classes, and the other classes are described by their tracial rank functions. The hypothesis on the models requires no common bound on the nuclear dimensions of their local approximants. For basic homogeneous models, the approximation and strict comparison already supply the required rank realization. This gives the following criterion.

Theorem H (Theorem 5.19)**.** *Let*

$$\mathcal{H} = \big\{\mathrm{M}_n(\mathrm{C}_0(X)) : n \in \mathbb{N},\ X \text{ locally compact and metrizable}\big\}$$

be the class of basic homogeneous C^**-algebras. Let* A *be a separable, simple, unital, nuclear, non-type-I* C^**-algebra. If* $A \in \mathrm{TA}_0(\mathcal{H})$*, then*

$$A \cong A \otimes \mathcal{Z} \quad \Longleftrightarrow \quad A \text{ has strict comparison of positive elements.}$$

For crossed products, the hereditary models in Theorem C allow regularity of the coefficient algebra to enter the argument. Corollary 5.8 shows that, for a separable, simple, unital, infinite-dimensional A with simple crossed product B, WRP transfers weak tracial $\mathcal{Z}$-absorption from A to B. This supplies approximately central order zero maps from matrix algebras with tracially small complement. Strict comparison in B strengthens this to Cuntz control of the complement, and nuclearity then gives $\mathcal{Z}$-stability.

If A is nuclear and $\mathcal{Z}$-stable, the required comparison in B follows from the results above: the action has COS, the hereditary matrix models have almost unperforated Cuntz semigroups, and the TA_1 comparison theorem gives strict comparison in B. This yields the following permanence theorem.

Theorem I (Theorem 5.23)**.** *Let* A *be a separable, simple, unital, nuclear,* $\mathcal{Z}$*-stable* C^**-algebra, and let* Γ *be a countable discrete amenable group acting on* A*. If* (A, Γ) *has WRP, then*

$$B = A \rtimes_{\alpha,r} \Gamma$$

is simple and $\mathcal{Z}$*-stable. In particular, the conclusion holds for a separable, simple, unital, infinite-dimensional coefficient algebra of finite nuclear dimension with WRP.*

The theorem requires no separate strict comparison assumption on the crossed product. Preservation of $\mathcal{Z}$-stability under the weak tracial Rokhlin property was obtained in [52, Corollary 8.13]. That formulation controls the defect in the coefficient algebra in the Cuntz order; our WRP uses uniform control over invariant traces. For the present WRP, the conclusion also follows from [30, Corollary 5.6], using the implication between the two Rokhlin conditions described above. The sources of WRP in the preceding subsection can therefore be combined with $\mathcal{Z}$-stable coefficients to obtain simple $\mathcal{Z}$-stable crossed products.

1.1.5 Real rank and stable rank

Comparison also allows rank properties to pass from the local models to the ambient algebra. Recall that real rank zero means that self-adjoint elements can be approximated by finite-spectrum self-adjoint elements, and stable rank one means that invertible elements are dense in the unitization. Chapter 6 first proves permanence results for these two properties, and then uses the tower structure to obtain stable rank one for crossed products.

For the permanence results, weak tracial approximation supplies the desired approximations on a large positive compression. Strict comparison controls the remaining defect by comparison with positive elements in the model. This allows the approximation to be completed in the whole algebra.

Theorem J (Theorem 6.1)**.** *Let $\mathcal{S}$ be a class of* C*-*algebras of real rank zero, and let A be a simple unital* C*-*algebra in* $\mathrm{TA}_0(\mathcal{S})$*. If A has strict comparison of positive elements, then A has real rank zero.*

The real-rank conclusion requires no stable-rank-one assumption on the models. The corresponding stable-rank permanence theorem is as follows.

Theorem K (Theorem 6.4)**.** *Let $\mathcal{S}$ be a class of* C*-*algebras of stable rank one. Let A be a simple unital* C*-*algebra in* $\mathrm{TA}_0(\mathcal{S})$*. If A has strict comparison of positive elements, then A has stable rank one.*

The models in these permanence theorems need not be unital or simple. For crossed products, the dynamics give a further stable-rank result without requiring stable rank one of the coefficient algebra or of the hereditary tower models. Minimality and WRP provide orthogonal positive coefficient elements that are equivalent in the crossed product. Choosing enough such elements allows the fixed comparison multiplicity in COS to be absorbed. Together with the matrix structure of the towers, this gives the approximations by invertibles needed for the following theorem. The argument extends the stable-rank methods of [42] to noncommutative coefficient algebras.

Theorem L (Theorem 6.17)**.** *Let (A, Γ) be a minimal* C*-*dynamical system with WRP, where A is unital and separable, $\mathrm{T}(A) \neq \varnothing$, and Γ is countably infinite, discrete, and amenable. If (A, Γ) has (λ, m)-COS for some $\lambda \in (0, 1]$ and $m \in \mathbb{N}$, then*

$$\mathrm{tsr}(A \rtimes_{\alpha,r} \Gamma) = 1.$$

The coefficient algebra in Theorem L may be nonsimple; minimality excludes nonzero proper invariant ideals. The theorem requires neither nuclearity nor strict comparison of the whole crossed product. Its comparison hypothesis is COS, tested on coefficient elements and implemented in the crossed product.

Rank properties also combine with the Jiang–Su stability results to give classification structure. Corollary 5.9 shows that a WRP crossed product is tracially AF, and is isomorphic to a simple

unital AH algebra with slow dimension growth and real rank zero, under its additional hypotheses. These include real rank zero and weak tracial $\mathcal{Z}$-absorption for the simple coefficient algebra, and strict comparison, nuclearity, and the UCT for the crossed product. The stable-rank theorem will also be used with the quantitative comparison estimates in the next subsection.

1.1.6 Orbit radius of comparison

To obtain quantitative comparison estimates, we introduce the *orbit radius of comparison*, $\operatorname{orc}(A, \alpha)$, in Section 4.3. It can be regarded as a C*-*algebraic analogue of mean dimension*: comparison of positive matrices over A takes the place of dimension estimates for open covers along finite orbit sets. The rank functions are normalized by the size of the orbit set, just as covering dimensions are normalized in the definition of mean dimension. The comparison tests are made over A before the Rokhlin towers are chosen.

The tower approximation transfers these tests to the crossed product. Let a countably infinite discrete amenable group act minimally on a unital separable C*-algebra A with $\mathrm{T}(A) \neq \emptyset$. If the action has WRP and COS and $B = A \rtimes_{\alpha,r} \Gamma$ is exact, then

$$\operatorname{rc}(A \rtimes_{\alpha,r} \Gamma) \leq \operatorname{orc}(A, \alpha) \qquad \text{(Theorem 4.27)}.$$

For $A = \mathrm{C}(X)$, with X compact and metrizable, the relation to mean dimension is made explicit by

$$\operatorname{orc}(\mathrm{C}(X), \alpha) \leq \tfrac{1}{2} \operatorname{mdim}(X, \Gamma) \qquad \text{(Proposition 4.24)}.$$

This bound itself requires neither freeness nor minimality nor a Rokhlin hypothesis. In the noncommutative setting, a fixed finite coefficient comparison radius becomes negligible after normalization by growing orbit size: if A is stably finite and exact and $\operatorname{rc}(A) < \infty$, then $\operatorname{orc}(A, \alpha) = 0$ for an infinite amenable group (Proposition 4.22). This parallels the vanishing of mean dimension for actions on finite-dimensional compact spaces. The averaging statement does not require WRP or COS; those hypotheses control the passage to the crossed product.

There are also tower conditions that supply the required complement comparison directly. WRP with Cuntz control in the crossed product supplements WRP by requiring sufficiently invariant central sequence towers whose complement is Cuntz below any prescribed nonzero positive element of B, with the comparison made in B_ω (Definition 4.29). Strong URP uses central projection towers covering the unit, so their complement is zero (Definition 4.28 and Lemma 4.30). For the minimal actions above, either condition together with zero orbit radius gives $(1, 1)$-COS and, when B is exact, strict comparison (Proposition 4.31). Thus these stronger tower conditions can replace a separate COS assumption.

Combining these comparison results with Theorem L gives strict comparison and stable rank one under the hypotheses below. Stable rank one supplies positive elements realizing the strictly positive lower semicontinuous affine rank functions. Locally finite nuclear dimension then allows the Jiang–Su stability criterion of the preceding subsections to be applied. The coefficient algebra need not already be $\mathcal{Z}$-stable.

Theorem M (Corollary 6.20)**.** *Let A be a separable, unital, stably finite* C*-*algebra with* $\mathrm{T}(A) \neq \varnothing$*, locally finite nuclear dimension, and* $\mathrm{rc}(A) < \infty$*. Let a countably infinite discrete amenable group* Γ *act minimally on A. Suppose that one of the following holds:*

1. *the action has WRP and (λ, m)-COS for some $\lambda \in (0, 1]$ and $m \in \mathbb{N}$;*
2. *the action has strong URP;*
3. *the action has WRP with Cuntz control in the crossed product.*

Then

$$B = A \rtimes_{\alpha,r} \Gamma$$

is simple and $\mathcal{Z}$-stable.

In particular, Theorem M gives a *noncommutative extension of the finite-dimensional-space case* of [47, Theorem 4.9]. If $A = \mathrm{C}(X)$ with X finite-dimensional, compact, and metrizable, then A has finite nuclear dimension and finite radius of comparison, by [72, Proposition 2.4] and [26]. Topological URP agrees with WRP, so the theorem recovers $\mathcal{Z}$-stability for free minimal actions of countably infinite discrete amenable groups with URP and COS. The general statement replaces finite covering dimension by finite coefficient comparison radius together with locally finite nuclear dimension.

The coefficient algebra may therefore be nonsimple, have positive finite radius of comparison, and have infinite nuclear dimension; no common bound is imposed on the nuclear dimensions of its local approximating algebras. For infinite groups, the theorem also extends Theorem I: its $\mathcal{Z}$-stable coefficient algebras have finite nuclear dimension and comparison radius zero, simplicity makes the action minimal, and Corollary 4.16 supplies COS. Theorem I additionally covers finite groups.

1.2 Notation and preliminaries

We fix the notation for the approximation and comparison arguments that follow. We begin with positive elements, traces, and rank functions, then specify the conventions for actions and central sequences. The final subsection recalls the rank and regularity properties used in the main results. Hypotheses such as simplicity, nuclearity, and the existence of tracial states are specified in each statement. The coefficient algebras in our dynamical results are unital; the local models and hereditary subalgebras may be nonunital.

1.2.1 C*-algebras, cutoffs, and order zero maps

For a C*-algebra A, let A_+ and A_{sa} denote its positive cone and self-adjoint part. A positive contraction is an element of A_+ with norm at most one. We write $\widetilde{A} = A$ when A is unital and

use the minimal unitization otherwise. For a unital algebra, $\mathrm{U}(A)$ and $\mathrm{GL}(A)$ are the unitary group and the group of invertible elements. The hereditary subalgebra generated by $a \in A_+$ is

$$\mathrm{Her}(a) = \overline{aAa}.$$

We write $\mathcal{K}$ for the compact operators on a separable infinite-dimensional Hilbert space, $\mathrm{M}_n(A)$ for the algebra of n by n matrices over A, and $\mathrm{M}_\infty(A) = \bigcup_n \mathrm{M}_n(A)$ for the algebraic union under corner embeddings $x \mapsto \mathrm{diag}(x, 0)$. Tensor products are minimal unless specified otherwise. The notation $x \in_\varepsilon C$ means $\mathrm{dist}(x, C) < \varepsilon$, and $[x, y] = xy - yx$. The multiplier algebra of D is denoted by $\mathrm{Mult}(D)$.

For $a \in A_+$ and $\varepsilon > 0$, the positive part $(a - \varepsilon)_+$ is defined by continuous functional calculus. The approximation arguments repeatedly use the following continuous cutoff, for $s > 0$:

$$g_s(t) = \begin{cases} 0, & 0 \le t \le s/2, \\ 2t/s - 1, & s/2 \le t \le s, \\ 1, & t \ge s. \end{cases} \tag{1.1}$$

In particular, $g_s(a) \in \mathrm{Her}(a)$ is a positive contraction. When A is nonunital, functional calculus by a function vanishing at zero takes values in A.

The matrix structure of positive compressions will often be expressed through order zero maps. A map is *c.p.c.* if it is completely positive and contractive. A completely positive map $\phi : D \to A$ has *order zero* if $\phi(x)\phi(y) = 0$ whenever $x, y \in D_+$ and $xy = 0$. For a c.p.c. order zero map with unital domain, the structure theorem of [71, Theorem 3.3] gives

$$\phi(x) = h\pi(x) = \pi(x)h, \qquad h = \phi(1_D),$$

where $\pi : D \to \mathrm{Mult}(\mathrm{C}^*(\phi(D)))$ is a supporting *-homomorphism. Equivalently, such a map determines a *-homomorphism from $\mathrm{C}_0((0, 1]) \otimes D$, taking $t \otimes x$ to $\phi(x)$, where t is the coordinate function on $(0, 1]$ [71, Corollary 4.1]. A unital order zero map is a *-homomorphism. We use the order zero and cone descriptions interchangeably.

1.2.2 Traces and Cuntz comparison

The comparison estimates use ranks measured by traces. We write $\mathrm{T}(A)$ for the tracial state space of A. For unital A this space is weak* compact, and it is metrizable when A is separable. We extend a tracial state to $\mathrm{M}_n(A)$ using the *unnormalized* matrix trace, and to $A \otimes \mathcal{K}$ as the associated lower semicontinuous trace. The resulting rank function is

$$\mathrm{d}_\tau(a) = \lim_{m \to \infty} \tau(a^{1/m}) \in [0, \infty], \qquad a \in (A \otimes \mathcal{K})_+.$$

Thus $\mathrm{d}_\tau(1_{\mathrm{M}_n(A)}) = n$ when A is unital. For $a \in \mathrm{M}_n(A)_+$, the rank is at most n; for a projection p, $\mathrm{d}_\tau(p) = \tau(p)$. Every tracial state on a simple unital algebra is faithful.

We compare positive elements using Cuntz subequivalence. For $a, b \in (A \otimes \mathcal{K})_+$, write $a \precsim b$ if there are $v_j \in A \otimes \mathcal{K}$ such that $v_j^* b v_j \to a$ in norm. Write $a \sim b$ when both comparisons hold, and use a subscript to specify the ambient algebra when needed. The Cuntz semigroup $\mathrm{Cu}(A)$ consists of these equivalence classes, with order induced by $\precsim$ and addition by orthogonal direct sum. Its finite-matrix version is denoted by $\mathrm{W}(A)$; $\mathrm{V}(A)$ denotes the Murray–von Neumann semigroup of projections in $\mathrm{M}_\infty(A)$. We use $[a]$ for a Cuntz class and, when the context specifies $\mathrm{V}(A)$, for a projection class.

The following cutoff facts allow norm approximations to be used in comparison:

$$\|a - b\| < \varepsilon \implies (a - \varepsilon)_+ \precsim b, \qquad a \precsim b \iff (a - \varepsilon)_+ \precsim b \text{ for every } \varepsilon > 0;$$

see [12, Proposition 2.1(ii)–(iii)]. Increasing sequences in $\mathrm{Cu}(A)$ have suprema. The relation $x \ll y$ means that, whenever $y \le \sup_n y_n$ for an increasing sequence, one has $x \le y_n$ for some n. In particular, $[(a - \varepsilon)_+] \ll [a]$. We refer to [16] for the ordered-semigroup conventions.

Two comparison properties will be used repeatedly. An ordered semigroup is *almost unperforated* if $(n+1)x \le ny$ implies $x \le y$ for every positive integer n. A Cuntz semigroup S is *almost divisible* if, for every $x' \ll x$ and $k \ge 1$, there is $y \in S$ such that

$$ky \le x, \qquad x' \le (k+1)y.$$

It is *pure* if it has both properties.

Definition 1.1 (Strict comparison)**.** Let A be a unital C*-algebra with $\mathrm{T}(A) \neq \emptyset$. We say that A has *strict comparison by tracial states* if

$$\mathrm{d}_\tau(a) < \mathrm{d}_\tau(b) \quad (\tau \in \mathrm{T}(A)) \quad \implies \quad a \precsim b, \qquad a, b \in \mathrm{M}_\infty(A)_+.$$

The phrase *strict comparison by tracial states on A_+* denotes the same implication restricted to $a, b \in A_+$. Unless expressly qualified, strict comparison in this paper is understood to be by tracial states.

The usual quasitracial formulation replaces $\mathrm{T}(A)$ by the normalized bounded 2-quasitraces on A and uses the same unnormalized matrix extensions. For exact algebras these 2-quasitraces are traces [35, Theorem 5.11], so the two formulations agree. For a unital stably finite exact algebra with nonempty tracial state space, the radius of comparison $\mathrm{rc}(A)$ is the infimum of $r \ge 0$ such that

$$\mathrm{d}_\tau(a) + r < \mathrm{d}_\tau(b) \quad (\tau \in \mathrm{T}(A)) \quad \implies \quad a \precsim b, \qquad a, b \in \mathrm{M}_\infty(A)_+.$$

The infimum is ∞ if no such r exists. Gap functions on the unnormalized trace cone, and the pointed comparison radius for nonunital models, are introduced in Chapter 4.

1.2.3 Affine functions and rank functions

Rank functions connect comparison with affine functions on the trace space. For a compact convex set S, $\mathrm{Aff}(S)$ is the real Banach space of continuous affine functions, with the uniform norm. The

set $\mathrm{LAff}_{++}(S)$ consists of strictly positive lower semicontinuous affine functions $S \to (0,\infty]$. The function $\widehat{a}(\tau) = \mathrm{d}_\tau(a)$ is the rank function of a positive element; for a projection p, $\widehat{p}(\tau) = \tau(p)$. We denote the K-groups by $\mathrm{K}_0(A)$ and $\mathrm{K}_1(A)$. The trace pairing on $\mathrm{K}_0(A)$ is

$$\rho_A([p]-[q])(\tau) = \tau(p) - \tau(q).$$

We write $\partial_e S$ for the extreme boundary of S and dim for covering dimension. A Bauer simplex is a Choquet simplex whose extreme boundary is compact.

For a simple unital stably finite C*-algebra with strict comparison by tracial states, the expression $\mathrm{V}(A) \sqcup \mathrm{LAff}_{++}(\mathrm{T}(A))$ carries the usual addition and order on each summand, together with

$$\begin{aligned} &[p] + f = \widehat{p} + f, \\ &[p] \le f \iff \widehat{p}(\tau) < f(\tau) \quad \text{for every } \tau \in \mathrm{T}(A), \\ &f \le [p] \iff f \le \widehat{p}. \end{aligned}$$

When we assert a *canonical* Cuntz-semigroup identification with this model, projection classes retain their projection data and nonprojection classes are represented by their rank functions. The existence of such an identification, including its comparison and rank-realization conclusions, is proved under the hypotheses of the relevant theorem. See [12, 16] for the underlying conventions.

1.2.4 Actions, crossed products, and tilings

We now fix the dynamical conventions used to form the tower models. Groups Γ are discrete, with identity e. The Rokhlin and orbit-radius results concern countable amenable groups, whereas the coefficient comparison criterion in Section 4.2 allows arbitrary discrete groups. An action is written $\alpha : \Gamma \to \mathrm{Aut}(A)$, with $\alpha_g\alpha_k = \alpha_{gk}$. We also write $g(a) = \alpha_g(a)$. For the reduced crossed product $B = A \rtimes_{\alpha,r} \Gamma$, the canonical unitaries satisfy

$$u_g u_k = u_{gk}, \qquad u_g a u_g^* = \alpha_g(a), \qquad \mathbb{E}\Big(\sum_g a_g u_g\Big) = a_e.$$

Here $\mathbb{E} : B \to A$ is the canonical faithful conditional expectation. The algebra $\mathrm{C_c}(\Gamma, A)$ of finite Fourier sums is dense in B. For amenable groups the full and reduced crossed products agree, and we sometimes omit the subscript r. The action is *minimal* if A has no nonzero proper α-invariant closed two-sided ideals.

The traces relevant to the action form the invariant trace space, for which we use the notation

$$S = \mathrm{T}^\alpha(A) = \mathcal{M}_1(A,\Gamma) = \{\tau \in \mathrm{T}(A) : \tau \circ \alpha_g = \tau \text{ for all } g \in \Gamma\}.$$

If A is unital, $\mathrm{T}(A) \neq \emptyset$, and Γ is amenable, then $S \neq \emptyset$. For a finite nonempty $F \subseteq \Gamma$, the averaging convention is

$$\tau_F = \frac{1}{|F|}\sum_{g\in F} \tau \circ \alpha_{g^{-1}}.$$

For a left action on a compact space X, the induced action on $\mathrm{C}(X)$ is $\alpha_g(f)(x) = f(g^{-1}x)$, and invariant tracial states are invariant Borel probability measures.

To specify tower shapes, for finite $K, F \subseteq \Gamma$ with $e \in K$, put

$$\mathrm{Int}_K(F) = \{g \in F : Kg \subseteq F\}.$$

A nonempty finite F is (K, ε)-invariant if $|F \setminus \mathrm{Int}_K(F)| < \varepsilon|F|$. An exact tiling with shapes $D_1, \dots, D_R$ is a disjoint decomposition

$$\Gamma = \bigsqcup_{s=1}^{R} \bigsqcup_{c \in \mathcal{C}_s} D_s c,$$

where the shapes are finite and nonempty and the sets $\mathcal{C}_s$ are their centers. We may normalize shapes to contain e. Thus the tiles are *right* translates, whereas the invariance and boundary estimates use *left* multiplication. The prescribed-tiling quantifier is part of WRP in Definition 3.4.

1.2.5 Ultrapowers and trace-kernel ideals

The tower constructions use central sequences and uniform tracial estimates. Fix a free ultrafilter ω on $\mathbb{N}$. Throughout the paper, A_ω denotes the *norm* ultrapower

$$A_\omega = \ell^\infty(\mathbb{N}, A)/c_\omega(A), \qquad c_\omega(A) = \{(a_n) : \lim_{n\to\omega} \|a_n\| = 0\}.$$

Constant sequences identify A with a subalgebra of A_ω; its central sequence algebra is $A_\omega \cap A'$. Actions induce coordinatewise actions on these algebras. For nonempty $S \subseteq \mathrm{T}(A)$, define

$$\|a\|_{2,S} = \sup_{\tau \in S} \tau(a^*a)^{1/2}, \qquad \|[(a_n)]\|_{2,S} = \lim_{n\to\omega} \|a_n\|_{2,S}, \qquad J_S = \{x \in A_\omega : \|x\|_{2,S} = 0\}.$$

Taking the supremum before the ultralimit makes this condition uniform over S, including sequences of varying traces. The set J_S is a closed two-sided ideal, and $q_S : A_\omega \to A_\omega/J_S$ is the quotient map. Each sequence $\tau_n \in S$ defines a limit trace $\tau_\omega([(a_n)]) = \lim_{n\to\omega} \tau_n(a_n)$, which annihilates J_S.

The lifting arguments also use the following ideal property. An ideal $I \lhd D$ is a σ-ideal if, for every separable subalgebra $C \subseteq D$, there is a positive contraction $z \in I \cap C'$ with $zb = b$ for $b \in I \cap C$. The trace-kernel ideals J_S have this property [40, Remark 4.7]; it permits lifts from quotient relative commutants that commute with a prescribed separable subalgebra [40, Proposition 4.5(iii)]. The detailed lifting statements are collected in Proposition 7.25.

For von Neumann algebras we use a tracial ultrapower in the finite case. If M is finite with faithful normal tracial state τ, then

$$M^\omega = \ell^\infty(\mathbb{N}, M)/\{(x_n) : \lim_{n\to\omega} \|x_n\|_{2,\tau} = 0\}, \qquad M_\omega = M' \cap M^\omega.$$

For general von Neumann algebras, M^ω denotes the Ocneanu ultrapower and M_ω the asymptotic centralizer, as defined in Definition 7.5. For $\tau \in \mathrm{T}(A)$, write $M_\tau = \pi_\tau(A)''$ and $\bar{\tau}$ for its faithful normal trace. If $\tau \in \mathrm{T}^\alpha(A)$, the action induces a normal trace-preserving action α_τ on M_τ, with $(\alpha_\tau)_g(\pi_\tau(a)) = \pi_\tau(\alpha_g(a))$. W*-central freeness means central freeness of every such action; see Definition 7.11.

1.2.6 Rank and regularity conventions

Finally, we recall the regularity properties used in the rank and absorption results. Real rank zero, written $\mathrm{RR}(A) = 0$, means that finite-spectrum self-adjoint elements are dense in $\widetilde{A}_{\mathrm{sa}}$. Stable rank one, written $\mathrm{tsr}(A) = 1$, means that $\mathrm{GL}(\widetilde{A})$ is dense in $\widetilde{A}$. The Jiang–Su algebra is denoted by $\mathcal{Z}$, and $\mathcal{Z}$-stability means $A \cong A \otimes \mathcal{Z}$. The UCT is the universal coefficient theorem for KK-theory.

We use nuclear dimension in the sense of [72, Definition 2.1]. The condition $\dim_{\mathrm{nuc}}(A) \leq d$ means that finite sets can be approximated by completely positive factorizations

$$A \xrightarrow{\psi} F_0 \oplus \cdots \oplus F_d \xrightarrow{\phi} A,$$

where each F_i is a finite-dimensional C*-algebra, ψ is c.p.c., and each $\phi|_{F_i}$ is c.p.c. and order zero. The sum map ϕ need not be contractive. An algebra has *locally finite nuclear dimension*, or is *locally approximated by finite nuclear dimension algebras*, if, for every finite set and tolerance, a C*-subalgebra of finite nuclear dimension approximates every element of the set within that tolerance. The dimension bound may depend on the finite set and the tolerance.

Chapter 2

Weak tracial approximation properties

2.1 Definitions and equivalent formulations

For a unital C*-algebra A, write $\mathrm{T}(A)$ for its tracial state space. If $\tau \in \mathrm{T}(A)$ and $a \in A_+$, set

$$\mathrm{d}_\tau(a) = \lim_{m\to\infty} \tau(a^{1/m}).$$

For $a, b \in A_+$, the notation $a \precsim b$ means Cuntz subequivalence: there are $v_j \in A$ such that $v_j^* b v_j \to a$ in norm. We write $x \in_\varepsilon C$ when $\mathrm{dist}(x, C) < \varepsilon$. Classes of C*-algebras are understood up to isomorphism; the approximating subalgebras need not be unital.

Definition 2.1. Let $\mathcal{S}$ be a class of C*-algebras, and let A be unital. We say that A is *weakly tracially approximated by* $\mathcal{S}$, and write $A \in \mathrm{TA}_0(\mathcal{S})$, if $\mathrm{T}(A) \neq \emptyset$ and, for every finite set $\mathcal{F} = \{x_1, \ldots, x_n\} \subseteq A$ and every $\varepsilon > 0$, there are a C*-subalgebra $C \subseteq A$ with $C \in \mathcal{S}$, elements $x_1', \ldots, x_n' \in A$, and a positive contraction $h \in C$ with $\|h\| = 1$ such that

(1) $\|x_i - x_i'\| < \varepsilon$, $1 \leq i \leq n$;

(2) $\|[h, x_i']\| < \varepsilon$, $1 \leq i \leq n$;

(3) $hx_i'h \in C$, $1 \leq i \leq n$;

(4) $\mathrm{d}_\tau(1 - h) < \varepsilon$ for every $\tau \in \mathrm{T}(A)$.

We say that A is *tracially approximated by* $\mathcal{S}$, and write $A \in \mathrm{TA}_1(\mathcal{S})$, if, for every such $\mathcal{F}$ and ε and every nonzero positive contraction $a_0 \in A$, the same data can be chosen to satisfy (1)–(3) and

$$(4') \qquad 1 - h \precsim a_0.$$

The definition of $\mathrm{TA}_1(\mathcal{S})$ does not require $\mathrm{T}(A) \neq \emptyset$.

Example 2.2 (Motivation from topological dynamics). Topological Rokhlin towers give the positive compressions in Definition 2.1. Let a countably infinite amenable group Γ act on a nonempty

compact metrizable space X, and let $\mathcal{C}_X$ be the class of finite direct sums of algebras $\mathrm{M}_n(\mathrm{C}_0(U))$, where $U \subseteq X$ is open. If the action has the topological uniform Rokhlin property, then

$$\mathrm{C}(X) \rtimes \Gamma \in \mathrm{TA}_0(\mathcal{C}_X).$$

By Theorem 7.2 and Proposition 3.6, topological URP implies WRP. Theorem 3.15 then gives approximation by matrix algebras over hereditary subalgebras of $\mathrm{C}(X)$. For $b \in \mathrm{C}(X)_+$,

$$\overline{b\mathrm{C}(X)b} = \mathrm{C}_0\big(\{x \in X : b(x) > 0\}\big).$$

Amenability ensures the existence of invariant probability measures, which induce tracial states of the crossed product. The earlier tracial approximation construction appears in [48, Theorem 3.9].

An open tower base need not be compact, and its algebra $\mathrm{C}_0(U)$ may therefore be nonunital. Positive functions supported on the towers define the compression. For each invariant probability measure, the dimension function of the defect is the measure of the set on which the compression is less than one. Thus Definition 2.1 accommodates these nonunital models and positive compressions.

If the action is also free and minimal and has Cuntz comparison of open sets (COS), then $\mathrm{C}(X) \rtimes \Gamma \in \mathrm{TA}_1(\mathcal{C}_X)$. To prove the required comparison, put $B = \mathrm{C}(X) \rtimes \Gamma$, fix $0 \neq a_0 \in B_+$, and let (λ, m) be COS constants as in [48, Definition 4.1]. The algebra B is simple and infinite dimensional. Choose m nonzero orthogonal Cuntz equivalent positive elements $b_1, \dots, b_m \in \overline{a_0 B a_0}$, using [55, Lemma 2.4]. By [48, Lemma 4.2], there is $0 \neq c \in \mathrm{C}(X)_+$ with $c \precsim b_1$. Consequently

$$c \otimes 1_m \precsim \sum_{j=1}^{m} b_j \precsim a_0.$$

For $V = \{c > 0\}$, minimality gives $\beta = \inf_\mu \mu(V) > 0$, where μ ranges over invariant probability measures: finitely many translates of V cover X. Choose the approximation data in Theorem 3.15 so that $\mu(\{h < 1\}) < \lambda\beta$ for every such μ. This is possible because $\tau_\mu = \mu \circ \mathbb{E}$ is a tracial state of B and $\mathrm{d}_{\tau_\mu}(1 - h) = \mu(\{h < 1\})$. Since $h \in \mathrm{C}(X)$, COS then gives

$$1 - h \precsim c \otimes 1_m \precsim a_0.$$

The TA_1 conclusion follows from Lemma 2.8.

Example 2.3 (Classifiable C*-algebras). Let $\mathcal{I}_{\mathrm{h}}$ and $\mathcal{I}_{\mathrm{o}}$ be the classes of finite direct sums of matrix algebras over $\mathrm{C}_0([0,1))$ and $\mathrm{C}_0((0,1))$, respectively. Every unital, separable, simple, infinite-dimensional, nuclear, stably finite, $\mathcal{Z}$-stable C*-algebra A satisfying the UCT belongs to

$$\mathrm{TA}_0(\mathcal{I}_{\mathrm{h}}) \cap \mathrm{TA}_0(\mathcal{I}_{\mathrm{o}}).$$

Thus every such classifiable C*-algebra admits weak tracial approximation by either half-open or open interval algebras. Theorem 7.1 gives the stronger conclusion

$$A \in \mathrm{TA}_1(\mathcal{I}_{\mathrm{h}}) \cap \mathrm{TA}_1(\mathcal{I}_{\mathrm{o}}),$$

and Remark 2.5 gives the stated weak tracial approximation. The proof uses the Gong–Lin–Niu models, whose traces concentrate on the point quotients. Compression to endpoint collars gives half-open interval models, and $\mathcal{Z}$-absorption gives open interval models; see Section 7.1.

Remark 2.4 (Comparison with generalized tracial approximation). Let A be simple and unital with $\mathrm{T}(A) \neq \emptyset$. Then

$$A \in \mathrm{TA}_1(\mathcal{S}) \quad \Longleftrightarrow \quad A \in \mathrm{gTA}(\mathcal{S}),$$

where gTA is the nonunital generalized tracial approximation property of Elliott–Fan–Fang [22, Definition 1.2 and Theorem 2.11]. When $\mathcal{S}$ consists of unital algebras, these conditions also agree with their $\mathrm{WTA}(\mathcal{S})$ [21, Definition 1.1]; see also [22, Theorem 2.12]. In that formulation, the approximating algebra has unit $p = 1_C$, and the compression is $p - g$, with $0 \leq g \leq p$. The property called weak tracial approximation in that reference therefore corresponds to TA_1.

We verify the equivalence, including the one-sided approximation and norm preservation conditions in gTA. Include the prescribed nonzero positive contraction a_0 in the finite set. By Lemma 2.10, the TA_1 approximation data can be chosen with $\|ha_0h\|$ arbitrarily close to $\|a_0\|$. For $0 < s < 1$, put $k = g_s(h)$ and $\ell = g_{s/2}(h)$. Then $k\ell = k$ and

$$kx_i'\ell = \frac{g_s(h)}{h}\,(hx_i'h)\,\frac{g_{s/2}(h)}{h} \in C,$$

where the quotients are continuous functions extended by zero at the origin. Likewise $\ell x_i'k \in C$. It follows that the distances of kx_i' and $x_i'k$ from C are bounded by $\|[\ell, x_i']\|$. These commutators are small by continuous functional calculus. Moreover,

$$1 - k \leq (1 - s)^{-1}(1 - h), \qquad \|ka_0k\| \geq \|ha_0h\| - 2s\|a_0\|.$$

The first estimate gives $1 - k \precsim a_0$. The second follows from $\|h(1 - k)\| \leq s$ and proves the required norm estimate. For sufficiently small s and initial tolerance, these estimates give the formulation of gTA in [22, Theorem 2.11]. They also imply the conditions of Definition 1.2, since, for every $y \in A_+$,

$$(y^2 - yky - \varepsilon)_+ \precsim y(1 - k)y \precsim 1 - k \precsim a_0.$$

Rescaling gives the same conclusion for arbitrary nonzero positive comparison elements.

Conversely, suppose that A satisfies Definition 1.2. Fix $0 \neq a_0 \in A_+$, put $a = a_0/\|a_0\|$, and use $y = 1_A$ in Definition 1.2 with a tolerance $0 < \delta < 1/4$. There are an approximating algebra $C \in \mathcal{S}$ and a positive contraction $k_0 \in C$ such that

$$(1 - k_0 - \delta)_+ \precsim a, \qquad \|k_0ak_0\| \geq 1 - \delta,$$

and the one-sided approximation and commutator estimates hold with tolerance δ. Set

$$h = \min\{k_0/(1 - \delta), 1_A\}.$$

Continuous functional calculus defines this minimum, and the scalar function vanishes at zero. Thus $h \in C$, including when C is nonunital. Since $\|k_0\| \geq \sqrt{1 - \delta} > 1 - \delta$, we have $\|h\| = 1$. Moreover, $\|h - k_0\| \leq \delta$, and

$$1 - h = (1 - \delta)^{-1}(1 - k_0 - \delta)_+ \precsim a \sim a_0.$$

Multiplication by k_0 of an approximation to k_0x in C gives $\operatorname{dist}(k_0xk_0, C) < \delta$. Hence, for every x in the prescribed finite set,

$$\|[h, x]\| < \delta(1 + 2\|x\|), \qquad \operatorname{dist}(hxh, C) < \delta(1 + 2\|x\|).$$

Taking δ sufficiently small and applying Lemma 2.8 proves $A \in \mathrm{TA}_1(\mathcal{S})$. When C is unital, setting $g = p - h$ identifies these conditions with those of WTA.

The defect in TA_0 is controlled by dimension functions. Remark 2.5 gives $\mathrm{TA}_1 \Rightarrow \mathrm{TA}_0$. If A has strict comparison by traces, the reverse implication also holds. For a nonzero positive contraction a_0, faithfulness and compactness give

$$m = \min_{\tau \in \mathrm{T}(A)} \tau(a_0) > 0, \qquad \mathrm{d}_\tau(a_0) \geq \tau(a_0) \geq m.$$

Choosing h so that $\mathrm{d}_\tau(1 - h) < m$ for every τ gives $1 - h \precsim a_0$. Thus strict comparison by tracial states on A_+ suffices for the reverse implication.

Remark 2.5. If A is simple and $\mathrm{T}(A) \neq \emptyset$, then

$$A \in \mathrm{TA}_1(\mathcal{S}) \quad \Longrightarrow \quad A \in \mathrm{TA}_0(\mathcal{S}).$$

Moreover, the TA_1 approximation data can be chosen with uniformly small dimension-function defect and $1 - h \precsim a_0$ for a prescribed a_0.

Suppose first that A is infinite dimensional. Fix $a_0 \neq 0$ and $\eta > 0$, and choose N with $1/N < \eta$. Such an A is not of type I, since a simple unital type-I algebra is a matrix algebra. By [55, Lemma 2.4], there are nonzero, pairwise orthogonal, Cuntz equivalent positive elements $b_1, \ldots, b_N$ in $\overline{a_0Aa_0}$. Thus $b_1 \precsim a_0$ and

$$N\mathrm{d}_\tau(b_1) = \mathrm{d}_\tau\left(\sum_{j=1}^{N} b_j\right) \leq 1, \qquad \tau \in \mathrm{T}(A).$$

After rescaling b_1 to be a contraction, apply the TA_1 property with b_1 in place of a_0. The resulting defect satisfies both $1 - h \precsim a_0$ and $\mathrm{d}_\tau(1 - h) < \eta$ for every τ.

Now suppose $A \cong \mathrm{M}_r$. Choose a finite family $\mathcal{U}$ of unitaries whose conjugation average is $x \mapsto \mathrm{tr}_r(x)1_A$, where tr_r is the normalized matrix trace. Apply the approximation property to $\mathcal{F} \cup \mathcal{U}$ with tolerance $\delta < \min\{\varepsilon, 1/12\}$. Conditions (1) and (2) give $\|[h, u]\| < 3\delta$ for $u \in \mathcal{U}$, so

$$\|h - \mathrm{tr}_r(h)1_A\| < 3\delta.$$

Since $\|h\| = 1$, we have $\mathrm{tr}_r(h) > 1 - 3\delta$, and therefore $h > (1 - 6\delta)1_A > \frac{1}{2}1_A$. Consequently $g_{1/2}(h) = 1_A \in C$, and $h^{-1} \in C$. Condition (3) then gives $x_i' \in C$ for the original finite set. Replacing h by 1_A makes the defect zero and preserves the required approximation.

Remark 2.6. In Definition 2.1, the TA_0 condition (4) can equivalently be replaced by

$$\tau(1 - h) < \varepsilon, \qquad \tau \in \mathrm{T}(A).$$

The equivalence allows a decrease in the initial tolerance; the two estimates need not be equivalent for a fixed h.

The dimension-function estimate implies the trace estimate because $0 \leq 1-h \leq 1_A$. Conversely, choose the approximation data with a smaller trace tolerance and put $k = g_{1/2}(h)$. Then $k \in C$ and $\|k\| = 1$. In the tracial von Neumann algebra associated to τ,

$$\mathrm{d}_\tau(1-k) = \tau\big(1_{[0,1/2)}(h)\big) \leq 2\tau(1-h).$$

Define $r(t) = g_{1/2}(t)/t$ for $t > 0$, with $r(0) = 0$. This is continuous on $[0,1]$, and

$$kx_i'k = r(h)(hx_i'h)r(h) \in C.$$

The commutators $[k, x_i']$ are small when $[h, x_i']$ are sufficiently small. This follows by polynomial approximation to $g_{1/2}$, using the bound $\|x_i'\| \leq \|x_i\| + 1$. A sufficiently small initial tolerance therefore makes k satisfy the required conditions.

Example 2.7. Requiring C to be unital and taking $h = 1_C$ gives compression by the projection $p = 1_C \in A$. With the Cuntz comparison condition of TA_1, this recovers the tracial approximation property of [25]. The TA_0 condition gives the corresponding property with tracial control of the defect. Allowing general positive contractions also permits nonunital approximating subalgebras, as in Example 2.2.

The following equivalent formulation uses the original finite set rather than the auxiliary perturbations x_i'.

Lemma 2.8. *Let A be a unital C^*-algebra. Then $A \in \mathrm{TA}_0(\mathcal{S})$ if and only if $\mathrm{T}(A) \neq \emptyset$ and, for every finite set $\{x_1, \ldots, x_n\} \subseteq A$ and every $\varepsilon > 0$, there are a C^*-subalgebra $C \subseteq A$ with $C \in \mathcal{S}$ and a positive contraction $h \in C$ with $\|h\| = 1$ such that*

$$\|[h, x_i]\| < \varepsilon, \qquad \mathrm{dist}(hx_ih, C) < \varepsilon, \qquad 1 \leq i \leq n,$$

and $\mathrm{d}_\tau(1-h) < \varepsilon$ for every $\tau \in \mathrm{T}(A)$.

Likewise, $A \in \mathrm{TA}_1(\mathcal{S})$ if and only if, for every such finite set and ε and every nonzero positive contraction $a_0 \in A$, the same data can be chosen with $1-h \precsim a_0$ in place of the dimension-function estimate.

Proof. The approximation data in Definition 2.1 with tolerance δ satisfy

$$\|[h, x_i]\| < 3\delta, \qquad \mathrm{dist}(hx_ih, C) < \delta.$$

The defect condition is unchanged.

Conversely, fix the finite set and $\varepsilon > 0$, and put $\rho = \varepsilon/16$. Choose C and k satisfying the stated conditions with tolerance δ, where δ is sufficiently small. Set

$$u = g_{1/4}(k), \qquad v = g_{1/2}(k)^{1/2}, \qquad w = u^{1/2}, \qquad z = (1-u)^{1/2}.$$

Polynomial approximation in continuous functional calculus allows δ to be chosen so that

$$\|[w, x_i]\| < \rho, \qquad \|[z, x_i]\| < \rho, \qquad \|[v, x_i]\| < \rho$$

for all i. Define the continuous function

$$r(t) = \begin{cases} g_{1/4}(t)^{1/2}/t, & t > 0, \\ 0, & t = 0. \end{cases}$$

Choose $y_i \in C$ with $\|kx_ik - y_i\| < \delta$ and put $c_i = r(k)y_ir(k) \in C$. Since $r(k)k = w$,

$$\|c_i - wx_iw\| \leq \|r\|^2\delta < \rho$$

after decreasing δ further. Set

$$x_i' = zx_iz + c_i.$$

Using $w^2 + z^2 = 1_A$ gives

$$\|wx_iw + zx_iz - x_i\| \leq \|[w, x_i]\| + \|[z, x_i]\| < 2\rho.$$

Hence $\|x_i' - x_i\| < 3\rho$ and $\|[v, x_i']\| < 7\rho < \varepsilon$. Moreover, $vz = 0$, so $vx_i'v = vc_iv \in C$, and $v \in C$ is a positive contraction of norm one.

For TA_0, the zero set of $1 - v$ contains that of $1 - k$, so

$$\mathrm{d}_\tau(1 - v) \leq \mathrm{d}_\tau(1 - k) < \delta.$$

For TA_1, the scalar inequality $1 - g_{1/2}(t)^{1/2} \leq 2(1 - t)$ on $[0, 1]$ gives

$$1 - v \precsim 1 - k \precsim a_0.$$

Taking also $\delta < \varepsilon$ proves both converse implications. □

2.2 Norm preservation under tracial cut-downs

A small tracial defect gives a small compression error in the tracial 2-norm. In a simple algebra with tracial states, this estimate yields norm preservation for the compressions in Definition 2.1. The corresponding results for asymptotically order zero maps and relatively central compressions apply to individual Rokhlin towers.

For a trace τ, write $\|x\|_{2,\tau} = \tau(x^*x)^{1/2}$.

Lemma 2.9. *Let A be unital, let $h \in A$ be a positive contraction, and let $\tau \in \mathrm{T}(A)$. For every $x \in A$,*

$$\|hxh - x\|_{2,\tau} \leq 2\|x\|\tau(1 - h)^{1/2}$$

and

$$|\tau(hxh) - \tau(x)| \leq \|x\|\tau(1 - h^2) \leq 2\|x\|\tau(1 - h).$$

Proof. Write $hxh - x = hx(h-1) + (h-1)x$. The inequalities $\|ab\|_{2,\tau} \leq \|a\|\|b\|_{2,\tau}$ and $\|ab\|_{2,\tau} \leq \|a\|_{2,\tau}\|b\|$ give

$$\|hxh - x\|_{2,\tau} \leq 2\|x\|\|1-h\|_{2,\tau} \leq 2\|x\|\tau(1-h)^{1/2}.$$

By traciality, $\tau(hxh) - \tau(x) = -\tau(x(1-h^2))$. For $b \geq 0$, the positive functional $x \mapsto \tau(b^{1/2}xb^{1/2})$ has norm $\tau(b)$. Taking $b = 1 - h^2$ proves the second estimate, since $1 - h^2 \leq 2(1-h)$. □

Lemma 2.10. *Let A be simple and unital with $\mathrm{T}(A) \neq \emptyset$, and suppose $A \in \mathrm{TA}_0(\mathcal{S})$ or $A \in \mathrm{TA}_1(\mathcal{S})$. The approximation data in Definition 2.1 can also be chosen so that*

$$\|hx_i'h\| > \|x_i\| - \varepsilon \quad \textit{and} \quad \|hx_i'h\| > \|x_i'\| - \varepsilon, \qquad 1 \leq i \leq n.$$

In the TA_1 case, comparison with the prescribed a_0 is retained.

Proof. Fix $\tau_0 \in \mathrm{T}(A)$, which is faithful by simplicity. Faithfulness implies, for every $x \in A$,

$$\lim_{p\to\infty} \tau_0((x^*x)^p)^{1/(2p)} = \|x\|.$$

Thus, for each i, choose an integer $p_i \geq 1$ so that

$$\tau_0((x_i^*x_i)^{p_i})^{1/(2p_i)} > \|x_i\| - \varepsilon/4.$$

Choose the approximation data with tolerance δ. In the TA_0 case, $\tau_0(1-h) < \delta$ follows from condition (4). In the TA_1 case, Remark 2.5 gives the same estimate while preserving $1 - h \precsim a_0$. Put $y_i = hx_i'h$. Lemma 2.9 gives

$$\|y_i - x_i\|_{2,\tau_0} \leq \delta + 2\|x_i\|\delta^{1/2}.$$

The elements x_i and y_i lie in a fixed norm-bounded set, on which the function $x \mapsto \tau_0((x^*x)^p)$ is continuous in the tracial 2-norm: if $\|x\|, \|y\| \leq M$ and $p \geq 1$ is an integer, then

$$\left|\tau_0((x^*x)^p) - \tau_0((y^*y)^p)\right| \leq 2pM^{2p-1}\|x - y\|_{2,\tau_0}.$$

Indeed, $\|x^*x - y^*y\|_{2,\tau_0} \leq 2M\|x-y\|_{2,\tau_0}$; expanding the difference of the pth powers and applying Cauchy–Schwarz to each term proves the estimate. Therefore, choosing δ sufficiently small gives

$$\|y_i\| \geq \tau_0((y_i^*y_i)^{p_i})^{1/(2p_i)} > \|x_i\| - \varepsilon/2$$

for all i. Choosing also $\delta < \varepsilon/2$ and using $\|x_i'\| < \|x_i\| + \delta$ proves both required inequalities. □

A completely positive map has *order zero* if it preserves orthogonality of positive elements. A sequence (ϕ_n) of completely positive contractions is asymptotically order zero if $\|\phi_n(a)\phi_n(b)\| \to 0$ whenever $a, b \geq 0$ and $ab = 0$.

Lemma 2.11. *Let A be unital, let ω be a free ultrafilter on $\mathbb{N}$, and let $\phi_n : A \to D_n$ be completely positive contractions. Suppose $\tau_n \in \mathrm{T}(D_n)$ and*

$$\lim_{n\to\infty} \|\phi_n(a)\phi_n(b)\| = 0 \quad \text{whenever } a, b \in A_+ \text{ and } ab = 0, \tag{2.1}$$

$$\lim_{n\to\infty} \tau_n(\phi_n(1_A)) = 1. \tag{2.2}$$

If the induced state

$$\tau(a) = \lim_{n\to\omega} \tau_n(\phi_n(a)), \qquad a \in A,$$

is faithful, then

$$\lim_{n\to\omega} \|\phi_n(a)\| = \|a\|, \qquad a \in A.$$

The state τ is automatically tracial. If every tracial state of A is faithful (in particular, if A is simple), then the ordinary limit satisfies

$$\lim_{n\to\infty} \|\phi_n(a)\| = \|a\|, \qquad a \in A.$$

Proof. Let $\widetilde{D}_n = D_n$ if D_n is unital, and let $\widetilde{D}_n$ be its unitization otherwise. Extend τ_n to a tracial state $\widetilde{\tau}_n$ on $\widetilde{D}_n$, and form the norm ultraproduct

$$Q_\omega = \prod_n \widetilde{D}_n / \{(d_n) : \lim_{n\to\omega} \|d_n\| = 0\}.$$

It carries the tracial state

$$\tau_\omega([(d_n)]) = \lim_{n\to\omega} \widetilde{\tau}_n(d_n).$$

Condition (2.1) makes the induced completely positive contraction $\Phi : A \to Q_\omega$ an order zero map. Put $h = \Phi(1_A)$. By (2.2), $\tau_\omega(h) = 1$, and

$$\tau_\omega((1-h)^2) \le \tau_\omega(1-h) = 0.$$

Since τ_ω is tracial, its GNS representation π_{τ_ω} annihilates $1-h$. Thus $\Pi = \pi_{\tau_\omega} \circ \Phi$ is a unital completely positive order zero map, hence a *-homomorphism by [71, Theorem 3.3]. In particular, $\tau = \tau_\omega \circ \Phi$ is tracial. Faithfulness of τ makes Π injective, so

$$\|a\| = \|\Pi(a)\| \le \|\Phi(a)\| = \lim_{n\to\omega} \|\phi_n(a)\| \le \|a\|.$$

This proves the first conclusion. If all tracial states of A are faithful, the argument applies to every free ultrafilter. Since a bounded scalar sequence whose limit along every free ultrafilter is the same must converge to that value, the ordinary norm limit follows.

The state normalization also controls all powers of $h_n = \phi_n(1_A)$: for $k \ge 1$,

$$0 \le \tau_n(h_n - h_n^k) \le (k-1)\big(1 - \tau_n(h_n)\big) \longrightarrow 0. \tag{2.3}$$

□

Lemma 2.12. *Let A and B_n be unital* C*-*algebras, and suppose every tracial state of A is faithful. Let $\iota_n : A \to B_n$ be unital *-homomorphisms and let $h_n \in B_n$ be positive contractions such that*

$$\|[h_n, \iota_n(a)]\| \longrightarrow 0, \qquad a \in A.$$

Suppose there are $\tau_n \in \mathrm{T}(B_n)$ with $\mathrm{d}_{\tau_n}(h_n) > 0$ and

$$\frac{\tau_n(h_n^2)}{\mathrm{d}_{\tau_n}(h_n)} \longrightarrow 1. \tag{2.4}$$

Then

$$\lim_{n\to\infty} \|h_n \iota_n(a) h_n\| = \|a\|, \qquad a \in A.$$

Proof. Set

$$D_n = \overline{h_n B_n h_n}, \qquad \phi_n(a) = h_n \iota_n(a) h_n, \qquad \widetilde{\tau}_n = \frac{\tau_n|_{D_n}}{\mathrm{d}_{\tau_n}(h_n)}.$$

The norm of $\tau_n|_{D_n}$ is $\mathrm{d}_{\tau_n}(h_n)$, as is seen by evaluating it on the approximate unit $h_n^{1/j}$. Thus $\widetilde{\tau}_n$ is a tracial state of D_n. Each ϕ_n is completely positive and contractive. If $a, b \in A_+$ and $ab = 0$, then

$$\phi_n(a)\phi_n(b) = h_n \iota_n(a)[h_n^2, \iota_n(b)] h_n,$$

whose norm tends to zero by relative centrality. Finally, $\phi_n(1_A) = h_n^2$, and (2.4) gives

$$\widetilde{\tau}_n(\phi_n(1_A)) = \frac{\tau_n(h_n^2)}{\mathrm{d}_{\tau_n}(h_n)} \longrightarrow 1.$$

Lemma 2.11 now gives the asserted norm limit. □

Remark 2.13. Condition (2.4) also implies

$$\frac{\tau_n(h_n^k)}{\mathrm{d}_{\tau_n}(h_n)} \longrightarrow 1 \quad \text{for every } k \geq 1.$$

Indeed, integrate $0 \leq 1 - t^k \leq k(1 - t^2)$ over the nonzero spectrum of h_n and divide by $\mathrm{d}_{\tau_n}(h_n)$.

Remark 2.14. In Lemma 2.12, the support traces $\mathrm{d}_{\tau_n}(h_n)$ may tend to zero. The normalized traces on the hereditary subalgebras must concentrate at 1, as required for individual Rokhlin towers. For compressions whose defects have uniformly small trace, Lemma 2.10 gives the corresponding norm estimate on a finite set.

Chapter 3

Rokhlin properties and weak tracial approximation

Rokhlin properties for actions of discrete amenable groups on unital C*-algebras give matrix models over hereditary coefficient subalgebras. The weak Rokhlin property thereby gives weak tracial approximation of the crossed product. We also establish the freeness conditions and lifting results underlying these tower constructions.

Throughout this chapter, we write the action as

$$\alpha_g(a) = g(a) = u_g a u_g^*, \qquad a \in A,\ g \in \Gamma,$$

where $(u_g)_{g\in\Gamma}$ are the canonical unitaries and $u_g u_k = u_{gk}$. For finite subsets $K, F \subseteq \Gamma$, put

$$\mathrm{Int}_K(F) = \{g \in F : Kg \subseteq F\}.$$

All invariance and collar estimates below therefore use left multiplication.

3.1 Rokhlin properties for C*-dynamical systems

3.1.1 A finite-stage sufficient condition: uniform Rokhlin property

For a topological action on X, we use the left action convention $\alpha_g(f)(x) = f(g^{-1}x)$ on $\mathrm{C}(X)$. Invariance is understood with respect to left multiplication, as above.

Definition 3.1 (Topological URP)**.** Let (X, Γ) be a topological dynamical system, where X is a compact Hausdorff space and Γ is a discrete amenable group. The system has the *uniform Rokhlin property* (URP) if, for every $\varepsilon > 0$ and every finite set $K \subseteq \Gamma$, there are (K, ε)-invariant finite sets $\Gamma_1, \Gamma_2, ..., \Gamma_S \subseteq \Gamma$ and open sets

$$U_s \subseteq X, \quad s = 1, ..., S$$

such that

1. γU_s, $\gamma \in \Gamma_s$, $s = 1, 2, ..., S$, are mutually disjoint, and

2. $\mu(X \setminus \bigcup_{s=1}^{S} \bigcup_{\gamma \in \Gamma_s} \gamma U_s) < \varepsilon$ for every invariant probability measure μ.

The corresponding property for C*-dynamical systems is defined as follows.

Definition 3.2. Let (A, Γ) be a C*-dynamical system, where A is unital, Γ is countable and amenable, and $\mathrm{T}(A) \neq \varnothing$. A finite set $F \subseteq \Gamma$ is (K, ε)-invariant if $|F \setminus \{g \in F : Kg \subseteq F\}| < \varepsilon |F|$; we may assume $e \in K$. The action has the *uniform Rokhlin property* (URP) if, for every $\varepsilon > 0$ and all finite sets $\mathcal{F} \subseteq A$ and $\mathcal{K} \subseteq \Gamma$, there are $(\mathcal{K}, \varepsilon)$-invariant finite sets $\Gamma_1, \Gamma_2, ..., \Gamma_S \subseteq \Gamma$ and positive elements

$$c_s \in A, \quad s = 1, ..., S$$

such that

1. $\|c_s\| = 1$, $s = 1, ..., S$,

2. $\gamma(c_s)$, $\gamma \in \Gamma_s$, $s = 1, 2, ..., S$, are mutually orthogonal,

3. $\displaystyle\sum_{s=1}^{S} \sum_{\gamma \in \Gamma_s} \|[\gamma(c_s), f]\| < \varepsilon$ for every $f \in \mathcal{F}$,

4. $$\mathrm{d}_\tau(1 - \sum_{s=1}^{S} \sum_{\gamma \in \Gamma_s} \gamma(c_s)) < \varepsilon, \quad \tau \in \mathrm{T}(A \rtimes \Gamma).$$

Remark 3.3. The aggregate commutator estimate in (3) is stronger than a levelwise bound, since the number of tower levels is not bounded in advance. For $A = \mathrm{C}(X)$, all commutators vanish. Cutting finite URP towers along an exact tiling gives central sequence towers with the prescribed shapes (Proposition 3.6).

3.1.2 Weak Rokhlin property

Weak Rokhlin-type properties for positive towers were studied by Sato for automorphisms of the Jiang–Su algebra [59] and by Wang for actions of countable discrete amenable groups [69]. We use the following prescribed-tiling formulation, with the remainder controlled uniformly over invariant tracial states.

Definition 3.4. Let (A, Γ) be a C*-dynamical system, where A is a unital separable C*-algebra with $\mathrm{T}(A) \neq \varnothing$ and Γ is a countable discrete amenable group. The action has the *weak Rokhlin property* (WRP) if, for every exact tiling $\mathcal{T}$ of Γ with shapes $K_1, \ldots, K_S$, there are positive contractions $c_s \in A' \cap A_\omega$, $s = 1, ..., S$ such that

1. $\gamma(c_s)$, $\gamma \in K_s$, $s = 1, 2, ..., S$, are mutually orthogonal, and

2.
$$1-\sum_{s=1}^{S}\sum_{\gamma\in K_s}\gamma(c_s)\in J_{\mathcal{M}_1(A,\Gamma)},$$
where $\mathcal{M}_1(A,\Gamma)$ is the simplex of invariant tracial states, and $J_{\mathcal{M}_1(A,\Gamma)}$ is the corresponding trace-kernel ideal (Definition 7.23).

The weak Rokhlin property admits the following local formulation for each exact tiling.

Lemma 3.5. *WRP is equivalent to the following condition: for every exact tiling with shapes $D_1,\dots,D_R$, every finite $\mathcal{F}\subseteq A$, and every $\varepsilon>0$, there are positive contractions $c_s\in A$ such that*

1. *$\|\gamma(c_s)\gamma'(c_{s'})\|<\varepsilon$ for distinct levels $(s,\gamma)\neq(s',\gamma')$, with $\gamma\in D_s$ and $\gamma'\in D_{s'}$;*
2. *$\|[c_s,f]\|<\varepsilon$ for $s\le R$ and $f\in\mathcal{F}$;*
3. $$\sup_{\rho\in\mathcal{M}_1(A,\Gamma)}\rho\left(\left(1-\sum_{s=1}^{R}\sum_{\gamma\in D_s}\gamma(c_s)\right)^2\right)<\varepsilon.$$

Proof. Suppose first that WRP holds. Lift the positive contractions in A_ω to sequences of positive contractions. The norm conditions and the uniform tracial 2-norm condition hold simultaneously at some coordinate.

Conversely, apply the local condition to an increasing sequence of finite sets with dense union in A and to tolerances tending to zero. The resulting sequences define central positive contractions in A_ω with orthogonal translates and complement in the trace-kernel ideal. At a finite stage, their sum need not be at most one; condition (3) therefore uses the square of the complement. □

Proposition 3.6. *For separable A, the URP of Definition 3.2 implies the prescribed-tiling WRP of Definition 3.4.*

Proof. Fix an exact tiling $\Gamma=\bigsqcup_{s=1}^{R}\bigsqcup_{t\in C_s}D_st$. After translating each shape and its centers, assume $e\in D_s$. Set $L=\bigcup_s D_sD_s^{-1}$, and choose $\eta_n\downarrow 0$ and increasing finite sets $\mathcal{F}_n\subset A$ with dense union. Apply URP with $K=L$, $\mathcal{F}=\mathcal{F}_n$, and tolerance η_n. Write the resulting shapes as $F_{r,n}$, their bases as $b_{r,n}$, and their levels as $p_{r,g,n}=g(b_{r,n})$. Define
$$c_{s,n}=\sum_{r}\sum_{\substack{t\in C_s\\ D_st\subset F_{r,n}}}t(b_{r,n})\in A_+.$$

These finite sums are contractions because $e\in D_s$ and the summands are distinct levels of the URP towers. For distinct (s,k), $k\in D_s$, the levels $k(c_{s,n})$ are orthogonal: within each $F_{r,n}$ they comprise disjoint tiles of the prescribed tiling. The aggregate URP estimate gives $\|[c_{s,n},f]\|<\eta_n$ for $f\in\mathcal{F}_n$.

Put $q_n=\sum_{r,g\in F_{r,n}}p_{r,g,n}$ and $q_n'=\sum_{s,k\in D_s}k(c_{s,n})$. If $g\in F_{r,n}$ satisfies $Lg\subset F_{r,n}$, its unique tiling piece D_st lies in $F_{r,n}$: writing $g=kt$, $k\in D_s$, one has $D_st\subset Lg$. Hence at most $\eta_n|F_{r,n}|$

levels of each URP tower are omitted from q'_n. For any invariant trace ρ, $\rho \circ E$ is a trace on the crossed product, and subadditivity of dimension functions yields

$$\begin{aligned} \mathrm{d}_\rho(1-q'_n) &\leq \mathrm{d}_\rho(1-q_n) + \sum_r \sum_{g \text{ omitted from } F_{r,n}} \mathrm{d}_\rho(p_{r,g,n}) \\ &< \eta_n + \eta_n \sum_r |F_{r,n}| \mathrm{d}_\rho(b_{r,n}) \leq 2\eta_n. \end{aligned}$$

Thus $\sup_\rho \rho((1-q'_n)^2) \leq 2\eta_n$. The classes $[(c_{s,n})] \in A_\omega \cap A'$ are the required WRP bases for the prescribed tiling. □

Proposition 3.7. *Let a countable discrete amenable group Γ act on a nonempty compact metrizable space X. Then topological URP is equivalent to WRP for the induced action on $\mathrm{C}(X)$.*

Proof. Suppose first that the topological system has URP, and discard empty bases. For the finite family of open towers, choose open sets V_s with $\overline{V_s} \subset U_s$ such that the union of their translates has measure greater than $1-\varepsilon$ for every invariant measure. To make this choice uniformly, regularity gives such sets for each measure, and lower semicontinuity of the measure of an open set gives neighborhoods covering the compact invariant measure space. A finite subcover and unions within each U_s give the required V_s. Choose norm-one positive contractions $c_s \in \mathrm{C}(X)$ supported in U_s and equal to one on $\overline{V_s}$. The translated functions are orthogonal, and the dimension function of $1 - \sum_{s,g} g(c_s)$ is below ε for every invariant measure. The aggregate commutator condition is automatic. Thus URP holds, and Proposition 3.6 applies.

Conversely, given K and ε, choose an exact tiling with (K,ε)-invariant shapes D_s and use WRP for this tiling. For infinite Γ, this follows from [19, Theorem 4.3 and Lemma 2.6]; for finite Γ, use the single shape $D_1 = \Gamma$. Let $c_{s,n} \in \mathrm{C}(X)_+$ be representatives by positive contractions, put $M = \sum_s |D_s|$, and choose $t < 1/(2M)$. At a suitable coordinate, products of distinct translated functions have norm below t^2 and

$$\sup_\mu \mu\left(\left(1 - \sum_{s, g \in D_s} g(c_{s,n})\right)^2\right) < \varepsilon/4.$$

The open sets $\{c_{s,n} > t\}$ form disjoint towers. Outside their union the sum of all translated functions is at most $Mt < 1/2$, so the complement has measure below ε for every invariant measure. □

3.1.3 Invariant traces

For a C*-dynamical system (A,Γ) with WRP, restriction to A identifies the tracial state simplex of $A \rtimes \Gamma$ with $\mathcal{M}_1(A,\Gamma)$.

Proposition 3.8. *Let (A,Γ) be a C*-dynamical system, where A is a unital separable C*-algebra with $\mathrm{T}(A) \neq \varnothing$ and Γ is a countable discrete amenable group. Let $\mathbb{E}\colon A \rtimes \Gamma \to A$ be the canonical conditional expectation. If (A,Γ) has WRP, then the map*

$$\theta : \mathcal{M}_1(A,\Gamma) \ni \tau \mapsto \tau \circ \mathbb{E} \in \mathrm{T}(A \rtimes \Gamma),$$

is an affine homeomorphism.

Proof. The restriction of a trace $\sigma \in \mathrm{T}(A \rtimes \Gamma)$ to A is invariant; conversely each invariant trace ρ extends to $\rho \circ \mathbb{E}$. Fix $k \neq e$, $a \in A$, and $\eta > 0$. Choose an exact tiling with shapes D_s satisfying $|D_s \setminus \{g \in D_s : kg \in D_s\}| < \eta\,|D_s|$. Let $c_s \in A_\omega \cap A'$ be WRP bases, $p_{s,g} = g(c_s)$, and $q = \sum_{s,g\in D_s} p_{s,g}$. The trace-kernel condition gives $\sigma_\omega(1-q) = 0$ for $\sigma \in \mathrm{T}(A \rtimes \Gamma)$, where σ_ω is its ultralimit trace. Hence q acts as the unit in the GNS representation associated with σ_ω. Off-diagonal terms vanish because $\sigma_\omega(p_{s,g}au_kp_{t,j}) = \sigma_\omega(p_{t,j}p_{s,g}au_k) = 0$ for distinct levels. Thus orthogonality and traciality give

$$\sigma(au_k) = \sigma_\omega(qau_kq) = \sum_{s,g\in D_s} \sigma_\omega(p_{s,g}au_kp_{s,g}).$$

For $kg \in D_s$, centrality with a and orthogonality of $p_{s,g}$ and $k(p_{s,g})$ make the summand zero. On the remaining levels, Cauchy–Schwarz and invariance of $\sigma|_A$ give

$$|\sigma(au_k)| \leq \|a\| \sum_s |D_s \setminus \{g : kg \in D_s\}|\, \sigma_\omega(c_s) \leq \eta\,\|a\|\,\sigma_\omega(q) \leq \eta\,\|a\|\,.$$

Let η decrease to zero. Fourier polynomials are dense, so $\sigma = \sigma|_A \circ \mathbb{E}$. The affine bijection is a homeomorphism by compactness. □

Corollary 3.9. *Let (A, Γ) be a C^*-dynamical system with WRP. If projections of A separate its tracial states, then projections of $A \rtimes \Gamma$ separate its tracial states.*

Proof. Tracial states of $A \rtimes \Gamma$ are determined by their restrictions to A, by Proposition 3.8. Distinct restrictions are therefore distinguished by a projection in $A \subseteq A \rtimes \Gamma$. □

Corollary 3.10. *Let (A, Γ) be a minimal C^*-dynamical system with WRP. Then every tracial state of $A \rtimes \Gamma$ is faithful.*

Proof. Let $\tau \in \mathrm{T}(A \rtimes \Gamma)$, and let $a \in A \rtimes \Gamma$ be a positive element with $\tau(a) = 0$.

By Proposition 3.8, $0 = \tau(a) = \tau(\mathbb{E}(a))$. The set $I = \{x \in A : \tau(xx^*) = 0\}$ is a closed two-sided ideal of A, since τ is tracial. Invariance of $\tau|_A$ makes I Γ-invariant. Since τ is nonzero, minimality gives $I = \{0\}$, so $\mathbb{E}(a) = 0$. Faithfulness of $\mathbb{E}$ now implies $a = 0$. □

Remark 3.11. Under the same hypotheses, $A \rtimes \Gamma$ is simple (Corollary 3.32).

3.2 Weak Rokhlin property and weak tracial approximation

For separable A, Proposition 3.6 shows that URP implies WRP. The approximation theorem in this section therefore also applies to systems with URP.

Let (A,Γ) be a C*-dynamical system with WRP, and let $\Gamma_1,...,\Gamma_S$ be the shapes of an exact tiling of Γ, with $e\in\Gamma_s$, $s=1,...,S$. There are $c_s\in A'\cap A_\omega$, $s=1,...,S$, such that

$$\gamma(c_s),\quad \gamma\in\Gamma_s,\ s=1,...,S,$$

are mutually orthogonal. For each s, index the matrix units of $\mathrm{M}_{|\Gamma_s|}$ by Γ_s. The elements

$$v_{s,\gamma}:=c_su_\gamma^*,\qquad \gamma\in\Gamma_s,\quad 1\le s\le S,$$

determine a *-homomorphism

$$\phi\colon \bigoplus_{s=1}^{S}\mathrm{M}_{|\Gamma_s|}(\mathrm{C}_0((0,1]))\longrightarrow(A\rtimes\Gamma)_\omega,\qquad \phi(t\otimes e_{e,\gamma}^{(s)})=v_{s,\gamma}.$$

Its image is the C*-subalgebra generated by these elements.

Projectivity of the cone gives a lift of ϕ. The following relative form of Loring's construction gives a lift whose diagonal belongs to the coefficient algebra.

Lemma 3.12 (Relative cone lifting)**.** *Let $\Gamma_0=\{\gamma_1=e,\gamma_2,\dots,\gamma_n\}$, let $c\in A_\omega$ be a positive contraction, and suppose that*

$$\gamma_i(c):=u_{\gamma_i}cu_{\gamma_i}^*,\qquad 1\le i\le n,$$

are mutually orthogonal. Let

$$\phi\colon \mathrm{M}_n(\mathrm{C}_0((0,1]))\longrightarrow(A\rtimes\Gamma)_\omega\quad \textit{be given by}\quad \phi(t\otimes e_{1i})=cu_{\gamma_i}^*.$$

Then ϕ has a lift

$$\widetilde{\phi}\colon \mathrm{M}_n(\mathrm{C}_0((0,1]))\longrightarrow\ell^\infty(A\rtimes\Gamma)$$

such that

$$\widetilde{\phi}(t\otimes e_{ii})\in\ell^\infty(A),\qquad 1\le i\le n.$$

Moreover, the elements $\widetilde{\phi}(t\otimes e_{i1})$ are generalized normalizers of $\ell^\infty(A)$ in $\ell^\infty(A\rtimes\Gamma)$.

Proof. Put $B=A\rtimes\Gamma$, $E=\ell^\infty(B)$, and $D=\ell^\infty(A)\subseteq E$, and denote the quotient map by $q\colon E\to B_\omega$. We use the same notation for the constant sequences $u_{\gamma_i}\in E$.

The standard lifting lemma for a finite family of orthogonal positive contractions gives mutually orthogonal positive contractions $k_1,\dots,k_n\in D$ such that

$$q(k_i)=\gamma_i(c),\qquad 1\le i\le n.$$

This lifting lemma follows by applying the usual finite orthogonalization perturbation to arbitrary positive lifts at each coordinate; the perturbation tends to zero because the products of distinct lifts tend to zero along ω. This is also the commutative instance of the cone-lifting argument in [44, Section 10].

Write $k_i = (k_{i,m})_m$ and choose numbers $\eta_m \searrow 0$. Set

$$h_{i,m} = (k_{i,m} - \eta_m)_+,$$

and let $g_{i,m} = f_m(k_{i,m})$, where $0 \le f_m \le 1$, $f_m = 0$ on $[0, \eta_m/2]$, and $f_m = 1$ on $[\eta_m, 1]$. With $h_i = (h_{i,m})_m$ and $g_i = (g_{i,m})_m$, we have

$$\begin{aligned} &q(h_i) = \gamma_i(c), \qquad g_i h_i = h_i, \\ &h_i h_j = g_i g_j = 0 \quad (i \neq j). \end{aligned} \tag{3.1}$$

Put $p_{n+1} = h_1^2$ and, recursively for $i = n, n-1, \ldots, 2$, define

$$b_i = u_{\gamma_i} p_{i+1}^{1/2}, \qquad a_i = g_i b_i, \qquad p_i = a_i^* a_i. \tag{3.2}$$

Since conjugation by u_{γ_i} preserves D, we have $p_i \in D$ and

$$0 \le p_2 \le p_3 \le \cdots \le p_{n+1} = h_1^2. \tag{3.3}$$

Moreover, an induction using (3.1) gives

$$q(a_i) = u_{\gamma_i} c, \qquad q(p_i) = c^2, \qquad 2 \le i \le n. \tag{3.4}$$

Indeed, if $q(p_{i+1}) = c^2$, then $q(b_i) = u_{\gamma_i} c$; and $u_{\gamma_i} c = \gamma_i(c) u_{\gamma_i}$, while $q(g_i)\gamma_i(c) = \gamma_i(c)$.

All the p_i belong to $\overline{h_1 D h_1}$. Since $g_j h_1 = 0$ for $j \ge 2$, it follows that

$$b_i g_j = 0 \quad (2 \le i, j \le n), \qquad a_i^* a_j = 0 \quad (i \neq j). \tag{3.5}$$

Each $a_i = g_i u_{\gamma_i} p_{i+1}^{1/2}$ is also a generalized normalizer of D in E:

$$a_i D a_i^* \subseteq D, \qquad a_i^* D a_i \subseteq D. \tag{3.6}$$

For $\lambda > 0$, define

$$z_i(\lambda) = a_i(\lambda + p_i)^{-1/2} p_2^{1/2}, \qquad 2 \le i \le n.$$

Because $p_2 \le p_i$, the elements $z_i(\lambda)$ converge in norm as $\lambda \searrow 0$; denote the limit by z_i. Indeed, the elements

$$e_\lambda = p_i(\lambda + p_i)^{-1}$$

form an approximate identity for $\overline{p_i D p_i}$, which contains p_2. Thus

$$z_i(\lambda)^* z_i(\lambda) = p_2^{1/2} e_\lambda p_2^{1/2} \longrightarrow p_2.$$

The same calculation with two parameters shows that $z_i(\lambda)$ is Cauchy: its squared difference is obtained by sandwiching

$$e_\lambda + e_\mu - 2(e_\lambda e_\mu)^{1/2} = (e_\lambda^{1/2} - e_\mu^{1/2})^2$$

between two copies of $p_2^{1/2}$. Consequently,

$$z_i^* z_i = p_2, \qquad 2 \le i \le n. \tag{3.7}$$

By (3.6), $z_i(\lambda) z_i(\lambda)^* \in D$, and hence

$$z_i z_i^* \in D, \qquad 2 \le i \le n. \tag{3.8}$$

The right-hand factor $(\lambda + p_i)^{-1/2} p_2^{1/2}$ belongs to $\overline{h_1 D h_1}$. Equations (3.5) therefore give, after taking limits,

$$z_i z_j = 0, \qquad z_i^* z_j = \delta_{ij} p_2, \qquad 2 \le i, j \le n. \tag{3.9}$$

Finally, (3.4) yields

$$q(z_i) = \lim_{\lambda \searrow 0} u_{\gamma_i} c (\lambda + c^2)^{-1/2} c = u_{\gamma_i} c.$$

The last convergence is in norm, since

$$\sup_{0 \le s \le 1} \left| \frac{s^2}{(\lambda + s^2)^{1/2}} - s \right| \longrightarrow 0.$$

Equation (3.7) also gives $\|z_i\|^2 = \|p_2\| \le 1$. Thus the relations in (3.9) are Loring's cone relations: the universal C*-algebra generated by contractions $x_2, \ldots, x_n$ satisfying

$$x_i x_j = 0, \qquad x_i^* x_j = \delta_{ij} x_2^* x_2$$

is $\mathrm{M}_n(\mathrm{C}_0((0,1])) $, with $x_i = t \otimes e_{i1}$; see [44, Sections 3 and 10]. Hence there is a *-homomorphism $\widetilde{\phi}$ with

$$\widetilde{\phi}(t \otimes e_{i1}) = z_i, \qquad 2 \le i \le n.$$

It lifts ϕ, because these generators generate the cone and $q(z_i) = \phi(t \otimes e_{i1})$. Furthermore,

$$\widetilde{\phi}(t \otimes e_{11}) = (z_2^* z_2)^{1/2} = p_2^{1/2} \in D$$

and, for $2 \le i \le n$,

$$\widetilde{\phi}(t \otimes e_{ii}) = (z_i z_i^*)^{1/2} \in D.$$

The generalized normalizer assertion follows from (3.6) and the norm-limit construction of the z_i. When $n = 1$, any lift of c by a positive contraction gives the assertion. □

Applying the construction simultaneously to all pairs (s, γ) gives the corresponding lift for the finite direct sum above. Indeed, the collar elements can be chosen mutually orthogonal across all summands, so the images of distinct summands are orthogonal.

Lemma 3.13 (Relative linking algebra)**.** *Let $A \subseteq B$ be C^*-algebras and let*

$$\psi : \mathrm{M}_n(\mathrm{C}_0((0,1])) \longrightarrow B$$

*be a *-homomorphism. Suppose that*

$$d_i = \psi(t \otimes e_{ii}) \in A, \qquad 1 \leq i \leq n,$$

and that $x_i = \psi(t \otimes e_{i1})$ is a generalized normalizer of A in B for every i. Put $H = \overline{d_1 A d_1}$. Then there is a C^*-subalgebra $C \subseteq B$ such that

$$C \cong \mathrm{M}_n(H), \qquad d_i \in C \cap A, \qquad C_{ii} = \overline{d_i A d_i}.$$

Proof. Write $x_i = w_i d_1$ for the polar decomposition in B^{**}, taking w_1 to be the support projection of d_1. The cone relations give

$$w_i^* w_j = \delta_{ij} s(d_1)$$

on the support of d_1, and $w_i d_1 w_i^* = d_i$. Define

$$\Theta : \mathrm{M}_n(H) \longrightarrow B^{**}, \qquad \Theta((a_{ij})) = \sum_{i,j=1}^{n} w_i a_{ij} w_j^*.$$

These relations show that Θ is an injective *-homomorphism.

To locate the image and its diagonal, note that $d_1 A d_1$ is dense in H and that, for $b \in A$,

$$w_i (d_1 b d_1) w_j^* = x_i b x_j^* \in B.$$

For $i = j$ this element belongs to A, since x_i is a generalized normalizer. Passing to norm limits proves $\Theta(\mathrm{M}_n(H)) \subseteq B$ and $\Theta(e_{ii} \otimes H) \subseteq A$. Every element of this diagonal corner is supported on $s(d_i)$, so the corner lies in $\overline{d_i A d_i}$. Conversely, for $b \in A$, the inverse-normalizer assumption gives $x_i^* b x_i \in A$. Its support lies in $s(d_1)$, hence $x_i^* b x_i \in H$, and

$$\Theta(e_{ii} \otimes x_i^* b x_i) = d_i b d_i.$$

Density proves the reverse inclusion. Taking $C = \Theta(\mathrm{M}_n(H))$ proves the lemma. □

Remark 3.14. The relative lift places the coordinate diagonals in A, and the linking lemma enlarges each lifted matrix cone to a matrix algebra over a hereditary C^*-subalgebra of A. The trace-kernel condition gives a uniform tracial estimate on the complement. A functional-calculus cutoff then gives the dimension-function estimate, as in Remark 2.6.

Theorem 3.15 (WRP approximation)**.** *Let (A, Γ) be a C^*-dynamical system with WRP, and put $B = A \rtimes_{\alpha,r} \Gamma$. Let $\mathcal{S}$ be the class of finite direct sums of matrix algebras over hereditary C^*-subalgebras of A. For every finite set $\mathcal{F} \subseteq B$ and every $\varepsilon > 0$, there are a C^*-subalgebra*

$$C \cong \bigoplus_{s=1}^{S} \mathrm{M}_{n_s}(H_s), \qquad H_s \subseteq A \text{ hereditary},$$

and a norm-one positive contraction $h \in A \cap C$ such that

1. $\|[h,x]\| < \varepsilon$ *for* $x \in \mathcal{F}$*;*

2. $\operatorname{dist}(hxh, C) < \varepsilon$ *for* $x \in \mathcal{F}$*, and the approximating element may be chosen in* $A \cap C$ *when* $x \in A$*;*

3. $\mathrm{d}_\tau(1-h) < \varepsilon$ *for every* $\tau \in \mathrm{T}(B)$*.*

Consequently,

$$B \in \mathrm{TA}_0(\mathcal{S}).$$

Proof. By Fourier approximation and a preliminary reduction of the tolerance, it is enough to treat elements au_k, where a belongs to a fixed finite subset $\mathcal{E}$ of the unit ball of A and k belongs to a finite symmetric set $N \subseteq \Gamma$ containing e.

Choose a large integer L and a small number $\beta > 0$. If Γ is infinite, the exact tiling theorem [19, Theorem 4.3 and Lemma 2.6] gives an exact tiling whose shapes $D_1, \dots, D_S$ satisfy

$$|D_s \setminus \operatorname{Int}_{N^L}(D_s)| < \beta |D_s|, \qquad 1 \le s \le S. \tag{3.10}$$

Moreover, the shapes may be required to have arbitrarily large cardinality. Indeed, enlarge the invariance set by a finite set K_0 of the desired cardinality and take the invariance tolerance below one. Then $\operatorname{Int}_{K_0}(D_s) \neq \varnothing$, so some translate of K_0 is contained in D_s. After right translating a shape and its centers, we may assume $e \in D_s$. If Γ is finite, take the one shape $D_1 = \Gamma$, for which the boundary in (3.10) is empty.

Apply WRP to this tiling. We obtain positive contractions $c_s \in A_\omega \cap A'$ such that the elements

$$\alpha_g(c_s), \qquad g \in D_s, \quad 1 \le s \le S,$$

are mutually orthogonal and

$$1 - \sum_{s=1}^{S} \sum_{g \in D_s} \alpha_g(c_s) \in J_{\mathcal{M}_1(A,\Gamma)}. \tag{3.11}$$

They determine the cone homomorphism

$$\phi : \bigoplus_{s=1}^{S} \mathrm{M}_{|D_s|}(\mathrm{C}_0((0,1])) \longrightarrow B_\omega, \qquad \phi(t \otimes e^{(s)}_{e,g}) = c_s u_g^*.$$

Let $\widetilde{\phi} = (\widetilde{\phi}_m)_m$ be the lift supplied by Lemma 3.12, where

$$\widetilde{\phi}_m : \bigoplus_{s=1}^{S} \mathrm{M}_{|D_s|}(\mathrm{C}_0((0,1])) \longrightarrow B$$

is the mth coordinate homomorphism. Put

$$d_{s,g,m} = \widetilde{\phi}_m(t \otimes e^{(s)}_{g,g}) \in A, \qquad q_m = \sum_{s,g} d_{s,g,m}.$$

The elements $d_{s,g,m}$ are exactly orthogonal and $0 \le q_m \le 1$. Equation (3.11) says that

$$\lim_{m\to\omega} \sup_{\rho\in\mathcal{M}_1(A,\Gamma)} \rho((1-q_m)^2) = 0. \tag{3.12}$$

For each s, m, apply Lemma 3.13 to the sth cone summand. We obtain mutually orthogonal C*-subalgebras

$$C_{s,m} \cong \mathrm{M}_{|D_s|}(H_{s,m}), \qquad H_{s,m} = \overline{d_{s,e,m} A d_{s,e,m}},$$

whose gth diagonal corner is $\overline{d_{s,g,m} A d_{s,g,m}}$. Put $C_m = \bigoplus_s C_{s,m}$, omitting zero summands. Also set

$$y_{s,g,m} = \widetilde{\phi}_m(t \otimes e^{(s)}_{g,e}) \in C_{s,m}.$$

In the quotient one has

$$[(d_{s,g,m})_m] = \alpha_g(c_s), \qquad [(y_{s,g,m})_m] = u_g c_s.$$

Fix an auxiliary tolerance $\eta > 0$. By (3.12), centrality, and the quotient identities, the following finite collection of conditions holds at some coordinate m:

$$\sup_{\rho\in\mathcal{M}_1(A,\Gamma)} \rho((1-q_m)^2) < \eta^2, \tag{3.13}$$

$$\sum_{s,g} \|[d_{s,g,m}, a]\| < \eta, \qquad a \in \mathcal{E}, \tag{3.14}$$

$$\sum_{\substack{s,g \\ kg\in D_s}} \|u_k d_{s,g,m} u_k^* - d_{s,kg,m}\| < \eta, \qquad k \in N, \tag{3.15}$$

and, whenever $g, j \in D_s$, $k \in N$, and $g = kj$,

$$\sum_{s,g,j,a,k} \left\| d_{s,g,m} a d_{s,g,m} u_k - y_{s,g,m} \alpha_{g^{-1}}(a) y^*_{s,j,m} \right\| < \eta. \tag{3.16}$$

The last sum ranges only over the indicated finite collection. Its summands tend to zero along ω, since their images in B_ω both equal

$$u_g c_s \alpha_{g^{-1}}(a) c_s u_j^*.$$

For $g \in D_s$, define

$$r_s(g) = \max\{0 \le r \le L : N^r g \subseteq D_s\}, \qquad \lambda_s(g) = r_s(g)/L,$$

and extend λ_s by zero off D_s. Symmetry of N gives

$$|\lambda_s(kg) - \lambda_s(g)| \le L^{-1}, \qquad k \in N. \tag{3.17}$$

Also, $\lambda_s(g) > 0$ implies $Ng \subseteq D_s$, and $\lambda_s = 1$ on $\mathrm{Int}_{N^L}(D_s)$. Define

$$h_0 = \sum_{s,g} \lambda_s(g) d_{s,g,m} \in A \cap C_m.$$

Equations (3.14), (3.15), and (3.17) give

$$\|[h_0, a]\| < \eta, \qquad \|[h_0, u_k]\| < L^{-1} + \eta \tag{3.18}$$

for $a \in \mathcal{E}$ and $k \in N$.

For the compression estimate, fix $k \in N$ and put $\mu_{s,g} = \lambda_s(k^{-1}g)$, again using the zero extension. Equation (3.15) gives

$$\alpha_k(h_0) \approx_\eta \sum_{s,g} \mu_{s,g} d_{s,g,m}.$$

Using exact orthogonality and (3.14), we get

$$\begin{aligned} h_0 a u_k h_0 &= h_0 a \alpha_k(h_0) u_k \\ &\approx_{2\eta} \sum_{s,g} \lambda_s(g) \mu_{s,g} d_{s,g,m} a d_{s,g,m} u_k. \end{aligned}$$

Writing $D_\mu = \sum_{s,g} \mu_{s,g} d_{s,g,m}$, the centrality error is

$$\begin{aligned} h_0 a D_\mu - \sum_{s,g} \lambda_s(g) \mu_{s,g} d_{s,g,m} a d_{s,g,m} \\ = \sum_{s,g} \mu_{s,g} (h_0 - \lambda_s(g) d_{s,g,m}) [a, d_{s,g,m}], \end{aligned}$$

whose norm is below η. The covariance error is also below η. When a coefficient in this sum is nonzero, $j = k^{-1}g \in D_s$, and (3.16) replaces the corresponding summand, within total error η, by

$$y_{s,g,m} \alpha_{g^{-1}}(a) y_{s,j,m}^* \in C_{s,m}.$$

Thus

$$\operatorname{dist}(h_0 a u_k h_0, C_m) < 3\eta. \tag{3.19}$$

For $k = e$, the displayed approximating element belongs to $A \cap C_m$.

It remains to control the defect. If $\tau \in \mathrm{T}(B)$, then $\rho = \tau|_A$ belongs to $\mathcal{M}_1(A, \Gamma)$, and (3.13) gives

$$\tau(1 - q_m) \le \tau((1 - q_m)^2)^{1/2} < \eta.$$

Under the identification $C_{s,m} \cong \mathrm{M}_{|D_s|}(H_{s,m})$, the elements $d_{s,g,m}$ are identical diagonal copies of the same positive element. Consequently, they have the same value under every trace on B. Therefore

$$\begin{aligned} \tau(1 - h_0) &= \tau(1 - q_m) + \sum_{s,g} (1 - \lambda_s(g)) \tau(d_{s,g,m}) \\ &< \eta + \beta \sum_{s,g} \tau(d_{s,g,m}) \le \eta + \beta. \end{aligned} \tag{3.20}$$

Finally, set

$$h = g_{1/2}(h_0) \in A \cap C_m.$$

Taking $\eta + \beta < 1/2$ in (3.20) shows that $\|h_0\| > 1/2$, and hence $\|h\| = 1$. Moreover,

$$\mathrm{d}_\tau(1 - h) = \tau(1_{[0,1/2)}(h_0)) \leq 2\tau(1 - h_0) < 2(\eta + \beta).$$

If $r(t) = g_{1/2}(t)/t$, extended continuously by zero at the origin, then

$$hxh = r(h_0)(h_0 x h_0) r(h_0).$$

Here $r(h_0) \in A \cap C_m$ and $\|r(h_0)\| \leq 2$. Consequently, (3.19) gives $\mathrm{dist}(hxh, C_m) < 12\eta$ for the prescribed Fourier terms. Continuous functional calculus, together with (3.18), gives the required commutator estimate for h after decreasing the preliminary tolerances. When $x \in A$, the approximating element remains in $A \cap C_m$. Choose L sufficiently large, then β sufficiently small, and finally η and the coordinate tolerances sufficiently small. The initial Fourier approximation gives the required estimates for $\mathcal{F}$. The trace space of B is nonempty because amenability supplies an invariant trace on A, whose composition with the canonical conditional expectation is a trace on B. □

Corollary 3.16. *Let (A, Γ) be a C^*-dynamical system with WRP, where A has real rank zero and Γ is a countable discrete amenable group. Put $B = A \rtimes_{\alpha,r} \Gamma$, and let*

$$\rho_B : \mathrm{K}_0(B) \longrightarrow \mathrm{Aff}(\mathrm{T}(B))$$

be the usual trace-pairing map. Then

$$\overline{\mathrm{span}_{\mathbb{R}}\, \rho_B(\mathrm{K}_0(B))} = \mathrm{Aff}(\mathrm{T}(B)).$$

If Γ is infinite, then, in addition, for every projection $p \in A$, every $r \in [0, 1]$, and every $\varepsilon > 0$, there is a projection $q \in A$ such that

$$\sup_{\tau \in \mathrm{T}(B)} |\tau(q) - r\tau(p)| < \varepsilon.$$

Consequently, in this case,

$$\overline{\rho_B(\mathrm{K}_0(B))} = \mathrm{Aff}(\mathrm{T}(B)).$$

Proof. Real rank zero implies that projections of A separate its tracial states. Corollary 3.9 therefore shows that the trace functions of projections in A, viewed as projections in B, separate the points of $\mathrm{T}(B)$. The real linear subspace

$$V = \mathrm{span}_{\mathbb{R}}\, \rho_B(\mathrm{K}_0(B)) \subseteq \mathrm{Aff}(\mathrm{T}(B))$$

contains the constant functions and separates points. The affine Stone–Weierstrass theorem, in the form [34, Corollary 7.4], gives $\overline{V} = \mathrm{Aff}(\mathrm{T}(B))$. Alternatively, if the closure were proper, the Hahn–Banach and Riesz representation theorems would give a nonzero functional on $\mathrm{Aff}(\mathrm{T}(B))$

represented by a signed measure annihilating V. Since $1 \in V$, its positive and negative parts have equal positive mass. Their normalized barycenters agree because V separates points. The signed measure therefore annihilates every continuous affine function, a contradiction.

Assume now that Γ is infinite. Fix a projection $p \in A$, $r \in [0,1]$, and $\varepsilon > 0$. Choose $N \in \mathbb{N}$ and $\eta, \delta > 0$ such that

$$N^{-1} + \eta + 2\delta < \varepsilon.$$

Use the proof of Theorem 3.15, with its preliminary tolerances sufficiently small, and require all the tiling shapes to have cardinality $n_s = |D_s| \geq N$. At the chosen coordinate that proof supplies mutually orthogonal algebras

$$C_s \cong \mathrm{M}_{n_s}(H_s), \qquad H_s = \overline{d_{s,e} A d_{s,e}},$$

and identical diagonal copies $d_{s,g}$, $g \in D_s$, of $d_{s,e}$, such that, with

$$z = \sum_{s=1}^{S} \sum_{g \in D_s} d_{s,g},$$

one has

$$\tau(1 - z) < \delta, \qquad \tau \in \mathrm{T}(B). \tag{3.21}$$

Every H_s has real rank zero and hence has an approximate identity of projections. Put $M = \sum_s n_s$. For each nonzero summand choose a projection $e_s \in H_s$ such that

$$\|d_{s,e} - e_s d_{s,e} e_s\| < \delta / M,$$

and denote its identical diagonal copies in C_s by $e_{s,g}$. Set

$$P = \sum_{s,g} e_{s,g}.$$

For every $\tau \in \mathrm{T}(B)$,

$$\tau(d_{s,g}) \leq \tau(e_{s,g} d_{s,g} e_{s,g}) + \delta/M \leq \tau(e_{s,g}) + \delta/M.$$

After summing and using (3.21), we obtain

$$\tau(1 - P) < 2\delta, \qquad \tau \in \mathrm{T}(B). \tag{3.22}$$

Fix identifications $C_s \cong \mathrm{M}_{n_s}(H_s)$, and denote the induced isomorphism from H_s onto the gth diagonal corner by $\theta_{s,g}$. Thus $e_{s,g} = \theta_{s,g}(e_s)$. Define

$$a_{s,g} = \theta_{s,g}^{-1}(e_{s,g} p e_{s,g}) \in (H_s)_+, \qquad a_s = \frac{1}{n_s} \sum_{g \in D_s} a_{s,g}.$$

These are positive contractions. Since H_s has real rank zero, there is a positive finite-spectrum contraction

$$b_s = \sum_{j=1}^{m_s} \lambda_{s,j} f_{s,j} \in H_s, \qquad 0 < \lambda_{s,j} \leq 1,$$

such that $\|b_s - a_s\| < \eta$, where the $f_{s,j}$ are mutually orthogonal projections. For each s, j, choose an integer $k_{s,j}$ such that

$$0 \leq k_{s,j} \leq n_s, \qquad |k_{s,j} - r\lambda_{s,j} n_s| \leq 1,$$

and choose a subset $E_{s,j} \subseteq D_s$ with $|E_{s,j}| = k_{s,j}$. Then

$$Q = \sum_{s=1}^{S} \sum_{j=1}^{m_s} \sum_{g \in E_{s,j}} \theta_{s,g}(f_{s,j})$$

is a projection in $A \cap C$: each summand belongs to the diagonal corner $\overline{d_{s,g} A d_{s,g}}$.

Let $\tau \in \mathrm{T}(B)$. Its restrictions to the diagonal corners of $C_s \cong \mathrm{M}_{n_s}(H_s)$ induce the same bounded trace σ_s on H_s. Since $P = \sum_{s,g} e_{s,g}$, traciality gives

$$\tau(pP) = \sum_{s,g} \tau(e_{s,g} p e_{s,g}) = \sum_s n_s \sigma_s(a_s). \tag{3.23}$$

Moreover,

$$\begin{aligned} |\tau(Q) - r\tau(pP)| &\leq \sum_{s,j} |k_{s,j} - r\lambda_{s,j} n_s| \sigma_s(f_{s,j}) + r \sum_s n_s |\sigma_s(b_s - a_s)| \\ &\leq \sum_{s,j} \sigma_s(f_{s,j}) + \eta \sum_s n_s \|\sigma_s\|. \end{aligned}$$

The sum of the norms of the restrictions of τ to the mutually orthogonal summands C_s is at most one. Under the matrix identification, the norm of $\tau|_{C_s}$ is $n_s \|\sigma_s\|$. Hence

$$\sum_s n_s \|\sigma_s\| \leq 1.$$

On the other hand,

$$F = \sum_{s,j} \sum_{g \in D_s} \theta_{s,g}(f_{s,j})$$

is a projection in B, and therefore

$$\sum_{s,j} \sigma_s(f_{s,j}) \leq \frac{1}{N} \sum_{s,j} n_s \sigma_s(f_{s,j}) = \frac{1}{N} \tau(F) \leq \frac{1}{N}.$$

It follows from (3.23) that

$$|\tau(Q) - r\tau(pP)| \leq N^{-1} + \eta. \tag{3.24}$$

Finally, $\tau(pP) = \tau(pPp)$, so

$$0 \leq \tau(p) - \tau(pP) = \tau(p(1-P)p) = \tau((1-P)p(1-P)) \leq \tau(1-P) < 2\delta.$$

Together with (3.24), this gives

$$\sup_{\tau \in \mathrm{T}(B)} |\tau(Q) - r\tau(p)| < N^{-1} + \eta + 2\delta < \varepsilon.$$

Set $G = \overline{\rho_B(\mathrm{K}_0(B))}$. The preceding construction shows that $r\widehat{p} \in G$ for every projection $p \in A$ and every $r \in [0,1]$, where $\widehat{p}(\tau) = \tau(p)$. Since G is a closed additive subgroup, it contains every real scalar multiple of $\widehat{p}$, and hence the real linear span of the trace functions of projections in A. This span contains the constants and separates the points of $\mathrm{T}(B)$. The affine Stone–Weierstrass theorem therefore gives $G = \mathrm{Aff}(\mathrm{T}(B))$. □

Remark 3.17. For infinite Γ, the relative averaging in the proof gives approximate divisibility of projection trace functions and hence density of the K_0-range. Approximation of constant functions alone would not give this conclusion. No non-elementarity assumption on A is required, but infinitude of Γ is essential: for $\Gamma = \{e\}$ and $A = \mathrm{M}_n$, WRP and real rank zero hold, whereas $B = \mathrm{M}_n$ and $\rho_B(\mathrm{K}_0(B)) = n^{-1}\mathbb{Z}$, which is not dense in $\mathrm{Aff}(\mathrm{T}(B)) \cong \mathbb{R}$.

If the hypotheses of Corollary 4.6 hold and A has real rank zero, then Theorem 3.15 gives $B \in \mathrm{TA}_0(\mathcal{S})$ with real-rank-zero models, while Corollary 4.6 gives strict comparison for B. Theorem 6.1 therefore gives real rank zero for B. This conclusion uses both approximation by real-rank-zero models and strict comparison.

3.3 Towers and freeness conditions

This section develops norm estimates for tower cut-downs and relates WRP to Kishimoto's and Elliott's freeness conditions. Equivariant lifting of central sequence towers from finite von Neumann algebras gives a further criterion for Elliott's condition.

3.3.1 Towers in a C*-dynamical system

A tower in a topological dynamical system consists of a base and finitely many pairwise disjoint translates. The following central-sequence formulation is its C*-algebraic analogue.

Definition 3.18. Let (A, Γ) be a unital C*-dynamical system. A *Rokhlin tower* is a pair (c, F), where $F \subseteq \Gamma$ is finite and nonempty and $c \in (A' \cap A_\omega)_+$ is a contraction of norm one, such that the elements

$$g(c), \qquad g \in F,$$

are mutually orthogonal. Its support is the positive contraction

$$h_{(c,F)} := \sum_{g \in F} g(c) \in A' \cap A_\omega.$$

Let D be a unital C*-algebra containing A_ω, let $\tau \in \mathrm{T}(D)$ satisfy $\mathrm{d}_\tau(c) > 0$, and let $\delta > 0$. The tower is (δ, τ)-*supported in* D if

$$\frac{\tau(c^l)}{\mathrm{d}_\tau(c)} > 1 - \delta, \qquad l \in \mathbb{N}. \tag{3.25}$$

We omit the ambient algebra when it is clear from the context.

A functional-calculus cut-down of the base gives (3.25) whenever the base has nonzero dimension function for the chosen trace.

Lemma 3.19. *Let (c, F) be a tower, let $D \supseteq A_\omega$ be unital, and let $\tau \in \mathrm{T}(D)$ satisfy $\mathrm{d}_\tau(c) > 0$. For every $\delta > 0$ there is a continuous function $f\colon [0,1] \to [0,1]$, with $f(0) = 0$ and $f(1) = 1$, such that $(f(c), F)$ is a (δ, τ)-supported tower.*

Proof. Let μ be the finite measure induced by τ on $\mathrm{C}^*(1, c)$. Then $\mathrm{d}_\tau(c) = \mu((0,1])$. Choose $\eta > 0$ such that

$$\mu([\eta, 1]) > (1 - \delta)\mu((0,1]),$$

and choose f to be strictly positive on $(0, \eta)$ and equal to one on $[\eta, 1]$. Thus $\mathrm{d}_\tau(f(c)) = \mathrm{d}_\tau(c)$ and, for every $l \in \mathbb{N}$,

$$\tau(f(c)^l) \geq \mu([\eta, 1]) > (1 - \delta)\mathrm{d}_\tau(f(c)).$$

Functional calculus preserves centrality and the orthogonality of the translates, so $(f(c), F)$ is again a tower. □

Remark 3.20. A tower base also defines partial-orbit averages. If $\tau \in \mathrm{T}(A_\omega)$ and $\tau(c^2) > 0$, then

$$\rho_{c,F,\tau}(a) = \frac{1}{|F|\tau(c^2)} \sum_{g \in F} \tau\big(c\, g^{-1}(a) c\big), \qquad a \in A,$$

is a tracial state. Moreover, for $k \in \Gamma$,

$$|\rho_{c,F,\tau}(k(a)) - \rho_{c,F,\tau}(a)| \leq \|a\| \frac{|F \triangle k^{-1}F|}{|F|}. \tag{3.26}$$

Indeed, each summand is tracial because c commutes with A, and (3.26) follows by reindexing the two finite sums. Consequently, weak* limit points of such states along a left Følner sequence are invariant.

3.3.1.1 Cut-downs by towers

The following finite-set cutoff gives almost invariant tower compressions. Recall that $\mathrm{Int}_K(F) = \{g \in F : Kg \subseteq F\}$.

Lemma 3.21. *Let $K \subseteq \Gamma$ be finite and symmetric with $e \in K$, let $m \in \mathbb{N}$, and let $F \subseteq \Gamma$ be finite. If $\mathrm{Int}_{K^m}(F)$ is nonempty, there is a function $w\colon \Gamma \to [0,1]$ such that*

1. $\mathrm{supp}(w) \subseteq \mathrm{Int}_K(F)$;

2. $w = 1$ *on* $\mathrm{Int}_{K^m}(F)$; *and*

3. $|w(kg) - w(g)| \leq 1/m$ *for* $k \in K$ *and* $g \in \Gamma$.

If (c, F) *is a tower and*

$$q_w := \sum_{g \in F} w(g) g(c),$$

then $\|q_w\| = 1$, $0 \leq q_w \leq h_{(c,F)}$, *and*

$$\|k(q_w) - q_w\| \leq 1/m, \qquad k \in K. \tag{3.27}$$

If $B = A \rtimes \Gamma$, $\tau \in \mathrm{T}(B_\omega)$, $0 < \delta < 1$, *the tower is* (δ, τ)*-supported, and* F *is* (K^m, δ)*-invariant, then*

$$\frac{\tau(q_w^l)}{\mathrm{d}_\tau(q_w)} > (1 - \delta)^2, \qquad l \in \mathbb{N}. \tag{3.28}$$

Proof. Give Γ the extended graph metric whose edges join g to kg for $k \in K$, and put $d_F(g) = \mathrm{dist}(g, \Gamma \setminus F)$, allowing the value $+\infty$. The function

$$w(g) = \max\left\{0, \min\left\{1, \frac{d_F(g) - 1}{m}\right\}\right\}$$

is understood to be one when $d_F(g) = +\infty$. It satisfies (1)–(3), since distance to a fixed set is one-Lipschitz. In particular, a nonzero coefficient of either q_w or $k(q_w)$ occurs at a level belonging to F. Orthogonality of the levels therefore gives (3.27), as well as $\|q_w\| = 1$ and $q_w \leq h_{(c,F)}$.

Every trace on B_ω is invariant under the automorphisms implemented by the constant unitaries u_g. Orthogonality therefore gives

$$\mathrm{d}_\tau(q_w) = |\,\mathrm{supp}(w)|\mathrm{d}_\tau(c)$$

and

$$\tau(q_w^l) = \sum_{g \in F} w(g)^l \tau(c^l) \geq |\,\mathrm{Int}_{K^m}(F)|\tau(c^l).$$

Since $|\,\mathrm{supp}(w)| \leq |F|$, the last two displays, (K^m, δ)-invariance, and (3.25) imply (3.28). □

Proposition 3.22. *Let* (A, Γ) *be a unital* C^**-dynamical system, where* Γ *is countable, discrete, and amenable, and put* $B = A \rtimes \Gamma$. *Assume* $\mathrm{T}(B) \neq \emptyset$ *and every tracial state of* B *is faithful. Let* $a \in B_+$ *have norm one. For every* $\varepsilon > 0$ *there are a finite set* $L \subseteq \Gamma$ *and* $\delta > 0$ *such that, whenever* (c, F) *is a tower,* F *is* (L, δ)*-invariant, and* (c, F) *is* (δ, τ)*-supported for some* $\tau \in \mathrm{T}(B_\omega)$, *one has*

$$\|h_{(c,F)} a h_{(c,F)}\| > 1 - \varepsilon.$$

Here and below, elements of B *are regarded as constant sequences in* B_ω.

Proof. Suppose otherwise. Then there is $\varepsilon_0 > 0$ for which the conclusion fails for every pair (L, δ). Choose finite symmetric sets

$$e \in K_1 \subseteq K_2 \subseteq \cdots \subseteq \Gamma, \qquad \bigcup_n K_n = \Gamma,$$

and numbers $\delta_n \downarrow 0$. Applying the assumed failure with $L = K_n^n$, choose a (K_n^n, δ_n)-invariant tower (c_n, F_n) and $\tau_n \in \mathrm{T}(B_\omega)$ such that the tower is (δ_n, τ_n)-supported and

$$\|h_{(c_n,F_n)} a h_{(c_n,F_n)}\| \leq 1 - \varepsilon_0. \tag{3.29}$$

After discarding finitely many terms, the interiors $\mathrm{Int}_{K_n^n}(F_n)$ are nonempty. Apply Lemma 3.21, with $K = K_n$ and $m = n$, and denote the resulting positive contraction by q_n.

The elements q_n commute with A. By (3.27), they asymptotically commute with every constant unitary u_g, and hence

$$\lim_{n\to\infty} \|[q_n, x]\| = 0, \qquad x \in B.$$

Equation (3.28) also gives, for every $l \in \mathbb{N}$,

$$\lim_{n\to\infty} \frac{\tau_n(q_n^l)}{\mathrm{d}_{\tau_n}(q_n)} = 1.$$

Lemma 2.12, applied to the constant embeddings $B \to B_\omega$, now yields

$$\limsup_{n\to\infty} \|q_n a q_n\| = 1. \tag{3.30}$$

The diagonal form of the cutoff gives a norm comparison with the tower support. In the bidual of the abelian C*-algebra generated by the orthogonal levels, there is a positive contraction z_n with value $w_n(g)$ on the g-summand, such that

$$q_n = z_n h_{(c_n,F_n)} = h_{(c_n,F_n)} z_n.$$

Consequently,

$$\|q_n a q_n\| = \|z_n h_{(c_n,F_n)} a h_{(c_n,F_n)} z_n\| \leq \|h_{(c_n,F_n)} a h_{(c_n,F_n)}\|,$$

contradicting (3.29) and (3.30). □

For positive elements of the coefficient algebra, the compression by a tower support is an orthogonal sum of level compressions. For general crossed-product elements, off-diagonal matrix coefficients between levels may contribute to the norm.

Corollary 3.23. *Under the hypotheses of Proposition 3.22, let $a \in A_+$ have norm one. For every $\varepsilon > 0$ there are a finite set $L \subseteq \Gamma$ and $\delta > 0$ such that, whenever (c, F) is a (δ, τ)-supported tower for some $\tau \in \mathrm{T}((A \rtimes \Gamma)_\omega)$ and F is (L, δ)-invariant, there is $g \in F$ such that*

$$\|g(c) a g(c)\| > 1 - \varepsilon.$$

Proof. Put $h = h_{(c,F)}$. Since every level $g(c)$ commutes with a and the levels are mutually orthogonal,

$$hah = \sum_{g \in F} g(c)ag(c)$$

is an orthogonal sum of positive elements. Hence

$$\|hah\| = \max_{g \in F} \|g(c)ag(c)\|,$$

and the result follows from Proposition 3.22. □

3.3.1.2 Order zero maps and towers

Lemma 3.24. *Let (c, F) be a tower and enumerate $F = \{g_1, \ldots, g_n\}$. There is a c.p.c. order zero map*

$$\phi \colon \mathrm{M}_n(\mathbb{C}) \longrightarrow (A \rtimes \Gamma)_\omega$$

such that

$$\phi(e_{ij}) = u_{g_i} c u_{g_j}^*, \qquad 1 \leq i, j \leq n, \tag{3.31}$$

and $\phi(1) = h_{(c,F)}$.

Proof. On elementary tensors define

$$\Theta(f \otimes e_{ij}) = u_{g_i} f(c) u_{g_j}^*, \qquad f \in \mathrm{C}_0((0,1]).$$

If $j \neq k$, tower orthogonality gives

$$f(c)u_{g_j}^* u_{g_k} \widetilde{f}(c) = 0 \qquad (f, \widetilde{f} \in \mathrm{C}_0((0,1])).$$

The usual matrix multiplication therefore shows that Θ extends to a *-homomorphism

$$\mathrm{C}_0((0,1]) \otimes \mathrm{M}_n(\mathbb{C}) \longrightarrow (A \rtimes \Gamma)_\omega.$$

If $\iota(t) = t$ is the canonical positive contraction in $\mathrm{C}_0((0,1])$, then $\phi(x) = \Theta(\iota \otimes x)$ is c.p.c. and order zero, and (3.31) follows directly. □

3.3.2 Kishimoto's condition, Elliott's condition, and WRP

Kishimoto-type conditions are noncommutative freeness conditions. We use the following simultaneous finite-set form to cut down Fourier polynomials.

Definition 3.25. Let (A, Γ) be a C*-dynamical system, where $A \neq 0$ and Γ is discrete. We say that (A, Γ) satisfies *Kishimoto's condition* if, for every $a \in A_+$ with $\|a\| = 1$, finite sets $K \subseteq \Gamma$ and $\mathcal{F} \subseteq A$, and $\varepsilon > 0$, there is a norm-one positive contraction $c \in A$ such that

1. $\|cac\| > 1 - \varepsilon$; and
2. $\|cb\gamma(c)\| < \varepsilon$ for every $\gamma \in K \setminus \{e\}$ and $b \in \mathcal{F}$.

Remark 3.26. Definition 3.25 is a simultaneous, norm-preserving version of the condition. The literature also uses "Kishimoto's condition" for the pointwise hereditary-subalgebra condition on each nontrivial automorphism. No equivalence between those formulations is needed here.

Elliott's condition abstracts the conclusion of [20, Lemma 3.3].

Definition 3.27. A C*-dynamical system (A, Γ) satisfies *Elliott's condition* if, for every $a \in A_+$, finite sets $\mathcal{F} \subseteq A$ and $K \subseteq \Gamma \setminus \{e\}$, and $\varepsilon > 0$, there is a norm-one positive contraction $c \in A$ such that

1. $\|cac\| > \|a\| - \varepsilon$;
2. $\|cb - bc\| < \varepsilon$ for every $b \in \mathcal{F}$; and
3. $\|c\gamma(c)\| < \varepsilon$ for every $\gamma \in K$.

Lemma 3.28. *Elliott's condition implies Kishimoto's condition.*

Proof. Fix the data in Definition 3.25, and put $M = \max(\{\|b\| : b \in \mathcal{F}\} \cup \{0\})$. Apply Elliott's condition with a tolerance

$$0 < \eta < \frac{\varepsilon}{1 + M}.$$

For $b \in \mathcal{F}$ and $\gamma \in K \setminus \{e\}$, the resulting positive contraction c satisfies

$$\|cb\gamma(c)\| \leq \|(cb - bc)\gamma(c)\| + \|bc\gamma(c)\| < (1 + \|b\|)\eta \leq (1 + M)\eta < \varepsilon.$$

After also requiring $\eta < \varepsilon$, the compression estimate is the one in Definition 3.25. □

For separable coefficient algebras, Elliott's condition is precisely the existence of finite central-sequence towers that preserve the norm of a prescribed positive element.

Proposition 3.29. *Let A be a nonzero, separable, unital* C*-*algebra. A* C*-*dynamical system (A, Γ) satisfies Elliott's condition if and only if, for every finite set $F \subseteq \Gamma$ containing e and every $a \in A_+$, there is a tower (c, F) such that*

$$\|cac\| = \|a\|.$$

Proof. Assume first that Elliott's condition holds. Let $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots$ be finite sets whose union is dense in A. For each n, apply the condition with $\mathcal{F}_n$, tolerance $1/n$, and

$$K = F^{-1}F \setminus \{e\}.$$

Denote the resulting norm-one positive contraction by c_n. Then $c = [(c_n)] \in A' \cap A_\omega$ is a norm-one positive contraction. If $g, h \in F$ are distinct, then

$$g(c)h(c) = g\big(c\,(g^{-1}h)(c)\big) = 0,$$

so (c, F) is a tower. The compression estimates and the reverse inequality $\|c_n a c_n\| \leq \|a\|$ give $\|cac\| = \|a\|$.

Conversely, fix a, $\mathcal{F}$, K, and ε as in Definition 3.27, and apply the stated tower property to $F = K \cup \{e\}$. Represent the tower base by norm-one positive contractions (c_n) in A. Centrality, tower orthogonality, and $\|cac\| = \|a\|$ imply that the three finite-stage estimates hold on a set of indices belonging to ω. Any coordinate in this set verifies Elliott's condition. □

Lemma 3.30. *Let (A, Γ) be a minimal C^*-dynamical system with WRP, where A is unital and separable and Γ is countable, discrete, and amenable. If $\mathrm{T}(A) \neq \emptyset$, then (A, Γ) satisfies Elliott's condition and hence Kishimoto's condition.*

Proof. Put $B = A \rtimes_{\mathrm{r}} \Gamma$. By amenability, the action of Γ on the nonempty compact convex set $\mathrm{T}(A)$ has a fixed point. Its composition with the canonical conditional expectation is a tracial state on B, so $\mathrm{T}(B) \neq \emptyset$. Corollary 3.10 shows that every tracial state of B is faithful.

We first prove Elliott's condition for a norm-one element $a \in A_+$. Fix finite sets $K \subseteq \Gamma \setminus \{e\}$ and $\mathcal{F} \subseteq A$, and fix $\varepsilon > 0$. Enlarge $K \cup \{e\}$ to a finite symmetric set K_0. Let (L, δ) be supplied by Corollary 3.23 for a and $\varepsilon/2$, and enlarge L so that $e \in L$. Choose $0 < \eta < 1$ so that $\eta/(1-\eta) < \delta$. By the exact tiling theorem [19, Theorem 4.3 and Lemma 2.6], there is an exact tiling whose shapes $D_1, \ldots, D_S$ are $(K_0 L, \eta)$-invariant. An expansion-invariance tolerance below $\eta/|K_0 L|$ gives this interior estimate. For finite Γ, take the tiling with a single shape $D_1 = \Gamma$. Set

$$F_s = \mathrm{Int}_{K_0}(D_s), \qquad 1 \leq s \leq S.$$

Since

$$\mathrm{Int}_{K_0 L}(D_s) \subseteq \mathrm{Int}_L(F_s) \quad \text{and} \quad |F_s| \geq |\,\mathrm{Int}_{K_0 L}(D_s)| > (1-\eta)|D_s|,$$

we have

$$\frac{|F_s \setminus \mathrm{Int}_L(F_s)|}{|F_s|} < \frac{\eta}{1-\eta} < \delta.$$

Thus every F_s is (L, δ)-invariant.

Apply WRP to this tiling, and let $c_s \in A' \cap A_\omega$ be the resulting bases. Fix $\sigma \in \mathrm{T}(B)$ and let $\tau = \sigma_\omega$ be the induced trace on B_ω. Put

$$h = \sum_{s=1}^{S} \sum_{g \in D_s} g(c_s).$$

The WRP defect condition gives $\tau((1-h)^2) = 0$. By Cauchy–Schwarz, $\tau(1-h) = 0$, and hence

$$1 = \tau(h) = \sum_{s=1}^{S} |D_s| \tau(c_s).$$

Choose s with $\tau(c_s) > 0$. In particular, $\mathrm{d}_\tau(c_s) > 0$ and $c_s \neq 0$. Apply Lemma 3.19 to $c_s/\|c_s\|$. It gives a function f such that, with

$$q = f(c_s/\|c_s\|),$$

the tower (q, D_s) is (δ, τ)-supported. Its restriction (q, F_s) has the same support property. Corollary 3.23 therefore gives $g_0 \in F_s$ such that

$$\|g_0(q) a g_0(q)\| > 1 - \varepsilon/2.$$

Set $d = g_0(q)$. Then d is a norm-one positive contraction in $A' \cap A_\omega$. For $k \in K$, the definition of F_s gives $kg_0 \in D_s$; since $k \neq e$, tower orthogonality gives

$$d\, k(d) = g_0(q)\, (kg_0)(q) = 0.$$

Represent d by norm-one positive contractions (d_n) in A. Centrality, the preceding orthogonality relations, and the compression estimate hold simultaneously on a set of indices belonging to ω. Thus some d_n satisfies the three estimates in Elliott's condition with tolerance ε.

For a nonzero $a \in A_+$ of arbitrary norm, apply the norm-one case to $a/\|a\|$ with a smaller tolerance and rescale the compression estimate. When $a = 0$, apply the construction to 1_A; the compression estimate for a is automatic. This proves Elliott's condition; Lemma 3.28 gives the final assertion. □

Lemma 3.31. *Let (A, Γ) be a C*-dynamical system satisfying Kishimoto's condition. For every nonzero $x \in (A \rtimes_{\mathrm{r}} \Gamma)_+$, there is a nonzero element $h \in A_+$ such that*

$$h \precsim x \quad \text{in } A \rtimes_{\mathrm{r}} \Gamma.$$

Proof. Put $B = A \rtimes_{\mathrm{r}} \Gamma$, and let $\mathbb{E}\colon B \to A$ be the canonical conditional expectation. Its faithfulness gives $\mathbb{E}(x) \neq 0$. Multiplying x by a positive scalar does not change its Cuntz class, so we may assume

$$\|\mathbb{E}(x)\| = 1.$$

Fix $0 < \eta < 1/3$. Density of the algebraic crossed product gives a Fourier polynomial

$$y = \sum_{\gamma \in F} a_\gamma u_\gamma, \qquad e \in F,$$

such that

$$\|x - y\| < \eta \quad \text{and} \quad a_e = \mathbb{E}(x). \tag{3.32}$$

Indeed, after choosing an arbitrary sufficiently close Fourier polynomial, replace its identity coefficient by $\mathbb{E}(x)$; the contractivity of $\mathbb{E}$ controls the additional error.

Set $N = \max\{1, |F \setminus \{e\}|\}$. Apply Kishimoto's condition to a_e, the group set $F \setminus \{e\}$, the coefficient set $\{a_\gamma : \gamma \in F \setminus \{e\}\}$, and tolerance η/N. We obtain a norm-one positive contraction $c \in A$ such that

$$\|c a_e c\| > 1 - \eta \quad \text{and} \quad \|c a_\gamma \gamma(c)\| < \frac{\eta}{N}, \qquad \gamma \in F \setminus \{e\}.$$

Using $u_\gamma c = \gamma(c)u_\gamma$ and (3.32), we get

$$\begin{aligned}\|cxc - ca_e c\| &\leq \|c(x-y)c\| + \sum_{\gamma \in F\setminus\{e\}} \|ca_\gamma \gamma(c)\| \\ &< 2\eta.\end{aligned}$$

The standard perturbation lemma for positive elements gives

$$h := (ca_e c - 2\eta)_+ \precsim cxc.$$

Moreover, $h \neq 0$ because $\|ca_e c\| > 1 - \eta > 2\eta$. Finally, $cxc \precsim x$: indeed, $cxc = (x^{1/2}c)^*(x^{1/2}c)$ is Cuntz equivalent to $x^{1/2}c^2x^{1/2} \in \overline{xBx}$. Consequently $0 \neq h \in A_+$ and $h \precsim x$, as required. □

Corollary 3.32. *Let (A, Γ) be a minimal C^*-dynamical system with WRP, where A is unital and separable, $\mathrm{T}(A) \neq \emptyset$, and Γ is countable, discrete, and amenable. Then $A \rtimes_{\mathrm{r}} \Gamma$ is simple. Since Γ is amenable, the same statement holds without distinguishing the full and reduced crossed products.*

Proof. Put $B = A \rtimes_{\mathrm{r}} \Gamma$, and let $I \triangleleft B$ be a nonzero ideal. Choose $0 \neq x \in I_+$. Lemma 3.30 and Lemma 3.31 give a nonzero $h \in A_+$ with $h \precsim_B x$. Since ideals are closed under Cuntz subequivalence, $h \in I$. Thus $I \cap A$ is a nonzero Γ-invariant ideal of A. Minimality gives $I \cap A = A$, so $1_A \in I$ and hence $I = B$. □

3.3.3 Lifting towers and Elliott's condition

The following equivariant form of the central-sequence lifting argument of Kirchberg and Rørdam lifts orthogonality in a quotient to exact orthogonality in the norm ultrapower while preserving centrality.

Lemma 3.33 (Equivariant orthogonal lifting)**.** *Let $A \subseteq N$ be a unital inclusion of C^*-algebras, with A separable, and let $\Gamma \curvearrowright N$ be an action which leaves A invariant. Let $J \triangleleft \ell^\infty(N)$ be a Γ-invariant closed ideal, and put*

$$N^\omega = \ell^\infty(N)/J.$$

Suppose that

1. *$c_\omega(A) \subseteq J$ and $J \cap N = \{0\}$, where N is viewed as the algebra of constant sequences;*
2. *the homomorphism induced by the inclusion,*

$$\Phi\colon A_\omega = \ell^\infty(A)/c_\omega(A) \longrightarrow N^\omega,$$

   is surjective; and

3. *there is a norm-continuous function $\theta\colon N \to [0, \infty)$, bounded on bounded subsets, such that*

$$J = \{(x_n) \in \ell^\infty(N) : \lim_{n\to\omega} \theta(x_n) = 0\}.$$

Then the following statements hold.

(i) Let $p_1, \dots, p_S$ be positive contractions in $N' \cap N^\omega$, and let $E_1, \dots, E_S$ be finite subsets of Γ. If

$$\{\gamma(p_s) : 1 \le s \le S,\ \gamma \in E_s\}$$

is pairwise orthogonal, then there are positive contractions $c_1, \dots, c_S \in A' \cap A_\omega$ such that

$$\Phi(c_s) = p_s \quad (1 \le s \le S)$$

and

$$\{\gamma(c_s) : 1 \le s \le S,\ \gamma \in E_s\}$$

is pairwise orthogonal. In particular,

$$N' \cap N^\omega \subseteq \Phi(A' \cap A_\omega).$$

(ii) Let $p \in N' \cap N^\omega$ be a positive contraction and let $K \subseteq \Gamma$ be finite. If

$$p\gamma(p) = 0 \quad (\gamma \in K),$$

then there is a positive contraction $c \in A' \cap A_\omega$ such that

$$\Phi(c) = p \quad \textit{and} \quad c\gamma(c) = 0 \quad (\gamma \in K).$$

Proof. Write

$$q_J \colon \ell^\infty(N) \longrightarrow N^\omega \quad \text{and} \quad q_A \colon \ell^\infty(A) \longrightarrow A_\omega$$

for the quotient maps. We prove (i). By surjectivity and positive lifting, for each s there is a sequence $b^{(s)} = (b_n^{(s)})$ of positive contractions in A such that $q_J(b^{(s)}) = p_s$.

Let H be the subgroup of Γ generated by $\bigcup_s E_s$. It is countable. The C*-algebra

$$C = \mathrm{C}^*\big(A, \gamma(b^{(s)}) : \gamma \in H,\ 1 \le s \le S\big) \subseteq \ell^\infty(A)$$

is separable and H-invariant. Set $I = C \cap J$. Then I is an H-invariant ideal of C. Centrality of the p_s gives

$$[b^{(s)}, a] \in I \quad (a \in A,\ 1 \le s \le S). \tag{3.33}$$

and the assumed orthogonality gives

$$\gamma(b^{(s)})\eta(b^{(t)}) \in I \tag{3.34}$$

whenever $(s, \gamma) \neq (t, \eta)$, with $\gamma \in E_s$ and $\eta \in E_t$.

Choose a quasicentral approximate identity $(e^{(m)})_{m=1}^\infty$ for I in C, consisting of positive contractions. Since I and C are H-invariant, $(\gamma(e^{(m)}))_m$ is again a quasicentral approximate identity for I in C for every $\gamma \in H$. Consequently,

$$\lim_{m\to\infty} \big\|[(1 - e^{(m)})b^{(s)}(1 - e^{(m)}), a]\big\| = 0 \quad (a \in A). \tag{3.35}$$

Indeed, modulo terms containing $[e^{(m)}, a]$, the commutator in (3.35) is

$$(1 - e^{(m)})[b^{(s)}, a](1 - e^{(m)}),$$

which tends to zero by (3.33). Similarly, quasicentrality allows the cutoff factors to be moved past the finitely many elements occurring in (3.34); since every translate of $(e^{(m)})_m$ is an approximate identity for I, we obtain

$$\lim_{m\to\infty} \big\|\gamma((1 - e^{(m)})b^{(s)}(1 - e^{(m)}))\eta((1 - e^{(m)})b^{(t)}(1 - e^{(m)}))\big\| = 0 \tag{3.36}$$

for all the pairs in (3.34).

Apply Kirchberg's epsilon test [40, Lemma 3.1] as follows. Choose a dense sequence $(a_j)_{j=1}^{\infty}$ in the unit ball of A. At the nth coordinate, the variable ranges over the positive contractions of A, with test functions

$$x \longmapsto \theta(x),$$
$$x \longmapsto \big\|[(1 - x)b_n^{(s)}(1 - x), a_j]\big\|,$$

and, for every pair occurring in (3.34), the function

$$x \longmapsto \big\|\gamma((1 - x)b_n^{(s)}(1 - x))\eta((1 - x)b_n^{(t)}(1 - x))\big\|.$$

For a fixed m, use the nth coordinate of $e^{(m)}$ as the test variable. Since $e^{(m)} \in I \subseteq J$, the ultralimit of the first function is zero. Equations (3.35) and (3.36) show that, as $m \to \infty$, the ultralimits of every prescribed finite collection of the remaining functions tend to zero. The epsilon test gives a sequence $d = (d_n)$ of positive contractions in A such that

$$\lim_{n\to\omega} \theta(d_n) = 0$$

and all the commutator and orthogonality norms above have ultralimit zero.

Thus $d \in J$. Define

$$c_s = q_A\big((1 - d)b^{(s)}(1 - d)\big), \qquad 1 \le s \le S.$$

These are positive contractions. The centrality tests imply that $c_s \in A' \cap A_\omega$, while $q_J(d) = 0$ gives $\Phi(c_s) = p_s$. The remaining tests give the required pairwise orthogonality of the translates. The final inclusion in (i) follows by scaling positive elements and then writing an arbitrary element of $N' \cap N^\omega$ as a linear combination of four positive elements of that C*-algebra.

For (ii), use the same construction with one lift b of p and with only the orthogonality tests indexed by the pairs (e, γ), $\gamma \in K$. The conclusion of those tests is $c\gamma(c) = 0$ for every $\gamma \in K$, as required. □

Corollary 3.34. *Let N be a finite von Neumann algebra with faithful normal tracial state ρ, and let $A \subseteq N$ be a unital separable weakly dense* C*-*subalgebra. Suppose that $\Gamma \curvearrowright N$ is a ρ-preserving action which leaves A invariant. Form the tracial ultrapower*

$$N^\omega = \ell^\infty(N)/J_\rho, \qquad J_\rho = \{(x_n) \in \ell^\infty(N) : \lim_{n\to\omega} \|x_n\|_{2,\rho} = 0\}.$$

If $p \in N' \cap N^\omega$ is a positive contraction and $K \subseteq \Gamma$ is finite with

$$p\gamma(p) = 0 \quad (\gamma \in K),$$

then there is a positive contraction $c \in A' \cap A_\omega$ such that

$$\Phi(c) = p \quad \textit{and} \quad c\gamma(c) = 0 \quad (\gamma \in K),$$

where $\Phi\colon A_\omega \to N^\omega$ is induced by the inclusion.

Proof. The ideal J_ρ is Γ-invariant, contains $c_\omega(A)$, and has trivial intersection with the constant copy of N. Moreover, $\theta(x) = \|x\|_{2,\rho}$ satisfies condition (3) of Lemma 3.33. Finally, coordinatewise Kaplansky density gives the surjectivity of Φ: a bounded sequence in N can be approximated, coordinate by coordinate and without increasing its norm bound, by a sequence in A in the $\|\cdot\|_{2,\rho}$-norm. Lemma 3.33(ii) now applies. □

Ocneanu's central-freeness and paving arguments give the following local finite-set result.

Lemma 3.35 (Ocneanu's small-tower lemma)**.** *Let Γ be a countable discrete group acting on a finite von Neumann algebra N with separable predual. Suppose that ρ is a faithful normal Γ-invariant tracial state and that the action is centrally free. Given a finite set $K \subseteq \Gamma \setminus \{e\}$ and $\delta > 0$, there are projections $p_0, p_1, \dots, p_m \in N' \cap N^\omega$ such that*

$$\sum_{j=0}^{m} p_j = 1, \qquad \rho_\omega(p_0) < \delta,$$

and

$$p_j\gamma(p_j) = 0 \quad (1 \le j \le m,\ \gamma \in K).$$

Proof. If K is empty, take $p_0 = 0$ and $p_1 = 1$. Otherwise replace K by $K \cup K^{-1}$. The lemma in [50, Section 5.6] shows that the induced action on $R = N' \cap N^\omega$ is properly outer on $S' \cap R$ for every countably generated invariant von Neumann subalgebra $S \subseteq R$. Apply the local small-tower construction of [50, Section 6.3] with remainder tolerance $\delta/2$; see also the monograph [51]. This gives the stated strict bound and exact orthogonality. The local construction is valid for every countable discrete group under the stated hypotheses. Section 6.2 uses Popa's paving lemma in the finite reduced von Neumann crossed product, and Section 6.3 uses maximality and countable reindexing to obtain exact orthogonality from arbitrarily small overlaps. These steps do not require the amenability assumption of the full Rokhlin theorem. □

Theorem 3.36. *Let (A, Γ) be a C^*-dynamical system, where A is a nonzero separable unital C^*-algebra and Γ is a discrete group. Suppose that A has a faithful Γ-invariant tracial state τ. If the induced action on*

$$N = \pi_\tau(A)''$$

is centrally free, then (A, Γ) satisfies Elliott's condition.

Proof. Identify A with its image in N, and denote by ρ the faithful normal extension of τ to N. Fix $a \in A_+$, finite sets $\mathcal{F} \subseteq A$ and $K \subseteq \Gamma \setminus \{e\}$, and $\varepsilon > 0$. The subgroup $H = \langle K \rangle$ is countable, and its induced action on N retains central freeness and the invariant faithful trace.

First suppose that $a \neq 0$, and put

$$r = \max\{\|a\| - \varepsilon, 0\}.$$

Choose $f \in \mathrm{C}([0, \|a\|])_+$ such that

$$0 \leq f \leq 1, \qquad f|_{[0,r]} = 0, \qquad f(\|a\|) = 1,$$

and set $b = f(a)$. Then $b \neq 0$, and faithfulness gives $\rho(b) > 0$. If $a = 0$, instead put $b = 1_A$. In either case apply Lemma 3.35 with

$$\delta = \frac{\rho(b)}{2}.$$

We obtain a partition of unity $p_0, p_1, \ldots, p_m \in N' \cap N^\omega$ with $\rho_\omega(p_0) < \rho(b)/2$ and $p_j \gamma(p_j) = 0$ for $j \geq 1$ and $\gamma \in K$.

The projections commute with the constant copy of N, and hence with b. Therefore

$$\rho(b) = \sum_{j=0}^{m} \rho_\omega(p_j b p_j).$$

Since $0 \leq b \leq 1$,

$$\rho_\omega(p_0 b p_0) \leq \rho_\omega(p_0) < \frac{\rho(b)}{2}.$$

It follows that $p_j b p_j \neq 0$ for some $j \geq 1$; fix such a j and write $p = p_j$. If $a \neq 0$, functional calculus in the corner $pN^\omega p$ gives

$$pf(a)p = f(pap) \neq 0.$$

As f vanishes on $[0, r]$, this implies

$$\|pap\| > r \geq \|a\| - \varepsilon. \tag{3.37}$$

If $a = 0$, the required compression inequality holds because $0 > -\varepsilon$. In either case, p is a nonzero projection.

Apply Corollary 3.34 to p and K. It gives a positive contraction $c \in A' \cap A_\omega$ satisfying

$$\Phi(c) = p, \qquad c\gamma(c) = 0 \quad (\gamma \in K).$$

Since Φ is contractive and p is a nonzero projection, $\|c\| = 1$. Moreover,

$$\Phi(cac) = pap,$$

and hence, by (3.37),

$$\|cac\| \geq \|pap\| > \|a\| - \varepsilon.$$

Choose a representing sequence (c_n) of norm-one positive contractions for c. Such a representative exists because any representative by positive contractions has $\lim_{n\to\omega}\|c_n\| = 1$ and may be normalized without changing its class. Centrality, the relations $c\gamma(c) = 0$, and the last compression estimate imply that the sets of indices on which

$$\|c_n x - x c_n\| < \varepsilon \quad (x \in \mathcal{F}), \qquad \|c_n \gamma(c_n)\| < \varepsilon \quad (\gamma \in K),$$

and

$$\|c_n a c_n\| > \|a\| - \varepsilon$$

all belong to ω. Their finite intersection is nonempty, and any coordinate in it verifies Elliott's condition. □

Remark 3.37. The lifting lemmas and the local small-tower argument do not require amenability. Since Elliott's condition concerns only finitely many group elements at a time, Theorem 3.36 applies to arbitrary discrete groups. The prescribed-tiling WRP and the large-tower approximation arguments above use amenability.

Chapter 4

Comparison of positive elements

The tower constructions of Chapter 3 give local models for crossed products with uniformly small tracial complements. Comparison allows these complements to be absorbed into prescribed positive elements. This chapter establishes comparison results for tracially approximated C*-algebras, introduces comparison of coefficient elements through property (COS), and defines the orbit radius of comparison. The automatic COS criterion permits arbitrary discrete groups; the Rokhlin and orbit-radius arguments use amenability. Comparison of finite orbit matrices then passes through Rokhlin towers to the crossed product. The final section computes orbit radii and gives examples of property (COS).

4.1 Radius of comparison and strict comparison under weak tracial approximation

Lemma 4.1. *Let $a, b \in A_+$. Regarding a and b as constant sequences, one has $a \precsim b$ in A if and only if $a \precsim b$ in A_ω.*

Proof. The forward implication follows from the inclusion of A in A_ω. For the converse, suppose that $a \precsim b$ in A_ω, and fix $\varepsilon > 0$. There is $x \in A_\omega$ such that $\|a - xbx^*\| < \varepsilon$. Choose a bounded lift $(x_k)_k$ of x. On an ω-large set of indices,

$$\|a - x_k b x_k^*\| < \varepsilon.$$

For each such index, the norm perturbation lemma for Cuntz comparison gives $(a-\varepsilon)_+ \precsim x_k b x_k^* \precsim b$ in A. Since ε is arbitrary, $a \precsim b$ in A. □

Lemma 4.2. *Let A be a C*-algebra, and let $a = [(a_k)]$, $b = [(b_k)]$, and $c = [(c_k)]$ be positive elements of A_ω, represented by bounded sequences of positive elements of A. Suppose that there is an ω-large set $U \subseteq \mathbb{N}$ such that, for every $k \in U$,*

1. *$a_k \precsim b_k$, and*

2. $b_k c_k = b_k$.

Then $a \precsim c$ *in* A_ω.

Proof. Fix $\varepsilon > 0$. For $k \in U$, the usual cut-down form of Cuntz subequivalence gives $x_k \in A$ such that

$$(a_k - \varepsilon)_+ = x_k b_k x_k^*.$$

Put $s_k = x_k b_k^{1/2}$ for $k \in U$ and $s_k = 0$ otherwise. The sequence $(s_k)_k$ is bounded, since $s_k s_k^* = (a_k - \varepsilon)_+$ on U. As b_k and c_k are positive, $b_k c_k = b_k$ also gives $c_k b_k = b_k$, and hence $b_k^{1/2} c_k b_k^{1/2} = b_k$. Therefore, in A_ω,

$$[(s_k)]c[(s_k)]^* = (a - \varepsilon)_+.$$

Thus $(a - \varepsilon)_+ \precsim c$ for every $\varepsilon > 0$, and consequently $a \precsim c$. □

Convention 4.3 (Gap functions and pointed comparison)**.** Write $\mathrm{T}^+(C)$ for the cone of densely finite lower semicontinuous traces on C, including the zero trace, with the topology of pointwise convergence on the Pedersen ideal; see [28]. Matrix extensions are unnormalized. Following [27, Definition 5.4], a *gap function* for C is a finite-valued, nonnegative continuous affine function $\gamma : \mathrm{T}^+(C) \to [0, \infty)$, with $\gamma(0) = 0$, such that

$$\mathrm{d}_\tau(a) + \gamma(\tau) < \mathrm{d}_\tau(b) \quad (0 \neq \tau \in \mathrm{T}^+(C)) \quad \Longrightarrow \quad a \precsim b, \qquad a, b \in (C \otimes \mathcal{K})_+. \tag{4.1}$$

Denote the set of gap functions by G_C. Affinity and the condition at zero imply positive homogeneity. The zero trace is excluded from the strict inequality. Densely finite traces on nonunital models may be unbounded. For a simple unital algebra A, $\mathrm{T}^+(A)$ consists precisely of the nonnegative multiples of tracial states.

A *pointed* C*-*algebra* is a pair (C, e), where $e \in C_+$ is a norm-one full element. Following [7, Definition 3.3.2], its relative radius of comparison $r_{C,e}$ is the infimum of the numbers $s > 0$ for which the following condition holds:

$$\lambda(x) + s\lambda([e]) \leq \lambda(y) \quad (\lambda \in F(\mathrm{Cu}(C))) \quad \Longrightarrow \quad x \leq y. \tag{4.2}$$

We write $A \in \mathrm{TA}_1^\bullet(\mathcal{S}_\bullet)$ if the witnesses in Definition 2.1 can always be chosen so that (C, h) belongs to $\mathcal{S}_\bullet$. Thus the distinguished element compresses the finite set. In particular, this implies $A \in \mathrm{TA}_1(|\mathcal{S}_\bullet|)$ for the underlying class $|\mathcal{S}_\bullet| = \{C : (C, e) \in \mathcal{S}_\bullet\}$. For the affine-gap assertion below, the witnesses are required to have e strictly positive in C, so that $C = \overline{eCe}$. Strict positivity is stronger than fullness and permits norm errors to be estimated using $\mathrm{d}_\tau(e) = \|\tau\|$ for bounded traces on C.

Lemma 4.4. *Let* A *be a simple, separable, exact, unital, infinite-dimensional* C*-*algebra with* $\mathrm{T}(A) \neq \emptyset$.

(1) If $0 \neq b \in \mathrm{M}_l(A)_+$ and $\eta > 0$, there are orthogonal elements $b_0, b_1 \in \overline{b\mathrm{M}_l(A)b}_+$, with $b_1 \neq 0$, such that

$$\mathrm{d}_\tau(b_0) > \mathrm{d}_\tau(b) - \eta, \qquad \tau \in \mathrm{T}(A). \tag{4.3}$$

Moreover, for every $L \in \mathbb{N}$ there is $0 \neq z \in A_+$ such that

$$L[z] \leq [b_1] \quad \textit{in } \mathrm{Cu}(A). \tag{4.4}$$

(2) Put $e = [1_A]$. Suppose that $u, v \in \mathrm{Cu}(A)$ are bounded by finite multiples of e, that $v \neq 0$, and that, for some $s > 0$,

$$\lambda(u) + s < \lambda(v) \tag{4.5}$$

for every $\lambda \in F(\mathrm{Cu}(A))$ satisfying $\lambda(e) = 1$. If $0 \leq t < s$ and $u_0 \ll u$, then there are $N, M \in \mathbb{N}$ such that

$$\frac{M}{N} > t, \qquad (N+1)u_0 + Me \leq Nv. \tag{4.6}$$

(3) Let $\mathcal{S}$ be a class of C^*-algebras and suppose that $A \in \mathrm{TA}_1(\mathcal{S})$. Given $0 \neq z \in A_+$, there are $C_j \in \mathcal{S}$ and norm-one positive contractions $h_j \in C_j$ such that, with

$$h = [(h_j)] \in A_\omega, \qquad Q = \pi\left(\prod_{j=1}^{\infty} C_j\right) \subseteq A_\omega,$$

one has

$$h \in A' \cap A_\omega, \qquad h\mathrm{M}_l(A)h \subseteq \mathrm{M}_l(Q) \quad (l \in \mathbb{N}), \qquad 1 - h^2 \precsim z. \tag{4.7}$$

The c.p.c. order zero map

$$\Phi : A \longrightarrow Q, \qquad \Phi(a) = hah,$$

induces an additive, order-preserving map on Cuntz classes, denoted by $\mathrm{Cu}(\Phi)$. Compact containment in the argument below comes from spectral cutoffs of its images. For $x, y \in \mathrm{Cu}(A)$ represented by positive elements in finite matrix algebras over A, the following assertions hold. If $\mathrm{Cu}(C)$ is almost unperforated for every $C \in \mathcal{S}$, then

$$(k+1)x \leq ky \quad \Longrightarrow \quad \mathrm{Cu}(\Phi)(x) \leq \mathrm{Cu}(\Phi)(y). \tag{4.8}$$

For the quantitative assertion, suppose that the witnesses can be chosen from a class $\mathcal{S}_\bullet$ of pointed algebras (C, e), where e is full, with a fixed number $r \geq 0$. Assume that every $(C, e) \in \mathcal{S}_\bullet$ satisfies at least one of the following conditions:

(i) $r_{C,e} \leq r$;

(ii) *there is* $\gamma_C \in G_C$ *with* $\gamma_C(\tau) \leq r\mathrm{d}_\tau(e)$ *for every* $\tau \in \mathrm{T}^+(C)$; *when* $r = 0$, *this means* $\gamma_C = 0$.

If $A \in \mathrm{TA}_1^\bullet(\mathcal{S}_\bullet)$, *put* $L = \lfloor r \rfloor + 1$. *Identifying the images from* $\mathrm{Cu}(Q)$ *with their images in* $\mathrm{Cu}(A_\omega)$, *one then has*

$$(4.9) \qquad (k+1)x + m[1_A] \leq ky, \quad \frac{m}{k} > r \quad \Longrightarrow \quad \mathrm{Cu}(\Phi)(x) \leq \mathrm{Cu}(\Phi)(y) + L[1 - h^2].$$

No uniform cutoff modulus for the ranks of the compressing elements is required.

Proof. For (1), use the unnormalized matrix extensions of traces on A. The hereditary algebra

$$B = \overline{b\mathrm{M}_l(A)b}$$

is simple and infinite-dimensional. Every nonzero hereditary C*-subalgebra of a simple non-elementary C*-algebra contains arbitrarily long finite families of nonzero, mutually orthogonal, Cuntz equivalent positive elements, by the cut-down argument of [55, Section 2]. Choose an integer $q > l/\eta$ and norm-one elements $c_1, \ldots, c_q \in B_+$ with these properties. Then

$$q\mathrm{d}_\tau(c_1) = \sum_{i=1}^{q} \mathrm{d}_\tau(c_i) \leq \mathrm{d}_\tau(b) \leq l,$$

and hence $\mathrm{d}_\tau(c_1) < \eta$ for every $\tau \in \mathrm{T}(A)$.

Choose $\delta > 0$ such that $(c_1 - \delta)_+ \neq 0$, and choose $f \in \mathrm{C}([0,1])_+$ which is zero on $[0, \delta/2]$ and one on $[\delta, 1]$. With the unit taken in the multiplier algebra of B, put

$$t = 1 - f(c_1), \qquad b_0 = t\frac{b}{\|b\|}t, \qquad b_1 = (c_1 - \delta)_+.$$

Then $b_0, b_1 \in B_+$ and $b_0 b_1 = 0$. In the von Neumann algebra generated by B, the support projection of $b/\|b\|$ is the unit of B^{**} and the support projection of b_0 is the support projection of t. Since $t + f(c_1) = 1$, it follows that

$$\mathrm{d}_\tau(b_0) \geq \mathrm{d}_\tau(b) - \mathrm{d}_\tau(f(c_1)) \geq \mathrm{d}_\tau(b) - \mathrm{d}_\tau(c_1) > \mathrm{d}_\tau(b) - \eta.$$

Applying the same argument in $\overline{b_1\mathrm{M}_l(A)b_1}$, and then taking a nonzero diagonal compression, gives, for any prescribed L, an element $0 \neq z \in A_+$ with $L[z] \leq [b_1]$. This proves (1).

For (2), choose integers p, q such that

$$t < \frac{p}{q} < s.$$

The separation argument in the proof of [7, Proposition 3.2.1], applied to $qu_0 + pe$ and qv using (4.5), gives integers K, K_1 and an element $w \in \mathrm{Cu}(A)$, with $w \leq K_3 v$ for some K_3, such that

$$Kqu_0 + Kpe + w + v \leq Kqv + w$$

and then

$$K_1Kqu_0 + K_1Kpe + v \leq K_1Kqv.$$

Regularization as in [7, Lemma 2.4.1] passes from additive order-preserving maps to functionals; the compact containment $u_0 \ll u$ ensures the required strict inequalities. The boundedness of u and v by finite multiples of e ensures that all the normalizations occurring in that separation argument are finite. Since A is simple and $v \neq 0$, v is full. Choose K_2 with $u_0 \leq K_2 v$. Multiplying the last display by K_2 and using this inequality gives

$$(K_2K_1Kq + 1)u_0 + K_2K_1Kp\, e \leq K_2K_1Kq\, v.$$

Thus (4.6) holds with $N = K_2K_1Kq$ and $M = K_2K_1Kp$. Moreover, $M/N = p/q > t$.

For (3), choose $\theta > 0$ with $(z-\theta)_+ \neq 0$ and increasing finite subsets whose union is dense in the unit ball of A. Apply the relevant tracial approximation property to these subsets, with tolerances tending to zero and comparison element $(z-\theta)_+$, rescaled to be a contraction if necessary. Let C_j, h_j be the resulting approximation data. The compression and commutator conditions give

$$h \in A' \cap A_\omega, \qquad hAh \subseteq Q.$$

The matrix-level assertion follows entry by entry. Moreover,

$$1 - h_j \precsim (z-\theta)_+, \qquad (z-\theta)_+ g_\theta(z) = (z-\theta)_+.$$

Lemma 4.2, applied with the fixed target cutoff $g_\theta(z)$, gives $1-h \precsim g_\theta(z) \precsim z$. Since $1-h^2 \sim 1-h$, this proves (4.7). Centrality of h makes Φ order zero.

Suppose first that $(k+1)x \leq ky$ and the model Cuntz semigroups are almost unperforated. Write

$$X = \mathrm{Cu}(\Phi)(x), \qquad Y = \mathrm{Cu}(\Phi)(y),$$

and represent these classes by bounded positive sequences $(x_j), (y_j)$ in common finite matrix algebras over C_j. Denote their images in Q by $\widetilde{x}, \widetilde{y}$. Fix $\rho > 0$. Compact containment gives a fixed $\sigma > 0$ with

$$(k+1)[(\widetilde{x} - \rho/2)_+] \leq k[(\widetilde{y} - \sigma)_+].$$

Indeed, the left side is compactly contained in $(k+1)X \leq kY$, and the classes of the spectral cutoffs of $\widetilde{y}$ increase to Y. Lifting a Cuntz witness to the product and applying the norm perturbation lemma gives, on an ω-large set of indices,

$$(k+1)[(x_j - \rho)_+] \leq k[(y_j - \sigma)_+].$$

Almost unperforation therefore gives $(x_j - \rho)_+ \precsim (y_j - \sigma)_+$ there. The exact support identity

$$(y_j - \sigma)_+ g_\sigma(y_j) = (y_j - \sigma)_+$$

permits the argument of Lemma 4.2 to be applied in $\prod_j \mathrm{M}_l(C_j)$. Multiplying the Cuntz witnesses by $(y_j - \sigma)_+^{1/2}$ bounds their norms by the square root of the source norm. Consequently

$$(\widetilde{x} - \rho)_+ \precsim g_\sigma(\widetilde{y}) \precsim \widetilde{y} \quad \text{in } Q.$$

Letting ρ decrease to zero proves (4.8).

For the quantitative assertion, choose the approximation data from $\mathcal{S}_\bullet$. Put $E = [h^2] = [h]$ and $q = m/k$. Applying $\mathrm{Cu}(\Phi)$ to the hypothesized relation gives

$$(k+1)X + mE \leq kY.$$

Choose s with $r < s < \min\{q, L\}$, and fix $0 < \delta < 1/2$. For each $\rho > 0$, compact containment again supplies a fixed $\sigma > 0$ such that

$$(k+1)[(\widetilde{x} - \rho/2)_+] + m[(h^2 - \delta/2)_+] \leq k[(\widetilde{y} - \sigma)_+].$$

Lifting a Cuntz witness and applying a perturbation smaller than $\min\{\rho/2, \delta/2\}$ gives, on an ω-large set of indices,

$$(k+1)[(x_j - \rho)_+] + m[(h_j^2 - \delta)_+] \leq k[(y_j - \sigma)_+]. \tag{4.10}$$

Set

$$e_j = h_j^2, \qquad p_j = (e_j - \delta)_+, \qquad v_j = e_j(1 - g_{2\delta}(e_j)), \qquad a = 1 - 2\delta > 0.$$

The supports of p_j and v_j cover the support of e_j. Functional calculus therefore gives

$$[e_j] \leq [p_j] + [v_j], \qquad v_j g_a(1 - e_j) = v_j.$$

For every $\lambda \in F(\mathrm{Cu}(C_j))$, (4.10) implies

$$\begin{aligned} \lambda([(x_j - \rho)_+]) + s\lambda([h_j]) &\leq \lambda([(x_j - \rho)_+]) + s\lambda([p_j]) + s\lambda([v_j]) \\ &\leq \lambda([(y_j - \sigma)_+]) + L\lambda([v_j]). \end{aligned}$$

These inequalities hold in $[0, \infty]$ and require no subtraction of infinite values. Under condition (i), since $s > r \geq r_{C_j, h_j}$, pointed comparison yields

$$(x_j - \rho)_+ \precsim (y_j - \sigma)_+ \oplus v_j^{\oplus L}. \tag{4.11}$$

Under condition (ii), the same functional inequality holds for d_τ. The source rank $\mathrm{d}_\tau((x_j - \rho)_+)$ is finite: the cutoff $g_\rho(x_j)$ lies in the Pedersen ideal, every densely defined lower semicontinuous trace is finite there, and

$$\mathrm{d}_\tau((x_j - \rho)_+) \leq \tau(g_\rho(x_j)).$$

Fullness of h_j gives $\mathrm{d}_\tau(h_j) > 0$ for each nonzero trace. If this value is finite, $s > r$ gives the strict inequality

$$\mathrm{d}_\tau((x_j - \rho)_+) + \gamma_{C_j}(\tau) < \mathrm{d}_\tau((y_j - \sigma)_+) + L\mathrm{d}_\tau(v_j).$$

If $\mathrm{d}_\tau(h_j) = \infty$, the target rank is infinite, while the source rank and the gap are finite; the same strict inequality holds. Gap comparison again proves (4.11).

Finally put

$$b_j = (y_j - \sigma)_+ \oplus v_j^{\oplus L}, \qquad c_j = g_\sigma(y_j) \oplus g_a(1 - e_j)^{\oplus L}.$$

Here c_j is a positive contraction in a matrix algebra over the ambient algebra A, and $b_j c_j = b_j$ exactly. The cutoff $g_a(1 - e_j)$ need not belong to the nonunital model C_j, so the resulting comparison takes place in A_ω. Applying Lemma 4.2 at matrix level gives

$$\begin{aligned}[(\widetilde{x} - \rho)_+] &\leq [g_\sigma(\widetilde{y})] + L[g_a(1 - h^2)] \\ &\leq Y + L[1 - h^2] \qquad \text{in } \mathrm{Cu}(A_\omega).\end{aligned}$$

The fixed target cutoff σ and low-spectrum cutoff a ensure that the coordinate witnesses form a bounded sequence. Letting ρ decrease to zero proves (4.9). □

Theorem 4.5. *Let A be a simple, separable, exact, unital, infinite-dimensional* C*-*algebra with* $\mathrm{T}(A) \neq \emptyset$.

1. *Suppose that $\mathcal{S}$ is a class of* C*-*algebras such that* $\mathrm{Cu}(C)$ *is almost unperforated for every $C \in \mathcal{S}$. If $A \in \mathrm{TA}_1(\mathcal{S})$, then A has strict comparison of positive elements.*

2. *Let $r \geq 0$, and let $\mathcal{S}_\bullet$ be a class of pointed* C*-*algebras. For each $(C, e) \in \mathcal{S}_\bullet$, suppose that at least one of the following conditions holds:*

   (a) $r_{C,e} \leq r$;

   (b) there is $\gamma_{C,e} \in G_C$ such that

   $$\gamma_{C,e}(\tau) \leq r\mathrm{d}_\tau(e), \qquad \tau \in \mathrm{T}^+(C). \tag{4.12}$$

   When $r = 0$, this means $\gamma_{C,e} = 0$.

   If $A \in \mathrm{TA}_1^\bullet(\mathcal{S}_\bullet)$, then

   $$\big(\tau \mapsto r\tau(1_A)\big) \in G_A, \qquad \mathrm{rc}(A) \leq r.$$

   In particular, condition (b) holds if $\gamma_{C,e}(\tau) \leq r\tau(e^2)$.

3. *Let $H : \mathrm{T}^+(A) \to [0, \infty)$ be continuous and affine, with $H(0) = 0$, and suppose that $A \in \mathrm{TA}_1(\mathcal{S})$ for a class $\mathcal{S}$ of* C*-*algebras. Suppose that, for every $0 < \eta < 1$, there is $c_\eta \in A_+$ satisfying*

   $$\tau(c_\eta) \leq H(\tau) + \eta\tau(1_A), \qquad \tau \in \mathrm{T}^+(A), \tag{4.13}$$

   and that the $\mathrm{TA}_1(\mathcal{S})$ witnesses can always be chosen with h strictly positive in C, a positive element $c' \in C$, and $\gamma_C \in G_C$ satisfying

   $$\|c' - hc_\eta h\| < \varepsilon, \qquad \gamma_C(\tau) \leq \tau(c') + \eta\mathrm{d}_\tau(h), \quad \tau \in \mathrm{T}^+(C). \tag{4.14}$$

   These data are required for every finite set, every tolerance $\varepsilon > 0$, and every prescribed nonzero defect-comparison element. Then $H \in G_A$.

Proof. Every trace on A is faithful, and therefore A is stably finite. Exactness implies that every bounded 2-quasitrace is a trace. Thus the normalized functionals on $\mathrm{Cu}(A)$ are precisely the dimension functions coming from $\mathrm{T}(A)$.

The comparison arguments reduce to finite matrix algebras. Indeed, fix $a, b \in (A \otimes \mathcal{K})_+$ with a strict gap and $\varepsilon > 0$. Norm approximation in finite matrix corners and spectral cutoffs give u_1 with $[(a-\varepsilon)_+] \ll u_1 \ll [a]$, represented by a_1 in a finite matrix algebra. Choose $\rho > 0$ with $[(a-\varepsilon)_+] \leq [(a_1-\rho)_+]$. The continuous function $\tau \mapsto \tau(g_{\rho/2}(a_1))$ is at most $\mathrm{d}_\tau(a)$. Compactness of $\mathrm{T}(A)$, increasing cut-downs of b, and finite-corner norm approximation give $v_1 \ll [b]$, represented by a finite-matrix element b', with

$$\tau(g_{\rho/2}(a_1)) + H(\tau) < \mathrm{d}_\tau(b') \quad (\tau \in \mathrm{T}(A)),$$

where H is zero in (1), the scalar gap in (2), and the given affine function in (3). The finite-matrix argument applied to $(a_1 - \rho/2)_+$ and b' compares $(a-\varepsilon)_+$ with b. Letting ε decrease to zero proves the stabilized assertion.

For (1), let $a, b \in \mathrm{M}_l(A)_+$ satisfy

$$\mathrm{d}_\tau(a) < \mathrm{d}_\tau(b), \qquad \tau \in \mathrm{T}(A), \tag{4.15}$$

and fix $\varepsilon > 0$. Put

$$x_0 = [(a-\varepsilon)_+], \qquad x_1 = [(a-\varepsilon/2)_+].$$

Then $x_0 \ll x_1$. Since

$$\mathrm{d}_\tau((a-\varepsilon/2)_+) \leq \tau(g_{\varepsilon/2}(a)) \leq \mathrm{d}_\tau(a),$$

the lower semicontinuous function

$$\tau \longmapsto \mathrm{d}_\tau(b) - \tau(g_{\varepsilon/2}(a))$$

is strictly positive on the compact space $\mathrm{T}(A)$. Let $3\eta > 0$ be no larger than its minimum. Lemma 4.4(1), applied to b and η, gives orthogonal $b_0, b_1 \in \overline{b\mathrm{M}_l(A)b}_+$ such that $b_1 \neq 0$ and

$$\mathrm{d}_\tau(x_1) + 2\eta < \mathrm{d}_\tau(b_0), \qquad \tau \in \mathrm{T}(A).$$

Lemma 4.4(2), with $s = \eta$ and $t = 0$, gives an integer $k \geq 1$ such that

$$(k+1)x_0 \leq k[b_0]. \tag{4.16}$$

Here the positive multiple of $[1_A]$ in (4.6) has been omitted.

Choose $0 \neq z \in A_+$ so that

$$l[z] \leq [b_1], \tag{4.17}$$

and apply Lemma 4.4(3) with this z. Write $e = h^2$. From (4.16) and (4.8),

$$h(a-\varepsilon)_+h \precsim hb_0h$$

in the appropriate matrix algebra over A_ω. Since e is central,

$$(a-\varepsilon)_+ = h(a-\varepsilon)_+h + (1-e)^{1/2}(a-\varepsilon)_+(1-e)^{1/2},$$

and hence, in $\mathrm{Cu}(A_\omega)$,

$$\begin{aligned}[(a-\varepsilon)_+] &\le [h(a-\varepsilon)_+h] + l[1-e] \\ &\le [hb_0h] + l[z] \\ &\le [b_0] + [b_1] \le [b].\end{aligned}$$

Lemma 4.1, applied at matrix level, gives the same comparison in A. Since $\varepsilon > 0$ was arbitrary, $a \precsim b$. This proves strict comparison.

For (2), let $a, b \in \mathrm{M}_l(A)_+$ satisfy

$$\mathrm{d}_\tau(a) + r < \mathrm{d}_\tau(b), \qquad \tau \in \mathrm{T}(A),$$

and fix $\varepsilon > 0$. As in (1), the continuous upper bound $\tau(g_{\varepsilon/2}(a))$ and compactness give a uniform positive margin after replacing a by $(a-\varepsilon/2)_+$. Using Lemma 4.4(1), choose orthogonal b_0, b_1 in $\overline{b\mathrm{M}_l(A)b}$, with $b_1 \neq 0$, and numbers s, t such that

$$r < t < s, \qquad \mathrm{d}_\tau((a-\varepsilon/2)_+) + s < \mathrm{d}_\tau(b_0) \quad (\tau \in \mathrm{T}(A)).$$

Lemma 4.4(2) gives $N, M \in \mathbb{N}$ with

$$(N+1)[(a-\varepsilon)_+] + M[1_A] \le N[b_0], \qquad M/N > t > r. \tag{4.18}$$

Put $L = \lfloor r \rfloor + 1$, and choose $0 \neq z \in A_+$ with

$$(l+L)[z] \le [b_1].$$

Lemma 4.4(3), with pointed approximation data, gives (4.9), and hence

$$[h(a-\varepsilon)_+h] \le [hb_0h] + L[1-h^2] \quad \text{in } \mathrm{Cu}(A_\omega).$$

Centrality of h and the positive decomposition used in (1) give

$$[(a-\varepsilon)_+] \le [hb_0h] + (l+L)[1-h^2] \le [b_0] + [b_1] \le [b].$$

The comparison holds in A by Lemma 4.1. Letting ε decrease to zero gives $a \precsim b$. This proves the scalar gap assertion for finite matrices and hence, by the preceding cut-down reduction, for $A \otimes \mathcal{K}$. The definition of radius of comparison then gives $\mathrm{rc}(A) \le r$.

For (3), suppose that $a, b \in \mathrm{M}_l(A)_+$ satisfy

$$\mathrm{d}_\tau(a) + H(\tau) < \mathrm{d}_\tau(b), \qquad \tau \in \mathrm{T}(A).$$

Fix $\varepsilon > 0$, and put $a_0 = (a - \varepsilon)_+$ and $a_1 = (a - \varepsilon/2)_+$. Compactness of $\mathrm{T}(A)$, continuity of H and $\tau \mapsto \tau(g_{\varepsilon/2}(a))$, and lower semicontinuity of $\tau \mapsto \mathrm{d}_\tau(b)$ give $0 < \mu < 1/8$ such that

$$\mathrm{d}_\tau(b) - \tau(g_{\varepsilon/2}(a)) - H(\tau) > 8\mu \quad (\tau \in \mathrm{T}(A)).$$

By Lemma 4.4(1), choose orthogonal $b_0, b_1 \in \mathrm{Her}_{\mathrm{M}_l(A)}(b)_+$, with $b_1 \neq 0$, such that $\mathrm{d}_\tau(b_0) > \mathrm{d}_\tau(b) - \mu$ for all normalized traces. Choose $0 < \eta < \mu/4$, and take the element $c = c_\eta$ from the hypothesis. Then

$$\mathrm{d}_\tau(a_1) + \tau(c) + 6\mu < \mathrm{d}_\tau(b_0), \qquad \tau \in \mathrm{T}(A). \tag{4.19}$$

Choose an integer $R \geq \max\{1, \|c\|\}$, an integer n with $2/n < \mu/4$, and $0 < \beta < 1/n$. For $1 \leq i \leq nR$, put

$$c_i = (c - i/n - \beta)_+, \qquad f_i(t) = \min\{1, \max\{0, n(t - i/n - \beta)\}\}.$$

Elementary scalar inequalities give

$$c \leq \frac{2}{n} 1_A + \frac{1}{n} \sum_{i=1}^{nR} f_i(c), \qquad f_i(c) \leq n c_i, \qquad \frac{1}{n} \sum_{i=1}^{nR} \mathrm{d}_\tau((c - i/n)_+) \leq \tau(c). \tag{4.20}$$

In $\mathrm{Cu}(A)$, set

$$u = n[a_1] + \sum_i [(c - i/n)_+], \quad u_0 = n[a_0] + \sum_i [c_i], \quad v = n[b_0].$$

Then $u_0 \ll u$, and (4.19) and (4.20) imply $\lambda(u) + 6n\mu < \lambda(v)$ for every normalized functional. Apply Lemma 4.4(2), with $t = 2n\mu$. For the resulting integers K, P, $(K+1)u_0 + P[1_A] \leq Kv$ and $P/K > 2n\mu$. Set $N = Kn$ and $M = P$. Omitting the additional positive terms gives

$$(N+1)[a_0] + \frac{N}{n} \sum_i [c_i] + M[1_A] \leq N[b_0], \qquad q := M/N > 2\mu. \tag{4.21}$$

Choose $0 \neq z \in A_+$ with $(l+1)[z] \leq [b_1]$, and choose $\theta > 0$ with $z_\theta = (z - \theta)_+ \neq 0$. Apply the compatibility hypothesis with this fixed η, comparison element z_θ, and tolerances tending to zero, to increasing finite sets with dense union containing the matrix entries of a_0, b_0 and the elements $c, c_i, f_i(c)$. This gives models C_j, strictly positive contractions $h_j \in C_j$, elements $c'_j \in (C_j)_+$, and finite-valued gap functions γ_j satisfying the compatibility inequality. Put $e_j = h_j^2$ and

$$h = [(h_j)], \quad e = h^2, \quad Q = \pi\Big(\prod_j C_j\Big) \subseteq A_\omega, \qquad \Phi(x) = hxh.$$

Then h commutes with A, and $\Phi : A \to Q$ is c.p.c. order zero. Moreover, $1 - e_j \sim 1 - h_j \precsim z_\theta$. Since $z_\theta g_\theta(z) = z_\theta$, Lemma 4.2 gives

$$1 - e \precsim g_\theta(z) \precsim z, \qquad (l+1)[1 - e] \leq [b_1]. \tag{4.22}$$

Choose positive representatives $x_j, y_j \in \mathrm{M}_l(C_j)$ and $U_{i,j} \in C_j$ for $\Phi(a_0), \Phi(b_0), \Phi(c_i)$, respectively, and positive contractions $F_{i,j} \in C_j$ for $\Phi(f_i(c))$. They may be chosen to approximate their corresponding compressed elements in norm, while c'_j approximates $h_j c h_j$. Consequently (4.20) gives numbers $\xi_j \to 0$ such that, in the unitization of C_j,

$$F_{i,j} \le nU_{i,j} + \xi_j 1, \qquad c'_j \le \frac{2}{n} e_j + \frac{1}{n} \sum_i F_{i,j} + \xi_j 1.$$

Fix $\rho > 0$ with $Rn\rho < \mu/4$. For a bounded trace τ on C_j, spectral splitting of $U_{i,j}$ at ρ gives

$$\tau(F_{i,j}) \le \mathrm{d}_\tau((U_{i,j} - \rho)_+) + (n\rho + \xi_j)\|\tau\|.$$

Strict positivity of h_j implies $\|\tau\| = \mathrm{d}_\tau(e_j)$. Hence, for all sufficiently large j, the compatibility bound on γ_j gives

$$\gamma_j(\tau) \le \frac{1}{n} \sum_i \mathrm{d}_\tau((U_{i,j} - \rho)_+) + \mu \mathrm{d}_\tau(e_j) \quad \text{for every bounded trace } \tau. \tag{4.23}$$

Indeed, the coefficient of $\mathrm{d}_\tau(e_j)$ is at most $\eta + 2/n + Rn\rho + (R+1)\xi_j < \mu$.

Fix $0 < \delta < 1/4$ and $\chi > 0$. Apply $\mathrm{Cu}(\Phi)$ to (4.21), and compactly cut down all terms on the left. A fixed target cut and the usual lifting of a Cuntz witness give $\sigma > 0$ and an ω-large set on which

$$(N+1)[(x_j - \chi)_+] + \frac{N}{n} \sum_i [(U_{i,j} - \rho)_+] + M[(e_j - \delta)_+] \le N[(y_j - \sigma)_+]. \tag{4.24}$$

To obtain this relation, apply the cutoffs $\chi/2, \rho/2, \delta/2$ in Q and choose a compactly contained target class. Lifting with norm errors smaller than the remaining spectral margins gives a cutoff σ independent of j.

Put $\ell_j = e_j(1 - g_{2\delta}(e_j))$. The supports of $(e_j - \delta)_+$ and ℓ_j cover the support of e_j, so

$$\mathrm{d}_\tau(e_j) \le \mathrm{d}_\tau((e_j - \delta)_+) + \mathrm{d}_\tau(\ell_j).$$

For every nonzero bounded trace, combining (4.23) and (4.24) gives

$$\mathrm{d}_\tau((x_j - \chi)_+) + \gamma_j(\tau) < \mathrm{d}_\tau((y_j - \sigma)_+) + \mathrm{d}_\tau(\ell_j). \tag{4.25}$$

The strictness follows from $q > 2\mu$, $\mu < 1$, and $\mathrm{d}_\tau(e_j) = \|\tau\| > 0$. The same strict inequality holds for every unbounded densely finite lower semicontinuous trace: fixed spectral cuts have finite dimension, while strict positivity gives $\mathrm{d}_\tau(e_j) = \infty$; therefore $\mathrm{d}_\tau(\ell_j) = \infty$. The left side is finite because γ_j is finite. The local comparison-gap property therefore yields

$$(x_j - \chi)_+ \precsim (y_j - \sigma)_+ \oplus \ell_j.$$

The target in this comparison is fixed by multiplication with

$$g_\sigma(y_j) \oplus g_{1-2\delta}(1 - e_j) :$$

the first summand is fixed by the cutoff at σ, and the second by the support condition $e_j < 2\delta$ for ℓ_j. Lemma 4.2 thus gives

$$(\Phi(a_0) - \chi)_+ \precsim g_\sigma(\Phi(b_0)) \oplus g_{1-2\delta}(1-e) \precsim \Phi(b_0) \oplus (1-e).$$

Letting $\chi \downarrow 0$ proves $\Phi(a_0) \precsim \Phi(b_0) \oplus (1-e)$. Finally, centrality of h and (4.22) give

$$[a_0] \leq [\Phi(a_0)] + l[1-e] \leq [\Phi(b_0)] + (l+1)[1-e] \leq [b_0] + [b_1] \leq [b] \quad \text{in } \mathrm{Cu}(A_\omega).$$

Comparison of constant elements descends to A. Since $\varepsilon > 0$ was arbitrary, $a \precsim b$. The stabilized reduction and homogeneity extend the comparison to all positive elements of $A \otimes \mathcal{K}$ and all nonzero traces on A. Thus $H \in G_A$.

□

Corollary 4.6. *Let A be a simple, separable, unital, infinite-dimensional* C*-*algebra with strict comparison and almost divisible Cuntz semigroup. Let Γ be a countable discrete amenable group acting on A. Suppose that the action has WRP and that $B = A \rtimes_{\alpha,r} \Gamma$ is simple and exact. Then B has strict comparison of positive elements.*

Proof. Amenability gives $\mathcal{M}_1(A,\Gamma) \neq \varnothing$. Proposition 4.12 gives (COS), and Proposition 4.9 gives $B \in \mathrm{TA}_1(\mathcal{S})$. Exactness passes from B to its C*-subalgebra A. Since WRP identifies $\mathrm{T}(B)$ with the nonempty compact set $\mathcal{M}_1(A,\Gamma)$, all hypotheses on the ambient algebra in Theorem 4.5(1) are satisfied. Strict comparison for the simple exact algebra A implies that $\mathrm{Cu}(A)$ is almost unperforated. Almost unperforation passes to matrix algebras, hereditary C*-subalgebras, and finite direct sums. Thus every member of $\mathcal{S}$ has almost unperforated Cuntz semigroup, and Theorem 4.5(1) applies to B. □

Remark 4.7. The strict inequality in (4.1) is essential: strict comparison does not imply comparison from equality of rank functions. The scalar assertion includes the endpoint r because a source cutoff produces a uniform positive margin.

The model gaps in (2) are bounded relative to the compression scale; they need not be constant on a normalized trace space. The affine bound $\gamma_{C,e}(\tau) \leq r\tau(e^2)$ is a sufficient condition. In (3), the gap functions are controlled by compressed positive representatives of the ambient affine function. This compatibility does not follow from the existence or continuity of gap functions on individual models, since model traces need not extend to traces on A.

The positive representatives required in (4.13) exist for every H in (3). Indeed, [17, Theorem 4.3] gives $c_\eta \in A_+$ with

$$\tau(c_\eta) = H(\tau) + \eta\tau(1_A), \qquad \tau \in \mathrm{T}^+(A).$$

Thus the hypothesis in (3) concerns approximation data satisfying (4.14) for such representatives. For example, if $c \in A_+$ and the witnesses have $\gamma_C(\tau) \leq \tau(c')$ with $c' \approx hch$, then $\tau \mapsto \tau(c)$ is a gap function of A.

Strict positivity of h localizes the model to $\overline{hCh}$. The proof also applies to these hereditary subalgebras when they belong to the approximation class and admit the required gap functions.

A gap function on C alone does not ensure these hereditary comparison properties. Throughout, model gap functions are finite-valued on the whole cone of densely finite traces. Allowing infinite gap values would require a different comparison convention, since $\infty < \infty$ is false.

4.2 Property (COS) for C^*-dynamical systems

The following condition extends comparison of open sets to C^*-dynamical systems.

Definition 4.8. Let A be unital, let $\alpha : \Gamma \to \mathrm{Aut}(A)$ be an action of a discrete group, and suppose that $\mathcal{M}_1(A,\Gamma) \neq \varnothing$. The system (A,Γ) has property (COS) if there are $\lambda \in (0,1]$ and $m \in \mathbb{N}$ such that, whenever $a, b \in A_+$ satisfy

$$\mathrm{d}_\rho(a) < \lambda \mathrm{d}_\rho(b), \qquad \rho \in \mathcal{M}_1(A,\Gamma),$$

one has

$$a \precsim_{A \rtimes_r \Gamma} b^{\oplus m}.$$

We specify the constants by saying that the action has (λ, m)-COS.

Proposition 4.9. *Let A be a simple, separable, unital, infinite-dimensional C^*-algebra, let Γ be a countable discrete amenable group acting on A, and put $B = A \rtimes_{\alpha,r} \Gamma$. Suppose that the action has WRP and (COS), and that B is simple. If $\mathcal{S}$ is the class from Theorem 3.15, then*

$$B \in \mathrm{TA}_1(\mathcal{S}).$$

Proof. Let λ and m be constants for (COS), and fix $0 \neq x \in B_+$. Since B is simple and infinite-dimensional, $\overline{xBx}$ contains nonzero pairwise orthogonal positive elements $x_1, \ldots, x_m$. Simplicity of A makes the action minimal. Lemmas 3.30 and 3.31 therefore give nonzero elements $b_j \in A_+$ with $b_j \precsim_B x_j$.

A standard hereditary-subalgebra argument, applied finitely many times, gives $0 \neq b \in A_+$ such that $b \precsim_A b_j$ for every j. It follows that

$$b^{\oplus m} \precsim_B x_1 \oplus \cdots \oplus x_m \precsim_B x. \tag{4.26}$$

Every invariant trace on the simple algebra A is faithful. Since $\mathcal{M}_1(A,\Gamma)$ is compact and $\rho \mapsto \mathrm{d}_\rho(b)$ is lower semicontinuous,

$$\kappa := \min_{\rho \in \mathcal{M}_1(A,\Gamma)} \mathrm{d}_\rho(b) > 0.$$

Apply Theorem 3.15 with the prescribed finite set and with the defect tolerance smaller than $\lambda\kappa$. It gives $C \in \mathcal{S}$ and a norm-one positive contraction $h \in A \cap C$ with the required commutator and compression estimates. If $\rho \in \mathcal{M}_1(A,\Gamma)$, then $\rho \circ \mathbb{E}$ is a trace on B; hence

$$\mathrm{d}_\rho(1-h) < \lambda\kappa \leq \lambda \mathrm{d}_\rho(b).$$

Property (COS) and (4.26) give

$$1 - h \precsim_B b^{\oplus m} \precsim_B x.$$

The tracial-approximation version of Lemma 2.8 therefore gives $B \in \mathrm{TA}_1(\mathcal{S})$. □

Remark 4.10. Every minimal free system $(\mathrm{C}(X), \mathbb{Z}^d)$ has URP and COS by [49].

For a Cuntz semigroup S with a Γ-action, write

$$S_\Gamma = \gamma(S/\Gamma).$$

Here S/Γ is the dynamical quotient in the category of precompleted Cuntz semigroups, and γ denotes Cu-completion. By [10, Corollary 9.8(ii)], the canonical map $q : S \to S_\Gamma$ is the universal invariant Cu-morphism: every invariant Cu-morphism from S to a Cuntz semigroup factors uniquely through q. Thus S_Γ is the colimit in the category Cu of the action diagram $B\Gamma \to \mathrm{Cu}$, where $B\Gamma$ is the one-object category with morphisms Γ. For a C*-algebra action, this is the complete dynamical Cuntz semigroup $\mathrm{Cu}_\Gamma(A)$ when $S = \mathrm{Cu}(A)$ [10, Corollary 9.10].

Lemma 4.11. *Let S be a pure Cuntz semigroup and let a discrete group Γ act on S. Then S_Γ is pure.*

Proof. Put $\mathsf{Z} = \mathrm{Cu}(\mathcal{Z})$. By [1, Theorems 7.1.13 and 7.3.11], pureness of S is equivalent to the canonical map

$$\eta_S : S \longrightarrow \mathsf{Z} \otimes_{\mathrm{Cu}} S, \qquad x \longmapsto 1_{\mathsf{Z}} \otimes x,$$

being an isomorphism. These maps are natural in S, so η_S intertwines the given action with $\mathrm{id}_{\mathsf{Z}} \otimes \alpha$. The category Cu is closed symmetric monoidal [2, Theorems 5.10 and 5.11]; hence $\mathsf{Z} \otimes_{\mathrm{Cu}} -$ is a left adjoint and preserves colimits. Applying it to the action diagram gives

$$\mathsf{Z} \otimes_{\mathrm{Cu}} S_\Gamma \cong \operatorname*{colim}_{B\Gamma}(\mathsf{Z} \otimes_{\mathrm{Cu}} S) \cong \operatorname*{colim}_{B\Gamma} S = S_\Gamma.$$

Under these identifications the colimit of the maps η_S is η_{S_Γ}. This canonical map is therefore an isomorphism, and the same characterization proves that S_Γ is pure. □

The amenable-group case of the following automatic comparison criterion was proved by Orovitz, Phillips, and Wang [52, Proposition 5.4]. The dynamical Cuntz semigroup permits comparison for arbitrary discrete groups.

Proposition 4.12. *Let A be a simple unital C*-algebra with strict comparison by tracial states and with almost divisible Cuntz semigroup. Let $\alpha : \Gamma \to \mathrm{Aut}(A)$ be an action of a discrete group, and suppose that $\mathcal{M}_1(A, \Gamma) \neq \varnothing$. For every $n \in \mathbb{N}$ and $a, b \in \mathrm{M}_n(A)_+$,*

$$\mathrm{d}_\rho(a) < \mathrm{d}_\rho(b), \qquad \rho \in \mathcal{M}_1(A, \Gamma), \quad \Longrightarrow \quad a \precsim_{A \rtimes_{\alpha, r} \Gamma} b.$$

In particular, (A, Γ) has $(1, 1)$-COS.

Proof. Strict comparison by traces makes $S = \mathrm{Cu}(A)$ almost unperforated. Indeed, the assertion first holds in $\mathrm{W}(A)$: if $(k+1)[c] \leq k[d]$ and $d \neq 0$, faithfulness of every tracial state gives

$$\mathrm{d}_\tau(c) \leq \frac{k}{k+1} \mathrm{d}_\tau(d) < \mathrm{d}_\tau(d), \qquad \tau \in \mathrm{T}(A),$$

and strict comparison applies. If $d = 0$, the assertion is immediate. For $x, y \in S$ with $(k+1)x \leq ky$, fix $x' \ll x$ and choose a finite-matrix class z with $x' \leq z \ll x$. Write $y = \sup_j w_j$ for an increasing sequence of finite-matrix classes. Then $(k+1)z \ll ky$, so $(k+1)z \leq kw_j$ for some j. The finite-matrix case gives $x' \leq z \leq w_j \leq y$. Thus $x \leq y$, proving almost unperforation of S. Together with almost divisibility, this makes S pure. Lemma 4.11 shows that $T = S_\Gamma$ is pure as well.

Write $q : S \to T$ for the canonical map. A functional on a Cuntz semigroup is an additive, order-preserving map to $[0, \infty]$ which preserves zero and increasing suprema. By [10, Proposition 11.2 and Paragraph 11.6], pullback by q identifies functionals on T with invariant functionals on S. A functional normalized at $q([1_A])$ pulls back to the rank function of an invariant normalized bounded 2-quasitrace on A. Strict comparison by tracial states makes this quasitrace a trace [46, Theorem 3.6 and Remark 3.7]. Thus functionals normalized at $q([1_A])$ correspond exactly to $\mathcal{M}_1(A, \Gamma)$.

Fix $a, b \in \mathrm{M}_n(A)_+$ with the stated rank inequalities. Nonemptiness of $\mathcal{M}_1(A, \Gamma)$ implies that $b \neq 0$. Since A is simple and unital, b is full. Compact containment in $[1_A] \ll [1_A] \leq \sup_j j[b]$ gives an integer L with $[1_A] \leq L[b]$. Put

$$x = q([a]), \qquad y = q([b]), \qquad u = q([1_A]).$$

Then $x, y \leq nu$ and $u \leq Ly$, so $x \leq nLy$. If λ is a functional on T with $\lambda(y) = 1$, then

$$\frac{1}{n} \leq \lambda(u) \leq L.$$

Consequently $\lambda \circ q/\lambda(u) = \mathrm{d}_\rho$ for some $\rho \in \mathcal{M}_1(A, \Gamma)$, and therefore

$$\lambda(x) = \lambda(u)\mathrm{d}_\rho(a) < \lambda(u)\mathrm{d}_\rho(b) = \lambda(y).$$

Almost unperforation of T now gives $x \leq y$, by [1, Proposition 5.2.20(3)]: x belongs to the ideal generated by y, and the required strict inequality holds for every functional normalized at y.

Finally, the coefficient inclusion $\iota : A \to B = A \rtimes_{\alpha,r} \Gamma$ induces an invariant Cu-morphism, since the canonical unitaries implement α. The universal property gives a Cu-morphism $\kappa : T \to \mathrm{Cu}(B)$ such that $\mathrm{Cu}(\iota) = \kappa \circ q$. Applying κ to $x \leq y$ proves $a \precsim_B b$. □

Remark 4.13. Almost divisibility in Proposition 4.12 is understood in the stabilized sense: for $x' \ll x$ and $k \geq 1$, there is $y \in \mathrm{Cu}(A)$ with $ky \leq x$ and $x' \leq (k+1)y$. The stronger condition $ky \leq x \leq (k+1)y$ is sufficient. The proof does not require a positive representative of y inside A itself, nor orthogonal orbit translates of such representatives.

Remark 4.14. For amenable Γ, the condition $\mathrm{T}(A) \neq \varnothing$ ensures $\mathcal{M}_1(A, \Gamma) \neq \varnothing$ by averaging. For arbitrary groups, nonemptiness of the invariant tracial state space is an explicit hypothesis. Neither separability of A nor countability of Γ is needed in Proposition 4.12.

Remark 4.15. Proposition 4.12 compares coefficient elements at every finite matrix level. It requires neither freeness of the action nor simplicity of the crossed product. It does not assert strict

comparison or pureness of the entire crossed product. The examples in Section 4.4 also include nonsimple coefficient algebras; their comparison follows from separate tower or crossed-product arguments.

Corollary 4.16. *Let A be a simple, unital, stably finite, exact, $\mathcal{Z}$-stable C^*-algebra. Every action of a discrete group Γ on A with $\mathcal{M}_1(A,\Gamma) \neq \varnothing$ has $(1,1)$-COS, including comparison at every finite matrix level.*

Proof. $\mathcal{Z}$-stability gives $\mathrm{Cu}(A)$ a $\mathrm{Cu}(\mathcal{Z})$-multiplication, so it is almost unperforated and almost divisible [1, Proposition 7.1.5(2) and Theorem 7.3.11]. Exactness ensures that 2-quasitraces are traces, so almost unperforation gives strict comparison by tracial states. Proposition 4.12 applies. □

4.3 Orbit radius of comparison

The orbit radius of comparison measures comparison of finite orbit matrices over the coefficient algebra. It replaces the dimension–rank ratio of topological tower models when the coefficient algebra is noncommutative. The comparison condition is imposed on prescribed Fourier polynomials before the Rokhlin bases are chosen, independently of WRP and of comparison in the crossed product.

Throughout this section, A is unital, $\mathrm{T}(A) \neq \emptyset$, $\alpha : \Gamma \to \mathrm{Aut}(A)$ is an action of a countable discrete amenable group, and $B = A \rtimes_{\alpha,r} \Gamma$. We use the left Følner convention of Section 1.2. Matrix traces are unnormalized unless a normalization is displayed. Every finite invariance set K is understood to contain e.

Notation 4.17 (Orbit matrices and scalar collars)**.** For a nonempty finite set $F \subseteq \Gamma$, regular compression gives a unital completely positive map

$$\Pi_F : B \longrightarrow \mathrm{M}_F(A), \qquad \Pi_F(x) = \big[\mathbb{E}(u_g^* x u_h)\big]_{g,h\in F}.$$

We use the same notation for its matrix amplifications. If $x = \sum_k x_k u_k \in \mathrm{M}_l(B)$ is a Fourier polynomial, then

$$\Pi_F(x) = \big[\alpha_{g^{-1}}(x_{gh^{-1}})\big]_{g,h\in F} \in \mathrm{M}_F(\mathrm{M}_l(A)). \tag{4.27}$$

In particular, positivity of x implies positivity of $\Pi_F(x)$.

A *scalar collar* on F is a diagonal matrix $\Lambda = \mathrm{diag}(\lambda_g)_{g\in F}$ with $0 \leq \lambda_g \leq 1$. It acts identically on the external M_l factor. Put

$$R_{F,\Lambda}(x) = \Lambda \Pi_F(x) \Lambda, \qquad D(\Lambda) = \{g \in F : \lambda_g \neq 1\}.$$

Thus Λ is independent of the coefficient algebra, and, when $A = \mathrm{C}(X)$, independent of the point of X. For $z \in \mathrm{M}_F(\mathrm{M}_l(A))_+$ and $\tau \in \mathrm{T}(A)$, write

$$\widehat{\mathrm{d}}_{\tau,F}(z) = \frac{1}{|F|} \mathrm{d}_{\mathrm{Tr}_F \otimes \mathrm{Tr}_l \otimes \tau}(z). \tag{4.28}$$

Only the orbit factor is normalized.

Lemma 4.18 (Finite-section trace formula)**.** *Let $a \in \mathrm{M}_l(B)_+$, and let f be continuous on an interval containing $[0, \|a\|]$. Given $\zeta > 0$, there are a finite $K \subseteq \Gamma$ and $\eta > 0$ such that, for every nonempty (K, η)-invariant F and every scalar collar with $|D(\Lambda)| < \eta|F|$,*

$$\left| \frac{(\mathrm{Tr}_F \otimes \mathrm{Tr}_l \otimes \tau)(f(R_{F,\Lambda}(a)))}{|F|} - (\mathrm{Tr}_l \otimes \bar{\tau}_F)(\mathbb{E}^{(l)}(f(a))) \right| < \zeta \tag{4.29}$$

for every $\tau \in \mathrm{T}(A)$, where $\bar{\tau}_F = |F|^{-1} \sum_{g \in F} \tau \circ \alpha_{g^{-1}}$.

Proof. First let a be a Fourier polynomial with support in a finite symmetric set N containing the identity. If $k \geq 1$ and $g \in \mathrm{Int}_{N^k}(F)$, all the intermediate indices in the gth diagonal entry of $\Pi_F(a)^k$ stay in F. Consequently that entry equals $\alpha_{g^{-1}}(\mathbb{E}^{(l)}(a^k))$. The remaining diagonal entries in both expressions have norm at most $\|a\|^k$. It follows that

$$\left| \frac{(\mathrm{Tr}_F \otimes \mathrm{Tr}_l \otimes \tau)(\Pi_F(a)^k)}{|F|} - (\mathrm{Tr}_l \otimes \bar{\tau}_F)(\mathbb{E}^{(l)}(a^k)) \right| \leq 2l\|a\|^k \frac{|F \setminus \mathrm{Int}_{N^k}(F)|}{|F|}.$$

The equality for the constant polynomial is immediate.

To insert the collar, put $P = \Pi_F(a)$ and $D = 1 - \Lambda$. Then $P - \Lambda P \Lambda = DP + \Lambda P D$. Telescoping the kth powers and using traciality gives

$$\frac{1}{|F|} \left| (\mathrm{Tr}_F \otimes \mathrm{Tr}_l \otimes \tau)(P^k - (\Lambda P \Lambda)^k) \right| \leq 2kl\|a\|^k \frac{|D(\Lambda)|}{|F|}.$$

Here $0 \leq D \leq 1_{D(\Lambda)}$, and the bound $|\sigma(Dz)| \leq \|z\|\sigma(D)$ is applied to each term. Polynomial approximation of f proves (4.29) when a is a positive Fourier polynomial. The normalized trace in the first term has norm l, independent of F.

For arbitrary positive a, approximate $a^{1/2}$ by Fourier polynomials and take their squares, rescaling if necessary so their norms do not exceed $\|a\|$. Contractivity of regular compression and norm continuity of continuous functional calculus on a bounded interval, obtained by polynomial approximation, control both sides of (4.29) uniformly in F and τ. This proves the assertion. □

Definition 4.19 (Orbit radius of comparison)**.** Let $r \geq 0$. The system (A, α) has *orbit r-comparison* if, for every $l \in \mathbb{N}$, every pair of positive Fourier polynomials $x, y \in \mathrm{M}_l(B)$, every $\varepsilon, \delta > 0$, and every $t > r$, there are a finite $K \subseteq \Gamma$ and $\eta \in (0, 1)$ with the following property. For every nonempty (K, η)-invariant F and every scalar collar Λ with $|D(\Lambda)| < \eta|F|$, the inequalities

$$\widehat{\mathrm{d}}_{\tau,F}((R_{F,\Lambda}(x) - \varepsilon)_+) + t < \widehat{\mathrm{d}}_{\tau,F}((R_{F,\Lambda}(y) - \delta)_+), \qquad \tau \in \mathrm{T}(A), \tag{4.30}$$

imply

$$(R_{F,\Lambda}(x) - 2\varepsilon)_+ \precsim_{\mathrm{M}_\infty(A)} (R_{F,\Lambda}(y) - \delta/2)_+. \tag{4.31}$$

The *orbit radius of comparison* is

$$\mathrm{orc}(A, \alpha) = \inf\{r \geq 0 : (A, \alpha) \text{ has orbit } r\text{-comparison}\},$$

with the infimum of the empty set equal to ∞.

Remark 4.20. This definition uses traces and is available without exactness. Whenever the ordinary radius of comparison is used below, exactness ensures that bounded 2-quasitraces are traces. The set of admissible radii is upward closed. Quantification over finite invariance sets and tolerances makes the definition independent of a selected Følner sequence. An equivariant isomorphism transports the orbit matrices, traces, and comparison witnesses, so the orbit radius is an invariant of the C*-dynamical system.

Only matrices arising from fixed Fourier inputs are tested; comparison of all positive elements of the whole algebra $\mathrm{M}_F(A)$ is not required. The spectral cutoffs in (4.31) give bounded witnesses before hereditary localization. The normalization therefore does not depend on the possibly noncompact scale of a tower base. This notion differs from the dynamical radius based on coefficient elements and dynamical Cuntz comparison in [6, Definition 3.4].

Lemma 4.21 (From invariant traces to orbit rank gaps)**.** *Let $a, b \in \mathrm{M}_l(B)_+$ and $s \geq 0$ satisfy*

$$\mathrm{d}_{\rho\circ\mathbb{E}}(a) + s < \mathrm{d}_{\rho\circ\mathbb{E}}(b), \qquad \rho \in \mathcal{M}_1(A, \Gamma).$$

For every $\varepsilon > 0$ there are $\delta, \kappa, \nu > 0$, a finite $K \subseteq \Gamma$, and $\eta > 0$ such that

$$\widehat{\mathrm{d}}_{\tau,F}((R_{F,\Lambda}(x) - \varepsilon)_+) + s + \kappa < \widehat{\mathrm{d}}_{\tau,F}((R_{F,\Lambda}(y) - \delta)_+) \tag{4.32}$$

for every $\tau \in \mathrm{T}(A)$, every nonempty (K, η)-invariant F, every scalar collar with $|D(\Lambda)| < \eta|F|$, and every positive $x, y \in \mathrm{M}_l(B)$ with $\|x - a\|, \|y - b\| < \nu$.

Proof. The function $\rho \mapsto (\rho \circ \mathbb{E})(g_\varepsilon(a))$ is continuous and no larger than $\mathrm{d}_{\rho\circ\mathbb{E}}(a)$. The lower semicontinuous difference

$$\mathrm{d}_{\rho\circ\mathbb{E}}(b) - (\rho \circ \mathbb{E})(g_\varepsilon(a)) - s$$

has a positive minimum on the compact invariant trace space. Choose $\kappa > 0$ sufficiently small. Since $g_{2\delta}(b)$ increases to the support of b as $\delta \downarrow 0$, compactness of the invariant trace space, applied to the increasing open sets defined by the strict inequalities, gives a fixed $\delta > 0$ such that

$$(\rho \circ \mathbb{E})(g_\varepsilon(a)) + s + 4\kappa < (\rho \circ \mathbb{E})(g_{2\delta}(b)) \tag{4.33}$$

for all invariant ρ.

For all sufficiently invariant F, the same inequality with 3κ holds for $\bar{\tau}_F$, uniformly in $\tau \in \mathrm{T}(A)$. Indeed, otherwise a sequence of increasingly invariant sets and traces violating the inequality would have orbit-average traces converging to an invariant trace, contradicting (4.33). Separability is not needed here: one may take a weak* convergent subnet of these states. The inverse convention in $\bar{\tau}_F$ is compatible with left invariance.

Apply Lemma 4.18 to g_ε and $g_{2\delta}$, with error smaller than $\kappa/4$ in each application. The inequalities

$$1_{(\varepsilon,\infty)} \leq g_\varepsilon, \qquad g_{2\delta} \leq 1_{(\delta,\infty)}$$

give (4.32) for $x = a, y = b$ with a strictly positive margin. Regular compression and scalar compression are contractive. Uniform continuity of the two fixed continuous cutoff functions, and the trace bound l, give a $\nu > 0$ for which the same estimates hold for x, y as stated. The target cutoff δ is therefore fixed before these approximations are chosen. □

Proposition 4.22 (Averaging finite radius of comparison)**.** *Suppose that A is stably finite and exact, and that* $\mathrm{rc}(A) < \infty$*. If* Γ *is infinite, then every action satisfies*

$$\mathrm{orc}(A, \alpha) = 0.$$

If Γ *is finite of cardinality* q*, then*

$$\mathrm{orc}(A, \alpha) \leq \frac{\mathrm{rc}(A)}{q}.$$

Proof. Choose $R > \mathrm{rc}(A)$ for which A has R-comparison. For infinite Γ, fix $t > 0$ and choose a finite set K with $|K| > R/t$. Take $0 < \eta < 1$. If F is nonempty and (K, η)-invariant, then $\mathrm{Int}_K(F) \neq \emptyset$, so $|F| \geq |K|$. Under (4.30), the unnormalized matrix traces give

$$\mathrm{d}_\tau((R_{F,\Lambda}(x) - \varepsilon)_+) + R < \mathrm{d}_\tau((R_{F,\Lambda}(y) - \delta)_+), \qquad \tau \in \mathrm{T}(A).$$

Ordinary comparison over A gives the stronger conclusion

$$(R_{F,\Lambda}(x) - \varepsilon)_+ \precsim (R_{F,\Lambda}(y) - \delta)_+.$$

This implies (4.31) and proves orbit 0-comparison. Exactness justifies testing ordinary comparison on traces.

For finite Γ, take $K = \Gamma$ and $\eta < 1/q$. The invariance condition forces $F = \Gamma$, and $|D(\Lambda)| < \eta q < 1$ forces $\Lambda = 1$. The same proof gives orbit R/q-comparison. Let $R \downarrow \mathrm{rc}(A)$. □

Remark 4.23. The infinite-group assertion requires no minimality, WRP, or COS. It is the comparison analogue of the vanishing of mean dimension for actions of infinite amenable groups on finite-dimensional compact spaces: a fixed comparison obstruction is divided by a growing orbit height. Even a trivial infinite-group action can have orbit radius zero. The additional tower and complement-comparison hypotheses are used in the crossed-product theorem below.

Proposition 4.24 (The topological bound)**.** *Let X be a nonempty compact metrizable space, and let an infinite countable amenable group* Γ *act on X. Then*

$$\mathrm{orc}(\mathrm{C}(X), \alpha) \leq \tfrac{1}{2} \mathrm{mdim}(X, \Gamma).$$

No freeness, minimality, or Rokhlin hypothesis is required here.

Proof. There is nothing to prove if the mean dimension is infinite. Put $d = \mathrm{mdim}(X, \Gamma)$ and fix $t > d/2$, positive Fourier polynomials $x, y \in \mathrm{M}_l(B)$, and $\varepsilon, \delta > 0$. Choose $\xi > 0$ with $d + \xi < 2t$ and $0 < \nu < \min\{\varepsilon/2, \delta/4\}$.

Choose a finite open cover $\mathcal{U}$ controlling the finitely many coefficient functions of x and y so closely that membership in a common atom of

$$\mathcal{U}_F = \bigvee_{g \in F} g^{-1}\mathcal{U}$$

implies norm variation less than ν for both orbit matrices. The estimate is independent of F: every row and column has at most the fixed number of Fourier-support blocks, and each coefficient block has uniformly small variation. It also remains valid after multiplication by any fixed scalar collar, since $\|\Lambda\| \leq 1$.

For all sufficiently invariant F, choose a refinement $\mathcal{V}_F = \{V_i\}_{i=1}^n$ of $\mathcal{U}_F$ whose nerve P_F satisfies

$$\dim P_F < (d+\xi)|F| < 2t|F|. \tag{4.34}$$

This uniform statement follows from the Følner-independent limit for the cover-dimension function [15, Theorem 2.1 and Lemma 3.1]. If uniformity failed, sets violating the dimension estimate along a countable exhaustion would form a Følner sequence contradicting that limit.

Choose $z_i \in V_i$ and a partition-of-unity map $\varphi : X \to P_F$ subordinate to $\mathcal{V}_F$. Write $R_x = R_{F,\Lambda}(x)$ and $R_y = R_{F,\Lambda}(y)$, and define positive matrix functions on P_F by

$$A_F(v) = \sum_i v_i R_x(z_i), \qquad B_F(v) = \sum_i v_i R_y(z_i).$$

These are positive because they are convex combinations of positive matrices. For a point v in a simplex with vertices I, choose $z \in \bigcap_{i\in I} V_i$. Each z_i, $i \in I$, lies in a common atom of $\mathcal{U}_F$ with z. Hence

$$\|A_F(v) - R_x(z)\| < \nu, \qquad \|B_F(v) - R_y(z)\| < \nu.$$

If (4.30) holds, apply it to the evaluation trace at z. Eigenvalue perturbation then gives

$$\operatorname{rank}((A_F(v) - 3\varepsilon/2)_+) + t|F| < \operatorname{rank}((B_F(v) - 3\delta/4)_+).$$

By (4.34) and homogeneous comparison [64, Theorem 4.6],

$$(A_F - 3\varepsilon/2)_+ \precsim (B_F - 3\delta/4)_+ \quad \text{in } \mathrm{M}_{l|F|}(\mathrm{C}(P_F)).$$

Pull this comparison back by φ. By construction,

$$\|A_F \circ \varphi - R_x\| < \nu, \qquad \|B_F \circ \varphi - R_y\| < \nu.$$

The estimates

$$\|R_x - (A_F \circ \varphi - 3\varepsilon/2)_+\| < \nu + 3\varepsilon/2 < 2\varepsilon,$$

and

$$\|B_F \circ \varphi - (R_y - \delta/2)_+\| < \nu + \delta/2 < 3\delta/4$$

give, by the norm perturbation lemma for Cuntz comparison,

$$(R_x - 2\varepsilon)_+ \precsim (A_F \circ \varphi - 3\varepsilon/2)_+ \precsim (B_F \circ \varphi - 3\delta/4)_+ \precsim (R_y - \delta/2)_+.$$

This proves the required orbit comparison for $r = d/2$. This argument applies without a bound on the number of tapered levels. The same finite-input factorization occurs in [48, Theorem 3.9]. □

Lemma 4.25 (Bounded spectral witnesses)**.** *Let $p, q \in \mathrm{M}_n(D)_+$ for a C^*-algebra D, let $a \geq 0$ and $b > 0$, and suppose that $(p-a)_+ \precsim (q-b)_+$. For every $\gamma > 0$ there is $v \in \mathrm{M}_n(D)$ such that*

$$vqv^* = (p-a-\gamma)_+, \qquad \|v\| \leq \sqrt{\|p\|/b}.$$

Proof. The cut-down factorization, compressed to the $n \times n$ corner when comparison is taken in the stabilization, gives $w \in \mathrm{M}_n(D)$ with $w(q-b)_+w^* = (p-a-\gamma)_+$. Put $z = w(q-b)_+^{1/2}$; then $\|z\|^2 \leq \|p\|$. Let $f(t) = b^{-1/2}$ on $[0,b]$ and $f(t) = t^{-1/2}$ on $[b,\infty)$. The element $v = zf(q)$ belongs to $\mathrm{M}_n(D)$, with functional calculus taken in its unitization if necessary. Its norm has the stated bound, and

$$(q-b)_+^{1/2} f(q) q f(q) (q-b)_+^{1/2} = (q-b)_+$$

proves the required identity. □

Lemma 4.26 (Hereditary localization of orbit comparison)**.** *Suppose that A is separable and the action has WRP. Let x_j, y_j be positive Fourier polynomials in $\mathrm{M}_l(B)$ with $x_j \to a$, $y_j \to b_0$, and $\|x_j\| \leq M$. Fix $\varepsilon, \delta > 0$. For each j, suppose that an exact tiling has shapes $F_{s,j}$ and scalar collars $\Lambda_{s,j}$ such that*

$$(R_{F_{s,j},\Lambda_{s,j}}(x_j) - 2\varepsilon)_+ \precsim (R_{F_{s,j},\Lambda_{s,j}}(y_j) - \delta/2)_+.$$

Assume that the collars are chosen as in Theorem 3.15, with boundary fractions tending to zero and widths tending to infinity sufficiently fast for the Fourier supports and a prescribed increasing dense family of finite subsets of B, and that the shapes satisfy the invariance conditions required for these collars. Then there are tower algebras $C_j = \bigoplus_s \mathrm{M}_{|F_{s,j}|}(H_{s,j}) \subseteq B$, contractions $h_{0,j} \in A \cap C_j$, and $T_j \in \mathrm{M}_l(C_j)$ such that

$$\begin{aligned} &\|T_j\| \leq C := \sqrt{2M/\delta}, \\ &\|T_j h_{0,j} y_j h_{0,j} T_j^* - h_{0,j} x_j h_{0,j}\| \leq 3\varepsilon + \alpha_j, \qquad \alpha_j \longrightarrow 0. \end{aligned} \tag{4.35}$$

For $h_j = g_{1/2}(h_{0,j})$ one may also arrange

$$\|[h_j, z]\| \longrightarrow 0 \quad (z \in B), \qquad \sup_{\rho \in \mathcal{M}_1(A,\Gamma)} \mathrm{d}_\rho(1-h_j) \longrightarrow 0. \tag{4.36}$$

Proof. Fix j and suppress it in the notation. Let N be a finite symmetric set containing e, the Fourier supports of x, y, and the finite set used in the collar construction. The assumed collar construction gives

$$\lambda_s(g) > 0 \implies Ng \subseteq F_s, \qquad |\lambda_s(kg) - \lambda_s(g)| \leq L^{-1} \quad (k \in N),$$

with $L = L_j \to \infty$ and the collars equal to one off a boundary whose relative cardinality tends to zero. The exact tiling theorem supplies these shapes before the orbit comparisons are applied.

Lemma 4.25 gives matrices V_s over A with

$$V_s R_{F_s,\Lambda_s}(y) V_s^* = (R_{F_s,\Lambda_s}(x) - 3\varepsilon)_+, \qquad \|V_s\| \leq C. \tag{4.37}$$

With the matrices V_s fixed, WRP gives bases $c_s \in A' \cap A_\omega$. These commute with every entry of the matrices, and all the levels $\alpha_g(c_s)$ are mutually orthogonal. Set

$$h_0 = \sum_{s,g} \lambda_s(g)\alpha_g(c_s).$$

For each s, the map

$$\psi_s(Z) = \sum_{g,h\in F_s} u_g c_s Z_{gh} c_s u_h^*$$

is c.p.c. order zero, since c_s commutes with A and its levels are orthogonal. The Fourier-interior condition gives the exact identities

$$\sum_s \psi_s(R_{F_s,\Lambda_s}(x)) = h_0 x h_0, \qquad \sum_s \psi_s(R_{F_s,\Lambda_s}(y)) = h_0 y h_0. \tag{4.38}$$

Indeed, in a Fourier term with index $k \in N$, a nonzero source collar level $h \in F_s$ has $kh \in F_s$. Orthogonality leaves only the target level $g = kh$ in the same tower.

The relative cone lifts and linking-algebra isomorphisms of Lemmas 3.12 and 3.13 give coordinate base diagonals $d_{s,m}$ and linking isomorphisms:

$$\Theta_{s,m} : \mathrm{M}_{F_s}(\mathrm{Her}_A(d_{s,m})) \longrightarrow C_{s,m}$$

For $f_n(t) = \min\{1, nt\}$ define

$$T_{s,n,m} = \Theta_{s,m}^{(l)}\big((1_{F_s} \otimes 1_l \otimes f_n(d_{s,m}))V_s(1_{F_s} \otimes 1_l \otimes f_n(d_{s,m}))\big).$$

This is an element of $\mathrm{M}_l(C_{s,m})$ of norm at most C. Its quotient image is

$$\sum_{g,h\in F_s} u_g f_n(c_s)(V_s)_{gh} f_n(c_s) u_h^*.$$

The formula follows from the linking identities because $f_n(t)/t$ extends continuously at zero.

Let $q_s = \sum_{g\in F_s} \alpha_g(c_s)$ and denote the quotient image of $T_{s,n,m}$ by $T_{s,n}$. Centrality of the base and (4.37) give

$$T_{s,n}\psi_s(R_{F_s,\Lambda_s}(y))T_{s,n}^* = f_n(q_s)^4 \psi_s((R_{F_s,\Lambda_s}(x) - 3\varepsilon)_+).$$

The order zero map has support element $\psi_s(1) = q_s^2$. Thus the truncation error on the sth tower is bounded by

$$M \sup_{0\le t\le 1} t^2(1 - f_n(t)^4) \le M/n^2.$$

Replacing the source by its 3ε cutoff adds at most 3ε to the error. The tower summands are orthogonal, so their combined norm error is the maximum of their errors. The sum of the coordinate witnesses also has norm at most C.

Choose $n = n_j \to \infty$ and then one sufficiently accurate coordinate $m = m_j$. As in the proof of Theorem 3.15, put $h_{0,j} = \sum_{s,g} \lambda_s(g) d_{s,g,m_j}$. The linking identities and (4.38) give (4.35), with $\alpha_j \to 0$. The coordinate m_j can also be chosen to satisfy the commutator, covariance, linking, and uniform 2-norm defect estimates of that theorem. The shrinking boundary fractions and increasing collar widths give (4.36) after applying $g_{1/2}$. Each step imposes only finitely many conditions, so m_j can be chosen after the witnesses and their truncations have been fixed. □

Theorem 4.27 (Comparison in the crossed product)**.** *Let (A,Γ) be minimal with WRP and (λ,m)-COS, where $\lambda\in(0,1]$, $m\in\mathbb{N}$, A is unital and separable, $\mathrm{T}(A)\neq\emptyset$, and Γ is countably infinite, discrete, and amenable. If $B=A\rtimes_{\alpha,r}\Gamma$ is exact, then*

$$\mathrm{rc}(B)\leq\mathrm{orc}(A,\alpha).$$

Proof. The crossed product is simple by Corollary 3.32, and Lemma 6.16 shows that every nonzero hereditary coefficient corner is infinite-dimensional. WRP identifies its traces with invariant coefficient traces by Proposition 3.8. They are faithful, and B is infinite-dimensional and stably finite. Exactness identifies the normalized 2-quasitraces with traces.

Let r be an admissible orbit radius, and let $q>r$. Suppose that $a,b\in\mathrm{M}_l(B)_+$ satisfy

$$\mathrm{d}_\tau(a)+q<\mathrm{d}_\tau(b),\qquad \tau\in\mathrm{T}(B).$$

Fix $\zeta>0$ and put $p=(a-\zeta/2)_+$. Choose s with $r<s<q$. By Lemma 4.4(1), applied to B, choose orthogonal $b_0,b_1\in\mathrm{Her}_{\mathrm{M}_l(B)}(b)_+$ with $b_1\neq0$ and $\mathrm{d}_\tau(b_0)>\mathrm{d}_\tau(b)-(q-s)/2$ for all traces. Thus

$$\mathrm{d}_\tau(p)+s<\mathrm{d}_\tau(b_0),\qquad \tau\in\mathrm{T}(B),\qquad b_0\oplus b_1\precsim b.$$

A nonzero diagonal compression of b_1, followed by Lemma 3.31, gives $0\neq c\in A_+$ with $c\precsim_B b_1$. Choose $\sigma>0$ with $c'=(c-\sigma)_+\neq0$; then $g_\sigma(c)c'=c'$. Apply Lemma 6.9 to c' and $g_\sigma(c)$, with lm columns. Its orthogonal range squares belong to $\mathrm{Her}_A(c)$, so the common source $z\in A_+$ satisfies

$$z^{\oplus lm}\precsim_B b_1. \tag{4.39}$$

Choose $\theta>0$ with $z_\theta=(z-\theta)_+\neq0$. Minimality makes invariant traces faithful; hence

$$\kappa_z=\min_{\rho\in\mathcal{M}_1(A,\Gamma)}\mathrm{d}_\rho(z_\theta)>0.$$

Choose one $\varepsilon>0$ with $48\varepsilon<\zeta/2$. Lemma 4.21, applied to p,b_0,s, gives fixed $\delta,\kappa,\nu>0$ and a Følner condition. Choose positive Fourier polynomials $x_j\to p$ and $y_j\to b_0$ within the specified neighborhood, with $\|x_j\|\leq M$. Fix t with $s<t<s+\kappa$. For each j, use orbit r-comparison for $x_j,y_j,\varepsilon,\delta,t$. Choose the shape invariance and collar boundary fractions to satisfy this comparison condition, Lemma 4.21, and the finite centrality conditions at step j. The exact tiling theorem and the collar construction of Theorem 3.15 give compatible shapes and collars. Their orbit rank gaps imply the internal comparisons in Lemma 4.26.

That lemma gives $h_{0,j},T_j$, with $\|T_j\|\leq C=\sqrt{2M/\delta}$, and put

$$h_j=g_{1/2}(h_{0,j}),\qquad k_j=\frac{g_{1/2}(h_{0,j})}{h_{0,j}},\qquad S_j=k_jT_j.$$

The quotient defining k_j extends continuously by zero at the origin; the bound $\|k_j\|\leq4$ suffices. Consequently $\|S_j\|\leq4C$ and

$$\begin{aligned}&\|S_jh_{0,j}b_0h_{0,j}S_j^*-h_jph_j\|\\&\quad\leq48\varepsilon+16\alpha_j+16C^2\|y_j-b_0\|+\|x_j-p\|.\end{aligned}$$

Here ε and δ remain fixed as $j \to \infty$. Their dependence on ζ therefore does not affect boundedness of the witnesses as j varies.

For large j, (4.36) gives

$$\mathrm{d}_\rho(1-h_j^2) = \mathrm{d}_\rho(1-h_j) < \lambda\kappa_z, \qquad \rho \in \mathcal{M}_1(A,\Gamma).$$

COS yields $1-h_j^2 \precsim_B z_\theta^{\oplus m}$. Put $h = [(h_j)] \in B_\omega$, pad $1-h_j^2$ with zeroes to an element $P_j \in \mathrm{M}_m(B)_+$, and put $Z = z^{\oplus m}$. Lemma 4.25, with the fixed target cutoff θ, gives $v_j \in \mathrm{M}_m(B)$ with

$$v_j Z v_j^* = (P_j - 1/j)_+, \qquad \|v_j\| \le \theta^{-1/2}.$$

The bounded sequence (v_j) therefore implements $1-h^2 \precsim_{B_\omega} z^{\oplus m}$. Thus, in $\mathrm{Cu}(B_\omega)$,

$$l[1-h^2] \le [z^{\oplus lm}] \le [b_1]. \tag{4.40}$$

Let $h_0 = [(h_{0,j})]$ and $S = [(S_j)]$. The element h commutes with p. Put

$$X = Sh_0b_0h_0S^*, \qquad R = (1-h^2)^{1/2}p(1-h^2)^{1/2}.$$

Then $X \precsim b_0$, $R \precsim (1-h^2)^{\oplus l} \precsim b_1$, and $p = hph + R$. The preceding norm estimate gives

$$\|p - (X+R)\| \le 48\varepsilon < \zeta/2.$$

Norm perturbation and (4.40) therefore give

$$(a-\zeta)_+ = (p-\zeta/2)_+ \precsim X + R \precsim X \oplus R \precsim b_0 \oplus b_1 \precsim b \quad \text{in } \mathrm{M}_\infty(B_\omega).$$

Comparison between constant elements descends to B by Lemma 4.1, applied to matrix algebras. Letting $\zeta \downarrow 0$ proves $a \precsim b$. Hence $\mathrm{rc}(B) \le r$ for every admissible r, and taking the infimum proves the theorem. □

Definition 4.28 (Strong URP)**.** Let A be unital and separable. We say that (A,Γ,α) has *strong URP* if, for every finite $K \subseteq \Gamma$ and every $\eta > 0$, there are nonempty finite (K,η)-invariant sets $F_1,\dots,F_S$ and projections $p_s \in A_\omega \cap A'$ such that the levels $\alpha_g(p_s)$, $g \in F_s$, are mutually orthogonal and

$$\sum_{s=1}^{S}\sum_{g\in F_s} \alpha_g(p_s) = 1 \quad \text{in } A_\omega.$$

The shapes are fixed for the central-sequence towers in this condition; they may depend on K and η.

Definition 4.29 (WRP with Cuntz control in the crossed product)**.** Let A be unital and separable, with $\mathrm{T}(A) \neq \emptyset$, and put $B = A \rtimes_{\alpha,r} \Gamma$. We say that the action has *WRP with Cuntz control in the crossed product* if it has WRP and the following condition holds. For every finite $K \subseteq \Gamma$, every

$\eta > 0$, and every $0 \neq z \in B_+$, there are nonempty finite (K,η)-invariant shapes $F_1, \dots, F_S$ and positive contractions $c_s \in A_\omega \cap A'$ whose levels $\alpha_g(c_s)$, $g \in F_s$, are mutually orthogonal, such that

$$q = \sum_{s=1}^{S} \sum_{g \in F_s} \alpha_g(c_s), \qquad 1 - q \in J_{\mathcal{M}_1(A,\Gamma)}, \qquad 1 - q \precsim_{B_\omega} z.$$

Here A_ω is viewed as a subalgebra of B_ω, and z denotes its constant image. The shapes may depend on K, η, z, but are fixed within the central-sequence towers. The reserve z is prescribed before choosing these towers.

The additional comparison condition is imposed in the norm ultrapower of the crossed product, in addition to the usual uniform tracial control of the complement. It differs from the weak tracial Rokhlin property of [69, Definition 2.1], where comparison with a prescribed coefficient element is imposed pointwise inside A.

Lemma 4.30 (Full towers and Cuntz control)**.** *If A is unital and separable with $\mathrm{T}(A) \neq \emptyset$, then strong URP implies WRP with Cuntz control in the crossed product.*

Proof. Fix an exact tiling with prescribed shapes D_s. Normalize the shapes to contain the identity and put $L = \bigcup_s D_s D_s^{-1}$. Apply strong URP with $K = L$ and $\eta \downarrow 0$, and cut its projection towers along this prescribed tiling, retaining only tiles wholly contained in each F_r. The construction in the proof of Proposition 3.6 applies in A_ω. The new bases are central projections and their prescribed levels are orthogonal. At most $\eta|F_r|$ original levels are lost from the rth tower. For every ultralimit ρ_ω of a possibly varying sequence of invariant traces, their total trace is therefore at most

$$\eta \sum_r |F_r| \rho_\omega(p_r) = \eta.$$

Choosing representatives and diagonalizing as in Lemma 3.5 gives the prescribed-tiling WRP.

The original full towers from strong URP satisfy the additional condition in Definition 4.29. Their sum is $q = 1$ in A_ω, so $1 - q = 0 \precsim_{B_\omega} z$ for every prescribed nonzero $z \in B_+$. These towers need not be the recut towers used to establish prescribed-tiling WRP. □

Proposition 4.31 (Comparison from a Cuntz-small tower complement)**.** *Let (A, Γ, α) be minimal, where A is unital and separable, $\mathrm{T}(A) \neq \emptyset$, and Γ is countably infinite, discrete, and amenable. Suppose that the action has WRP with Cuntz control in the crossed product, and assume that $\mathrm{orc}(A, \alpha) = 0$. Then it has $(1,1)$-COS. If $B = A \rtimes_{\alpha,r} \Gamma$ is exact, it also has strict comparison of positive elements. In particular, these conclusions hold when A is stably finite and exact with $\mathrm{rc}(A) < \infty$.*

Strong URP is a special case, by Lemma 4.30. For strong URP, the implication from zero orbit radius to $(1,1)$-COS holds even without minimality and also for finite amenable groups.

Proof. Let $a, b \in A_+$ satisfy $\mathrm{d}_\rho(a) < \mathrm{d}_\rho(b)$ for every $\rho \in \mathcal{M}_1(A, \Gamma)$. Rescale b so that it is a contraction; this does not change its dimension functions or Cuntz class. Fix $\zeta > 0$ and put $p = (a - \zeta/2)_+$. Since $\rho(g_{\zeta/2}(a)) \leq \mathrm{d}_\rho(a)$, lower semicontinuity and compactness give

$$\mu = \min_{\rho \in \mathcal{M}_1(A,\Gamma)} \big(\mathrm{d}_\rho(b) - \rho(g_{\zeta/2}(a))\big) > 0.$$

Choose $N \in \mathbb{N}$ with $1/N < \mu/2$.

Lemma 6.16 shows that every nonzero hereditary C*-subalgebra of A is infinite-dimensional. This permits the choice of a reserve in the coefficient algebra. Choose $\sigma_0 > 0$ with $(b-\sigma_0)_+ \neq 0$ and apply Lemma 6.9 to $(b-\sigma_0)_+$ and $g_{\sigma_0}(b)$, with N columns. Their range squares $v_1, \ldots, v_N$ are nonzero, mutually orthogonal, belong to $\mathrm{Her}_A(b)$, and are Cuntz equivalent in B. Each invariant ρ extends to the trace $\rho \circ \mathbb{E}$ on B. Consequently

$$N\mathrm{d}_\rho(v_1) = \mathrm{d}_\rho\Big(\sum_{i=1}^{N} v_i\Big) \leq \mathrm{d}_\rho(b) \leq 1, \qquad \rho \in \mathcal{M}_1(A, \Gamma).$$

Choose $\sigma > 0$ so that $b_1 = (v_1 - \sigma)_+ \neq 0$, and put

$$h = g_\sigma(v_1), \qquad b_0 = (1-h)b(1-h).$$

Then $hb_1 = b_1$, so $b_0 b_1 = 0$, and $b_0, b_1 \in \mathrm{Her}_A(b)$ give $b_0 \oplus b_1 \precsim_A b$. The decomposition

$$b^{1/2} = (1-h)b^{1/2} + hb^{1/2}$$

gives $b \leq 2b_0 + 2hbh$. Dimension-function subadditivity and $hbh \in \mathrm{Her}_A(h)$ therefore imply $\mathrm{d}_\rho(b) \leq \mathrm{d}_\rho(b_0) + \mathrm{d}_\rho(h)$. Since $h \in \mathrm{Her}_A(v_1)$, it follows that

$$\mathrm{d}_\rho(b_0) \geq \mathrm{d}_\rho(b) - 1/N > \mathrm{d}_\rho(p) + \mu/2.$$

The nonzero element b_1 is fixed before the towers are chosen and will absorb their complement.

Apply Lemma 4.21 to $p, b_0, s = 0$, with $\varepsilon = \zeta/4$ and scalar collar 1. It gives $\delta, \kappa > 0$ and a Følner condition under which

$$\widehat{\mathrm{d}}_{\tau,F}((\Pi_F(p) - \varepsilon)_+) + \kappa < \widehat{\mathrm{d}}_{\tau,F}((\Pi_F(b_0) - \delta)_+) \quad (\tau \in \mathrm{T}(A)).$$

Choose $0 < t < \kappa$. Since the orbit radius is zero, choose an admissible orbit radius $r < t$ and strengthen this Følner condition to satisfy orbit r-comparison for $p, b_0, \varepsilon, \delta, t$. Coefficient elements are Fourier polynomials, and their orbit matrices are diagonal, so the resulting comparison is

$$\Pi_F((a-\zeta)_+) \precsim_{\mathrm{M}_\infty(A)} \Pi_F((b_0 - \delta/2)_+)$$

for every qualifying shape F.

Choose the towers of Definition 4.29 for this strengthened condition and the already fixed reserve $z = b_1$. Write $q_s = \sum_{g \in F_s} \alpha_g(c_s)$, so $q = \sum_s q_s$ and $1 - q \precsim_{B_\omega} b_1$. Define

$$\Psi_s : \mathrm{M}_{F_s}(A) \longrightarrow B_\omega, \qquad \Psi_s(Z) = \sum_{g,h \in F_s} u_g c_s^{1/2} Z_{gh} c_s^{1/2} u_h^*.$$

The row $V_s = (u_g c_s^{1/2})_{g \in F_s}$ satisfies $V_s^* V_s = 1_{F_s} \otimes c_s$, which commutes with coefficient matrices. Hence Ψ_s is c.p.c. and has order zero, with $\Psi_s(1) = q_s$. The ranges of the different Ψ_s are orthogonal. For $x \in A$, diagonal reconstruction gives

$$\Psi_s(\Pi_{F_s}(x)) = q_s x, \qquad \sum_s \Psi_s(\Pi_{F_s}(x)) = qx.$$

Order zero maps preserve Cuntz comparison [71, Corollary 4.5]. Apply them to the orbit comparisons and take the orthogonal sum to obtain, with $a_\zeta = (a-\zeta)_+$,

$$qa_\zeta \precsim q(b_0 - \delta/2)_+ \precsim b_0 \quad \text{in } B_\omega.$$

Since q commutes with A, the remaining part is positive and satisfies

$$(1-q)a_\zeta \precsim 1-q \precsim b_1.$$

Subadditivity of Cuntz classes gives

$$a_\zeta = qa_\zeta + (1-q)a_\zeta \precsim qa_\zeta \oplus (1-q)a_\zeta \precsim b_0 \oplus b_1 \precsim b.$$

Comparison of constant elements descends from B_ω to B by Lemma 4.1. Letting $\zeta \downarrow 0$ proves $a \precsim_B b$, which is $(1,1)$-COS.

Under strong URP, apply the same orbit comparison and order zero reconstruction with $b_0 = b$, $b_1 = 0$, and full projection towers. Fix $\zeta > 0$, put $p = (a - \zeta/2)_+$, and apply Lemma 4.21 directly to p, b with $s = 0$, $\varepsilon = \zeta/4$, and scalar collar 1. Zero orbit radius gives the diagonal orbit comparisons above. Choose full towers satisfying their Følner condition; then $q = 1$, so reconstruction gives $(a-\zeta)_+ \precsim b$ in B_ω. Lemma 4.1 gives the comparison in B. This argument requires no coefficient reserve and proves the COS assertion without minimality or infiniteness.

If A is stably finite and exact with finite radius of comparison and Γ is infinite, Proposition 4.22 gives orbit radius zero. Under the minimal, infinite-group hypotheses of the proposition, exactness of B permits an application of Theorem 4.27, giving $\mathrm{rc}(B) = 0$. For a strict tracial gap, fixing a source cutoff gives a uniform positive gap by compactness and continuous cutoff functions, as in Lemma 4.21. Since B has t-comparison for every $t > 0$, each source cutoff is below the target. Letting the cutoff tend to zero proves strict comparison. □

Corollary 4.32. *Assume the hypotheses of Theorem 4.27 except for COS, and suppose that* $\mathrm{rc}(A) < \infty$. *If the action has* (λ, m)*-COS for some* $\lambda \in (0,1]$, $m \in \mathbb{N}$, *strong URP in the sense of Definition 4.28, or WRP with Cuntz control in the crossed product in the sense of Definition 4.29, then* B *has strict comparison of positive elements.*

Proof. Exactness passes from B to A. Proposition 4.22 gives $\mathrm{orc}(A, \alpha) = 0$. In the strong URP and Cuntz-controlled cases, Proposition 4.31 supplies $(1,1)$-COS. Theorem 4.27 therefore gives $\mathrm{rc}(B) = 0$ in all three cases. A strict tracial gap becomes uniformly positive after fixing a source cutoff, by compactness and continuous cutoff functions as in Lemma 4.21. Since B has q-comparison for every $q > 0$, each source cutoff is below the target. Letting the cutoff tend to zero proves strict comparison. □

Remark 4.33. The proof only uses the additional tower condition for nonzero reserves $z \in A_+$. Thus this restricted condition suffices for Proposition 4.31. The ordinary WRP hypothesis supplies the coefficient reserves; the particular towers used for comparison need only have a Cuntz-small complement in B_ω. The reserve must be prescribed before the towers are chosen. Comparison of all complements with a single fixed large element does not suffice for this absorption argument.

Remark 4.34 (Exact coverage and the Fourier boundary). For full projection towers the untapered matrix tower algebra is unital. This is sufficient for the coefficient comparisons in Proposition 4.31. For general Fourier inputs, however, the localization in Lemma 4.26 still uses scalar collars vanishing on the tower boundary. If a nonzero projection level is discarded, its norm is one even when its trace is small. Thus exact coverage alone does not justify setting the localized cutoff equal to 1 in the proof of Theorem 4.27.

For example, let u implement a $\mathbb{Z}$-action with an exact cyclic projection tower $p_i = u^i p_0 u^{-i} \in A$, $0 \le i < N$, whose sum is 1. Its unital matrix tower algebra is

$$D = \left\{ \sum_{i,j=0}^{N-1} u^i a_{ij} u^{-j} : a_{ij} \in p_0 A p_0 \right\}.$$

For $x \in D$, Fourier coefficients give

$$\mathbb{E}(p_0 x p_{N-1} u^*) = 0, \qquad \mathbb{E}(p_0 u p_{N-1} u^*) = p_0.$$

Consequently $\operatorname{dist}(u, D) = 1$. The matrix entry joining the last level to the first requires the return unitary $p_0 u^N p_0$, which belongs to the crossed-product corner rather than $p_0 A p_0$. For this reason the general positive-orbit-radius bound retains its COS hypothesis. In the zero-orbit-radius case, including finite coefficient radius when Γ is infinite, Proposition 4.31 makes that hypothesis automatic under strong URP.

4.4 Examples of orbit radius of comparison and COS

The following three families have zero orbit radius of comparison and satisfy $(1,1)$-COS. In the first two, the coefficient algebra may have positive finite radius of comparison; we compute the normalized comparison allowance in the orbit matrices and use full towers to obtain COS. In the third, the coefficient algebra has radius of comparison zero, and COS follows from strict comparison in the crossed product. These examples include nonsimple coefficient algebras, as allowed in Section 4.2.

Example 4.35 (Cantor skew products with finite radius of comparison). Let X be the Cantor set, let $h : X \to X$ be a minimal homeomorphism, and let D be a simple, separable, unital, exact, stably finite C*-algebra with $\mathrm{T}(D) \neq \emptyset$ and $\mathrm{rc}(D) < \infty$. For a point-norm continuous map $x \mapsto \beta_x$ from X to $\mathrm{Aut}(D)$, put

$$A = \mathrm{C}(X, D), \qquad \alpha(f)(x) = \beta_x\big(f(h^{-1}(x))\big).$$

Then the $\mathbb{Z}$-action generated by α is minimal, has strong URP, and has $(1,1)$-COS. Moreover,

$$\mathrm{rc}(A) = \mathrm{rc}(D), \qquad \mathrm{orc}(A, \alpha) = 0.$$

Proof. For every nonempty compact metrizable zero-dimensional space Y,

$$\mathrm{rc}(\mathrm{C}(Y, D)) = \mathrm{rc}(D). \tag{4.41}$$

Indeed, let $r > \mathrm{rc}(D)$, and let $a, b \in \mathrm{M}_\ell(\mathrm{C}(Y, D))_+$ satisfy $\mathrm{d}_\rho(a) + r < \mathrm{d}_\rho(b)$ for every tracial state ρ of $\mathrm{C}(Y, D)$. The evaluation traces $\sigma \circ \mathrm{ev}_y$, with $y \in Y$ and $\sigma \in \mathrm{T}(D)$, give

$$\mathrm{d}_\sigma(a(y)) + r < \mathrm{d}_\sigma(b(y)).$$

Consequently $a(y) \precsim_D b(y)$ for each y. Fix $\varepsilon > 0$. For every y, choose $v_y \in \mathrm{M}_\ell(D)$ such that

$$\|a(y) - v_y b(y) v_y^*\| < \varepsilon/2.$$

A stabilized comparison witness gives such a v_y after compression to the $\ell \times \ell$ corner. Continuity gives a neighborhood U_y on which the same norm is less than ε. Refine a finite subcover to a finite clopen partition $V_1, \ldots, V_q$, with $V_j \subseteq U_{y_j}$, and put $v = \sum_j 1_{V_j} \otimes v_{y_j}$. Then $\|a - vbv^*\| < \varepsilon$. Letting ε tend to zero gives $a \precsim b$, so $\mathrm{rc}(\mathrm{C}(Y, D)) \leq \mathrm{rc}(D)$. Conversely, r-comparison for $\mathrm{C}(Y, D)$ implies r-comparison for D: apply it to constant functions, using restriction of traces and evaluation at any $y \in Y$. This proves the reverse inequality.

To construct full central towers, fix a finite $K \subseteq \mathbb{Z}$ and $\eta > 0$. Choose H so large that every interval $\{0, \ldots, l-1\}$ with $l \geq H$ is (K, η)-invariant. Since h is aperiodic, there is a nonempty clopen set $Y \subseteq X$ with $Y \cap h^j(Y) = \emptyset$ for $1 \leq j < H$. The first-return function to Y is locally constant and bounded, by minimality and compactness. Its finitely many values $l_1, \ldots, l_S$ are at least H. Let Y_s be the clopen set on which the return time is l_s. The Kakutani–Rokhlin partition [56, proof of Lemma 3.1] gives

$$X = \bigsqcup_{s=1}^{S} \bigsqcup_{j=0}^{l_s - 1} h^j(Y_s).$$

For $p_s = 1_{Y_s} \otimes 1_D \in Z(A)$, the fibre automorphisms fix scalar functions, so

$$\alpha^j(p_s) = 1_{h^j(Y_s)} \otimes 1_D, \qquad \sum_{s=1}^{S} \sum_{j=0}^{l_s - 1} \alpha^j(p_s) = 1_A.$$

These central projection towers give strong URP (Definition 4.28).

Simplicity of D identifies the ideals of $\mathrm{C}(X, D)$ with open subsets of X. Minimality of h consequently makes the action minimal. We compute the orbit radius using (4.41). Fix $t > 0$ and $R > \mathrm{rc}(D)$, and choose a finite $L \subseteq \mathbb{Z}$ with $|L| > R/t$ and $0 < \eta < 1$. Every nonempty (L, η)-invariant F has $N := |F| \geq |L|$, since $\mathrm{Int}_L(F) \neq \varnothing$. For the Fourier inputs and scalar collars in Definition 4.19, put

$$a_F = (R_{F,\Lambda}(x) - \varepsilon)_+, \qquad b_F = (R_{F,\Lambda}(y) - \delta)_+.$$

If the normalized rank inequalities (4.30) hold, then the unnormalized matrix extensions satisfy

$$\mathrm{d}_\tau(a_F) + R < \mathrm{d}_\tau(a_F) + Nt < \mathrm{d}_\tau(b_F), \qquad \tau \in \mathrm{T}(A).$$

Since $R > \mathrm{rc}(A)$, this gives $a_F \precsim b_F$, and hence (4.31). Thus the normalized comparison allowance is R/N, which tends to zero with the orbit height. As $t > 0$ was arbitrary, $\mathrm{orc}(A, \alpha) = 0$. The strong URP case of Proposition 4.31 gives $(1, 1)$-COS.

By [64, Theorem 5.11], the fibre D can be a simple unital AH algebra with any prescribed finite positive radius of comparison. The same radius then occurs in A, while the system has $(1,1)$-COS. □

Example 4.36 (Profinite translations with fibres of finite radius of comparison). Let Γ be a countably infinite, residually finite, discrete amenable group, and choose a decreasing sequence of finite-index normal subgroups Γ_n with $\bigcap_n \Gamma_n = \{e\}$. Let D satisfy the hypotheses of Example 4.35, and set

$$K = \varprojlim \Gamma/\Gamma_n, \qquad A = \mathrm{C}(K, D), \qquad \alpha_g(f)(x) = f(g^{-1}x).$$

Then the translation action is free on K and minimal, has strong URP, and has $(1,1)$-COS. In addition, $\mathrm{rc}(A) = \mathrm{rc}(D)$ and $\mathrm{orc}(A,\alpha) = 0$.

Proof. The canonical map $\Gamma \to K$ is injective with dense image. Left translation is therefore free and minimal on K. Since D is simple, minimality also holds for the action on A. The space K is compact, metrizable, and zero-dimensional, so (4.41) gives the radius equality. For $R > \mathrm{rc}(D)$ and $t > 0$, choose a finite $L \subseteq \Gamma$ with $|L| > R/t$ and an invariance tolerance less than one. Every corresponding nonempty invariant set F then has $|F| \geq |L|$, so $R/|F| < t$. The orbit-matrix comparison calculation in Example 4.35 applies unchanged and proves $\mathrm{orc}(A,\alpha) = 0$.

By [18, Corollary 5.6], there are finite sets $F_n \subseteq \Gamma$, each a set of representatives for Γ/Γ_n, forming a Følner sequence. Fix a finite $L \subseteq \Gamma$ and $\eta > 0$, and choose n so that F_n is (L,η)-invariant in the left Følner convention. Let $\pi_n : K \to \Gamma/\Gamma_n$ be the quotient map and put

$$U_n = \pi_n^{-1}(\Gamma_n), \qquad p_n = 1_{U_n} \otimes 1_D \in Z(A).$$

The cosets gU_n, $g \in F_n$, partition K, and hence

$$\alpha_g(p_n)\alpha_{g'}(p_n) = 0 \quad (g \neq g'), \qquad \sum_{g \in F_n} \alpha_g(p_n) = 1_A.$$

For these towers the normalized comparison allowance is explicitly

$$\frac{R}{|F_n|} = \frac{R}{[\Gamma : \Gamma_n]} \longrightarrow 0, \qquad R > \mathrm{rc}(D),$$

because Γ is infinite and $\bigcap_n \Gamma_n = \{e\}$. Their constant images give a full central projection tower in A_ω, establishing strong URP. Together with the orbit-radius computation, strong URP gives $(1,1)$-COS by Proposition 4.31. Thus this construction also allows fibres of positive finite radius. □

Example 4.37 (Minimal skew products with pure fibres). Let X be an infinite compact metrizable space, let $h : X \to X$ be a minimal homeomorphism, and let D be a simple, separable, unital, exact, stably finite, pure C*-algebra with $\mathrm{T}(D) \neq \emptyset$. Here purity means that $\mathrm{Cu}(D)$ is almost unperforated and almost divisible. For any point-norm continuous map $x \mapsto \beta_x \in \mathrm{Aut}(D)$, define

$$A = \mathrm{C}(X, D), \qquad \alpha(f)(x) = \beta_x\big(f(h^{-1}(x))\big).$$

Then

$$\operatorname{rc}(A) = 0, \qquad \operatorname{orc}(A, \alpha) = 0,$$

and $(A, \mathbb{Z}, \alpha)$ has $(1,1)$-COS. No restriction on the covering dimension of X or the mean dimension of h is needed.

Proof. By [60, Corollary 5.9], the algebra $A = \mathrm{C}(X, D)$ is pure. We first show that its radius of comparison is zero. Suppose that $a, b \in \mathrm{M}_n(A)_+$ satisfy $\mathrm{d}_\tau(a) < \mathrm{d}_\tau(b)$ for all $\tau \in \mathrm{T}(A)$. Fix a tracial state on D, which is faithful by simplicity. The associated evaluation traces show that $b(z) \neq 0$ for every $z \in X$. Hence b is full in $\mathrm{M}_n(A)$, and there is $L \in \mathbb{N}$ with $[1_A] \leq L[b]$.

If λ is a functional on $\mathrm{Cu}(A)$ with $\lambda([b]) = 1$, then $1/n \leq \lambda([1_A]) \leq L$. After normalization at $[1_A]$, it is the rank functional of a bounded 2-quasitracial state, which is a trace because A is exact. Thus $\lambda([a]) < \lambda([b])$ for every such functional. Almost unperforation of $\mathrm{Cu}(A)$ and [1, Proposition 5.2.20(3)] give $a \precsim_A b$. This proves $\operatorname{rc}(A) = 0$. Proposition 4.22 now gives $\operatorname{orc}(A, \alpha) = 0$.

The action is free because h is minimal on the infinite space X. The crossed product $B = A \rtimes_\alpha \mathbb{Z}$ is simple by [4, Proposition 1.6] and pure by [3, Theorem 4.6]. Exactness of D and amenability of $\mathbb{Z}$ give exactness of B. Averaging a trace on A and composing with the canonical expectation gives a trace on B, faithful by simplicity. Thus B is stably finite, and purity and exactness give strict comparison by traces.

Suppose $a, b \in A_+$ satisfy $\mathrm{d}_\rho(a) < \mathrm{d}_\rho(b)$ for every $\rho \in \mathcal{M}_1(A, \mathbb{Z})$. Every $\tau \in \mathrm{T}(B)$ restricts to such an invariant trace, and functional calculus in $A \subseteq B$ gives

$$\mathrm{d}_\tau(a) = \mathrm{d}_{\tau|_A}(a) < \mathrm{d}_{\tau|_A}(b) = \mathrm{d}_\tau(b).$$

Strict comparison in B yields $a \precsim_B b$, proving $(1,1)$-COS.

For a concrete noncommutative Furstenberg transformation, take $X = S^1$, $h(z) = e^{2\pi i\theta} z$ with $\theta \in \mathbb{R} \setminus \mathbb{Q}$, and $D = C_r^*(\mathbb{F}_2)$. For its canonical generators u_1, u_2, let

$$\beta_z(u_i) = z u_i \quad (i = 1, 2), \qquad \alpha(f)(z) = \beta_z\big(f(e^{-2\pi i\theta} z)\big).$$

The maps β_z are the point-norm continuous gauge automorphisms arising from the characters of $\mathbb{F}_2$ with these values on the generators. The algebra D is pure by [3, Proposition 7.3], and this construction appears in [3, Example 7.8]. It gives a system with $\operatorname{orc}(A, \alpha) = 0$, $(1,1)$-COS, and nonnuclear fibre. □

Chapter 5

$\mathcal{Z}$-stability and weak tracial approximation

We study $\mathcal{Z}$-stability for C*-algebras with weak tracial approximation, where $\mathcal{Z}$ denotes the Jiang–Su algebra.

5.1 Weak tracial $\mathcal{Z}$-absorption

The results of this section give an abstract formulation of part of [47, Section 4].

The following decomposition uses three nested spectral cutoffs. The middle cutoff decomposes the given elements, while the inner and outer cutoffs give exact annihilation and support relations, respectively.

Lemma 5.1. *Let $\mathcal{S}$ be a class of C*-algebras, and let $A \in \mathrm{TA}_0(\mathcal{S})$. For every finite set $\{f_1, f_2, \ldots, f_n\} \subseteq A$ and every $\varepsilon > 0$, there are a C*-subalgebra $C \subseteq A$ with $C \in \mathcal{S}$, positive contractions $\tilde{h}, h \in C$ with $\|h\| = 1$, and elements $f_i^{(0)}, f_i^{(1)} \in A$, $1 \leq i \leq n$, such that*

1. $\left\| f_i - (f_i^{(0)} + f_i^{(1)}) \right\| < \varepsilon$;
2. $f_i^{(1)} \in C$;
3. $h f_i^{(0)} = f_i^{(0)} h = 0$;
4. $\max\{ \left\| [f_i^{(1)}, h] \right\|, \left\| [f_i^{(1)}, h^{1/2}] \right\| \} < \varepsilon$;
5. $\tau(1 - h^2) < \varepsilon$ *for every* $\tau \in \mathrm{T}(A)$; *and*
6. $\tilde{h} h = h$ *and* $\tilde{h} f_i^{(1)} = f_i^{(1)} = f_i^{(1)} \tilde{h}$.

Proof. Put $M = \max\{1, \|f_1\|, \ldots, \|f_n\|\}$ and $x_i = f_i / M$. Choose $0 < \eta < 1$ such that

$$3M\eta < \varepsilon \quad \text{and} \quad M\eta < \varepsilon.$$

For a positive contraction e, set

$$q(e) = g_{1/4}(e), \qquad h(e) = g_{1/2}(e).$$

By the standard functional-calculus commutator estimate, there is $\delta > 0$ such that, whenever e is a positive contraction, $\|x\| \leq 2$, and $\|[e, x]\| < \delta$, one has

$$\max\left\{ \left\|[q(e)^{1/2}, x]\right\|, \left\|[(1-q(e))^{1/2}, x]\right\|, \|[h(e), x]\|, \left\|[h(e)^{1/2}, x]\right\| \right\} < \eta. \tag{5.1}$$

Apply $A \in \mathrm{TA}_0(\mathcal{S})$ to $\{x_1, \ldots, x_n\}$ with tolerance smaller than $\min\{\delta, \eta, \varepsilon\}$. We obtain $C \subseteq A$ with $C \in \mathcal{S}$, a positive contraction $e \in C$ of norm one, and $x_1', \ldots, x_n' \in A$ such that

$$\|x_i - x_i'\| < \eta, \qquad \|[e, x_i']\| < \delta, \qquad ex_i'e \in C, \qquad \mathrm{d}_\tau(1-e) < \varepsilon \tag{5.2}$$

for all i and all $\tau \in \mathrm{T}(A)$. Since $\|x_i\| \leq 1$ and $\eta < 1$, we also have $\|x_i'\| < 2$.

Set

$$q = g_{1/4}(e), \qquad h = g_{1/2}(e), \qquad \tilde{h} = g_{1/8}(e),$$

and put

$$f_i^{(0)} = M(1-q)^{1/2} x_i' (1-q)^{1/2},$$
$$f_i^{(1)} = M q^{1/2} x_i' q^{1/2}.$$

The pointwise relations among the three taper functions give

$$qh = h, \qquad \tilde{h}q = q, \qquad \tilde{h}h = h. \tag{5.3}$$

The distinct cutoffs ensure these exact relations on the transition regions.

The function

$$r(t) = \begin{cases} g_{1/4}(t)^{1/2}/t, & t > 0, \\ 0, & t = 0, \end{cases}$$

is continuous, since it vanishes on a neighbourhood of zero. Hence

$$f_i^{(1)} = M r(e)(ex_i'e)r(e) \in C.$$

Moreover, by (5.1),

$$\begin{aligned} \left\| x_i' - \left((1-q)^{1/2} x_i' (1-q)^{1/2} + q^{1/2} x_i' q^{1/2} \right) \right\| \\ \leq \left\|[(1-q)^{1/2}, x_i']\right\| + \left\|[q^{1/2}, x_i']\right\| < 2\eta. \end{aligned}$$

Together with (5.2), this yields

$$\left\| f_i - (f_i^{(0)} + f_i^{(1)}) \right\| < 3M\eta < \varepsilon.$$

Since q, h, and $\tilde{h}$ commute, (5.3) also gives

$$hf_i^{(0)} = f_i^{(0)}h = 0, \qquad \tilde{h}f_i^{(1)} = f_i^{(1)} = f_i^{(1)}\tilde{h}.$$

For $s \in \{1/2, 1\}$, again using (5.1),

$$\left\| [f_i^{(1)}, h^s] \right\| = M \left\| q^{1/2}[x_i', h^s]q^{1/2} \right\| < M\eta < \varepsilon.$$

Finally, $\|h\| = 1$, and in the tracial von Neumann algebra associated to τ one has

$$1 - h^2 \leq 1_{[0,1)}(e) = s(1-e).$$

Consequently,

$$\tau(1 - h^2) \leq \mathrm{d}_\tau(1-e) < \varepsilon,$$

which completes the proof. □

Remark 5.2 (Nested taper functions). Lemma 5.1 is an abstract version of [47, Lemma 4.1]. The taper construction in the proof of the cited lemma requires a distinction between the outer and inner taper functions. Using the same coefficients for h_U and h_V introduces their squares in the product and does not give $h_U h_V = h_V$. Separate taper functions (a_r) and (b_r) satisfy the required identity when

$$a_r b_r = b_r$$

and both vary sufficiently slowly between adjacent tower layers. For example, for $L \in \mathbb{N}$ and depths $0 \leq r \leq 2L+1$, take

$$a_r = \min\{(r-1)_+/L, 1\}, \qquad b_r = \min\{(r-L-1)_+/L, 1\}.$$

Both successive increments are at most $1/L$, and $a_r = 1$ wherever $b_r \neq 0$. At depth $2L+1$ both functions are one. Choosing the Følner shapes for this depth preserves the commutator and trace estimates and gives the exact identity. The nested cutoffs $\tilde{h}$, q, and h in the proof above are the functional-calculus counterpart of this outer–inner separation.

Proposition 5.3 (cf. Proposition 4.7 of [47]). *Let $\mathcal{S}$ be a class such that every $C \in \mathcal{S}$ can be locally approximated by C^*-subalgebras of finite nuclear dimension. Let A be a separable, simple, unital C^*-algebra such that $A \in \mathrm{TA}_0(\mathcal{S})$, and suppose that A is tracially $\bar{m}$-almost divisible for some $\bar{m} \in \{0, 1, 2, \ldots\}$ in the sense of [70, Definition 3.5(ii)]. It suffices to assume the divisibility estimates in that definition only for tracial states. Then, for every finite set $\mathcal{F} \subseteq A$, every $\varepsilon > 0$, and every $k \in \mathbb{N}$, there is a c.p.c. order zero map $\phi : \mathrm{M}_k(\mathbb{C}) \to A$ such that*

1. *$\|[\phi(a), f]\| < \varepsilon$ for $f \in \mathcal{F}$ and $a \in \mathrm{M}_k(\mathbb{C})$ with $\|a\| \leq 1$;*
2. *$\mathrm{d}_\tau(1_A - \phi(1_k)) < \varepsilon$ for every $\tau \in \mathrm{T}(A)$.*

Proof. We first prove the corresponding statement with the second condition replaced by

$$\tau(1_A - \phi(1_k)) < \beta, \qquad \tau \in \mathrm{T}(A), \tag{5.4}$$

for arbitrary $\gamma, \beta > 0$, with commutators bounded by γ. By decreasing γ and β, we may assume that $\gamma, \beta < 1$.

Write $\mathcal{F} = \{f_1, \ldots, f_n\}$ and put $M = \max\{1, \|f_1\|, \ldots, \|f_n\|\}$. Choose $\kappa > 0$ such that

$$2M\kappa < \frac{\gamma}{8} \quad \text{and} \quad \kappa < \frac{\beta}{8}. \tag{5.5}$$

By weak stability of order zero maps from $\mathrm{M}_k(\mathbb{C})$ [41, Proposition 2.5], there is $\delta > 0$ such that every c.p.c. map $\rho : \mathrm{M}_k(\mathbb{C}) \to E$ into a C*-algebra E satisfying

$$\|\rho(a)\rho(b)\| < \delta$$

for orthogonal positive contractions $a, b \in \mathrm{M}_k(\mathbb{C})$ is within κ of a c.p.c. order zero map.

Apply Lemma 5.1 with a sufficiently small tolerance. It gives $C \in \mathcal{S}$, $h \in C_+$, and decompositions $f_i \approx f_i^{(0)} + f_i^{(1)}$ with $f_i^{(1)} \in C$ and $hf_i^{(0)} = f_i^{(0)}h = 0$. Since C is locally approximated by subalgebras of finite nuclear dimension, there are a subalgebra $D \subseteq C$ of finite nuclear dimension, a positive contraction $e \in D$, and $b_i \in D$ such that

$$\left\| f_i - f_i^{(0)} - b_i \right\| < \frac{\gamma}{16}, \tag{5.6}$$

$$\left\| e^{1/2} f_i^{(0)} \right\| + \left\| f_i^{(0)} e^{1/2} \right\| < \frac{\gamma}{16}, \tag{5.7}$$

$$\left\| [e^{1/2}, b_i] \right\| < \frac{\gamma}{32}, \tag{5.8}$$

$$\tau(1 - e) < \frac{\beta}{8}, \qquad \tau \in \mathrm{T}(A). \tag{5.9}$$

These inequalities follow by choosing the tolerance in Lemma 5.1 sufficiently small and approximating h and the $f_i^{(1)}$ in a subalgebra of C of finite nuclear dimension. Continuity of $t \mapsto t^{1/2}$ gives (5.7)–(5.8).

The unital algebra $D + \mathbb{C}1_A$ has finite nuclear dimension. Applying [70, Lemma 5.11] to it, with the finite set consisting of e and the b_i (rescaled to contractions), gives a c.p.c. order zero map $\varphi : \mathrm{M}_k(\mathbb{C}) \to A$ such that, for every contraction $a \in \mathrm{M}_k(\mathbb{C})$,

$$\|[\varphi(a), e]\| < \delta, \tag{5.10}$$

$$\|[\varphi(a), b_i]\| < \frac{\gamma}{8}, \tag{5.11}$$

$$\tau(1 - \varphi(1_k)) < \frac{\beta}{8}, \qquad \tau \in \mathrm{T}(A). \tag{5.12}$$

In the proof of the cited lemma, divisibility gives order zero maps in the hereditary subalgebras generated by coordinate positive contractions, with lower bounds on $\mathrm{T}(A)$. The remainder of that proof uses these bounds only for the induced limit traces $\mathrm{T}_\infty(A)$, together with order zero

lifting, norm estimates, and finite nuclear dimension of the approximating subalgebra. Thus the proof also applies when the divisibility estimates are assumed only for tracial states, without identifying quasitraces with traces. Define

$$\rho(a) = e^{1/2}\varphi(a)e^{1/2}, \qquad a \in \mathrm{M}_k(\mathbb{C}).$$

If a, b are orthogonal positive contractions, then

$$\rho(a)\rho(b) = e^{1/2}\varphi(a)[e, \varphi(b)]e^{1/2},$$

and hence (5.10) and the choice of δ give a c.p.c. order zero map $\psi : \mathrm{M}_k(\mathbb{C}) \to A$ with

$$\|\psi(a) - \rho(a)\| < \kappa, \qquad \|a\| \leq 1. \tag{5.13}$$

For $\|a\| \leq 1$, equations (5.6)– (5.11) give

$$\begin{aligned}\|[\rho(a), f_i]\| \leq 2\left\|f_i - f_i^{(0)} - b_i\right\| + \left\|e^{1/2}f_i^{(0)}\right\| + \left\|f_i^{(0)}e^{1/2}\right\| \\ + 2\left\|[e^{1/2}, b_i]\right\| + \|[\varphi(a), b_i]\|.\end{aligned}$$

Together with (5.5) and (5.13), this is strictly less than γ. Moreover, traciality and positivity give

$$\begin{aligned}1 - \tau(\rho(1_k)) &= 1 - \tau(e\varphi(1_k)) \\ &\leq \tau(1-e) + \tau(1 - \varphi(1_k)) < \frac{\beta}{4}.\end{aligned}$$

Thus (5.13) implies (5.4). This proves the trace version.

For the dimension-function estimate, choose $\eta > 0$ such that

$$\sqrt{\eta} < \varepsilon \quad \text{and} \quad 2M\sqrt{\eta} < \frac{\varepsilon}{2}.$$

Also require $\eta < 1$. Apply the trace version with centrality tolerance $\varepsilon/2$ and trace tolerance η, obtaining a map ψ. Following [47, Lemma 4.6], set $r = \sqrt{\eta}$, $v = \psi(1_k)$, and $f(t) = \min\{t/(1-r), 1\}$. If π_ψ is the supporting *-homomorphism, define

$$\phi(x) = \pi_\psi(x)f(v), \qquad x \in \mathrm{M}_k(\mathbb{C}).$$

This is a c.p.c. order zero map into A, since $f(t)/t$ extends continuously to $t = 0$. Moreover, $\|\phi - \psi\| \leq r$. Writing μ_τ for the spectral probability measure of v, we have

$$\mathrm{d}_\tau(1 - f(v)) = \mu_\tau([0, 1-r)) \leq \frac{\tau(1-v)}{r} < \frac{\eta}{r} = r.$$

Consequently,

$$\mathrm{d}_\tau(1 - \phi(1_k)) < \sqrt{\eta} < \varepsilon, \qquad \tau \in \mathrm{T}(A).$$

The choice of η preserves the required commutator estimate. □

Weak tracial $\mathcal{Z}$-absorption requires approximately central order zero maps whose images of the unit have uniformly large trace.

Definition 5.4. Let A be a unital C*-algebra with $\mathrm{T}(A) \neq \varnothing$. We say that A is *weakly tracially $\mathcal{Z}$-absorbing* if, for every $k \in \mathbb{N}$, every finite set $\mathcal{F} \subseteq A$, and every $\varepsilon > 0$, there is a c.p.c. order zero map $\phi : \mathrm{M}_k(\mathbb{C}) \to A$ such that

1. $\|[\phi(a), f]\| < \varepsilon$ for $f \in \mathcal{F}$ and $a \in \mathrm{M}_k(\mathbb{C})$ with $\|a\| \leq 1$;
2. $\tau(\phi(1_k)) > 1 - \varepsilon$ for every $\tau \in \mathrm{T}(A)$.

For hereditary approximation algebras, we use a relative nonunital version in which traces are normalized at a local unit c.

Definition 5.5. A C*-algebra C is *weakly tracially $\mathcal{Z}$-absorbing relative to a local unit* if, for every $k \in \mathbb{N}$, every $\varepsilon > 0$, every finite set $\mathcal{F} \subseteq C$, and every norm-one positive contraction $c \in C$ satisfying

$$cf = fc = f, \qquad f \in \mathcal{F},$$

there is a c.p.c. order zero map $\phi : \mathrm{M}_k(\mathbb{C}) \to C$ such that

1. $\|[\phi(a), f]\| < \varepsilon$ for $f \in \mathcal{F}$ and $a \in \mathrm{M}_k(\mathbb{C})$ with $\|a\| \leq 1$;
2. $|\tau(\phi(1_k)) - 1| < \varepsilon$ for every bounded positive trace τ on C satisfying $\tau(c) = 1$.

Lemma 5.6. *Let A be a simple unital C*-algebra that is weakly tracially $\mathcal{Z}$-absorbing in the sense of Definition 5.4. Then every nonzero hereditary C*-subalgebra of A satisfies Definition 5.5. Consequently, so does every finite direct sum of matrix algebras over hereditary C*-subalgebras of A.*

In particular, for a simple unital C-algebra with $\mathrm{T}(A) \neq \varnothing$, Definitions 5.4 and 5.5 are equivalent.*

Proof. Let $H \subseteq A$ be a nonzero hereditary C*-subalgebra, let $\mathcal{F} \subseteq H$ be finite, and let $c \in H$ be a norm-one positive contraction satisfying $cf = fc = f$ for $f \in \mathcal{F}$. Since c is full in A, there are $z_1, \ldots, z_l \in A$ such that

$$b := \sum_{j=1}^{l} z_j^* c z_j \geq \frac{1}{2} 1_A.$$

Every bounded positive trace τ on H has the standard lower semicontinuous extension $\tilde{\tau}$ to the ideal generated by H, which is all of A. If $\tau(c) = 1$, then this extension is bounded and

$$\|\tilde{\tau}\| = \tilde{\tau}(1_A) \leq 2\tilde{\tau}(b) \leq 2 \sum_{j=1}^{l} \|z_j\|^2 =: L. \tag{5.14}$$

This bound is uniform over traces normalized at c.

Put $M = \max\{1, \|f\| : f \in \mathcal{F}\}$. Choose $\kappa, \eta > 0$ such that

$$2M\kappa < \frac{\varepsilon}{3} \quad \text{and} \quad L(\kappa + \eta) < \frac{\varepsilon}{3}.$$

Choose an order zero stability tolerance $\delta > 0$ corresponding to κ. Definition 5.4, applied with a sufficiently small centrality tolerance to $\mathcal{F} \cup \{c, c^{1/2}\}$ and with trace tolerance η, gives a c.p.c. order zero map $\varphi' : \mathrm{M}_k(\mathbb{C}) \to A$ such that

$$\begin{aligned} \|[\varphi'(a), c]\| &< \delta, \\ \|[\varphi'(a), f]\| + M \left\|[\varphi'(a), c^{1/2}]\right\| &< \frac{\varepsilon}{3}, \end{aligned}$$

for $f \in \mathcal{F}$ and $\|a\| \leq 1$, and

$$\sigma(1 - \varphi'(1_k)) < \eta, \qquad \sigma \in \mathrm{T}(A).$$

The c.p.c. map

$$\rho(a) = c^{1/2}\varphi'(a)c^{1/2} \in \overline{cAc} \subseteq H$$

is δ-order zero: if a, b are orthogonal positive contractions, then

$$\rho(a)\rho(b) = c^{1/2}\varphi'(a)[c, \varphi'(b)]c^{1/2}.$$

Weak stability therefore gives a c.p.c. order zero map $\phi : \mathrm{M}_k(\mathbb{C}) \to \overline{cAc}$ with $\|\phi - \rho\| < \kappa$. Since $c^{1/2}f = f = fc^{1/2}$,

$$\|[\rho(a), f]\| \leq \|[\varphi'(a), f]\| + \|f\| \left\|[\varphi'(a), c^{1/2}]\right\|,$$

so $\|[\phi(a), f]\| < \varepsilon$.

Let τ be a bounded positive trace on H with $\tau(c) = 1$. If $s = \|\tilde{\tau}\|$, then $s^{-1}\tilde{\tau} \in \mathrm{T}(A)$, and (5.14) gives

$$0 \leq 1 - \tau(\rho(1_k)) = \tilde{\tau}\big(c(1 - \varphi'(1_k))\big) \leq \tilde{\tau}(1 - \varphi'(1_k)) < L\eta.$$

The perturbation from ρ to ϕ changes this value by at most $L\kappa$, proving Definition 5.5 for H.

Applying Definition 5.4 simultaneously to all matrix entries and using $a \mapsto \varphi'(a) \otimes 1_r$ shows that $\mathrm{M}_r(A)$ satisfies Definition 5.4. The first part therefore applies to every hereditary subalgebra of $\mathrm{M}_r(A)$, in particular to $\mathrm{M}_r(H)$. For finite direct sums, let $c = (c_j)$ be the prescribed local unit. Apply the relative property to $c_j / \|c_j\|$ when $c_j \neq 0$ and multiply the resulting map by $\|c_j\|$. Homogeneity of the normalized trace estimate and $\sum_j \tau_j(c_j) = 1$ give the required estimate. If $\|c_j\| < 1$, the relation $c_j f_j = f_j$ implies $f_j = 0$.

Finally, Definition 5.5 with $c = 1_A$ implies Definition 5.4; the converse is the case $H = A$ just proved. □

Proposition 5.7. *Let $\mathcal{S}$ be a class of* C*-*algebras, not necessarily unital, each of which satisfies Definition 5.5. If the unital* C*-*algebra A belongs to $\mathrm{TA}_0(\mathcal{S})$, then, for every finite set $\mathcal{F} \subseteq A$, every $\varepsilon > 0$, and every $k \in \mathbb{N}$, there is a c.p.c. order zero map $\phi : \mathrm{M}_k(\mathbb{C}) \to A$ such that*

1. $\|[\phi(a), f]\| < \varepsilon$ *for* $f \in \mathcal{F}$ *and* $a \in \mathrm{M}_k(\mathbb{C})$ *with* $\|a\| \leq 1$;

2. $\mathrm{d}_\tau(1_A - \phi(1_k)) < \varepsilon$ *for every* $\tau \in \mathrm{T}(A)$.

In particular, A is weakly tracially $\mathcal{Z}$-absorbing in the sense of Definition 5.4.

Proof. As in Proposition 5.3, we first obtain commutators bounded by γ and tracial defects bounded by β, for arbitrary $\gamma, \beta > 0$. We may assume that $\gamma, \beta < 1$. Write $\mathcal{F} = \{f_1, \ldots, f_n\}$, $M = \max\{1, \|f_1\|, \ldots, \|f_n\|\}$, and choose $\kappa > 0$ with

$$2M\kappa < \frac{\gamma}{8} \quad \text{and} \quad \kappa < \frac{\beta}{8}.$$

Let $\delta > 0$ be an order zero stability tolerance corresponding to κ.

Apply Lemma 5.1 with a sufficiently small tolerance. We obtain $C \in \mathcal{S}$, positive contractions $h, \tilde{h} \in C$, and $f_i^{(0)}, f_i^{(1)} \in A$ such that

$$\left\| f_i - f_i^{(0)} - f_i^{(1)} \right\| < \frac{\gamma}{16}, \tag{5.15}$$

$$h^{1/2} f_i^{(0)} = f_i^{(0)} h^{1/2} = 0, \tag{5.16}$$

$$\left\| [h^{1/2}, f_i^{(1)}] \right\| < \frac{\gamma}{32}, \tag{5.17}$$

$$\tau(1-h) < \frac{\beta}{8}, \qquad \tau \in \mathrm{T}(A), \tag{5.18}$$

and

$$\tilde{h}h = h = h\tilde{h}, \qquad \tilde{h} f_i^{(1)} = f_i^{(1)} = f_i^{(1)} \tilde{h}.$$

Apply Definition 5.5 in C, with local unit $\tilde{h}$ and with the finite set $\{h, f_i^{(1)} : 1 \leq i \leq n\}$. Decreasing its tolerance if necessary, we obtain a c.p.c. order zero map $\varphi : \mathrm{M}_k(\mathbb{C}) \to C$ such that, for $\|a\| \leq 1$,

$$\|[\varphi(a), h]\| < \delta, \tag{5.19}$$

$$\left\| [\varphi(a), f_i^{(1)}] \right\| < \frac{\gamma}{8}, \tag{5.20}$$

and

$$|\sigma(\varphi(1_k)) - 1| < \frac{\beta}{8} \tag{5.21}$$

for every bounded positive trace σ on C with $\sigma(\tilde{h}) = 1$.

Fix $\tau \in \mathrm{T}(A)$ and set $s = \tau(\tilde{h})$. By (5.18), $s \geq \tau(h) > 1 - \beta/8$, so $s > 0$; hence $s^{-1}\tau|_C$ is one of the traces in (5.21). It follows that

$$\tau(1 - \varphi(1_k)) \leq \tau(1 - \tilde{h}) + \left| \tau(\tilde{h}) - \tau(\varphi(1_k)) \right| < \frac{\beta}{4}. \tag{5.22}$$

Set $\rho(a) = h^{1/2}\varphi(a)h^{1/2}$. By (5.19), this is a δ-order zero c.p.c. map, and there is a c.p.c. order zero map $\psi : \mathrm{M}_k(\mathbb{C}) \to A$ with $\|\psi - \rho\| < \kappa$. Equations (5.15)– (5.20) give, for $\|a\| \leq 1$,

$$\|[\rho(a), f_i]\| \leq 2 \left\| f_i - f_i^{(0)} - f_i^{(1)} \right\| + 2 \left\| [h^{1/2}, f_i^{(1)}] \right\| + \left\| [\varphi(a), f_i^{(1)}] \right\|.$$

The contribution from $f_i^{(0)}$ vanishes by (5.16). The displayed bounds and $\|\psi - \rho\| < \kappa$ give $\|[\psi(a), f_i]\| < \gamma$. Furthermore,

$$1 - \tau(\rho(1_k)) \leq \tau(1-h) + \tau(1 - \varphi(1_k)) < \frac{3\beta}{8},$$

so $\tau(1 - \psi(1_k)) < \beta$.

Finally, apply this trace version with centrality tolerance $\varepsilon/2$ and with a trace tolerance $\eta > 0$ satisfying $\sqrt{\eta} < \varepsilon$ and $2M\sqrt{\eta} < \varepsilon/2$ and $\eta < 1$. The cutoff in the last paragraph of the proof of Proposition 5.3 gives a c.p.c. order zero map ϕ with $\|\phi - \psi\| \leq \sqrt{\eta}$ and $\mathrm{d}_\tau(1 - \phi(1_k)) < \sqrt{\eta}$ for all $\tau \in \mathrm{T}(A)$. The choice of η gives both conclusions. □

Corollary 5.8. *Let A be a separable, simple, unital, infinite-dimensional* C*-*algebra, let Γ be a countable discrete amenable group acting on A, and put $B = A \rtimes_{\alpha,r} \Gamma$. Suppose that the action has WRP and that B is simple. If A is weakly tracially $\mathcal{Z}$-absorbing, then B is weakly tracially $\mathcal{Z}$-absorbing. If, in addition, B has strict comparison, then B is tracially $\mathcal{Z}$-absorbing; if B is also nuclear, then $B \cong B \otimes \mathcal{Z}$.*

Proof. By Theorem 3.15, $B \in \mathrm{TA}_0(\mathcal{S})$, where $\mathcal{S}$ consists of finite direct sums of matrix algebras over hereditary C*-subalgebras of A. Lemma 5.6 and Proposition 5.7 show that B is weakly tracially $\mathcal{Z}$-absorbing.

Now assume that B has strict comparison, and let $0 \neq b \in B_+$. The function $\tau \mapsto \mathrm{d}_\tau(b)$ is strictly positive and lower semicontinuous on the compact space $\mathrm{T}(B)$; hence

$$\mu := \min_{\tau \in \mathrm{T}(B)} \mathrm{d}_\tau(b) > 0.$$

Apply Proposition 5.7 with defect tolerance less than μ. Strict comparison gives $1 - \phi(1_k) \precsim b$, so B is tracially $\mathcal{Z}$-absorbing. In the nuclear case, [37] then gives $B \cong B \otimes \mathcal{Z}$. □

Corollary 5.9. *Let A be a separable, simple, unital, infinite-dimensional* C*-*algebra which has real rank zero and is weakly tracially $\mathcal{Z}$-absorbing. Let Γ be a countably infinite discrete amenable group acting on A, and put $B = A \rtimes_{\alpha,r} \Gamma$. Suppose that the action has WRP and that B is simple, has strict comparison, is nuclear, and satisfies the UCT. Then B is tracially AF, equivalently,* $\mathrm{TR}(B) = 0$. *In fact, B is isomorphic to a simple unital AH algebra with slow dimension growth and real rank zero.*

Proof. Corollary 5.8 gives $B \cong B \otimes \mathcal{Z}$. Since WRP requires $\mathrm{T}(A) \neq \emptyset$ and Γ is amenable, the fixed-point theorem gives $\mathcal{M}_1(A, \Gamma) \neq \emptyset$. Proposition 3.8 identifies this simplex with $\mathrm{T}(B)$. Since B is simple, every trace on B is faithful; in particular, B is stably finite.

Since Γ is infinite and A has real rank zero, Corollary 3.16 gives

$$\overline{\rho_B(\mathrm{K}_0(B))} = \mathrm{Aff}(\mathrm{T}(B)).$$

The algebra B is exact because it is nuclear. Therefore [58, Theorem 7.2], applied to the simple, unital, finite, $\mathcal{Z}$-stable algebra B, shows that $\mathrm{RR}(B) = 0$.

Finiteness and $\mathcal{Z}$-stability give stable rank one and weak unperforation, while real rank zero gives the refinement property for projections. Thus $(\mathrm{K}_0(B), \mathrm{K}_0(B)_+, [1_B])$ is a countable simple weakly unperforated ordered Riesz group with order unit. The range theorem allows torsion in $\mathrm{K}_0(B)$ and does not require this group to be a dimension group. Moreover, the canonical map

$$\mathrm{T}(B) \longrightarrow \mathrm{S}(\mathrm{K}_0(B), [1_B])$$

is an affine homeomorphism. Indeed, injectivity follows from the density of $\rho_B(\mathrm{K}_0(B))$. Conversely, a state on the ordered K_0-group extends to a quasitrace by the state-extension theorem of [8], and this quasitrace is a trace because B is exact [35].

This homeomorphism preserves extreme points. Moreover, density of $\rho_B(\mathrm{K}_0(B))$ excludes $\mathrm{K}_0(B)/\operatorname{Inf}(\mathrm{K}_0(B)) \cong \mathbb{Z}$, where $\operatorname{Inf}(\mathrm{K}_0(B)) = \ker \rho_B$: a cyclic pairing range containing the constant function one is discrete. Thus the hypotheses of the range theorem for simple AH algebras with slow dimension growth [68] are satisfied; see also [43]. There is therefore a simple unital AH algebra C with slow dimension growth and an isomorphism

$$\mathrm{Ell}(C) \cong \mathrm{Ell}(B).$$

The density of the K_0-range implies that C is infinite-dimensional. The algebra C is therefore $\mathcal{Z}$-stable by [65]; it is also nuclear and satisfies the UCT. By [13], both B and C have finite nuclear dimension. By [23, Theorem 4.9] and quasidiagonality of traces [63], both stably finite algebras belong to the class $\mathcal{N}_1$. The classification theorem for this class [32, 33] therefore yields $B \cong C$. Consequently, C has real rank zero. Finally, for a simple unital AH algebra whose K_0-range is dense, $\mathcal{Z}$-stability is equivalent to being tracially AF by [67, Theorem 3.11]. Thus B is tracially AF and is isomorphic to an AH algebra with slow dimension growth and real rank zero. □

Theorem 5.10. *Let $\mathcal{S}$ be a class such that every $C \in \mathcal{S}$ can be locally approximated by C^*-subalgebras of finite nuclear dimension. Let A be a separable, simple, unital, infinite-dimensional, nuclear C^*-algebra with $A \in \mathrm{TA}_0(\mathcal{S})$. Then $A \cong A \otimes \mathcal{Z}$ if and only if*

1. *A has strict comparison of positive elements; and*
2. *for every $f \in \mathrm{LAff}_{++}(\mathrm{T}(A))$ there is $x \in (A \otimes \mathcal{K})_+$ such that $\mathrm{d}_\tau(x) = f(\tau)$ for every $\tau \in \mathrm{T}(A)$.*

Consequently, within this class, $\mathcal{Z}$-stability is also equivalent to the canonical ordered-semigroup description

$$\mathrm{Cu}(A) \cong \mathrm{V}(A) \sqcup \mathrm{LAff}_{++}(\mathrm{T}(A)).$$

Proof. If A is $\mathcal{Z}$-stable, then strict comparison and the stated description of the Cuntz semigroup are standard; see [58, 12, 16]. We prove the converse.

We show that A is tracially 0-almost divisible. Let $a \in \mathrm{M}_N(A)_+$ be a positive contraction, let $k \in \mathbb{N}$, and let $\varepsilon > 0$. The case $a = 0$ is immediate; assume that $a \neq 0$. Choose

$$0 < \theta < \min\left\{\frac{1}{2}, \frac{\varepsilon}{2N}\right\}.$$

The function

$$f(\tau) = \frac{1-\theta}{k} \mathrm{d}_\tau(a), \qquad \tau \in \mathrm{T}(A),$$

belongs to $\mathrm{LAff}_{++}(\mathrm{T}(A))$. By (2), there is a positive contraction $x \in A \otimes \mathcal{K}$ such that

$$\mathrm{d}_\tau(x) = \frac{1-\theta}{k} \mathrm{d}_\tau(a), \qquad \tau \in \mathrm{T}(A). \tag{5.23}$$

Since A is nuclear, every normalized quasitrace is a trace. Thus $k\mathrm{d}_\tau(x) < \mathrm{d}_\tau(a)$ for every normalized quasitrace, and strict comparison gives

$$k[x] \leq [a] \quad \text{in } \mathrm{Cu}(A). \tag{5.24}$$

For $0 < \delta < 1/2$, let $g_{\delta,2\delta}$ be zero on $[0,\delta]$, one on $[2\delta,1]$, and linear in between. As $\delta \searrow 0$,

$$k\tau(g_{\delta,2\delta}(x)) \nearrow k\mathrm{d}_\tau(x).$$

Since $\mathrm{d}_\tau(a) \geq \tau(a)$ and $\mathrm{d}_\tau(a) \leq N$, equation (5.23) gives

$$k\mathrm{d}_\tau(x) = (1-\theta)\mathrm{d}_\tau(a) > \tau(a) - \frac{\varepsilon}{2}.$$

For each fixed δ, the element $g_{\delta,2\delta}(x)$ belongs to the Pedersen ideal of $A \otimes \mathcal{K}$, so the function $\tau \mapsto \tau(g_{\delta,2\delta}(x))$ is continuous. Compactness of $\mathrm{T}(A)$ and monotonicity as $\delta \searrow 0$ therefore give one $\delta > 0$ such that

$$k\tau(g_{\delta,2\delta}(x)) > \tau(a) - \varepsilon, \qquad \tau \in \mathrm{T}(A). \tag{5.25}$$

Apply [70, Proposition 2.12] in the ambient algebra $A \otimes \mathcal{K}$ to (5.24), with the function $g_{\delta,2\delta}$. Its hereditary target is $\mathrm{her}(a) \subseteq \mathrm{M}_N(A)$. We obtain a c.p.c. order zero map

$$\psi : \mathrm{M}_k(\mathbb{C}) \longrightarrow \mathrm{her}(a)$$

such that $\psi(e_{11})$ is Murray–von Neumann equivalent to $g_{\delta,2\delta}(x)$. Since a trace composed with an order zero map is a trace,

$$\tau(\psi(1_k)) = k\tau(\psi(e_{11})) = k\tau(g_{\delta,2\delta}(x)) > \tau(a) - \varepsilon$$

for every $\tau \in \mathrm{T}(A)$. This proves that A is tracially 0-almost divisible.

Proposition 5.3, with $\bar{m} = 0$, gives approximately central order zero maps with uniformly small dimension functions on their complements. Given $0 \neq b \in A_+$, simplicity and compactness of $\mathrm{T}(A)$ give $\min_{\tau \in \mathrm{T}(A)} \mathrm{d}_\tau(b) > 0$. Choosing the defect smaller than this minimum and using strict comparison yields $1 - \phi(1_k) \precsim b$. Thus A is tracially $\mathcal{Z}$-absorbing, and nuclearity gives $A \cong A \otimes \mathcal{Z}$ by [37].

Finally, the canonical Cuntz-semigroup description implies (1) and (2), while $\mathcal{Z}$-stability gives that description by the results cited in the first paragraph. □

Since locally finite nuclear dimension passes to hereditary subalgebras, matrix algebras, and finite direct sums, Theorems 3.15 and 5.10 give the following consequence.

Theorem 5.11. *Let A be a separable, unital C^*-algebra with $\mathrm{T}(A) \neq \varnothing$ and locally finite nuclear dimension, and let Γ be a countably infinite discrete amenable group acting on A. Suppose that the action has WRP and that $B = A \rtimes_{\alpha,r} \Gamma$ is simple. Then B is $\mathcal{Z}$-stable if and only if*

$$\mathrm{Cu}(B) \cong \mathrm{V}(B) \sqcup \mathrm{LAff}_{++}(\mathrm{T}(B))$$

as ordered semigroups, via the canonical identification.

Proof. Locally finite nuclear dimension passes to hereditary subalgebras as follows. Let $H \subseteq A$ be hereditary and let $\mathcal{F} \subseteq H$ be finite. Choose a positive contraction $h \in H$ such that $h^{1/2}xh^{1/2}$ is close to x for $x \in \mathcal{F}$. For $0 < \eta < \delta < 1$, local approximation gives a subalgebra $D \subseteq A$ of finite nuclear dimension, a positive contraction $d \in D$ with $\|d - h\| < \eta$, and $x_D \in D$ close to each $x \in \mathcal{F}$. Put

$$q = (d-\delta)_+, \qquad f(t) = \begin{cases} \sqrt{(t-\delta)_+/t}, & t > 0, \\ 0, & t = 0, \end{cases} \qquad z = h^{1/2} f(d), \qquad a_0 = z^* z.$$

Since $q = f(d)df(d)$, we have

$$(1-\eta/\delta)q \leq a_0 \leq (1+\eta/\delta)q, \qquad \mathrm{her}_A(a_0) = \mathrm{her}_A(q).$$

Let $z = va_0^{1/2}$ be its polar decomposition in A^{**}. Writing $c = 1 - \eta/\delta > 0$, the inequality $cq \leq a_0$ gives

$$\left\| \left(v - z(a_0 + 1/n)^{-1/2}\right) q^{1/2} \right\|^2 \leq \frac{1}{cn}, \qquad n \in \mathbb{N}.$$

Thus $vq^{1/2} \in A$ by norm approximation. In particular, polar conjugation embeds $E = \mathrm{her}_D(q)$ into $\mathrm{her}_A(zz^*) \subseteq H$. Moreover,

$$\left\| vq^{1/2} - q^{1/2} \right\| \leq \left\| a_0^{1/2} - q^{1/2} \right\| + \left\| z - q^{1/2} \right\| \leq 2\sqrt{\eta}.$$

Consequently, $v(q^{1/2}x_D q^{1/2})v^*$ approximates x if $\left\|h^{1/2}xh^{1/2} - x\right\|$, δ, and then η are sufficiently small, since $\|q - h\| \leq \delta + \eta$. The algebra E has finite nuclear dimension by [72, Proposition 2.5], so its image in H is the required local approximating subalgebra. Matrix amplifications and finite direct sums preserve this local property by entrywise approximation.

Theorem 3.15 gives $B \in \mathrm{TA}_0(\mathcal{S})$, where $\mathcal{S}$ consists of finite direct sums of matrix algebras over hereditary C^*-subalgebras of A. Every member of $\mathcal{S}$ locally has finite nuclear dimension. Moreover, A is nuclear, and amenability of Γ implies that B is nuclear. Averaging a tracial state on A gives an invariant trace, which extends to B and is faithful by simplicity. The canonical group unitaries make B infinite dimensional. Theorem 5.10 gives the conclusion. □

5.2 Realization of strictly positive affine rank functions

Definition 5.12. Let A be a unital C*-algebra with $\mathrm{T}(A) \neq \varnothing$. We say that A has *property (E1)* if, for every $\varepsilon > 0$ and every $\alpha \in \mathrm{Aff}(\mathrm{T}(A))$ with $\alpha > 0$, there is $a \in \mathrm{M}_\infty(A)_+$ such that

$$|\mathrm{d}_\tau(a) - \alpha(\tau)| < \varepsilon, \qquad \tau \in \mathrm{T}(A).$$

We say that A has *property (E2)* if the following holds. Let $(\alpha_n)_{n=1}^\infty$ be a sequence in $\mathrm{Aff}(\mathrm{T}(A))$ such that

$$0 < \alpha_1 < \alpha_2 < \cdots$$

pointwise on $\mathrm{T}(A)$. Put

$$\delta_n = \min_{\tau \in \mathrm{T}(A)} \big(\alpha_{n+1}(\tau) - \alpha_n(\tau)\big) > 0$$

and

$$\eta_1 = \frac{\delta_1}{4}, \qquad \eta_n = \frac{1}{4}\min\{\delta_{n-1}, \delta_n\}, \quad n \geq 2.$$

Then there are $a_n \in \mathrm{M}_\infty(A)_+$ such that

$$|\mathrm{d}_\tau(a_n) - \alpha_n(\tau)| < \eta_n, \qquad \tau \in \mathrm{T}(A),\ n \in \mathbb{N},$$

and

$$a_1 \precsim a_2 \precsim \cdots.$$

Remark 5.13. Suppose that A has property (E1) and strict comparison by traces. Apply (E1) to α_n with tolerance η_n. The resulting elements satisfy, for every $\tau \in \mathrm{T}(A)$,

$$\mathrm{d}_\tau(a_n) < \alpha_n(\tau) + \frac{\delta_n}{4} < \alpha_{n+1}(\tau) - \frac{\delta_n}{4} < \mathrm{d}_\tau(a_{n+1}).$$

Strict comparison gives $a_n \precsim a_{n+1}$, so A has property (E2). If strict comparison is formulated using normalized 2-quasitraces, the same conclusion holds whenever every normalized 2-quasitrace on A is a trace; this is automatic when A is exact.

Lemma 5.14. *Let A be a separable unital* C*-*algebra satisfying property (E2). Then, for every $s \in \mathrm{LAff}_{++}(\mathrm{T}(A))$, there is $x \in (A \otimes \mathcal{K})_+$ such that*

$$\mathrm{d}_\tau(x) = s(\tau), \qquad \tau \in \mathrm{T}(A).$$

Proof. Fix $s \in \mathrm{LAff}_{++}(\mathrm{T}(A))$. By the standard description of lower semicontinuous affine functions on the metrizable compact simplex $\mathrm{T}(A)$, there are strictly positive continuous affine functions

$$\alpha_1 < \alpha_2 < \cdots, \qquad \sup_n \alpha_n(\tau) = s(\tau), \quad \tau \in \mathrm{T}(A).$$

Choose (a_n) as in property (E2). Thus $[a_1] \leq [a_2] \leq \cdots$ in $\mathrm{Cu}(A)$. By [16, Theorem 1(i) and Appendix 6], this sequence has a supremum in $\mathrm{Cu}(A)$, represented by some $x \in (A \otimes \mathcal{K})_+$:

$$[x] = \sup_n [a_n].$$

For a fixed τ, if $s(\tau) < \infty$, then

$$\eta_n \leq \frac{1}{4}\big(\alpha_n(\tau) - \alpha_{n-1}(\tau)\big) \longrightarrow 0.$$

If $s(\tau) = \infty$, then, for $n \geq 2$, $\eta_n < \alpha_n(\tau)/4$, and hence $\mathrm{d}_\tau(a_n) > 3\alpha_n(\tau)/4 \longrightarrow \infty$. Consequently,

$$\sup_n \mathrm{d}_\tau(a_n) = s(\tau)$$

in either case. Since the functional on $\mathrm{Cu}(A)$ induced by τ preserves suprema of increasing sequences,

$$\mathrm{d}_\tau(x) = \sup_n \mathrm{d}_\tau(a_n) = s(\tau), \qquad \tau \in \mathrm{T}(A).$$

□

Proposition 5.15. *For $\varepsilon \geq 0$, let $\mathcal{S}_\varepsilon$ be the class of C^*-algebras C with the following property. For every $c \in C_+$ for which there is $e \in C_+$ satisfying $ec = c$, there is $b \in \mathrm{M}_\infty(C)_+$ such that*

$$|\sigma(c) - \mathrm{d}_\sigma(b)| \leq \varepsilon, \qquad \sigma \in \mathrm{T}(C).$$

Such an element c is called compactly supported in C. *Thus $C \in \mathcal{S}_0$ means that every compactly supported positive element of C has its trace function realized exactly by a rank function. In particular,*

$$\mathcal{S}_0 \subseteq \mathcal{S}_\varepsilon \quad (\varepsilon > 0), \qquad \mathcal{S}_\varepsilon \subseteq \mathcal{S}_\delta \quad (0 \leq \varepsilon \leq \delta).$$

Let A be a simple unital C^*-algebra. If

$$A \in \mathrm{TA}_0(\mathcal{S}_\varepsilon) \quad \textit{for every } \varepsilon > 0,$$

then A has property (E1). Moreover, if $D \subseteq A$ is a C^*-subalgebra and the rank-approximating elements b from all local models can be chosen in $\mathrm{M}_\infty(D)$, then the elements realizing property (E1) can be chosen in $\mathrm{M}_\infty(D)$.

Proof. Fix $\alpha \in \mathrm{Aff}(\mathrm{T}(A))$ with $\alpha > 0$ and $\varepsilon > 0$. By [12, Corollary 3.10], simplicity of A gives $a' \in A_+$ such that

$$\tau(a') = \alpha(\tau), \qquad \tau \in \mathrm{T}(A).$$

Put $M = \max\{1, \|a'\|\}$ and choose $\gamma > 0$ such that $(2M + 4)\gamma < \varepsilon$.

Apply Lemma 2.8 to a' with tolerance γ. We obtain $C \in \mathcal{S}_\gamma$ and a positive contraction $h \in C$ of norm one such that

$$\mathrm{dist}(ha'h, C) < \gamma, \qquad \mathrm{d}_\tau(1 - h) < \gamma, \quad \tau \in \mathrm{T}(A).$$

There is $c_0 \in C_+$ with $\|ha'h - c_0\| < 2\gamma$. Set $c = (c_0 - \gamma)_+$. Then $\|c - c_0\| \leq \gamma$, and functional calculus gives $e \in C_+$ with $ec = c$; hence c is compactly supported in C. For $\tau \in \mathrm{T}(A)$,

$$\begin{aligned} 0 \leq \tau(a') - \tau(ha'h) &= \tau\big(a'^{1/2}(1 - h^2)a'^{1/2}\big) \\ &\leq M\tau(1 - h^2) \leq 2M\tau(1 - h) < 2M\gamma. \end{aligned}$$

It follows that

$$|\tau(c) - \alpha(\tau)| < (2M+3)\gamma, \qquad \tau \in \mathrm{T}(A). \tag{5.26}$$

Since $C \in \mathcal{S}_\gamma$, there is $b \in \mathrm{M}_\infty(C)_+$ whose rank function approximates the trace function of c. For $\tau \in \mathrm{T}(A)$, put $t = \|\tau|_C\| \leq 1$. If $t > 0$, applying the defining estimate to $t^{-1}\tau|_C$ and multiplying by t gives

$$|\mathrm{d}_\tau(b) - \tau(c)| \leq \gamma, \qquad \tau \in \mathrm{T}(A).$$

The same estimate holds if $t = 0$, since both terms vanish. Together with (5.26), this gives

$$|\mathrm{d}_\tau(b) - \alpha(\tau)| < (2M+4)\gamma < \varepsilon, \qquad \tau \in \mathrm{T}(A).$$

This proves property (E1). The final assertion follows by taking the same element b in $\mathrm{M}_\infty(D)$. □

Corollary 5.16. *Let a countably infinite discrete amenable group Γ act freely and minimally on a compact metrizable space X, and suppose that the action has WRP. Put $B = \mathrm{C}(X) \rtimes_{\mathrm{r}} \Gamma$. Then, for every strictly positive continuous affine function α on $\mathrm{T}(B)$ and every $\varepsilon > 0$, there are $m \in \mathbb{N}$ and $f_1, \ldots, f_m \in \mathrm{C}(X)_+$ such that, with $a = \mathrm{diag}(f_1, \ldots, f_m)$,*

$$|\mathrm{d}_\tau(a) - \alpha(\tau)| < \varepsilon, \qquad \tau \in \mathrm{T}(B).$$

Proof. Freeness and minimality imply that B is simple. Fix $\gamma > 0$. In the proof of Theorem 3.15, the tiling shapes may be chosen so that their cardinalities n_s are all greater than any prescribed integer. The resulting local model has the form

$$C = \bigoplus_{s=1}^{S} \mathrm{M}_{n_s}(H_s),$$

where every H_s is a hereditary subalgebra of $\mathrm{C}(X)$ and hence is isomorphic to $\mathrm{C}_0(U_s)$ for an open set $U_s \subseteq X$. Choose the shapes so large that $1/n_s < \gamma$ for every s. Lemma 5.18 and homogeneity of its estimate for bounded traces show that $C \in \mathcal{S}_\gamma$; the same estimate passes to finite direct sums by decomposing a trace over the summands. Consequently,

$$B \in \mathrm{TA}_0(\mathcal{S}_\gamma) \quad \text{for every } \gamma > 0.$$

The construction in Lemmas 5.17 and 5.18 uses diagonal scalar-valued functions. Under the tower embeddings in the proof of Theorem 3.15, all diagonal corners of each local model lie in $\mathrm{C}(X)$. Thus the rank-approximating elements required in Proposition 5.15 can be chosen as finite diagonal elements of $\mathrm{M}_\infty(\mathrm{C}(X))$. That proposition, applied with $D = \mathrm{C}(X)$, proves the assertion. □

Lemma 5.17. *Let X be a locally compact metrizable space, and let $\alpha : X \to \mathbb{N}_0$ be a bounded lower semicontinuous function whose support is compact. Then there are $m \in \mathbb{N}$ and a positive diagonal element $f \in \mathrm{M}_m(\mathrm{C}_0(X))$ such that*

$$\mathrm{rank}(f(x)) = \alpha(x), \qquad x \in X. \tag{5.27}$$

Proof. Let $r = \max_{x\in X} \alpha(x)$. If $r = 0$, take $f = 0 \in \mathrm{C}_0(X) = \mathrm{M}_1(\mathrm{C}_0(X))$. Otherwise, for $1 \leq k \leq r$, put

$$U_k = \{x \in X : \alpha(x) \geq k\}.$$

Since α is integer-valued and lower semicontinuous, each U_k is open. Moreover, U_k has compact closure because it is contained in the compact support of α. As X is metrizable, there is $\varphi_k \in \mathrm{C}_0(X)_+$ whose cozero set is exactly U_k. Set

$$f = \mathrm{diag}(\varphi_1, \ldots, \varphi_r).$$

At a point x, precisely $\alpha(x)$ of the functions $\varphi_k(x)$ are nonzero, which proves (5.27). □

Lemma 5.18. *Let $C = \mathrm{M}_n(\mathrm{C}_0(X))$, where X is a locally compact metrizable space. Then $C \in \mathcal{S}_{1/n}$. In fact, if $c \in C_+$ is compactly supported and σ is a nonzero bounded positive trace on C, the element b below can be chosen diagonal and satisfies*

$$|\mathrm{d}_\sigma(b) - \sigma(c)| < \frac{\|\sigma\|}{n}.$$

For the zero trace, both quantities are zero.

Proof. Let $c \in C_+$ be compactly supported, and choose $e \in C_+$ with $ec = c$. Define

$$q(x) = \lceil \mathrm{Tr}_n(c(x)) \rceil, \qquad x \in X.$$

The function q is bounded, integer-valued, and lower semicontinuous. It has compact support: if $c(x) \neq 0$, then $e(x)c(x) = c(x)$ forces $\|e(x)\| \geq 1$, and the set $\{x : \|e(x)\| \geq 1\}$ is compact. Lemma 5.17 therefore gives a positive diagonal matrix over $\mathrm{C}_0(X)$ whose rank at x is $q(x)$. After padding by zero entries to make its matrix size a multiple of n, regard it as an element $b \in \mathrm{M}_\infty(C)_+$.

First suppose that $\sigma \in \mathrm{T}(C)$. There is a Borel probability measure μ on X such that

$$\sigma(c) = \frac{1}{n}\int_X \mathrm{Tr}_n(c(x))\, d\mu(x), \qquad \mathrm{d}_\sigma(b) = \frac{1}{n}\int_X \mathrm{rank}(b(x))\, d\mu(x).$$

Since $0 \leq \lceil t \rceil - t < 1$ for $t \geq 0$, we obtain

$$0 \leq \mathrm{d}_\sigma(b) - \sigma(c) = \frac{1}{n}\int_X \big(q(x) - \mathrm{Tr}_n(c(x))\big)\, d\mu(x) < \frac{1}{n}.$$

The last inequality follows because the integrand is strictly less than one at every point. Normalization and homogeneity give the estimate for every nonzero bounded positive trace. □

Theorem 5.19. *Let*

$$\mathcal{H} = \big\{\mathrm{M}_n(\mathrm{C}_0(X)) : n \in \mathbb{N},\ X \text{ locally compact and metrizable}\big\}$$

be the class of basic homogeneous C^*-algebras. Let A be a separable, simple, unital, nuclear, non-type-I C^*-algebra. If $A \in \mathrm{TA}_0(\mathcal{H})$, then

$$A \cong A \otimes \mathcal{Z} \quad \Longleftrightarrow \quad A \text{ has strict comparison of positive elements.}$$

Proof. The forward implication is standard: A is finite because it has a faithful trace, and a simple, exact, finite, $\mathcal{Z}$-stable C*-algebra has strict comparison by [58]. We prove the converse.

For $N \in \mathbb{N}$, put

$$\mathcal{H}_{\geq N} = \{\mathrm{M}_n(\mathrm{C}_0(X)) \in \mathcal{H} : n \geq N\}.$$

We claim that

$$A \in \mathrm{TA}_0(\mathcal{H}_{\geq N}) \quad \text{for every } N \in \mathbb{N}. \tag{5.28}$$

Suppose otherwise. For some $N \geq 2$, there are a finite set $\mathcal{F}_0 \subseteq A$ and $\varepsilon_0 > 0$ for which the defining TA_0 approximation cannot be obtained with an algebra in $\mathcal{H}_{\geq N}$. Let $(a_j)_{j=1}^{\infty}$ be dense in the unit ball of A. Applying $A \in \mathrm{TA}_0(\mathcal{H})$ to

$$\mathcal{F}_j = \mathcal{F}_0 \cup \{a_1, \ldots, a_j\} \quad \text{with tolerance} \quad \delta_j < \min\{\varepsilon_0, 1/j\},$$

gives

$$C_j = \mathrm{M}_{n_j}(\mathrm{C}_0(X_j)) \subseteq A, \qquad h_j \in (C_j)_+, \qquad \|h_j\| = 1.$$

The choice of $(\mathcal{F}_0, \varepsilon_0)$ implies that $n_j < N$ for every j. For $j \geq k$, let $a'_{k,j}$ be the approximant to a_k in the defining TA_0 approximation. Then

$$\|[h_j, a_k]\| \leq \left\|[h_j, a'_{k,j}]\right\| + 2\left\|a_k - a'_{k,j}\right\| < 3\delta_j$$

and, since $h_j a'_{k,j} h_j \in C_j$,

$$\mathrm{dist}(h_j a_k h_j, C_j) \leq \left\|h_j(a_k - a'_{k,j})h_j\right\| < \delta_j.$$

Density of $(a_k)_k$ now gives

$$\begin{aligned} \|[h_j, a]\| &\longrightarrow 0, \\ \mathrm{dist}(h_j a h_j, C_j) &\longrightarrow 0 \end{aligned} \qquad (a \in A). \tag{5.29}$$

Fix a free ultrafilter ω, let $q_\omega : \ell^\infty(A) \to A_\omega$ be the quotient map, and set

$$h = q_\omega((h_j)_j), \qquad Q = q_\omega\left(\prod_{j=1}^{\infty} C_j\right) \subseteq A_\omega.$$

Equation (5.29) shows that $h \in A' \cap A_\omega$ and $hAh \subseteq Q$. Hence

$$\Phi : A \longrightarrow Q, \qquad \Phi(a) = hah,$$

is a c.p.c. order zero map. It is nonzero because $\Phi(1_A) = h^2$ and $\|h\| = 1$. The kernel of an order zero map is an ideal, so simplicity of A implies that Φ is injective.

Every C_j is $(N-1)$-subhomogeneous. Equivalently, the standard polynomial of degree $2(N-1)$ vanishes on C_j. The same polynomial therefore vanishes on Q and on $D = \mathrm{C}^*(\Phi(A))$. By the structure theorem for order zero maps [71], there is a supporting *-homomorphism

$$\pi_\Phi : A \longrightarrow \mathrm{Mult}(D) \subseteq D^{**} \quad \text{such that} \quad \Phi(a) = \pi_\Phi(a)\Phi(1_A).$$

It is faithful because Φ is faithful. Since the standard polynomial is multilinear, its identity passes to D^{**} by separate weak* continuity. Thus $\pi_\Phi(A)$ satisfies the same standard identity. Kaplansky's characterization [39] implies that A is $(N-1)$-subhomogeneous, hence type I. This contradicts the hypothesis and proves (5.28).

Let $\varepsilon > 0$ and choose N with $1/N < \varepsilon$. Lemma 5.18 and (5.28) give

$$A \in \mathrm{TA}_0(\mathcal{H}_{\geq N}) \subseteq \mathrm{TA}_0(\mathcal{S}_\varepsilon).$$

Since this holds for every $\varepsilon > 0$, Proposition 5.15 gives property (E1). Nuclearity implies exactness, so strict comparison and the remark following Definition 5.12 give property (E2). Lemma 5.14 then shows that every function in $\mathrm{LAff}_{++}(\mathrm{T}(A))$ is realized by the rank function of a positive element of $A \otimes \mathcal{K}$.

Finally, every basic homogeneous algebra locally has finite nuclear dimension. Indeed, a finite subset of $\mathrm{M}_n(\mathrm{C}_0(X))$ is contained in $\mathrm{M}_n(D)$, where $D \subseteq \mathrm{C}_0(X)$ is generated by finitely many matrix entries. The spectrum of the finitely generated commutative algebra D embeds in a finite-dimensional Euclidean space, so D, and hence $\mathrm{M}_n(D)$, has finite nuclear dimension. Theorem 5.10 now gives $A \cong A \otimes \mathcal{Z}$. □

Remark 5.20. The proof of the reverse implication in Theorem 5.19 gives tracial $\mathcal{Z}$-absorption without nuclearity if strict comparison is assumed with respect to tracial states. Properties (E1) and (E2) and the rank-realization argument remain valid. The proof of Theorem 5.10 then gives the tracial 0-almost-divisibility estimates for tracial states. The version of Winter's centralization lemma for tracial states used in Proposition 5.3 gives approximately central order zero maps. Nuclearity is needed only to pass from tracial $\mathcal{Z}$-absorption to tensorial $\mathcal{Z}$-stability.

Lemma 5.21. *Let A be a separable, simple, unital, exact, $\mathcal{Z}$-stable* C*-*algebra with* $\mathrm{T}(A) \neq \emptyset$. *Then*

$$\overline{cAc} \in \mathcal{S}_0, \qquad c \in A_+.$$

Proof. Put $H = \overline{cAc}$. The assertion is immediate if $c = 0$. Let $0 \neq a \in H_+$ be compactly supported in H; the case $a = 0$ is handled by taking $b = 0$. The function

$$f : \mathrm{T}(A) \longrightarrow (0, \infty), \qquad f(\tau) = \tau(a),$$

is continuous and strictly positive. The Cuntz-semigroup description for simple, unital, exact, finite, $\mathcal{Z}$-stable algebras [12, Theorem 5.5] gives $x \in (A \otimes \mathcal{K})_+$ such that

$$\mathrm{d}_\tau(x) = \tau(a), \qquad \tau \in \mathrm{T}(A). \tag{5.30}$$

Every trace on A is faithful. Thus $\tau \mapsto \mathrm{d}_\tau(c)$ is a strictly positive lower semicontinuous function on the compact space $\mathrm{T}(A)$, and consequently

$$\mu := \min_{\tau \in \mathrm{T}(A)} \mathrm{d}_\tau(c) > 0.$$

Choose $m \in \mathbb{N}$ with $m\mu > \|a\|$. Equation (5.30) gives

$$\mathrm{d}_\tau(x) = \tau(a) \leq \|a\| < m\mathrm{d}_\tau(c) = \mathrm{d}_\tau(c^{\oplus m}), \qquad \tau \in \mathrm{T}(A).$$

Strict comparison yields $x \precsim c^{\oplus m}$. Moreover, A has stable rank one by [58]. The Hilbert-module picture of the Cuntz semigroup [16] therefore gives a positive element

$$b \in \mathrm{her}(c^{\oplus m}) \subseteq \mathrm{M}_m(H) \quad \text{with} \quad [b] = [x].$$

We now consider tracial states on the possibly nonunital algebra H. Let $\sigma \in \mathrm{T}(H)$. Since H is full in the simple unital algebra A, let $\widetilde{\sigma}$ be the standard lower semicontinuous extension of σ to A. Fullness of $0 \neq q \in H_+$ gives finitely many $w_i \in A$ with $\sum_i w_i^* q w_i = 1_A$. Traciality gives

$$\widetilde{\sigma}(1_A) = \sum_i \sigma(q^{1/2} w_i w_i^* q^{1/2}) \leq \sum_i \|w_i\|^2 \, \sigma(q) < \infty.$$

Thus $\widetilde{\sigma}$ is bounded. Set $s = \widetilde{\sigma}(1_A) > 0$. Then $\tau = s^{-1}\widetilde{\sigma} \in \mathrm{T}(A)$. Using (5.30) and $[b] = [x]$, we obtain

$$\mathrm{d}_\sigma(b) = s\mathrm{d}_\tau(b) = s\mathrm{d}_\tau(x) = s\tau(a) = \sigma(a).$$

Thus the defining estimate for $\mathcal{S}_0$ holds with equality. □

Remark 5.22 (Hereditary rank realization). Hereditary subalgebras of a nonsimple C*-algebra of finite nuclear dimension need not belong to $\mathcal{S}_0$. For example, let $X = [0, 1]$. The pointwise rank of every element of $\mathrm{M}_\infty(\mathrm{C}(X))$ is integer-valued, whereas evaluation traces take the value $\frac{1}{2}$ on the constant positive element $\frac{1}{2}1$. Thus finite nuclear dimension alone does not imply the rank realization in Lemma 5.21 for nonsimple coefficient algebras.

The preservation of $\mathcal{Z}$-stability below was obtained by Orovitz, Phillips, and Wang under their weak tracial Rokhlin property [52, Corollary 8.13]. Their formulation controls the defect in the coefficient algebra in the Cuntz order. The following theorem proves the same conclusion under WRP, which controls the defect uniformly over invariant traces.

Theorem 5.23 (WRP preserves $\mathcal{Z}$-stability)**.** *Let A be a separable, simple, unital, nuclear, $\mathcal{Z}$-stable* C*-*algebra, and let Γ be a countable discrete amenable group acting on A. If (A, Γ) has WRP, then*

$$B = A \rtimes_{\alpha,r} \Gamma$$

is simple and $\mathcal{Z}$-stable. In particular, the conclusion holds for a separable, simple, unital, infinite-dimensional coefficient algebra of finite nuclear dimension with WRP.

Proof. WRP requires $\mathrm{T}(A) \neq \emptyset$, so simplicity implies that A is stably finite. It is infinite dimensional since it is $\mathcal{Z}$-stable. The action is minimal because A is simple, and Corollary 3.32 gives simplicity of B. Nuclearity of A and amenability of Γ give nuclearity of B.

Since A is stably finite, nuclear, and $\mathcal{Z}$-stable, it has strict comparison and almost divisible Cuntz semigroup, as in the proof of Corollary 4.16. Corollary 4.6 therefore gives strict comparison

for B. Furthermore, $\mathcal{Z}$-stability implies tracial $\mathcal{Z}$-absorption by [37, Proposition 2.2]. Choosing the Cuntz defect below a nonzero positive element with uniformly small dimension functions gives weak tracial $\mathcal{Z}$-absorption, as in Remark 2.5. Corollary 5.8 and the strict comparison and nuclearity of B give $B \cong B \otimes \mathcal{Z}$. For the final assertion, finite nuclear dimension gives nuclearity and [70, Corollary 7.3] gives $\mathcal{Z}$-stability of A.

The conclusion also follows from [30, Corollary 5.6], since WRP implies the weak Rokhlin condition in [30, Definition 5.2]. Indeed, choose exact tiling shapes D_s that are $(K^{-1} \cup \{e\}, \eta)$-invariant. Their common K-translate intersection is $\mathrm{Int}_{K^{-1}\cup\{e\}}(D_s)$ and has cardinality at least $(1-\eta)|D_s|$. The levels $\alpha_g(c_s)$ are central, orthogonal, and covariant; their defect lies in $J_{\mathcal{M}_1(A,\Gamma)}$ and has value zero under every invariant ultralimit trace. □

Remark 5.24 (Infinite groups). For infinite groups, Corollary 6.20 extends this conclusion to stably finite coefficient C*-algebras with locally finite nuclear dimension and finite radius of comparison. These algebras need not be simple or $\mathcal{Z}$-stable and may have positive radius of comparison. The action is assumed to be minimal and to satisfy at least one of the following conditions: WRP and COS, strong URP, or WRP with Cuntz control in the crossed product. Theorem 5.23 also applies to finite groups.

Chapter 6

Real rank and stable rank

6.1 Permanence of real rank zero in weak tracial approximation

Zhao and Fang [73] proved that real rank zero passes from a centrally large subalgebra to an infinite-dimensional simple unital C*-algebra, without assuming stable rank one for the subalgebra. The same permanence theorem appears in the earlier joint preprint [74, Theorem 3.11]. Its proof, particularly Step III, uses mutually equivalent projections and a self-adjoint matrix congruence to absorb the positive and negative parts of the defect. The following proof adapts this method to weak tracial approximation.

The approximating algebras may vary and need not be unital or simple, so the cited permanence theorems do not apply directly. Lifting an order zero map transfers fixed matrix units to an approximating algebra while preserving a uniform lower trace bound. Strict comparison then compares the positive and negative parts of the defect with the resulting projections, and a self-adjoint matrix congruence gives an invertible approximation.

Theorem 6.1. *Let $\mathcal{S}$ be a class of* C*-*algebras of real rank zero, and let A be a simple unital* C*-*algebra in* $\mathrm{TA}_0(\mathcal{S})$. *If A has strict comparison of positive elements, then A has real rank zero.*

Proof. Hereditary subalgebras of a C*-algebra of real rank zero have approximate identities of projections, and real rank zero passes to unitizations and corners [11]. The approximation hypothesis includes $\mathrm{T}(A) \neq \emptyset$. Since A is simple, every tracial state on A is faithful.

Step 1: matrix units in an annihilating hereditary algebra. We first prove that A has property (SP). Let $0 \neq a \in A_+$ be a contraction and put

$$m = \min_{\tau \in \mathrm{T}(A)} \tau(a) > 0.$$

Choose an approximation satisfying the TA_0 conditions with tolerance $\eta < m/16$. Taking a self-adjoint approximant to a and then the positive part of its compression gives $c \in C_+$ with

$$\|c - hah\| < 2\eta.$$

Since $0 \leq 1 - h^2 \leq 2(1-h)$, we have

$$\tau(hah) \geq \tau(a) - \tau(1-h^2) > m - 2\eta.$$

Consequently $\|c\| > m - 4\eta > m/2$, and the perturbation lemma for Cuntz comparison gives

$$(c - m/4)_+ \precsim_A hah \precsim_A a.$$

The nonzero hereditary algebra generated by $(c - m/4)_+$ inside C contains a nonzero projection. A projection Cuntz subequivalent to a is Murray–von Neumann equivalent to a projection in $\mathrm{Her}_A(a)$, so this hereditary algebra also contains a nonzero projection. This proves (SP).

If A is finite-dimensional, it has real rank zero. Suppose therefore that A is infinite-dimensional. Fix $x = x^* \in A$ with $\|x\| \leq 1$, and fix $\varepsilon > 0$. We may assume that x is noninvertible. Continuous functional calculus then provides a self-adjoint contraction x_0 and a nonzero positive contraction d such that

$$\|x - x_0\| < \varepsilon/4, \qquad x_0 d = d x_0 = 0.$$

Indeed, replace $t \mapsto t$ by a function vanishing on a sufficiently small neighborhood of zero, and choose d supported in that neighborhood.

There are four mutually orthogonal equivalent nonzero projections in $\mathrm{Her}_A(d)$. To prove this, choose four orthogonal nonzero positive elements in this simple non-elementary hereditary algebra. Property (SP) gives a nonzero projection in the hereditary algebra generated by each element. Simplicity gives a common nonzero positive Cuntz lower bound for these four projections: for two nonzero positive elements b, c, choose $0 \neq z \in b^{1/2} A c^{1/2}$ and use $zz^* \sim z^* z$, and iterate. Applying (SP) to this lower bound and using comparison of projections gives equivalent subprojections of the four original projections. Let

$$\iota \colon \mathrm{M}_4 \longrightarrow A$$

be the resulting homomorphism, and write $e_{ij} = \iota(E_{ij})$, where E_{ij} are the standard matrix units. Thus $x_0 e_{ij} = e_{ij} x_0 = 0$. Set

$$\delta = \min_{\tau \in \mathrm{T}(A)} \tau(e_{11}) > 0.$$

The matrix units and the constant δ are fixed before the approximating algebras are chosen.

Step 2: equivalent projections in the approximating algebras. Choose C_n, h_n satisfying the TA_0 conditions for the finite set $\{x_0, e_{ij} : 1 \leq i, j \leq 4\}$ with tolerances $\eta_n \searrow 0$. Choose self-adjoint approximants x_n to x_0 and approximants $e'_{ij,n}$ to e_{ij}, so that

$$\begin{aligned} h_n x_n h_n,\ h_n e'_{ij,n} h_n \in C_n, \qquad & \|x_n - x_0\|,\ \|e'_{ij,n} - e_{ij}\| \longrightarrow 0, \\ \|[h_n, x_0]\|,\ \|[h_n, e_{ij}]\| \longrightarrow 0, \qquad & \sup_{\tau \in \mathrm{T}(A)} \mathrm{d}_\tau(1 - h_n) < \eta_n. \end{aligned}$$

Put

$$q_n = g_{1/20}(h_n), \qquad r_n = g_{3/4}(h_n), \qquad D_n = \mathrm{Her}_{C_n}(r_n).$$

Consider the norm sequence algebra $A_\infty = \ell^\infty(A)/c_0(A)$. If $h = [(h_n)]$ and $r = [(r_n)]$, then h commutes with x_0 and all the matrix units. The map

$$\mu\colon \mathrm{M}_4 \longrightarrow \prod_n D_n \Big/ \bigoplus_n D_n \subseteq A_\infty, \qquad \mu(z) = r\iota(z),$$

is c.p.c. and order zero. Its range lies in the indicated quotient, since

$$r_n^{1/2} e'_{ij,n} r_n^{1/2} \in D_n :$$

the function $g_{3/4}(t)^{1/2}/t$, extended by zero at zero, is continuous, so the compression belongs to C_n and is supported in r_n.

The cone over M_4 is projective [44, Theorem 10.2.1]. Equivalently, c.p.c. order zero maps from M_4 lift across quotient maps. Hence μ lifts to c.p.c. order zero maps $\phi_n\colon \mathrm{M}_4 \to D_n$ satisfying

$$\|\phi_n(E_{ij}) - r_n^{1/2} e_{ij} r_n^{1/2}\| \longrightarrow 0. \tag{6.1}$$

Write $w_n = \phi_n(1_{\mathrm{M}_4})$ and $a_n = \phi_n(E_{11})$. The scalar inequality $1 - g_{3/4}(t) \leq 4(1-t)$ and (6.1) imply, uniformly over $\mathrm{T}(A)$,

$$\tau(a_n) \geq \delta - 4\eta_n - o(1). \tag{6.2}$$

Define self-adjoint elements

$$a_{0,n} = (1-q_n)^{1/2} x_n (1-q_n)^{1/2},$$
$$a'_{1,n} = q_n^{1/2} x_n q_n^{1/2} \in C_n.$$

Functional calculus and the commutator estimates give

$$\|x_0 - a_{0,n} - a'_{1,n}\| \longrightarrow 0.$$

Moreover, $a_{0,n} D_n = D_n a_{0,n} = 0$, because $(1-q_n) r_n = 0$. In the sequence algebra, $[(a'_{1,n})] = g_{1/20}(h) x_0$ and $[(w_n)] = r\iota(1)$, so

$$\|w_n a'_{1,n}\| + \|a'_{1,n} w_n\| \longrightarrow 0.$$

Fix $0 < \theta < \delta/32$ and put

$$a_{1,n} = (1 - g_\theta(w_n)) a'_{1,n} (1 - g_\theta(w_n)) \in C_n.$$

Since $g_\theta(t)/t$ is bounded and continuous at zero, it follows that

$$\|x_0 - a_{0,n} - a_{1,n}\| \longrightarrow 0. \tag{6.3}$$

Both summands annihilate $\mathrm{Her}_{C_n}((w_n - 2\theta)_+)$.

For all sufficiently large n, (6.2) gives $\tau(a_n) > 3\delta/4$ for every $\tau \in \mathrm{T}(A)$. Real rank zero gives a projection

$$p_{1,n} \in \mathrm{Her}_{C_n}((a_n - 2\theta)_+)$$

such that

$$\|(a_n - 2\theta)_+ - p_{1,n}(a_n - 2\theta)_+ p_{1,n}\| < \delta/8.$$

As a_n is a contraction, it follows that

$$\tau(p_{1,n}) \geq \tau((a_n - 2\theta)_+) - \delta/8 > \delta/2, \qquad \tau \in \mathrm{T}(A). \tag{6.4}$$

The supporting homomorphism of ϕ_n gives four orthogonal equivalent projections $p_{i,n}$. They belong to $\mathrm{Her}_{C_n}((w_n - 2\theta)_+)$. These projections belong to C_n: choose a continuous function f with $f(0) = 0$ and $f(t) = t^{-1}$ for $t \geq 2\theta$, and set

$$t_{i,n} = \phi_n(E_{i1}) f(a_n) p_{1,n}, \qquad p_{i,n} = t_{i,n} t_{i,n}^*.$$

The order zero relations give $t_{i,n}^* t_{i,n} = p_{1,n}$ and the asserted orthogonality and support. Thus $\tau(p_{i,n}) = \tau(p_{1,n}) > \delta/2$ for all four projections.

Step 3: an invertible self-adjoint approximation. Choose n sufficiently large that $\eta_n < \delta/2$ and the norm in (6.3) is less than $\varepsilon/4$. Suppress n from the notation and put

$$P = p_1 + p_2 + p_3 + p_4, \qquad Q = 1_A - P.$$

The elements a_0, a_1 belong to QAQ, with $a_1 \in C$. Also, the positive and negative parts of a_0 are supported in $1 - q$, so

$$\mathrm{d}_\tau((a_0)_\pm) \leq \mathrm{d}_\tau(1 - q) \leq \mathrm{d}_\tau(1 - h) < \eta_n < \tau(p_i).$$

Strict comparison therefore gives

$$(a_0)_+ \precsim_A p_1, \qquad (a_0)_- \precsim_A p_3.$$

For any $\kappa > 0$, choose $b_+ \in QAp_1$ and $b_- \in QAp_3$ such that

$$\|(a_0)_+ - b_+ b_+^*\| < \kappa, \qquad \|(a_0)_- - b_- b_-^*\| < \kappa.$$

These elements are obtained from the Cuntz comparison witnesses by multiplying on the left by Q and on the right by p_1 or p_3.

The algebra $B = C + \mathbb{C}1_A$ has real rank zero. If C is nonunital, B is a quotient of its unitization; if C has unit e, then $B = C \oplus \mathbb{C}(1_A - e)$, with the second summand omitted when $e = 1_A$. Hence QBQ also has real rank zero, since $P \in C$. If $Q = 0$, (6.3) shows that x is within $\varepsilon/2$ of zero, which gives the required invertible self-adjoint approximation. Otherwise choose an invertible self-adjoint $v \in QBQ$ with

$$\|v - a_1\| < \kappa.$$

Choose partial isometries $u_1, u_2 \in C$ such that

$$u_1^* u_1 = p_1, \quad u_1 u_1^* = p_2, \qquad u_2^* u_2 = p_3, \quad u_2 u_2^* = p_4.$$

For $t > 0$ define

$$\begin{aligned} R_t &= Q + b_+ + b_- + tP, \\ W_t &= v + p_1 - p_3 + t(u_1 + u_1^* + u_2 + u_2^*). \end{aligned}$$

The element R_t is invertible, with inverse

$$R_t^{-1} = Q + t^{-1}P - t^{-1}(b_+ + b_-).$$

The element W_t is self-adjoint and invertible, since its Q-corner is v and its two remaining matrix corners are

$$\begin{pmatrix} 1 & t \\ t & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} -1 & t \\ t & 0 \end{pmatrix}.$$

Consequently $R_t W_t R_t^*$ is invertible and self-adjoint. Multiplying the corners gives

$$R_t W_t R_t^* \longrightarrow v + b_+ b_+^* - b_- b_-^* \quad \text{in norm as } t \searrow 0.$$

All remaining terms contain a factor t: the off-diagonal terms are $t(b_+ - b_-) + t^2(b_+ u_1^* + b_- u_2^*)$ and their adjoints, while the P-corner is $t^2(p_1 - p_3) + t^3(u_1 + u_1^* + u_2 + u_2^*)$.

Choose $\kappa < \varepsilon/12$ and then t sufficiently small that the error in the preceding norm limit is less than $\varepsilon/4$. The approximation estimates give

$$\|x - R_t W_t R_t^*\| < \varepsilon/4 + \varepsilon/4 + 3\kappa + \varepsilon/4 < \varepsilon.$$

Invertible self-adjoint elements are therefore dense in A_{sa}, which proves real rank zero. □

Remark 6.2 (Local weak comparison). The proof uses only the two comparisons in Step 3. Thus Theorem 6.1 also holds under local weak comparison by tracial states: it suffices that there be a fixed $\gamma \geq 1$ such that

$$\gamma \sup_{\tau \in \mathrm{T}(A)} \mathrm{d}_\tau(a) < \inf_{\tau \in \mathrm{T}(A)} \mathrm{d}_\tau(b) \quad \Longrightarrow \quad a \precsim_A b.$$

Indeed, choose n with $\gamma \eta_n < \delta/2$. No separability, nuclearity, or exactness is required in the proof. If strict comparison is formulated using quasitraces rather than tracial states, exactness identifies quasitraces with traces and therefore makes that formulation compatible with Definition 2.1.

6.2 Permanence of stable rank one in weak tracial approximation

The following proposition isolates the use of tracial approximation. Its comparison hypothesis concerns only the positive elements in the subalgebra generated by the approximant to d and the compressing element.

Proposition 6.3. *Let $\mathcal{S}$ be a class of C^*-algebras of stable rank one. Let A be a simple finite unital C^*-algebra with $\mathrm{T}(A) \neq \varnothing$, and let $a \in A$ and $d \in A_+$ satisfy*

$$ad = da = 0, \qquad \|d\| = 1.$$

Suppose that, for every $\eta > 0$, there are a C^*-subalgebra $C \subseteq A$ with $C \in \mathcal{S}$, a positive contraction $h \in C$ of norm one, $a' \in A$, and a positive contraction $d' \in A$ such that

(a) $\|a - a'\| < \eta$ *and* $\|d - d'\| < \eta$;

(b) $ha'h, hd'h \in C$;

(c) $\|[h, a']\| < \eta$ *and* $\|[h, d']\| < \eta$;

(d) $\mathrm{d}_\tau(1 - h) < \eta$ *for every* $\tau \in \mathrm{T}(A)$;

(e) if $x, y \in \mathrm{C}^*(d', h, 1_A)_+$ *satisfy*

$$\mathrm{d}_\tau(x) < \mathrm{d}_\tau(y), \qquad \tau \in \mathrm{T}(A),$$

then $x \precsim_A y$.

Then $a \in \overline{\mathrm{GL}(A)}$.

Proof. Since A is simple, every tracial state on A is faithful. Compactness of $\mathrm{T}(A)$ therefore gives

$$\delta := \min_{\tau \in \mathrm{T}(A)} \tau\big(g_{1/2}(d)\big) > 0.$$

Fix $\varepsilon > 0$, and choose

$$0 < \theta < \frac{\delta}{64}.$$

Apply the hypothesis with $\eta > 0$ sufficiently small, as specified in the estimates that follow. Put

$$q = g_{1/20}(h), \qquad r = g_{3/4}(h),$$

and define

$$\begin{aligned} a_0 &= (1 - q)^{1/2} a' (1 - q)^{1/2}, \\ a_1' &= q^{1/2} a' q^{1/2}, \\ D &= r^{1/2} d' r^{1/2}. \end{aligned}$$

The functions $g_{1/20}(t)^{1/2}/t$ and $g_{3/4}(t)^{1/2}/t$, extended by zero at the origin, are continuous on $[0, 1]$. Consequently, condition (b) gives

$$a_1', D \in C.$$

Since $(1-q)r = 0$, we have

$$a_0 D = D a_0 = 0. \tag{6.5}$$

For the fixed value of θ, conditions (a)–(c), functional calculus, and $ad = da = 0$ give, for sufficiently small η,

$$\|a - (a_0 + a_1')\| < \frac{\varepsilon}{2}, \qquad \|g_\theta(D)a_1'\| + \|a_1' g_\theta(D)\| < \frac{\varepsilon}{4}. \tag{6.6}$$

The first estimate follows from $[q, a'] \approx 0$ and

$$(1-q)^{1/2} a' (1-q)^{1/2} + q^{1/2} a' q^{1/2} \approx a'.$$

For the second estimate, $Da_1' \approx 0 \approx a_1' D$ follows from $d'a' \approx 0 \approx a'd'$ and the commutator estimates. Since $g_\theta(t) = k_\theta(t)t$ for a continuous function k_θ with $\|k_\theta\| \le 2/\theta$, decreasing η gives the required bound.

The scalar inequality

$$1 - g_{3/4}(t) \le 4(1-t), \qquad 0 \le t \le 1,$$

together with condition (d), gives

$$\tau(D) = \tau(d'r) \ge \tau(d') - \tau(1-r) > \tau(d) - 5\eta.$$

Since $d \ge \frac{1}{2} g_{1/2}(d)$, we may decrease η so that

$$\tau(D) > \frac{\delta}{4}, \qquad \tau \in \mathrm{T}(A). \tag{6.7}$$

Set

$$a_1 = (1 - g_\theta(D))a_1'(1 - g_\theta(D)) \quad \text{and} \quad y = (D - 2\theta)_+.$$

Then $a_1 \in C$. The second estimate in (6.6) gives $\|a_1 - a_1'\| < \varepsilon/2$, and hence

$$\|a - (a_0 + a_1)\| < \varepsilon. \tag{6.8}$$

Moreover,

$$y(a_0 + a_1) = (a_0 + a_1)y = 0. \tag{6.9}$$

Indeed, this follows for a_0 from (6.5), and for a_1 from $y(1 - g_\theta(D)) = 0$.

Consider the positive contractions

$$e = 1 - g_{1/3}(h) \quad \text{and} \quad b = 2(e - \tfrac{1}{2})_+.$$

We have

$$\mathrm{d}_\tau(e) \le \mathrm{d}_\tau(1-h) < \eta,$$

whereas (6.7) gives

$$\mathrm{d}_\tau(y) \geq \tau(y) \geq \tau(D) - 2\theta > \frac{\delta}{8}.$$

Taking $\eta < \delta/8$, condition (e) gives $e \precsim_A y$. The standard cut-down form of Cuntz subequivalence provides $v_0 \in A$ such that

$$v_0^* v_0 = (e - \tfrac{1}{2})_+, \qquad v_0 v_0^* \in \mathrm{Her}_A(y).$$

With $v = \sqrt{2} v_0$, we therefore have

$$v^* v = b, \qquad p := v v^* \in \mathrm{Her}_A(y). \tag{6.10}$$

Because $b\,y = 0$, the positive contractions b and p are orthogonal.

Define

$$\rho(t) = \begin{cases} t^{-1/2} \sin(\pi t/2), & t > 0, \\ 0, & t = 0, \end{cases}$$

and

$$w = \cos\bigl(\tfrac{\pi}{2}(p+b)\bigr) + \rho(p) v - \rho(b) v^*. \tag{6.11}$$

By [42, Lemma 3.2], w is a unitary. Set

$$\begin{aligned} z_1 &= 1 - g_{1/10}(h), \\ z_3 &= g_{2\theta}(D), \\ z_2 &= 1 - z_1 - z_3 = g_{1/10}(h) - g_{2\theta}(D). \end{aligned}$$

Since D is supported in $r = g_{3/4}(h)$, the element $g_{1/10}(h)$ acts as the identity on $\mathrm{Her}(D)$. Thus z_2 is positive, and

$$z_1 + z_2 + z_3 = 1, \qquad z_1 z_3 = 0.$$

Also, $by = z_1 y = 0$, while $z_3 y = y$. It follows from (6.10) that

$$z_1 p = 0, \qquad z_3 p = p, \qquad z_2 p = 0. \tag{6.12}$$

Since $b z_1 = z_1$, the last assertion of [42, Lemma 3.2] gives

$$w z_1 w^* \in \mathrm{Her}_A(p). \tag{6.13}$$

Let

$$x = w^*(a_0 + a_1).$$

Equations (6.9) and (6.13) show that $z_1 x = 0$. The definitions of a_1 and z_3, together with (6.5), give $x z_3 = 0$. The middle compression belongs to $C + \mathbb{C}1_A$, as follows. Taking adjoints in (6.11), using $p(a_0 + a_1) = 0$, and using $z_2 v = 0$, which follows from (6.12), we obtain

$$\begin{aligned} z_2 x z_2 &= z_2 \cos\bigl(\tfrac{\pi}{2}(p+b)\bigr)(a_0 + a_1) z_2 \\ &= z_2 \cos\bigl(\tfrac{\pi}{2} b\bigr)(a_0 + a_1) z_2 \\ &= z_2 \cos\bigl(\tfrac{\pi}{2} b\bigr) a_1 z_2. \end{aligned}$$

The last equality follows from $b(1-q) = 1-q$ and hence $\cos(\pi b/2)(1-q) = 0$.

Put $B = C + \mathbb{C}1_A$. If C is nonunital, B is a quotient of its unitization. If C is unital, B is either C or the direct sum of C and a one-dimensional algebra. Thus B has stable rank one. Moreover, $z_1, z_2, z_3 \in B$, and the last display shows that $z_2 x z_2 \in B$. Lemma 5.4 of [5] therefore gives

$$x \in \overline{\mathrm{GL}(A)}.$$

Since w is unitary, $a_0 + a_1 \in \overline{\mathrm{GL}(A)}$. Equation (6.8) and the arbitrariness of ε prove the proposition. □

Theorem 6.4. *Let $\mathcal{S}$ be a class of* C*-*algebras of stable rank one. Let A be a simple unital* C*-*algebra in* $\mathrm{TA}_0(\mathcal{S})$. *If A has strict comparison of positive elements, then A has stable rank one.*

Proof. The tracial states on A are faithful, so A is finite. Let $x \in A$ and let $\varepsilon > 0$. We may assume that x is not invertible. Since a one-sided invertible element in a finite unital C*-algebra is invertible, x is not one-sided invertible.

Rørdam's zero-divisor reduction [57, Proposition 3.2] gives a two-sided zero divisor $x_0 \in A$ such that

$$\|x - x_0\| < \frac{\varepsilon}{2}.$$

The left and right annihilators of x_0 may be different. By [57, Lemma 3.5], there are a unitary $u \in A$ and a nonzero positive element $d \in A$ such that

$$d(ux_0) = (ux_0)d = 0.$$

After rescaling, we may suppose that $\|d\| = 1$.

Apply the $\mathrm{TA}_0(\mathcal{S})$ approximation to the finite set $\{ux_0, d\}$. The approximant to d may be chosen to be a positive contraction. Indeed, if $\|d - d_0'\| < \kappa$, put

$$s = \frac{d_0' + (d_0')^*}{2} \quad \text{and} \quad d' = \frac{s + \kappa 1_A}{1 + 2\kappa}.$$

Then $0 \le d' \le 1_A$, $\|d - d'\| < 4\kappa$, and $hd'h \in C$ because $hd_0'h \in C$ and $h^2 \in C$. The commutator estimate is preserved as well. Since κ is arbitrary, condition (a) holds with any prescribed tolerance. Strict comparison of positive elements gives condition (e) of Proposition 6.3, and the remaining conditions follow from $\mathrm{TA}_0(\mathcal{S})$. Hence

$$ux_0 \in \overline{\mathrm{GL}(A)}.$$

Multiplication by the unitary u shows that $x_0 \in \overline{\mathrm{GL}(A)}$. Consequently $\mathrm{dist}(x, \mathrm{GL}(A)) < \varepsilon$. Since x and ε were arbitrary, A has stable rank one. □

Remark 6.5. The proof of Proposition 6.3 uses only the comparison

$$1 - g_{1/3}(h) \precsim (r^{1/2} d' r^{1/2} - 2\theta)_+, \qquad r = g_{3/4}(h).$$

Thus the strict comparison hypothesis in Theorem 6.4 can be replaced by this local comparison condition. More generally, it suffices that there be a fixed $K \geq 1$ such that the relevant elements satisfy

$$K\mathrm{d}_\tau(e) < \mathrm{d}_\tau(y) \quad (\tau \in \mathrm{T}(A)) \quad \Longrightarrow \quad e \precsim_A y.$$

In the proof, $\mathrm{d}_\tau(e) < \eta$ and $\mathrm{d}_\tau(y) > \delta/8$, so choosing $\eta < \delta/(8K)$ suffices. This is a comparison with the single target y; comparison with $y^{\oplus m}$ alone does not imply the unitary rotation in that proof.

6.3 Zero-divisor reduction to the coefficient algebra

Rørdam's argument gives left and right annihilators in the crossed product. The results in this section replace them by a common positive annihilator in the coefficient algebra. In the norm ultrapower, the Rokhlin levels are exactly orthogonal, and Lemma 3.2 of [42] gives the required unitary rotations. Throughout the section, write $B = A \rtimes_{\alpha,r} \Gamma$ and denote the canonical conditional expectation by $\mathbb{E} : B \to A$.

Lemma 6.6 (One-sided coefficient reduction)**.** *Let (A, Γ) be a minimal C^*-dynamical system with WRP, where A is unital and separable, $\mathrm{T}(A) \neq \emptyset$, and Γ is countably infinite, discrete, and amenable. If $a \in B$ and $0 \neq b \in B_+$ satisfy $ba = 0$, then, for every $\varepsilon > 0$, there exist a unitary $u \in B$ and a norm-one positive contraction $h \in A$ such that*

$$\|hua\| < \varepsilon.$$

Proof. Faithfulness of $\mathbb{E}$ gives $\mathbb{E}(b) \neq 0$. After rescaling b, we may suppose that $\|\mathbb{E}(b)\| = 1$. Fix $\kappa > 0$. By approximating $b^{1/2}$ by a Fourier polynomial, then squaring and rescaling, choose

$$b_0 \in \mathrm{C_c}(\Gamma, A)_+, \qquad \|\mathbb{E}(b_0)\| = 1, \qquad \|b - b_0\| < \kappa.$$

Let $F \subseteq \Gamma$ contain the Fourier supports of b_0 and b_0^2. Since Γ is infinite, choose $\gamma_0 \notin F$.

By Lemma 3.30, the action satisfies Elliott's condition. Proposition 3.29, applied to $\mathbb{E}(b_0)^2$ and $F \cup \{\gamma_0\}$, gives a norm-one positive contraction $c \in A' \cap A_\omega$ such that its indicated translates are mutually orthogonal and

$$\|c\mathbb{E}(b_0)^2 c\| = 1. \tag{6.14}$$

Centrality and tower orthogonality give

$$cb_0^2 c = c\mathbb{E}(b_0^2)c \in A_\omega, \qquad \alpha_{\gamma_0}(c)b_0 c = 0. \tag{6.15}$$

Put $v = u_{\gamma_0} c b_0 \in B_\omega$. Then

$$vv^* = u_{\gamma_0} c\mathbb{E}(b_0^2)cu_{\gamma_0}^* \in A_\omega, \qquad v^*v = b_0 c^2 b_0,$$

and the second relation in (6.15) shows that $vv^* \perp v^*v$. Kadison's inequality and (6.14) give $\|v\| \geq 1$. Thus, for $x = v/\|v\|$,

$$xx^* \in A_\omega, \qquad xx^* \perp x^*x, \qquad \|x^*xa\| \leq \|b_0\| \, \|(b_0 - b)a\|. \tag{6.16}$$

Functional calculus gives a perturbation whose range modulus is identically one on a nonzero spectral region. Write the polar decomposition in B_ω^{**} as $x = (xx^*)^{1/2}s$. Choose a continuous function $f : [0,1] \to [0,1]$ which vanishes near zero, is identically one near one, and is uniformly as close to the identity function as desired. Then

$$t = f((xx^*)^{1/2})s \in B_\omega, \qquad tt^* \in A_\omega, \qquad tt^* \perp t^*t, \qquad \|t\| = 1.$$

Choose a continuous function $g : [0,1] \to [0,1]$ with $g(1) = 1$ whose support is contained in the region on which $f = 1$, and put

$$h = g((xx^*)^{1/2}) \in (A_\omega)_+.$$

Then $\|h\| = 1$ and $(tt^*)h = h(tt^*) = h$. Lemma 3.2 of [42] supplies a unitary $w \in B_\omega$ such that

$$(t^*t)(w^*hw) = w^*hw = (w^*hw)(t^*t).$$

Consequently,

$$\|hwa\| = \|(w^*hw)a\| \leq \|t^*ta\|.$$

By (6.16), the right-hand side is smaller than ε when first κ and then the functional-calculus error are chosen sufficiently small.

Choose positive-contraction representatives of h in A and representatives of w in B. Polar decomposition gives unitary representatives on an ω-large set. The required estimate holds on an ω-large set of coordinates; choosing one such coordinate and normalizing the positive contraction proves the lemma. □

The following lemma aligns two positive elements of the coefficient algebra by a unitary in the crossed product.

Lemma 6.7 (Alignment of coefficient supports)**.** *Let (A, Γ) satisfy Elliott's condition, where A is unital and separable and Γ is countable and discrete. Assume that $B = A \rtimes_{\alpha,r} \Gamma$ is simple and that every nonzero hereditary C^*-subalgebra of A is infinite-dimensional. If $h_0, h_1 \in A_+$ are norm-one positive contractions, then, for every $\varepsilon > 0$, there exist a norm-one positive contraction $k \in A$ and a unitary $u \in B$ such that*

$$\|h_0k - k\| < \varepsilon, \qquad \|h_1uku^* - uku^*\| < \varepsilon. \tag{6.17}$$

Proof. We first construct orthogonal hereditary subalgebras on which multiplication by h_0 and h_1, respectively, is close to the identity. Fix $0 < \eta < \varepsilon/2$ and put

$$e_i = (h_i - (1 - \eta))_+, \qquad i = 0, 1.$$

These elements are nonzero. To obtain nonzero orthogonal positive elements $d_i \in \overline{e_i A e_i}$, $i = 0, 1$, first observe that when $e_0 e_1 = 0$ the normalized elements e_0, e_1 suffice. Otherwise set

$$T = e_0^{1/2} e_1 e_0^{1/2}.$$

If $\mathrm{sp}(T) \cap (0, \infty)$ contains at least two points, choose nonzero functions $f_0, f_1 \in \mathrm{C}_0((0, \infty))_+$ with disjoint supports and with $T f_i(T) \neq 0$, and set

$$d_0' = e_0^{1/2} f_0(T) e_0^{1/2},$$
$$d_1' = e_1 e_0^{1/2} f_1(T) e_0^{1/2} e_1.$$

Then $d_0' d_1' = 0$, and both elements are nonzero. If the positive spectrum of T is the singleton $\{\lambda\}$, its spectral projection p belongs to A. The hereditary algebra pAp is infinite-dimensional, so it contains nonzero orthogonal positive elements x_0, x_1. The same formulas, with $f_i(T)$ replaced by x_i, again give nonzero orthogonal elements. After normalization we obtain d_0, d_1. Since the support projection of e_i is dominated by $1_{(1-\eta, 1]}(h_i)$,

$$\|(1 - h_i) d_i\| \leq \eta, \qquad i = 0, 1. \tag{6.18}$$

Simplicity of B gives $x \in \mathrm{C_c}(\Gamma, A)$ such that $v = d_0 x d_1 \neq 0$. Faithfulness of $\mathbb{E}$ gives $\mathbb{E}(vv^*) \neq 0$; rescale v so that $\|\mathbb{E}(vv^*)\| = 1$. Let F contain the Fourier support of vv^*. Proposition 3.29, applied to $\mathbb{E}(vv^*)^2$, gives a tower base $c \in A' \cap A_\omega$ with

$$\|c \mathbb{E}(vv^*)^2 c\| = 1.$$

As in (6.15), exact tower orthogonality gives

$$cvv^*c = c\mathbb{E}(vv^*)c \in A_\omega.$$

Put $z = cv \in B_\omega$. Since c commutes with d_0 and d_1,

$$zz^* \in \overline{d_0 A_\omega d_0}, \qquad z^* z \in \overline{d_1 B_\omega d_1}.$$

In particular, $zz^* \perp z^* z$. The preceding norm equality also implies $\|zz^*\| = 1$.

Applying the functional-calculus construction in the proof of Lemma 6.6 gives $t \in B_\omega$ and a norm-one positive contraction $k_\omega \in A_\omega$ such that

$$tt^* \perp t^* t, \qquad k_\omega \in \overline{d_0 A_\omega d_0}, \qquad (tt^*) k_\omega = k_\omega, \qquad t^* t \in \overline{d_1 B_\omega d_1}.$$

Lemma 3.2 of [42] gives a unitary $w \in B_\omega$ for which

$$w^* k_\omega w \in \overline{d_1 B_\omega d_1}.$$

Together with (6.18), this yields

$$\|h_0 k_\omega - k_\omega\| \leq \eta, \qquad \|h_1 w^* k_\omega w - w^* k_\omega w\| \leq \eta.$$

Choose representatives of k_ω and w^* in A and B, respectively, and use polar decomposition to obtain unitaries on an ω-large set. Choosing a coordinate at which the estimates hold and normalizing the positive element gives (6.17). □

Proposition 6.8 (Coefficient zero-divisor reduction)**.** *Let (A,Γ) be a minimal C^*-dynamical system with WRP, where A is unital and separable, $\mathrm{T}(A)\neq\emptyset$, and Γ is countably infinite, discrete, and amenable. Assume that every nonzero hereditary C^*-subalgebra of A is infinite-dimensional. Put $B = A\rtimes_{\alpha,r}\Gamma$. If $a\in B$ is noninvertible, then, for every $\varepsilon>0$, there are unitaries $u_0,u_1\in B$, an element $b\in \mathrm{C_c}(\Gamma,A)$, and a norm-one positive contraction $d\in A$ such that*

$$\|u_0 a u_1 - b\| < \varepsilon, \qquad db = bd = 0.$$

Proof. Corollary 3.32 shows that B is simple. It has a faithful tracial state: average a tracial state of A over Γ, compose the result with $\mathbb{E}$, and use Corollary 3.10. Thus B is finite, and a noninvertible element of B is neither left nor right invertible. By [57, Proposition 3.2], after an arbitrarily small perturbation of a we obtain a two-sided zero divisor $a_0\in B$ and nonzero elements $r_0,r_1\in B_+$ such that

$$r_0 a_0 = 0 = a_0 r_1.$$

Apply Lemma 6.6 to (a_0,r_0) and to (a_0^*,r_1). For any prescribed $\delta>0$, it gives unitaries $v_0,v_1\in B$ and norm-one positive contractions $h_0,h_1\in A$ such that

$$\|h_0 v_0 a_0\| < \delta, \qquad \|a_0 v_1^* h_1\| < \delta. \tag{6.19}$$

Lemma 6.7 gives a unitary $w\in B$ and a norm-one positive contraction $k\in A$ such that

$$\|h_0 k - k\| < \delta, \qquad \|h_1 wkw^* - wkw^*\| < \delta.$$

Using these relations and (6.19), we obtain

$$\|k v_0 a_0 v_1^* w\| < (1+\|a_0\|)\delta, \qquad \|v_0 a_0 v_1^* wk\| < (1+\|a_0\|)\delta. \tag{6.20}$$

Fix $0<\sigma<1/4$ and put $q = g_\sigma(k)$. Since $g_\sigma(t) = t\ell_\sigma(t)$ for a continuous function ℓ_σ, (6.20) shows that both $qv_0a_0v_1^*w$ and $v_0a_0v_1^*wq$ have arbitrarily small norm when δ is sufficiently small. Choose $z\in \mathrm{C_c}(\Gamma,A)$ close to $v_0a_0v_1^*w$ and set

$$b = (1-q)z(1-q), \qquad d = \frac{(k-\sigma)_+}{1-\sigma}.$$

Then d is a norm-one positive contraction, $d(1-q) = 0 = (1-q)d$, and hence $db = bd = 0$. Taking the perturbation of a, then δ, and then the Fourier approximation error sufficiently small gives $\|v_0 a v_1^* w - b\| < \varepsilon$. Taking $u_0 = v_0$ and $u_1 = v_1^* w$ proves the proposition. □

The following lemma gives orthogonal coefficient elements that are equivalent in the crossed product. They will absorb the multiplicity in COS.

Lemma 6.9 (Equivalent coefficient reserves)**.** *Let α be a minimal action on a unital C^*-algebra A, and suppose that every nonzero hereditary C^*-subalgebra of A is infinite-dimensional. Put $B = A\rtimes_{\alpha,r}\Gamma$. Let $0\neq h\in A_+$, and let $c\in A_+$ be a positive contraction with $ch = h$. For every $m\in\mathbb{N}$, there are a norm-one positive contraction $a\in A$, elements $g_1,\dots,g_m\in\Gamma$, and monomials $t_i\in Au_{g_i}$ such that*

$$t_i^* t_j = \delta_{ij} a, \qquad ct_i = t_i, \qquad t_i t_i^* \in \mathrm{Her}_A(h).$$

The range elements $t_it_i^*$ are mutually orthogonal.

Proof. Choose mutually orthogonal nonzero positive elements $h_1, \ldots, h_m \in \mathrm{Her}_A(h)$. Begin with $t_1 = h_1^{1/2}$ and $a = h_1$. Suppose that $t_1, \ldots, t_r$ satisfy $t_i^* t_i = a \neq 0$ and $t_i t_i^*$ belongs to the respective hereditary algebra $\mathrm{Her}_A(h_i)$.

Minimality gives $g \in \Gamma$ and $x \in A$ such that

$$y = ax\alpha_g(h_{r+1}) \neq 0.$$

Indeed, otherwise a annihilates the sum of the ideals generated by all the translates of h_{r+1}; its closure is a nonzero invariant ideal and hence is A, contradicting $a \neq 0$. Put $z = yy^* \in \mathrm{Her}_A(a)$. Write $t_i = V_i a^{1/2}$ for the polar decompositions in B^{**}, and replace each t_i by $V_i z^{1/2}$. These elements still belong to Au_{g_i}. To see this, approximate $z^{1/2}$ by $g_\varepsilon(a)z^{1/2}$; the element $V_i g_\varepsilon(a)$ equals $t_i f_\varepsilon(a)$, where $f_\varepsilon(t) = g_\varepsilon(t)t^{-1/2}$ for $t > 0$ and $f_\varepsilon(0) = 0$. The approximants belong to the closed monomial subspace Au_{g_i}.

The new elements have source z, and their range elements remain in the respective hereditary algebras. Add the monomial $u_g^* y^* \in Au_{g^{-1}}$. Its source is z and its range element is $\alpha_{g^{-1}}(y^*y) \in \mathrm{Her}_A(h_{r+1})$. Orthogonality of the hereditary algebras gives the off-diagonal relations. Induction produces all m elements. Divide the common source by its norm and all t_i by the square root of that norm. Since $ch = h$, multiplication by c fixes $\mathrm{Her}_A(h)$; the inclusion of every range element in this hereditary algebra then gives $ct_i = t_i$. □

Combining the coefficient reduction with the local comparison argument of Proposition 6.3 gives the following theorem. The equivalent coefficient elements are fixed before the approximation and absorb any prescribed comparison multiplicity.

Theorem 6.10. *Let (A, Γ) be a minimal C^*-dynamical system with WRP, where A is unital and separable, has stable rank one, and $\mathrm{T}(A) \neq \emptyset$, and where Γ is countably infinite, discrete, and amenable. Assume that every nonzero hereditary C^*-subalgebra of A is infinite-dimensional. If (A, Γ) has (λ, m)-COS for some $\lambda \in (0, 1]$ and $m \in \mathbb{N}$, then*

$$\mathrm{tsr}(A \rtimes_{\alpha,r} \Gamma) = 1.$$

Proof. Put $B = A \rtimes_{\alpha,r} \Gamma$. As in the preceding proposition, B is simple and finite. Let $x \in B$ be noninvertible. By Proposition 6.8, it suffices, up to multiplication by unitaries and an arbitrarily small perturbation, to consider an element $b \in B$ for which there is a norm-one positive contraction $d_0 \in A$ satisfying

$$d_0 b = b d_0 = 0.$$

Put $d = g_{1/2}(d_0)$ and $h_0 = (d_0 - 1/2)_+$. Then d has norm one, $db = bd = 0$, and $dh_0 = h_0$. Apply Lemma 6.9 to obtain a norm-one positive contraction $a \in A$ and fixed monomials $t_1, \ldots, t_m \in B$ such that

$$t_i^* t_j = \delta_{ij} a, \qquad dt_i = t_i.$$

All these elements are chosen before the tracial approximation.

Let $\mathcal{S}$ be the class in Theorem 3.15. Stable rank one passes to hereditary C^*-subalgebras, matrix algebras, and finite direct sums, so every member of $\mathcal{S}$ has stable rank one. The same

theorem gives $B \in \mathrm{TA}_0(\mathcal{S})$, with the approximating contraction in A and with the approximants to elements of A still in A.

It remains to establish the comparison used in the proof of Proposition 6.3. Since every trace on B is faithful, compactness gives fixed constants

$$\delta_d = \min_{\tau \in \mathrm{T}(B)} \tau(g_{1/2}(d)) > 0, \qquad \delta_a = \min_{\tau \in \mathrm{T}(B)} \tau(a) > 0.$$

Fix $0 < \theta < \delta_d/64$ and put $\kappa = \delta_a/4$. Apply Theorem 3.15 with the finite set enlarged to include $t_1, \ldots, t_m$. Use Lemma 2.8 to obtain the exact compressed approximants, with $d' \in A$ chosen to be a positive contraction as in the proof of Theorem 6.4. The functional-calculus cutoff in that lemma preserves arbitrarily small commutators with the fixed columns. Choose the tolerances sufficiently small for the estimates in the proof of Proposition 6.3 and also require

$$\eta < \min\{\delta_a/4, \lambda\delta_a/2\}.$$

Following the notation of Proposition 6.3, put

$$r = g_{3/4}(h), \qquad D = r^{1/2} d' r^{1/2}, \qquad e = 1 - g_{1/3}(h), \qquad y = (D - 2\theta)_+.$$

Both e and y belong to A_+. The enlarged finite set ensures small commutators of h with every t_i and, by functional calculus, of $r^{1/2} = g_{3/4}(h)^{1/2}$ with every t_i. We construct a coefficient element whose m-fold direct sum is Cuntz subequivalent to y.

Put

$$q_0 = 4(h - 3/4)_+, \qquad W = (t_1 q_0, \ldots, t_m q_0) \in \mathrm{M}_{1,m}(B).$$

Then $r^{1/2} q_0 = q_0$ and

$$W^* W = (q_0 a q_0)^{\oplus m}, \qquad \|W\| \leq 1.$$

For each column, the exact relation $d t_i = t_i$ gives

$$\|(1 - D) t_i q_0\| \leq 2\|[r^{1/2}, t_i]\| + \|d' - d\|.$$

With the columns t_i fixed, $\|(1 - D)W\|$ tends to zero with the approximation tolerances. Set

$$V = g_{4\theta}(D) W.$$

The cutoff vanishes on $[0, 2\theta]$, so $VV^* \in \mathrm{Her}_B(y)$. Moreover,

$$\|V - W\| \leq \frac{1}{1 - 4\theta} \|(1 - D)W\|.$$

Indeed, $(1 - g_{4\theta}(t))/(1 - t)$ extends continuously to $[0, 1]$ and has norm at most $(1 - 4\theta)^{-1}$. Since V and W are contractions, a further reduction of the preliminary tolerances therefore ensures

$$\|V^* V - W^* W\| < \kappa.$$

The positive-element perturbation lemma gives, with $z = (q_0 a q_0 - \kappa)_+ \in A_+$,

$$z^{\oplus m} \precsim_B V^*V \sim VV^* \precsim_B y.$$

All comparisons here are understood in matrix algebras over B.

The target z retains a uniform positive trace bound. Since q_0 is a contraction and its defining function takes the value one at one,

$$\tau(1 - q_0^2) \leq \mathrm{d}_\tau(1 - h) < \eta, \qquad \tau \in \mathrm{T}(B).$$

Consequently,

$$\mathrm{d}_\tau(z) \geq \tau(z) \geq \tau(q_0 a q_0) - \kappa > \delta_a - \eta - \kappa > \frac{\delta_a}{2}.$$

By the trace identification in Proposition 3.8, these bounds also hold for every invariant trace ρ on A. The defect estimate now gives

$$\mathrm{d}_\rho(e) < \eta < \lambda \frac{\delta_a}{2} < \lambda \mathrm{d}_\rho(z), \qquad \rho \in \mathcal{M}_1(A, \Gamma).$$

Property (λ, m)-COS and the preceding comparisons give

$$e \precsim_B z^{\oplus m} \precsim_B y.$$

Since e and y belong to B, this is also the unstabilized comparison $e \precsim_B y$. The unitary rotation in the proof of Proposition 6.3 therefore gives $b \in \overline{\mathrm{GL}(B)}$. Multiplying by the inverses of the two unitaries and letting the initial perturbation tend to zero proves that every noninvertible element of B lies in $\overline{\mathrm{GL}(B)}$. □

Remark 6.11. The columns and the constants δ_a, θ, κ are fixed before the approximation. Approximate commutation with the columns, followed by a fixed spectral cut, gives the comparison of the m-fold source with y. Theorem 6.17 uses inner subtowers to obtain the same conclusion without assuming stable rank one for the coefficient algebra. In Lemma 6.6, infinitude of Γ is used to choose a tower level outside the Fourier support of b_0.

6.4 Stable rank of $A \rtimes \Gamma$

For a unital C*-algebra D, write $\mathrm{T}_\omega(D)$ for the set of *limit traces* on D_ω, that is, the traces of the form

$$\tau_\omega([(x_j)]) = \lim_{j \to \omega} \tau_j(x_j), \qquad \tau_j \in \mathrm{T}(D).$$

The WRP defect is uniformly small over traces on D, so every limit trace annihilates it. An arbitrary trace on the norm ultrapower need not annihilate this defect; thus the estimates below concern $\mathrm{T}_\omega(D)$ rather than the possibly larger space $\mathrm{T}(D_\omega)$.

Lemma 6.12. *Let A be a unital C^*-algebra, let $a \in A_+$, and suppose that there are $x_1, \ldots, x_m \in A$ such that*

$$\sum_{i=1}^{m} x_i a x_i^* = 1.$$

For every $0 < \varepsilon < 1$ there is $\eta > 0$ with the following property. If $c \in A_+$ and $\tau \in \mathrm{T}(A)$ satisfy

$$\tau\big(|x_i c - c x_i|^2\big) \leq \eta \tau(c^2), \qquad 1 \leq i \leq m,$$

then

$$\tau(cac) \geq \frac{1-\varepsilon}{\sum_{i=1}^{m} \|x_i\|^2}\, \tau(c^2).$$

Proof. Put $S = \sum_i \|x_i\|^2$. The fullness identity implies $S > 0$. Choose $\eta > 0$ so small that

$$2\|a\|\eta^{1/2} \sum_{i=1}^{m} \|x_i\| \leq \varepsilon. \tag{6.21}$$

Fix c and τ as in the statement, and write $\|z\|_{2,\tau} = \tau(z^*z)^{1/2}$. For $r_i = cx_i$ and $s_i = x_i c$, traciality gives

$$\|r_i\|_{2,\tau},\ \|s_i\|_{2,\tau} \leq \|x_i\| \tau(c^2)^{1/2}, \qquad \|r_i - s_i\|_{2,\tau} \leq \eta^{1/2} \tau(c^2)^{1/2}.$$

Since

$$r_i a r_i^* - s_i a s_i^* = (r_i - s_i) a r_i^* + s_i a (r_i - s_i)^*,$$

the tracial Cauchy–Schwarz inequality yields

$$|\tau(c x_i a x_i^* c) - \tau(x_i cac x_i^*)| \leq 2\|a\| \|x_i\| \eta^{1/2} \tau(c^2).$$

Summing and applying (6.21) gives

$$\begin{aligned} S\tau(cac) &\geq \sum_{i=1}^{m} \tau(x_i cac x_i^*) \\ &\geq \sum_{i=1}^{m} \tau(c x_i a x_i^* c) - \varepsilon \tau(c^2) \\ &= (1-\varepsilon)\tau(c^2), \end{aligned}$$

which is the desired estimate. □

Lemma 6.13. *Let A be a unital C^*-algebra, let $\tau \in \mathrm{T}(A)$, and let $E \in A$ be a positive contraction satisfying $\tau(E - E^2) = 0$. If $a \in A_+$ commutes with E and $f \colon \mathbb{R} \to \mathbb{R}$ is continuous with $f(0) = 0$, then*

$$\tau(f(EaE)) = \tau(Ef(a)E).$$

Proof. Let π_τ be the GNS representation associated to τ. The positive element $E - E^2$ belongs to the trace kernel, and hence

$$p := \pi_\tau(E)$$

is a projection. Since E commutes with a, the projection p commutes with $\pi_\tau(a)$. Functional calculus and $f(0) = 0$ give

$$\pi_\tau(f(EaE)) = f(p\pi_\tau(a)p) = pf(\pi_\tau(a))p = \pi_\tau(Ef(a)E).$$

Applying the GNS trace proves the assertion. □

The following lemma is a grouped form of Lemma 3.24 and a noncommutative counterpart of [42, Lemma 2.18].

Lemma 6.14. *Let $F \subseteq \Gamma$ be finite and let $c \in A' \cap A_\omega$ be a positive contraction such that the translates $g(c)$, $g \in F$, are mutually orthogonal. Let $F_1, \ldots, F_r \subseteq F$ be pairwise disjoint nonempty sets of the same cardinality. There is a c.p.c. order zero map*

$$\phi\colon \mathrm{M}_r(\mathbb{C}) \longrightarrow (A \rtimes \Gamma)_\omega$$

such that

$$\phi(e_{ii}) = \sum_{g \in F_i} g(c), \qquad 1 \leq i \leq r.$$

Proof. Write $|F_i| = q$ and enumerate $F_i = \{g_{i,1}, \ldots, g_{i,q}\}$. If $c \neq 0$, apply Lemma 3.24 to the normalized base $c/\|c\|$ and the shape $\bigsqcup_i F_i$, ordered lexicographically, and multiply the resulting map by $\|c\|$. This gives a c.p.c. order zero map ψ from $\mathrm{M}_{rq}(\mathbb{C})$ with diagonal images $g_{i,j}(c)$. If $c = 0$, take $\psi = 0$ instead. Under the identification $\mathrm{M}_{rq} \cong \mathrm{M}_r \otimes \mathrm{M}_q$, let

$$\iota\colon \mathrm{M}_r \longrightarrow \mathrm{M}_r \otimes \mathrm{M}_q, \qquad \iota(x) = x \otimes 1_q.$$

Then $\phi = \psi \circ \iota$ is c.p.c. and order zero, and

$$\phi(e_{ii}) = \sum_{j=1}^{q} \psi(e_{(i,j),(i,j)}) = \sum_{j=1}^{q} g_{i,j}(c).$$

□

Lemma 6.15. *Let (A, Γ) be a unital C^*-dynamical system, where Γ is countably infinite, discrete, and amenable, and put $B = A \rtimes_{\alpha,r} \Gamma$. Let $a, b \in A_+$ and suppose that*

$$\sum_{i=1}^{p} x_i a x_i^* = 1, \qquad \sum_{j=1}^{q} y_j b y_j^* = 1$$

for some $x_1, \ldots, x_p, y_1, \ldots, y_q \in B$. Set

$$\delta = \frac{1}{2\left(\sum_{i=1}^{p} \|x_i\|^2 + \sum_{j=1}^{q} \|y_j\|^2\right)}.$$

Fix $\lambda \in (0,1]$ and a finite symmetric set $K \subseteq \Gamma$ containing e, and fix a finite set $G \subseteq \Gamma$ containing e. Put $\beta = \min\{\delta/16, 1/4\}$. Then there are a finite set $L \subseteq \Gamma$, a number $\eta > 0$, and $n \in \mathbb{N}$ such that whenever F is (L,η)-invariant and $c \in A' \cap A_\omega$ is a positive contraction whose F-translates are mutually orthogonal, there is a c.p.c. order zero map

$$\phi\colon \mathrm{M}_{n^2}(\mathbb{C}) \longrightarrow B_\omega.$$

Writing

$$h_F = \sum_{g\in F} g(c), \qquad e_i = \phi(e_{ii}), \qquad E_k = \sum_{j=1}^{n} e_{n(k-1)+j}, \qquad 1 \le k \le n,$$

one has $e_i \in A' \cap A_\omega$ and:

1. $$\phi(1) = \sum_{i=1}^{n^2} e_i \le h_F, \qquad \tau\left(h_F - \phi(1)\right) \le \frac{\lambda\delta}{16n}\tau(h_F), \qquad \tau \in \mathrm{T}_\omega(B);$$

2. *for $1 \le k \le n$ and $\tau \in \mathrm{T}_\omega(B)$,*

$$\tau(E_k a E_k) \ge \delta\tau(E_k^2), \qquad \tau(E_k b E_k) \ge \delta\tau(E_k^2);$$

3. *$E_{k_1} \perp \alpha_g(E_{k_2})$ whenever $g \in K$ and $k_1 \ne k_2$;*

4. *$\alpha_g(E_k) \le h_F$ for $g \in K$ and $1 \le k \le n$.*

5. *Put $R_k = \sum_{j=4}^{n} e_{n(k-1)+j}$. There are inner subtower sums $P_k, T_k \in A' \cap A_\omega$ such that*

$$P_k \le R_k, \qquad T_k \le E_k,$$

and, for every $g \in G$, the levels of $\alpha_g(P_k)$ belong to the subtower defining R_k, and those of $\alpha_g(T_k)$ belong to the subtower defining E_k. Moreover,

$$\tau(R_k - P_k) \le \beta\tau(E_k), \qquad \tau(E_k - T_k) \le \beta\tau(E_k), \qquad \tau \in \mathrm{T}_\omega(B).$$

Consequently,

$$g_{1/2}(R_k)\alpha_g(g_1(P_k)) = \alpha_g(g_1(P_k)), \qquad g_{1/2}(E_k)\alpha_g(g_1(T_k)) = \alpha_g(g_1(T_k)) \quad (g \in G).$$

The integer n may be required to exceed any prescribed bound.

Proof. We first derive a uniform commutator estimate from Følner invariance. The estimate applies to tower sums whose bases are arbitrary positive contractions, including zero. Given a finite set $\mathcal{X} \subseteq B$ and $\zeta > 0$, there are a finite symmetric set $L_0 \subseteq \Gamma$ and $\eta_0 > 0$ such that, whenever H is (L_0,η_0)-invariant and $c \in A' \cap A_\omega$ is a positive contraction with orthogonal H-translates, its support $h_H = \sum_{g\in H} g(c)$ satisfies

$$\tau\big(|[x,h_H]|^2\big) \le \zeta\tau(h_H^2), \qquad x \in \mathcal{X}, \quad \tau \in \mathrm{T}_\omega(B). \tag{6.22}$$

Indeed, first consider a Fourier monomial $x = du_s$. Since h_H commutes with A,

$$\|[du_s, h_H]\|_{2,\tau} \le \|d\| \, \|\alpha_s(h_H) - h_H\|_{2,\tau}.$$

If $s, s^{-1} \in L_0$, tower orthogonality and traciality give

$$\begin{aligned} \|\alpha_s(h_H) - h_H\|_{2,\tau}^2 &= 2\tau(h_H^2) - 2\tau(h_H \alpha_s(h_H)) \\ &\le 2\big(|H| - |H \cap sH|\big)\tau(c^2) \\ &\le 2\eta_0 |H| \tau(c^2) = 2\eta_0 \tau(h_H^2). \end{aligned}$$

The middle inequality uses the common levels indexed by $H \cap sH$; all other terms have nonnegative trace because they are products of positive elements. The estimate does not require h_H and $\alpha_s(h_H)$ to commute. Approximate the finitely many elements of $\mathcal{X}$ by Fourier polynomials. The preceding estimate, the triangle inequality, and

$$\|[x - z, h_H]\|_{2,\tau} \le 2\|x - z\| \tau(h_H^2)^{1/2}$$

then prove (6.22) after enlarging L_0 and decreasing η_0.

Apply Lemma 6.12, with $\varepsilon = 1/2$, first to $(a, x_1, \ldots, x_p)$ and then to $(b, y_1, \ldots, y_q)$. For the resulting tolerances, apply (6.22). Thus L_0 and η_0 may be chosen so that every (L_0, η_0)-invariant H and central positive contraction c with orthogonal H-translates satisfy

$$\tau(h_H a h_H) \ge \delta \tau(h_H^2), \qquad \tau(h_H b h_H) \ge \delta \tau(h_H^2), \qquad \tau \in \mathrm{T}_\omega(B). \tag{6.23}$$

Enlarge L_0 to contain G, and decrease η_0 so that $\eta_0 \le \beta$. These changes preserve (6.23).

Choose $n > 3$ such that

$$\frac{1}{n-3} < \frac{\delta}{24}, \qquad \frac{\lambda\delta}{16n} < \frac{1}{2}, \qquad \frac{3}{n} < \frac{\delta}{16}, \qquad \frac{32}{n} < \frac{\lambda\delta}{4}. \tag{6.24}$$

Put $\theta = \lambda\delta/(32n)$. We use the left-Følner, right-translate versions of Lemmas 7.1 and 7.2 of [42]. They follow from the stated right-Følner, left-translate versions by passing to inverses (equivalently, to the opposite group). In particular, the conclusion corresponding to Lemma 7.2 is $KH_i \subseteq F_i$, consistent with the fact that α_g moves a tower level indexed by r to the level indexed by gr.

By the exact tiling theorem [19, Theorem 4.3], in the same inverse convention there is a right-translate tiling of Γ with shapes $P_1, \ldots, P_t$ that are (L_0, η_0)-invariant. Apply the inverse version of [42, Lemma 7.2] to θ, n, K and these shapes, obtaining a pair (L_1, η_1). Use the exact tiling theorem again to obtain a right-translate tiling with (L_1, η_1)-invariant shapes $Q_1, \ldots, Q_s$. Finally, the inverse version of [42, Lemma 7.1], applied to θ, n and the Q_l, gives a pair (L, η).

Fix F and c as in the statement. Lemma 7.1 provides $R_1 \subseteq F$ such that

$$|R_1| < \theta |F|, \tag{6.25}$$

and $F \setminus R_1$ is tiled by right translates of the Q_l, with the multiplicity of every shape divisible by n. Distributing each kind of tile evenly gives a decomposition

$$F \setminus R_1 = F_1 \sqcup \cdots \sqcup F_n$$

in which the F_i are (L_1, η_1)-invariant and have equal cardinality. Lemma 7.2 then supplies sets $H_i \subseteq F_i$ such that

$$KH_i \subseteq F_i, \qquad |H_1| = \cdots = |H_n|, \qquad |H_i| > (1-\theta)|F_i|, \tag{6.26}$$

and every H_i is tiled by right translates of the P_l, with all multiplicities divisible by n. If

$$R_2 = (F \setminus R_1) \setminus \bigsqcup_{i=1}^{n} H_i,$$

then

$$|R_2| < \theta |F|. \tag{6.27}$$

Divide, for each i, the tiles of every shape occurring in H_i evenly into n groups. This gives

$$H_i = H_{i,1} \sqcup \cdots \sqcup H_{i,n}.$$

All n^2 sets $H_{i,j}$ have the same cardinality. Moreover, each is (L_0, η_0)-invariant, since left invariance is preserved by right translation and disjoint unions. Define

$$e_{n(i-1)+j} = \sum_{g \in H_{i,j}} g(c), \qquad 1 \leq i, j \leq n.$$

Lemma 6.14 gives a c.p.c. order zero map $\phi \colon \mathrm{M}_{n^2}(\mathbb{C}) \to B_\omega$ with these diagonal images. Each e_l belongs to $A' \cap A_\omega$.

For each cell define

$$J_{i,j} = \mathrm{Int}_G(H_{i,j}).$$

Since $G \subseteq L_0$ and $H_{i,j}$ is (L_0, η_0)-invariant, we have

$$|H_{i,j} \setminus J_{i,j}| < \beta |H_{i,j}|, \qquad GJ_{i,j} \subseteq H_{i,j}.$$

Set

$$P_i = \sum_{j=4}^{n} \sum_{g \in J_{i,j}} g(c), \qquad T_i = \sum_{j=1}^{n} \sum_{g \in J_{i,j}} g(c).$$

These are the required inner subtower sums. Every trace on B_ω has the same value on all translates of c, so the cardinality estimates give the trace bounds in (5). The absorption identities in (5) follow level by level from $g_{1/2}(t) g_1(t) = g_1(t)$ and tower orthogonality.

The levels omitted from $\phi(1)$ are indexed by $R_1 \sqcup R_2$. Thus, for $\tau \in \mathrm{T}_\omega(B)$, inner invariance of τ and (6.25)–(6.27) give

$$\begin{aligned}\tau(h_F - \phi(1)) &= |R_1 \sqcup R_2|\tau(c) \\ &\leq 2\theta|F|\tau(c) = \frac{\lambda\delta}{16n}\tau(h_F).\end{aligned}$$

For each i, one has $E_i = \sum_{g \in H_i} g(c) = h_{H_i}$. Equation (6.23) therefore gives

$$\tau(E_i a E_i) \geq \delta\tau(E_i^2), \qquad \tau(E_i b E_i) \geq \delta\tau(E_i^2),$$

and hence assertion (2).

Finally, (6.26) and the disjointness of the F_i show that gH_{k_2} is disjoint from H_{k_1} whenever $g \in K$ and $k_1 \neq k_2$. Tower orthogonality proves (3). The inclusion $gH_k \subseteq F_k \subseteq F$ gives (4). □

The hypothesis on hereditary subalgebras in Proposition 6.8 follows from minimality and WRP when Γ is infinite.

Lemma 6.16. *Let (A, Γ) be a minimal C^*-dynamical system with WRP, where A is unital and separable, $\mathrm{T}(A) \neq \varnothing$, and Γ is countably infinite, discrete, and amenable. Then every nonzero hereditary C^*-subalgebra of A is infinite-dimensional.*

Proof. Amenability gives an invariant tracial state ρ on A. The trace $\rho \circ \mathbb{E}$ on $A \rtimes_{\alpha,r} \Gamma$ is faithful by Corollary 3.10; in particular, ρ is faithful.

Suppose that $p \in A$ is a nonzero minimal projection. Given N, choose distinct elements $g_1 = e, g_2, \ldots, g_N \in \Gamma$. Lemma 3.30 and Proposition 3.29, applied to p and $F = \{g_1, \ldots, g_N\}$, give a tower (c, F) such that

$$\|cpc\| = 1.$$

Since $c \in A' \cap A_\omega$ and $pA_\omega p = (pAp)_\omega = \mathbb{C}p$, we have $cpc = p$ and hence $cp = pc = p$. Applying g_i gives

$$g_i(c)g_i(p) = g_i(p)g_i(c) = g_i(p).$$

The tower levels $g_i(c)$ are mutually orthogonal, so the projections $g_1(p), \ldots, g_N(p)$ are mutually orthogonal. Invariance of ρ therefore gives

$$N\rho(p) \leq 1.$$

This is impossible for arbitrarily large N, since $\rho(p) > 0$. Thus A has no minimal projections.

If a nonzero hereditary C^*-subalgebra $H \subseteq A$ were finite-dimensional, it would contain a minimal projection p. Since $pAp \subseteq H$, this projection would also be minimal in A, a contradiction. □

The equivalent coefficient elements also give the property (D) argument for arbitrary fixed comparison multiplicity.

Theorem 6.17. *Let (A,Γ) be a minimal C^*-dynamical system with WRP, where A is unital and separable, $\mathrm{T}(A)\neq\varnothing$, and Γ is countably infinite, discrete, and amenable. If (A,Γ) has (λ,m)-COS for some $\lambda\in(0,1]$ and $m\in\mathbb{N}$, then*

$$\operatorname{tsr}(A\rtimes_{\alpha,r}\Gamma)=1.$$

Proof. Put $B=A\rtimes_{\alpha,r}\Gamma$. By Corollary 3.32, B is simple. Let $z\in B$ be noninvertible and fix $\varepsilon>0$. Lemma 6.16 permits us to apply Proposition 6.8. We obtain unitaries $u_0,u_1\in B$, an element $x\in\mathrm{C_c}(\Gamma,A)$, and a norm-one positive contraction $b_0\in A$ such that

$$\|u_0zu_1-x\|<\varepsilon,\qquad b_0x=xb_0=0. \tag{6.28}$$

The hereditary algebra $\overline{b_0Ab_0}$ is infinite-dimensional, so it contains orthogonal norm-one positive contractions c_0,d_0. Put

$$c=g_{1/2}(c_0),\qquad d=g_{1/2}(d_0),\qquad h_c=(c_0-1/2)_+,\qquad h_d=(d_0-1/2)_+.$$

These satisfy $ch_c=h_c$ and $dh_d=h_d$; all four elements annihilate x on both sides. Apply Lemma 6.9 separately to these two corners. Before choosing any tower parameters, we obtain norm-one positive contractions $a,b\in A$ and fixed monomials

$$t_i\in Au_{\gamma_i},\qquad \ell_i\in Au_{\nu_i},\qquad 1\le i\le m,$$

such that

$$t_i^*t_j=\delta_{ij}a,\qquad ct_i=t_i,\qquad \ell_i^*\ell_j=\delta_{ij}b,\qquad d\ell_i=\ell_i.$$

Their range elements are mutually orthogonal within each family and belong to $\mathrm{Her}_A(h_c)$ or $\mathrm{Her}_A(h_d)$, respectively. The nonzero elements $g_{1/2}(a)$ and $g_{1/2}(b)$ are full in the simple unital algebra B. Hence there are $v_1,\dots,v_p,w_1,\dots,w_q\in B$ such that

$$\sum_{i=1}^{p}v_ig_{1/2}(a)v_i^*=1,\qquad \sum_{j=1}^{q}w_jg_{1/2}(b)w_j^*=1. \tag{6.29}$$

Put

$$\delta=\frac{1}{2\left(\sum_i\|v_i\|^2+\sum_j\|w_j\|^2\right)}.$$

Write

$$x=\sum_{g\in K}x_gu_g,$$

where K is finite, symmetric, and contains e.

Put

$$G=\{e,\gamma_1,\dots,\gamma_m,\nu_1,\dots,\nu_m\}.$$

Apply Lemma 6.15 to $g_{1/2}(a)$, $g_{1/2}(b)$, λ, K, and G, and denote the resulting data by L, η, n. By the choice made in (6.24),

$$\frac{3}{n} < \frac{\delta}{16}, \qquad \frac{32}{n^2} < \frac{\lambda\delta}{4n}. \tag{6.30}$$

Choose, by the exact tiling theorem [19, Theorem 4.3], an exact tiling whose shapes $F_1, \ldots, F_S$ are (L, η)-invariant. WRP gives bases $r_s \in A' \cap A_\omega$ such that all the levels $g(r_s)$, $g \in F_s$, are mutually orthogonal and, with

$$Q_s = \sum_{g \in F_s} g(r_s), \qquad Q = \sum_{s=1}^{S} Q_s,$$

one has $1 - Q \in J_{\mathcal{M}_1(A,\Gamma)}$.

For each s, Lemma 6.15 gives a c.p.c. order zero map

$$\Phi_s : \mathrm{M}_{n^2}(\mathbb{C}) \longrightarrow B_\omega.$$

The ranges of these maps are mutually orthogonal, so $\Phi = \sum_s \Phi_s$ is again c.p.c. and order zero. Put

$$e_i = \Phi(e_{ii}), \qquad H = \Phi(1), \qquad E_k = \sum_{j=1}^{n} e_{n(k-1)+j}.$$

Then $e_i \in A' \cap A_\omega$, $H \leq Q$, and, for every $\tau \in \mathrm{T}_\omega(B)$,

$$\tau(Q - H) \leq \frac{\lambda\delta}{16n}, \tag{6.31}$$

$$\tau(E_k g_{1/2}(a) E_k) \geq \delta\tau(E_k^2), \tag{6.32}$$

$$\tau(E_k g_{1/2}(b) E_k) \geq \delta\tau(E_k^2). \tag{6.33}$$

Proposition 3.8 identifies the coordinate traces with invariant traces on A.

The image of Q in the GNS representation of any such τ is the unit. Since the tower levels are orthogonal and $Qg(r_s) = g(r_s)^2$, every tower level is a projection in that representation. The same is therefore true of e_i, H, and E_k. A trace composed with an order zero map has equal values on the diagonal matrix units, and hence

$$\tau(E_k^2) = \frac{1}{n}\tau(H) \geq \frac{1}{n}\left(1 - \frac{\lambda\delta}{16n}\right) > \frac{1}{2n}. \tag{6.34}$$

For $0 < t < 1$, let $\Phi_t = g_t(\Phi)$ and write

$$e_i^{(t)} = \Phi_t(e_{ii}), \qquad E_k^{(t)} = \sum_{j=1}^{n} e_{n(k-1)+j}^{(t)}, \qquad R_k^{(t)} = \sum_{j=4}^{n} e_{n(k-1)+j}^{(t)}.$$

Set

$$\begin{aligned}
\phi &= \Phi_{1/4}, & h &= \phi(1),\\
s_k &= \sum_{j=1}^{3} e_{n(k-1)+j}^{(1/4)}, & \widehat{E}_k &= E_k^{(1/4)},\\
\widetilde{c}_k &= g_{1/2}(R_k^{(1/2)} c R_k^{(1/2)}), & c_k &= g_{1/4}(R_k^{(1/2)} c R_k^{(1/2)}),\\
\widetilde{d}_k &= g_{1/2}(E_k^{(1/2)} d E_k^{(1/2)}), & d_k &= g_{1/4}(E_k^{(1/2)} d E_k^{(1/2)}).
\end{aligned}$$

All these elements belong to A_ω except for the order zero map, whose range lies in B_ω.

For each WRP tower, take the inner subtower sums in Lemma 6.15(5). Summing over the towers gives $P_k \le R_k$ and $T_k \le E_k$, where $R_k = E_k - \sum_{j=1}^{3} e_{n(k-1)+j}$. Put

$$q_k^c = g_1(P_k), \qquad q_k^d = g_1(T_k), \qquad A_k = q_k^c a q_k^c, \qquad B_k = q_k^d b q_k^d.$$

Define the inner and outer cutoffs of the comparison targets by

$$\widetilde{a}_k = g_{1/2}(A_k), \qquad \widehat{a}_k = g_{1/4}(A_k), \qquad \widetilde{b}_k = g_{1/2}(B_k), \qquad \widehat{b}_k = g_{1/4}(B_k).$$

We first compare the m-fold direct sums with the hereditary subalgebras generated by c_k and d_k. Set

$$v_{i,k} = t_i q_k^c, \qquad w_{i,k} = \ell_i q_k^d, \qquad 1 \le i \le m.$$

The source relations are

$$v_{i,k}^* v_{j,k} = \delta_{ij} A_k, \qquad w_{i,k}^* w_{j,k} = \delta_{ij} B_k.$$

Each range element belongs to A_ω: if, for example, $t_i = z_i u_{\gamma_i}$ with $z_i \in A$, then

$$v_{i,k} v_{i,k}^* = z_i \alpha_{\gamma_i}((q_k^c)^2) z_i^*.$$

The range elements are orthogonal within each family. The inner subtower identities in Lemma 6.15(5), together with centrality of the tower sums relative to A, give

$$R_k^{(1/2)} v_{i,k} = v_{i,k}, \qquad c v_{i,k} = v_{i,k}, \qquad E_k^{(1/2)} w_{i,k} = w_{i,k}, \qquad d w_{i,k} = w_{i,k}.$$

Thus $R_k^{(1/2)} c R_k^{(1/2)}$ fixes every $v_{i,k}$, and $E_k^{(1/2)} d E_k^{(1/2)}$ fixes every $w_{i,k}$. Their cutoff functions also fix these columns. In particular, the orthogonal equivalent range elements belong to the hereditary algebras of $\widetilde{c}_k$ and $\widetilde{d}_k$, respectively. Writing the columns as row matrices gives

$$\widehat{a}_k^{\oplus m} \precsim A_k^{\oplus m} \sim \sum_{i=1}^{m} v_{i,k} v_{i,k}^* \precsim c_k, \qquad \widehat{b}_k^{\oplus m} \precsim B_k^{\oplus m} \sim \sum_{i=1}^{m} w_{i,k} w_{i,k}^* \precsim d_k.$$

These stabilized Cuntz comparisons have targets in the respective hereditary subalgebras.

For the lower trace bounds on the targets, observe that in the GNS representation of every limit trace, the tower sums are projections, so their cutoffs q_k^c, q_k^d have the same images as P_k, T_k.

All these projections commute with a and b. By Lemma 6.13, replacing a row by an inner subtower decreases its trace by at most the trace of the omitted levels. With $\beta = \delta/16$, Lemma 6.15(5), (6.32)–(6.34), and (6.30) therefore give

$$\begin{aligned} \tau(\widetilde{a}_k) &= \tau(P_k g_{1/2}(a) P_k) \\ &\geq \left(\delta - \frac{3}{n} - \beta\right) \tau(E_k^2) > \frac{7\delta}{16n} > \frac{\delta}{3n}, \end{aligned} \tag{6.35}$$

$$\begin{aligned} \tau(\widetilde{b}_k) &= \tau(T_k g_{1/2}(b) T_k) \\ &\geq (\delta - \beta)\tau(E_k^2) > \frac{15\delta}{32n} > \frac{\delta}{3n}. \end{aligned} \tag{6.36}$$

Since a, b are contractions of norm one, the fullness identities give $\delta \leq 1/4$ and the value of β in Lemma 6.15 is $\delta/16$.

The corresponding dimension-function estimates at the coordinates follow from the spectral cutoffs. Choose positive-contraction representatives $e_i = [(e_{i,j})]$ in A. Since

$$\mathrm{d}_\rho(g_{1/4}(e_{i,j})) \leq 8\rho(e_{i,j}),$$

subadditivity of dimension functions and equality of the diagonal traces of Φ imply

$$\lim_{j\to\omega} \sup_{\rho\in\mathcal{M}_1(A,\Gamma)} \mathrm{d}_\rho(s_{k,j}) \leq \frac{24}{n^2}, \tag{6.37}$$

where $(s_{k,j})$ is the corresponding representative of s_k. Equations (6.35) and (6.36) imply, for suitable positive representatives, that on an ω-large set of coordinates

$$\inf_{\rho\in\mathcal{M}_1(A,\Gamma)} \rho(\widetilde{a}_{k,j}) > \frac{\delta}{4n}, \qquad \inf_{\rho\in\mathcal{M}_1(A,\Gamma)} \rho(\widetilde{b}_{k,j}) > \frac{\delta}{4n}. \tag{6.38}$$

For the defect, choose positive-contraction representatives H_j of H. The support of $1 - g_{1/4}(H_j)$ is contained in the spectral region $[0, 1/4)$, and therefore

$$\mathrm{d}_\rho(1 - g_{1/4}(H_j)) \leq \frac{16}{9}\rho((1 - H_j)^2).$$

In every limit-trace GNS representation, H is a projection and (6.31) gives

$$\tau((1 - H)^2) = \tau(1 - H) \leq \frac{\lambda\delta}{16n}.$$

Consequently,

$$\lim_{j\to\omega} \sup_{\rho\in\mathcal{M}_1(A,\Gamma)} \mathrm{d}_\rho(1 - g_{1/4}(H_j)) \leq \frac{\lambda\delta}{9n}. \tag{6.39}$$

The uniform estimates in (6.37)–(6.39) follow by contradiction. Traces witnessing a failure at successive coordinates define a limit trace in $\mathrm{T}_\omega(B)$ that contradicts the ultrapower estimates.

By (6.30), (6.37), (6.38), and (6.39), on an ω-large set one has, simultaneously for every invariant trace ρ,

$$\mathrm{d}_\rho(s_{k,j}) < \lambda \mathrm{d}_\rho(\widetilde{a}_{k,j}), \qquad \mathrm{d}_\rho(1 - g_{1/4}(H_j)) < \lambda \mathrm{d}_\rho(\widetilde{b}_{k,j}).$$

Property (λ, m)-COS gives, coordinatewise in B,

$$s_{k,j} \precsim \widetilde{a}_{k,j}^{\oplus m}, \qquad 1 - g_{1/4}(H_j) \precsim \widetilde{b}_{k,j}^{\oplus m}.$$

Choose positive-contraction representatives of A_k, B_k, and obtain both their inner and outer cutoffs by coordinatewise functional calculus. The relations

$$\widehat{a}_{k,j}\widetilde{a}_{k,j} = \widetilde{a}_{k,j}, \qquad \widehat{b}_{k,j}\widetilde{b}_{k,j} = \widetilde{b}_{k,j}$$

then hold exactly at every coordinate. Apply Lemma 4.2 in $\mathrm{M}_m(B)$, identifying $(\mathrm{M}_m(B))_\omega$ with $\mathrm{M}_m(B_\omega)$ and embedding scalar sources in the first corner. It gives

$$s_k \precsim \widehat{a}_k^{\oplus m} \precsim c_k, \qquad 1 - h \precsim \widehat{b}_k^{\oplus m} \precsim d_k.$$

The last comparisons follow from the preceding m-fold comparisons. Since the first and last elements belong to B_ω, these stabilized comparisons are also comparisons in B_ω. Thus

$$s_k \precsim c_k, \qquad 1 - h \precsim d_k. \tag{6.40}$$

The remaining $\mathcal{D}_0$ relations of [42, Definition 5.1] follow from the cutoff identities and tower orthogonality. The cutoff functions satisfy

$$g_{1/4}(t)g_{1/2}(t) = g_{1/2}(t),$$

and hence

$$\widehat{E}_k c_k = c_k, \qquad \widehat{E}_k d_k = d_k.$$

The subtower defining c_k is orthogonal to the first three diagonal entries, so $c_k \perp s_k$; also $c_k \perp d_k$ because $c \perp d$. Both elements lie in $\overline{b_0 B_\omega b_0}$. The trace lower bounds show that A_k and B_k are nonzero; hence there are nonzero columns $v_{i,k}$ and $w_{i,k}$ fixed by c_k and d_k. Since c_k, d_k are contractions, this proves that they have norm one. Summing the relations in Lemma 6.15(3)–(4) over the WRP towers gives

$$\widehat{E}_{k_1}\alpha_g(\widehat{E}_{k_2}) = 0 \quad (g \in K,\ k_1 \neq k_2).$$

Since the diagonal entries commute with A, the Fourier expansion of x gives

$$\widehat{E}_{k_1} x \widehat{E}_{k_2} = 0 \quad (k_1 \neq k_2).$$

Finally, with $p = q = n$, $r = 3$, and $l = 1$, the numerical condition in the definition is $3n > 2n$. Together with (6.28) and (6.40), these relations show that the constant copy of x is a $\mathcal{D}_0$-element of B_ω. By [42, Proposition 5.4],

$$x \in \overline{\mathrm{GL}(B_\omega)}.$$

This ultrapower conclusion implies the same assertion in B. If an invertible $y \in B_\omega$ is close to x, lift y and y^{-1} to bounded sequences (y_j) and (z_j) in B. On an ω-large set, both $y_j z_j$ and $z_j y_j$ are within $1/2$ of the unit, so y_j is invertible; at one such coordinate it is as close to x as desired. Thus $x \in \overline{\mathrm{GL}(B)}$. Equation (6.28), followed by multiplication by the two unitaries, shows that z is arbitrarily close to an invertible element. Hence B has stable rank one. □

Remark 6.18. The equivalent coefficient elements and their monomial witnesses are chosen before the fullness constant δ and the integer n. Their group elements determine the inner subtowers. The comparison targets have trace at least $\delta/(3n)$, whereas s_k has dimension-function bound $24/n^2$. The m equivalent orthogonal copies lie in a single hereditary subalgebra. These comparisons absorb the multiplicity in (λ, m)-COS without cancellation of an amplified Cuntz comparison or an additional divisibility hypothesis.

6.5 $\mathcal{Z}$-stability of $A \rtimes \Gamma$ where $\mathrm{rc}(A) < \infty$

Corollary 4.32 and Theorem 6.17 give Cuntz-semigroup regularity for crossed products whose coefficient algebras have positive finite radius of comparison. Locally finite nuclear dimension then gives tensorial absorption.

Theorem 6.19 (Finite coefficient radius of comparison)**.** *Let A be a separable, unital, stably finite, nuclear C^*-algebra with $\mathrm{T}(A) \neq \varnothing$, and let a countably infinite discrete amenable group Γ act minimally on A. Suppose that at least one of the following holds:*

1. *the action has WRP and (λ, m)-COS for some $\lambda \in (0, 1]$ and $m \in \mathbb{N}$;*
2. *the action has strong URP;*
3. *the action has WRP with Cuntz control in the crossed product.*

If $\mathrm{rc}(A) < \infty$, then

$$B = A \rtimes_{\alpha,r} \Gamma$$

is simple, has strict comparison of positive elements, and has stable rank one. Every $f \in \mathrm{LAff}_{++}(\mathrm{T}(B))$ is the rank function of an element of $(B \otimes \mathcal{K})_+$. Moreover, the canonical inclusion $\iota : B \to B \otimes \mathcal{Z}$, $b \mapsto b \otimes 1$, induces an isomorphism of ordered semigroups

$$\mathrm{Cu}(\iota) : \mathrm{Cu}(B) \xrightarrow{\cong} \mathrm{Cu}(B \otimes \mathcal{Z}), \qquad \mathrm{Cu}(B) \cong \mathrm{V}(B) \sqcup \mathrm{LAff}_{++}(\mathrm{T}(B)).$$

Proof. Lemma 4.30 and Definition 4.29 show that each alternative includes WRP. In the second and third cases, Propositions 4.22 and 4.31 give $(1, 1)$-COS. Thus all three alternatives imply WRP and COS. Corollary 3.32 gives simplicity of B. Nuclearity of A and amenability of Γ give nuclearity, and hence exactness, of B. Amenability supplies an invariant trace on A, which extends to a trace on B. Simplicity makes this trace faithful, so B is stably finite. The infinitely many canonical group unitaries are linearly independent, so B is non-elementary.

Corollary 4.32 gives strict comparison, and Theorem 6.17 gives stable rank one. Exactness identifies normalized 2-quasitraces with tracial states. The rank-realization assertion is therefore [62, Theorem 8.11]; the canonical Cuntz-semigroup isomorphism and its displayed description follow from [62, Corollary 8.12]. □

Corollary 6.20 (Finite coefficient radius of comparison and $\mathcal{Z}$-stability)**.** *Let A be a separable, unital, stably finite* C*-*algebra with* $\mathrm{T}(A) \neq \varnothing$*, locally finite nuclear dimension, and* $\mathrm{rc}(A) < \infty$*. Let a countably infinite discrete amenable group Γ act minimally on A. Suppose that at least one of the following holds:*

1. *the action has WRP and (λ, m)-COS for some $\lambda \in (0, 1]$ and $m \in \mathbb{N}$;*
2. *the action has strong URP;*
3. *the action has WRP with Cuntz control in the crossed product.*

Then

$$B = A \rtimes_{\alpha,r} \Gamma$$

is simple and $\mathcal{Z}$-stable.

Proof. Locally finite nuclear dimension implies nuclearity of A. Theorem 6.19 therefore applies and gives simplicity of B and the canonical ordered-semigroup description

$$\mathrm{Cu}(B) \cong \mathrm{V}(B) \sqcup \mathrm{LAff}_{++}(\mathrm{T}(B)).$$

Theorem 5.11 gives $B \cong B \otimes \mathcal{Z}$. □

Remark 6.21 (Local dimension and the coefficient radius of comparison)*.* The nuclear dimension hypothesis in Corollary 6.20 is local: each finite set is approximated inside a subalgebra of finite nuclear dimension, with no common bound on these dimensions. The coefficient algebra may have positive finite radius of comparison and need not have strict comparison or be $\mathcal{Z}$-stable. The local hypothesis supplies the central divisibility step through Theorem 5.11; without it, Theorem 6.19 still gives Cuntz-semigroup regularity, but this argument does not establish tensorial absorption.

For an infinite group, this extends the conclusion of Theorem 5.23 beyond $\mathcal{Z}$-stable coefficients. Indeed, a simple, nuclear, stably finite, $\mathcal{Z}$-stable coefficient algebra has finite nuclear dimension by [13] and radius of comparison zero. Simplicity makes every action minimal, and Corollary 4.16 supplies COS. Theorem 5.23 also applies to finite groups.

Example 6.22 (A Villadsen algebra with positive radius and strong URP)*.* Motivated by Hirshberg's construction in [36], we construct a simple unital Villadsen algebra A of the first type and an automorphism α such that

$$0 < \mathrm{rc}(A) < 1, \qquad (A, \mathbb{Z}, \alpha) \text{ has strong URP.}$$

The algebra A is not $\mathcal{Z}$-stable, whereas $A \rtimes_\alpha \mathbb{Z}$ is simple and $\mathcal{Z}$-stable. The construction determines the algebra and action simultaneously, using one connected homogeneous building block at each stage.

Proof. The equivariant inductive system. Start with $X_0 = [0,1]^2$, $r_0 = q_0 = 1$, $\sigma_0 = \mathrm{id}$, and $u_0 = 1$. Inductively, write

$$A_n = \mathrm{M}_{r_n}(\mathrm{C}(X_n)), \qquad \alpha_n(a)(x) = u_n a(\sigma_n^{-1}x)u_n^*, \qquad \sigma_n^{q_n} = \mathrm{id}, \quad u_n^{q_n} = 1,$$

where σ_n permutes the copies of the seed cube and u_n is a constant unitary. Give all products the maximum product metric. Choose a finite σ_n-invariant $1/(n+1)$-net $E_n \subset X_n$ consisting of disjoint orbits of length q_n, and put $l_n = |E_n|$. Such nets exist: each nonidentity power σ_n^j, $0 < j < q_n$, is a coordinate permutation whose fixed-point set is closed with empty interior. The complement of their finite union is the dense full-period locus. Starting with σ_0, whose period is one, the coordinate cycle defining σ_{n+1} ensures that its period is k_n.

After choosing E_n, choose k_n divisible by q_n so large that

$$k_n \geq n+2, \qquad \frac{l_n}{k_n} < 2^{-n-2}. \tag{6.41}$$

Set $X_{n+1} = X_n^{k_n}$ and $r_{n+1} = r_n(k_n + l_n)$. List each evaluation orbit in the order $t_j = \sigma_n^{-j}(t_0)$, $0 \leq j < q_n$, and define

$$\phi_n(a)(x) = \mathrm{diag}\big(a(x_0), \ldots, a(x_{k_n-1}),\ (a(t))_{t\in E_n}\big). \tag{6.42}$$

Thus every coordinate projection occurs once, followed by the point evaluations. Define

$$(\sigma_{n+1}(x))_j = \sigma_n(x_{j+1}), \qquad j \in \mathbb{Z}/k_n\mathbb{Z}.$$

Let P_{n+1} be the permutation matrix shifting the k_n coordinate block indices by -1 and, separately, shifting the indices in each evaluation orbit by -1. With each block of size r_n, put

$$u_{n+1} = \mathrm{diag}(u_n, \ldots, u_n)P_{n+1}, \qquad q_{n+1} = k_n.$$

The two factors commute. Since q_n divides k_n, both $\sigma_{n+1}^{k_n} = \mathrm{id}$ and $u_{n+1}^{k_n} = 1$. Conjugation by P_{n+1} sends block $j+1$ to block j. For a coordinate block this gives $u_n a(\sigma_n^{-1}x_j)u_n^*$; for an evaluation block it gives $u_n a(t_{j+1})u_n^* = \alpha_n(a)(t_j)$. Hence

$$\alpha_{n+1}\phi_n = \phi_n\alpha_n. \tag{6.43}$$

The injective unital maps ϕ_n therefore define a Villadsen algebra $A = \varinjlim(A_n, \phi_n)$ and an automorphism α of A.

Simplicity and the coefficient radius of comparison. For each fixed earlier stage, every composite coordinate projection from X_n to that stage is surjective and 1-Lipschitz. Its image of E_n is consequently a $1/(n+1)$-net. The evaluation sets thus satisfy the density condition for simplicity of the Villadsen limit: every nonzero finite-stage positive element eventually acquires a nonzero constant diagonal summand and becomes full. In particular, A is simple. As a unital AH algebra it has tracial states, which are faithful by simplicity; thus A is stably finite. Moreover, A has locally finite nuclear dimension.

The cube is a solid seed space, so [24, Theorem 3.4] gives

$$\mathrm{rc}(A) = \frac{1}{2} \lim_{n\to\infty} \frac{\dim X_n}{r_n} = \prod_{n=0}^{\infty} \frac{k_n}{k_n + l_n} \in (0,1). \tag{6.44}$$

Positivity follows from (6.41), and the product is less than one because every l_n is positive. Since a simple stably finite $\mathcal{Z}$-stable algebra has strict comparison, A is not $\mathcal{Z}$-stable.

Central projection towers. Consider the units of all diagonal blocks at stage $n+1$, including the coordinate and point-evaluation blocks. The automorphism permutes them in cycles of lengths k_n and q_n, respectively, and both lengths tend to infinity. Every block-unit projection commutes with $\phi_n(A_n)$.

Fix $N \geq 2$. Every integer at least $N(N-1)$ is of the form $aN + b(N+1)$ with $a, b \geq 0$. For all sufficiently large n, partition each cycle into segments of lengths N and $N+1$, in the direction of the action. Sum the initial projections of the segments according to their length. The two resulting bases have shapes $\{0, \ldots, N-1\}$ and $\{0, \ldots, N\}$; all translated levels are orthogonal and their sum is exactly 1. With N fixed and $n \to \infty$, these bases commute with every fixed earlier stage, so they define projection towers in $A_\omega \cap A'$ with the same two shapes and exact coverage. For any finite $K \subset \mathbb{Z}$ and $\eta > 0$, choosing N large enough makes both interval shapes (K, η)-invariant. This proves strong URP in Definition 4.28.

Since A is simple, the action is minimal. Strong URP gives WRP by Lemma 4.30, while (6.44) and Propositions 4.22 and 4.31 give $(1,1)$-COS. Finally, the locally finite nuclear dimension of A and the strong URP case of Corollary 6.20 show that $B = A \rtimes_\alpha \mathbb{Z}$ is simple and $\mathcal{Z}$-stable. □

Remark 6.23 (Hirshberg's central projection towers). The underlying $\mathbb{Z}$-action in Hirshberg's construction [36, proof of Claim (A) in Theorem 1] has full cyclic towers given by the central projections of the finite-stage fibers, with lengths 2^n. Partitioning those cycles into segments of the same two fixed lengths gives strong URP by the same argument. His coefficient algebra also has finite radius of comparison: its homogeneous stages C_n have base dimension $2s(n)$ and matrix size $r(n)$, with $s(n)/r(n) \leq 1$, so [64, Theorem 5.1] gives

$$\mathrm{rc}(C_n) \leq \frac{2s(n) - 1}{2r(n)} \leq 1.$$

All quotients of the stages are stably finite, and the simple limit is stably finite. The inductive-limit inequality [7, Propositions 3.2.3 and 3.2.4(iii)] therefore gives $\mathrm{rc}(C) \leq 1$. Thus Proposition 4.31 gives COS from finite coefficient radius of comparison, as in Example 6.22.

Chapter 7

Examples

This chapter gives examples of C*-algebras with weak tracial approximation and C*-dynamical systems with the weak Rokhlin property. The main results are as follows.

First, any simple unital stably finite classified C*-algebra can be weakly tracially approximated by (non-unital) interval algebras:

Theorem (Theorem 7.1)**.** *Let A be a unital, separable, simple, infinite-dimensional, nuclear, stably finite, $\mathcal{Z}$-stable* C*-algebra satisfying the UCT. Let $\mathcal{I}_{\mathrm{h}}$ and $\mathcal{I}_{\mathrm{o}}$ be the classes of finite direct sums*

$$\bigoplus_{j=1}^{r} \mathrm{M}_{n_j}(\mathrm{C}_0([0,1))) \quad \text{and} \quad \bigoplus_{j=1}^{r} \mathrm{M}_{n_j}(\mathrm{C}_0((0,1))), \qquad r, n_1, \ldots, n_r \in \mathbb{N},$$

respectively. Then

$$A \in \mathrm{TA}_1(\mathcal{I}_{\mathrm{h}}) \cap \mathrm{TA}_1(\mathcal{I}_{\mathrm{o}}).$$

In particular, A can be weakly tracially approximated by either class.

Then we have examples of C*-dynamical systems which have the weak Rokhlin property. If the C*-algebra is a bundle over a topological space X and the action preserve $\mathrm{C}(X)$ and has the ordinary URP, the the action on the C*-algebra also has the URP.

Theorem (Theorem 7.2)**.** *Let A be a unital separable* C*-algebra with $\mathrm{T}(A) \neq \emptyset$, and let α be an action of a countable discrete amenable group Γ on A. Suppose that A contains an α-invariant unital subalgebra $\mathrm{C}(X) \subseteq Z(A)$, where X is compact Hausdorff. If the induced action on X has topological URP, then (A, Γ) has URP as in Definition 3.2 and hence WRP.*

In particular, this applies to a unital separable C*-algebra that is a continuous field over X and has a nonempty tracial state space, provided that the action preserves its canonical central copy of $\mathrm{C}(X)$ and the induced action on X has topological URP.*

If the action on a C*-algebra A is W*-free and has only finitely many extreme invariant tracial state, using Rokhlin property in W*-dynamical systems, we have that the action has the WRP.

Theorem (Theorem 7.20)**.** *Let A be a unital separable C^*-algebra with $\mathrm{T}(A) \neq \varnothing$, and let a countable discrete amenable group Γ act on A. If the action has a unique invariant tracial state and is W^*-centrally free, then (A, Γ) has WRP.*

If the coefficient C^*-algebra A is nuclear, the example above can be generalized to the case that the invariant states form a Bauer simplex with finite dimensional boundary. Note that in both cases, the C^*-algebra A is allowed to be a commutative C^*-algebra.

Theorem (Theorem 7.48)**.** *Let A be a unital separable nuclear C^*-algebra with $\mathrm{T}(A) \neq \emptyset$, and let $\alpha\colon \Gamma \to \mathrm{Aut}(A)$ be an action of a countable discrete amenable group. Suppose that α is W^*-centrally free and that $S = \mathrm{T}^\alpha(A)$ is a Bauer simplex with $\dim(\partial_e S) < \infty$. If Γ is infinite, then α has WRP. The same conclusion holds for finite Γ when A has no finite-dimensional quotients.*

7.1 Classifiable C^*-algebras

Theorem 7.1. *Let A be a unital, separable, simple, infinite-dimensional, nuclear, stably finite, $\mathcal{Z}$-stable C^*-algebra satisfying the UCT. Let $\mathcal{I}_{\mathrm{h}}$ and $\mathcal{I}_{\mathrm{o}}$ be the classes of finite direct sums*

$$\bigoplus_{j=1}^{r} \mathrm{M}_{n_j}(\mathrm{C}_0([0,1))) \quad \textit{and} \quad \bigoplus_{j=1}^{r} \mathrm{M}_{n_j}(\mathrm{C}_0((0,1))), \qquad r, n_1, \dots, n_r \in \mathbb{N},$$

respectively. Then

$$A \in \mathrm{TA}_1(\mathcal{I}_{\mathrm{h}}) \cap \mathrm{TA}_1(\mathcal{I}_{\mathrm{o}}).$$

In particular, A can be weakly tracially approximated by either class.

Proof. The half-open interval assertion follows from the inductive-limit model of [32]. The open interval assertion then follows from $\mathcal{Z}$-absorption.

(1) The inductive-limit model and its finite-dimensional quotient. By [13, Theorem A], A has finite nuclear dimension. Hence [23, Theorem 4.10] gives $\mathrm{gTR}(A \otimes \mathcal{Q}) \leq 1$, where $\mathcal{Q}$ is the universal UHF algebra. The range theorem [32, Theorem 13.50] applies because the Elliott invariant of A is a simple weakly unperforated invariant in the sense of [32, Subsection 13.7]. It provides a unital simple, $\mathcal{Z}$-stable model B with the same Elliott invariant as A and with $\mathrm{gTR}(B \otimes \mathcal{Q}) \leq 1$. Thus [33, Theorem 29.8] gives $A \cong B$. The model is a unital simple inductive limit

$$B = \varinjlim (B_n, \xi_{n,n+1})$$

with unital injective connecting maps. We use the building blocks in [32, Subsections 13.27–13.29] and the connecting maps constructed in [32, Subsections 13.43–13.47]. Write

$$F_n = P_n \mathrm{M}_{N_n}(\mathrm{C}(X_n)) P_n \oplus \bigoplus_{j=2}^{p_n} \mathrm{M}_{r_{n,j}}, \qquad Q_n = \bigoplus_{j=1}^{p_n} \mathrm{M}_{r_{n,j}}, \quad r_{n,1} = \mathrm{rank}(P_n).$$

Here $X_n = [0,1] \vee X'_n$, with 1 attached to X'_n and with distinguished point $\theta_n = 0$. The projection P_n is constant on the attached interval. Let

$$\pi_n : F_n \longrightarrow Q_n, \qquad \pi_n(g, a_2, \ldots, a_{p_n}) = (g(\theta_n), a_2, \ldots, a_{p_n}).$$

For a finite-dimensional algebra E_n and unital homomorphisms $\beta_{n,0}, \beta_{n,1} : Q_n \to E_n$, the building block is

$$B_n = \{(f,g) \in \mathrm{C}([0,1], E_n) \oplus F_n : f(e) = \beta_{n,e}(\pi_n(g)),\ e = 0,1\}.$$

Set $q_n(f,g) = \pi_n(g)$ and $J_n = \ker q_n$. The endpoint multiplicities in this construction can be chosen strictly positive, as allowed in [32, Subsections 13.12–13.13], so that $\beta_{n,0}$ is injective. In the notation of [32], F_n, Q_n, and q_n correspond to $A_{X,n}$, F_n, and $\pi^e_{A_n}$, respectively; J_n is the sum of the two coordinate ideals denoted there by I_n and J_n. This description includes the homogeneous summand required for general K-theory, so the blocks B_n need not be one-dimensional NCCW complexes.

(2) Uniform concentration at the point quotient. We claim that

$$\sup_{\tau \in \mathrm{T}(B)} \|\tau|_{J_n}\| \le \delta_n, \qquad \delta_n = 6 \cdot 2^{-2n} \longrightarrow 0, \tag{7.1}$$

where B_n is identified with its image in B. We derive this bound from the estimates for the connecting maps.

Let $\varphi_{n,n+1}$ be the connecting map before the small modification that makes the limit simple. This map induces a homomorphism $Q_n \to Q_{n+1}$ and hence maps J_n into J_{n+1}. For a homomorphism $\alpha : D \to D'$, write $\alpha^\sharp$ for the induced map $\mathrm{Aff}(\mathrm{T}(D)) \to \mathrm{Aff}(\mathrm{T}(D'))$. The claim in the proof of [32, Theorem 13.42] gives a positive unital linear map $\Gamma_{n+1} : \mathrm{Aff}(\mathrm{T}(Q_{n+1})) \to \mathrm{Aff}(\mathrm{T}(B_{n+1}))$ such that

$$\left\| \Gamma_{n+1} q^\sharp_{n+1} \varphi^\sharp_{n,n+1}(v) - \varphi^\sharp_{n,n+1}(v) \right\| \le 4 \cdot 2^{-2n} \|v\|.$$

For $x \in (J_n)_+$ with $\|x\| \le 1$, put $\widehat{x}(\rho) = \rho(x)$. Since $q_{n+1}\varphi_{n,n+1}(x) = 0$, the Γ_{n+1} term in the preceding estimate vanishes for $v = \widehat{x}$. Thus

$$\sigma(\varphi_{n,n+1}(x)) \le 4 \cdot 2^{-2n}, \qquad \sigma \in \mathrm{T}(B_{n+1}).$$

The modified map $\xi_{n,n+1}$ satisfies a similar uniform estimate on the positive unit ball of J_n. Write $\varphi_{n,n+1} = (\varphi_E, \varphi_X)$ and $\xi_{n,n+1} = (\xi_E, \xi_X)$ for their line and homogeneous coordinates. Equation (e13.77) of [32] gives

$$\left|\rho(\xi_X(x)) - \rho(\varphi_X(x))\right| \le 2 \cdot 2^{-2n}, \qquad \rho \in \mathrm{T}(F_{n+1}).$$

Composing such a trace with the homogeneous quotient map gives a trace on B_{n+1}, so $\rho(\xi_X(x)) \le 6 \cdot 2^{-2n}$. On the middle portion of each line, equation (e13.82) expresses $\xi_E(x)$ as a reparametrized evaluation of $\varphi_E(x)$; its normalized matrix traces are therefore at most $4 \cdot 2^{-2n}$. On the two endpoint collars, equations (e13.86) and (e13.87) bound the difference from the corresponding endpoint trace of $\varphi_E(x)$ by $2 \cdot 2^{-2n}$. Thus these traces are at most $6 \cdot 2^{-2n}$ as well. The estimates

for the modification hold on all of B_n, so the bounds above are uniform on the positive unit ball of J_n. Since every trace on the subhomogeneous block B_{n+1} is an integral of normalized traces of its irreducible representations, we obtain

$$\sigma(\xi_{n,n+1}(x)) \le \delta_n, \qquad \sigma \in \mathrm{T}(B_{n+1}), \quad x \in (J_n)_+, \ \|x\| \le 1.$$

Restricting a limit trace to B_{n+1} and taking the supremum over such x proves (7.1). Consequently, for every positive contraction $v \in J_n$,

$$\mathrm{d}_\tau(v) = \lim_{m\to\infty} \tau(v^{1/m}) \le \delta_n, \qquad \tau \in \mathrm{T}(B), \tag{7.2}$$

because all $v^{1/m}$ belong to J_n. The bound on the norm of the trace restriction controls all these roots simultaneously.

(3) The endpoint collar. Fix a finite subset $\mathcal{G} \subset B_n$ and a tolerance $\eta > 0$. Choose $0 < a < 1/4$ so small that the line coordinates of every $x = (f, g) \in \mathcal{G}$ are within η of their endpoint values on $[0, 2a]$ and $[1 - 2a, 1]$, and that the first coordinate of g on the attached interval $[0, 2a] \subset X_n$ is within η of $g(\theta_n)$. For each x, a scalar cutoff gives $x' \in B_n$ satisfying $\|x - x'\| < \eta$, $q_n(x') = q_n(x)$, and

$$f'(t) = \beta_{n,0}(q_n(x)) \quad (0 \le t \le a), \qquad f'(t) = \beta_{n,1}(q_n(x)) \quad (1 - a \le t \le 1),$$

while the first point coordinate is the constant matrix $q_n(x)_1$ on $[0, a] \subset X_n$. The modified coordinates agree with the originals outside the larger collars and satisfy the same boundary conditions.

There is an injective homomorphism

$$\iota : \mathrm{C}_0([0, 1), Q_n) \longrightarrow B_n$$

defined as follows, extending every $b \in \mathrm{C}_0([0, 1), Q_n)$ by $b(1) = 0$. Its line coordinate is

$$t \longmapsto \begin{cases} \beta_{n,0}(b(t/a)), & 0 \le t \le a, \\ 0, & a \le t \le 1 - a, \\ \beta_{n,1}(b((1 - t)/a)), & 1 - a \le t \le 1. \end{cases}$$

Its first point coordinate is $b_1(u/a)$ on the attached interval $0 \le u \le a$ and zero elsewhere on X_n; its remaining point coordinates are $b_j(0)$. These formulas are compatible at the endpoints and at the edges of the collars, and injectivity of $\beta_{n,0}$ implies injectivity of ι. Moreover,

$$q_n(\iota(b)) = b(0). \tag{7.3}$$

Choose a positive contraction $\chi \in \mathrm{C_c}([0, 1))$ with $\chi(0) = 1$ and put

$$C = \iota(\mathrm{C}_0([0, 1), Q_n)), \qquad h = \iota(\chi\, 1_{Q_n}).$$

Then $C \in \mathcal{I}_{\mathrm{h}}$, $\|h\| = 1$, and h is central in B_n: each of its fiber values is a scalar multiple of the fiber unit. The constant coordinates on the collars give

$$[h, x'] = 0, \qquad hx'h = \iota(\chi^2 q_n(x)) \in C, \qquad x \in \mathcal{G}. \tag{7.4}$$

Moreover, $q_n(h) = 1_{Q_n}$, so that $1 - h \in J_n$ and

$$\mathrm{d}_\tau(1 - h) \le \delta_n, \qquad \tau \in \mathrm{T}(B).$$

Fix a finite set $\mathcal{F} = \{b_1, \dots, b_m\} \subset B$ and $\varepsilon > 0$. Choose n so that $\delta_n < \varepsilon$ and choose $g_i \in B_n$ with $\|b_i - g_i\| < \varepsilon/4$. Apply the preceding construction to $\mathcal{G} = \{g_1, \dots, g_m\}$, with tolerance $\eta < \varepsilon/4$, and denote the resulting elements by g_i'. Then C, h, and g_i' satisfy

$$\|b_i - g_i'\| < \varepsilon/2, \qquad [h, g_i'] = 0, \qquad hg_i'h \in C, \qquad \mathrm{d}_\tau(1 - h) < \varepsilon \quad (\tau \in \mathrm{T}(B)).$$

These are the four conditions of Definition 2.1. Thus $B \in \mathrm{TA}_0(\mathcal{I}_{\mathrm{h}})$.

(4) Open intervals. Choose a positive contraction $z \in \mathcal{Z}$ with spectrum $[0, 1]$ whose spectral measure μ for the unique trace $\tau_{\mathcal{Z}}$ satisfies $\mu(\{0, 1\}) = 0$. To construct such an element, take a nonscalar selfadjoint element in the projectionless algebra $\mathcal{Z}$. Its spectrum is an interval. After normalization, choose two interior points $c < d$ that are not atoms of its spectral measure, and apply the piecewise linear function with successive values $1/2, 0, 1, 1/2$ at $0, c, d, 1$. Its only minimum and maximum occur at c and d, respectively.

For $b \in \mathrm{C}_0((0, 1), Q_n)$, extended by zero at both endpoints, define

$$\widetilde{\Theta}(b)(s) = b\big(s + (1 - s)z\big), \qquad 0 \le s \le 1.$$

Continuous functional calculus is applied entrywise in the Q_n coordinate. This is a homomorphism into $\mathrm{C}_0([0, 1), Q_n) \otimes \mathcal{Z}$, since its value at $s = 1$ is zero. Put

$$\Theta = (\iota \otimes \mathrm{id}_{\mathcal{Z}})\widetilde{\Theta}.$$

It is injective, since

$$(q_n \otimes \mathrm{id}_{\mathcal{Z}})(\Theta(b)) = b(z)$$

and z has full spectrum. Given $\alpha > 0$, choose a positive contraction $\gamma \in \mathrm{C}_{\mathrm{c}}((0, 1))$ of norm one which is equal to one on $[\nu, 1 - \nu]$, with $\nu > 0$ so small that

$$\mathrm{d}_{\tau_{\mathcal{Z}}}(1 - \gamma(z)) = \mu(\{t : \gamma(t) < 1\}) < \alpha.$$

Set

$$D = \Theta(\mathrm{C}_0((0, 1), Q_n)), \qquad k = \Theta(\gamma\, 1_{Q_n}).$$

The same constant coordinates on the collars give

$$[k, x' \otimes 1_{\mathcal{Z}}] = 0, \qquad k(x' \otimes 1_{\mathcal{Z}})k = \Theta(\gamma^2 q_n(x)) \in D, \quad x \in \mathcal{G}, \tag{7.5}$$

and $\|k\| = 1$.

Every trace on $B \otimes \mathcal{Z}$ is $\tau \otimes \tau_{\mathcal{Z}}$ for some $\tau \in \mathrm{T}(B)$. Its restriction to $B_n \otimes \mathcal{Z}$ splits into a positive trace supported on $J_n \otimes \mathcal{Z}$, of norm at most δ_n, and a positive trace factoring through $Q_n \otimes \mathcal{Z}$. This decomposition is induced by the central support of the ideal in the bidual. On the quotient, k is $1_{Q_n} \otimes \gamma(z)$. The dimension function is additive on the two trace components, and the quotient component has norm at most one. Hence

$$\mathrm{d}_{\tau\otimes\tau_{\mathcal{Z}}}(1-k) \leq \delta_n + \mathrm{d}_{\tau_{\mathcal{Z}}}(1-\gamma(z)) < \delta_n + \alpha. \tag{7.6}$$

To verify the approximation property, fix $\mathcal{F} = \{b_1, \ldots, b_m\} \subset B$ and $\varepsilon > 0$. Choose n and $g_i \in B_n$ such that $\delta_n < \varepsilon/2$ and $\|b_i - g_i\| < \varepsilon/4$. Apply the collar construction to $\mathcal{G} = \{g_1, \ldots, g_m\}$ with tolerance $\eta < \varepsilon/4$, obtaining g_i', and choose $0 < \alpha < \varepsilon/2$ in the construction of D and k. Since B is $\mathcal{Z}$-stable, [66, Theorem 2.2] gives an isomorphism $\rho : B \otimes \mathcal{Z} \to B$ with

$$\|\rho(b_i \otimes 1_{\mathcal{Z}}) - b_i\| < \varepsilon/4 \quad (1 \leq i \leq m).$$

Put $D' = \rho(D)$, $h_{\mathrm{o}} = \rho(k)$, and $b_i' = \rho(g_i' \otimes 1_{\mathcal{Z}})$. Then $D' \in \mathcal{I}_{\mathrm{o}}$, $\|h_{\mathrm{o}}\| = 1$, and

$$\|b_i - b_i'\| \leq \|b_i - \rho(b_i \otimes 1_{\mathcal{Z}})\| + \|b_i - g_i\| + \|g_i - g_i'\| < 3\varepsilon/4.$$

Equation (7.5) gives $[h_{\mathrm{o}}, b_i'] = 0$ and $h_{\mathrm{o}} b_i' h_{\mathrm{o}} \in D'$. Every pullback trace $\tau \circ \rho$ is a product trace, so (7.6) gives $\mathrm{d}_\tau(1 - h_{\mathrm{o}}) < \delta_n + \alpha < \varepsilon$ for every $\tau \in \mathrm{T}(B)$. Thus $B \in \mathrm{TA}_0(\mathcal{I}_{\mathrm{o}})$.

(5) Cuntz comparison of the defect. The algebra $A \cong B$ has strict comparison by [58, Corollary 4.6]. For a given nonzero positive contraction $a_0 \in A$, simplicity and compactness of $\mathrm{T}(A)$ give

$$m_0 = \min_{\tau \in \mathrm{T}(A)} \tau(a_0) > 0.$$

Given also a finite subset of A and a tolerance $\varepsilon > 0$, apply the approximation property for either interval class with tolerance $\min\{\varepsilon, m_0/2\}$. Write h for the resulting positive contraction, using h_{o} in the open-interval case. Then $\mathrm{d}_\tau(1-h) < m_0 \leq \mathrm{d}_\tau(a_0)$ for all $\tau \in \mathrm{T}(A)$, so strict comparison gives $1 - h \precsim a_0$. The other three defining conditions still hold with tolerance ε. Thus A belongs to both TA_1 classes. □

7.2 C^*-bundles

Theorem 7.2. *Let A be a unital separable C^*-algebra with $\mathrm{T}(A) \neq \emptyset$, and let α be an action of a countable discrete amenable group Γ on A. Suppose that A contains an α-invariant unital subalgebra $\mathrm{C}(X) \subseteq Z(A)$, where X is compact Hausdorff. If the induced action on X has topological URP, then (A, Γ) has URP as in Definition 3.2 and hence WRP.*

In particular, this applies to a unital separable C^*-algebra that is a continuous field over X and has a nonempty tracial state space, provided that the action preserves its canonical central copy of $\mathrm{C}(X)$ and the induced action on X has topological URP.

Proof. We identify $\mathrm{C}(X)$ with the given central subalgebra of A. Write the action on X on the left, so that $\alpha_g(f)(x) = f(g^{-1}x)$ and $\operatorname{supp}(\alpha_g(f)) = g\operatorname{supp}(f)$. Tower shapes follow the corresponding left-multiplication convention of Chapter 3.

Fix a finite set $K \subseteq \Gamma$, a finite set $\mathcal{F} \subseteq A$, and a tolerance $\varepsilon > 0$. It suffices to consider $0 < \varepsilon < 1$; enlarge K to contain the identity if necessary. Topological URP gives nonempty open bases $U_1, \dots, U_S$ and $(K, \varepsilon/2)$-invariant shapes $F_1, \dots, F_S$ such that the sets gU_s, $g \in F_s$, are pairwise disjoint and

$$\mu\left(X \setminus \bigcup_{s=1}^{S} \bigcup_{g \in F_s} gU_s\right) < \varepsilon/2, \qquad \mu \in \mathcal{M}_1(X, \Gamma).$$

Empty bases can be discarded.

We next choose compact subsets $B_s \subset U_s$ whose translated levels have complement of measure less than ε for every μ. For each invariant probability measure μ, regularity and normality give open sets V_s^μ with $\overline{V_s^\mu} \subset U_s$ and

$$\mu\left(\bigcup_{s=1}^{S} \bigcup_{g \in F_s} gV_s^\mu\right) > 1 - \varepsilon.$$

The measure of this open union is lower semicontinuous in the weak* topology. Thus this inequality holds on an open neighborhood of μ in $\mathcal{M}_1(X, \Gamma)$. Compactness of that space gives finitely many such families. Take B_s to be the union of their closures, adding a point of U_s if needed to make B_s nonempty. Then

$$\mu\left(X \setminus \bigcup_{s=1}^{S} \bigcup_{g \in F_s} gB_s\right) < \varepsilon, \qquad \mu \in \mathcal{M}_1(X, \Gamma).$$

Choose $f_s \in \mathrm{C}(X)_+$ with $0 \le f_s \le 1$, equal to one on B_s, and supported in U_s. Viewed in A, these are norm-one positive contractions. Their translated levels are mutually orthogonal, and

$$q = \sum_{s=1}^{S} \sum_{g \in F_s} \alpha_g(f_s)$$

is a positive contraction in $\mathrm{C}(X)$, equal to one on $\bigcup_{s=1}^{S} \bigcup_{g \in F_s} gB_s$. Every level is central in A, so the sum of commutator norms in Definition 3.2 is zero for every element of $\mathcal{F}$.

For $\tau \in \mathrm{T}(A \rtimes \Gamma)$, the restriction $\tau|_A$ is α-invariant. Its restriction to $\mathrm{C}(X)$ is therefore an invariant probability measure μ_τ. Functional calculus inside $\mathrm{C}(X)$ and monotone convergence give

$$\mathrm{d}_\tau(1 - q) = \lim_{m \to \infty} \int_X (1 - q)^{1/m} \, d\mu_\tau = \mu_\tau(\{x : q(x) < 1\}) < \varepsilon. \tag{7.7}$$

Set $c_s = f_s$ and $\Gamma_s = F_s$. The shapes are (K, ε)-invariant, the c_s have norm one, their translated levels are pairwise orthogonal, and the sum of commutator norms is zero. Equation (7.7) gives

the uniform dimension-function bound on the complement. Thus Definition 3.2 holds, and Proposition 3.6 gives WRP.

A unital continuous-field algebra has its canonical central $\mathrm{C}(X)$-action, so the stated special case follows. □

7.3 Central sequences and freeness

We introduce the von Neumann algebra conventions for the Rokhlin theorem in the next section. Throughout this section, M has separable predual, and ω is the fixed free ultrafilter on $\mathbb{N}$. Write $Z(M)$ for the center and M_* for the predual. All sequences in M are assumed to be bounded in operator norm. For $\psi \in M_*$ and $x, y \in M$, our module conventions are

$$(x\psi)(y) = \psi(yx), \qquad (\psi x)(y) = \psi(xy), \qquad [x, \psi] = x\psi - \psi x.$$

For a normal positive functional ϕ, put

$$\|x\|_\phi = \phi(x^*x)^{1/2}, \qquad \|x\|_\phi^\sharp = \left(\frac{\phi(x^*x) + \phi(xx^*)}{2}\right)^{1/2}.$$

If ϕ is faithful, $\|\cdot\|_\phi^\sharp$ induces the σ-strong* topology on bounded subsets of M.

7.3.1 Centralizing sequences and ultrapowers

Definition 7.3. A bounded sequence (x_n) in M is *ω-central* if

$$[x_n, a] \longrightarrow 0 \quad \sigma\text{-strong}^* \text{ along } \omega \qquad (a \in M),$$

and is *ω-centralizing* if

$$\lim_{n\to\omega} \|[x_n, \psi]\| = 0 \qquad (\psi \in M_*).$$

Replacing the ultralimit by the ordinary limit gives the usual notions of a central sequence and a centralizing sequence. Centralizing sequences are also called strongly central sequences.

Lemma 7.4. *Every ω-centralizing sequence is ω-central. If ϕ is a faithful normal state, an ω-central sequence is ω-centralizing precisely when $\lim_{n\to\omega} \|[x_n, \phi]\| = 0$. In particular, the two notions agree when M is finite. The same assertions hold for ordinary limits.*

Proof. Fix a faithful normal state ϕ. The linear space $M\phi = \{a\phi : a \in M\}$ is norm-dense in M_* by faithfulness of ϕ. The identity

$$[x_n, a]\phi = [x_n, a\phi] - a[x_n, \phi]$$

shows that its left side tends to zero in predual norm when (x_n) is centralizing. For $b \in M$, $\|b\|_\phi^2 \le \|b\|\,\|b\phi\|$; applying this to $b = [x_n, a]$ proves strong convergence of the commutators to

zero. Since $[x_n^*, \psi] = -[x_n, \psi^*]^*$, the adjoint sequence is also centralizing. Applying the same argument to x_n^* proves strong* centrality. Conversely,

$$\|[x_n, a\phi]\| \le \|[x_n, a]\|_\phi + \|a\|\,\|[x_n, \phi]\|$$

proves centralization on the dense space $M\phi$, and boundedness extends it to all of M_*. In the finite case choose a faithful normal tracial state, for which $[x_n, \phi] = 0$ identically. □

Definition 7.5. Inside $\ell^\infty(\mathbb{N}, M)$, define

$$\begin{aligned} I_\omega(M) &= \{(x_n) : x_n \longrightarrow 0 \ \sigma\text{-strong}^* \text{ along } \omega\}, \\ \mathcal{C}_\omega(M) &= \{(x_n) : (x_n) \text{ is } \omega\text{-centralizing}\}, \\ \mathcal{M}^\omega(M) &= \{x : xI_\omega(M) \subseteq I_\omega(M),\ I_\omega(M)x \subseteq I_\omega(M)\}. \end{aligned}$$

The last algebra is the two-sided normalizer of $I_\omega(M)$. One has $I_\omega(M) \subseteq \mathcal{C}_\omega(M) \subseteq \mathcal{M}^\omega(M)$, and $I_\omega(M)$ is a closed ideal of $\mathcal{M}^\omega(M)$. The *Ocneanu ultrapower* and the *asymptotic centralizer* are, respectively,

$$M^\omega = \mathcal{M}^\omega(M)/I_\omega(M), \qquad M_\omega = \mathcal{C}_\omega(M)/I_\omega(M).$$

The algebra M_ω is also called the central sequence algebra in Ocneanu's sense. Constant sequences identify M with a subalgebra of M^ω. For $\psi \in M_*$, write

$$\psi^\omega([(x_n)]) = \lim_{n\to\omega} \psi(x_n), \qquad \psi_\omega = \psi^\omega|_{M_\omega}.$$

Proposition 7.6 (Ocneanu)**.** *The algebra M^ω is a von Neumann algebra, and M and M_ω are von Neumann subalgebras. Moreover,*

$$M_\omega \subseteq M' \cap M^\omega, \qquad M \cap M_\omega = Z(M).$$

For every faithful normal state ϕ on M, ϕ^ω is a faithful normal state on M^ω, and ϕ_ω is a faithful normal tracial state on M_ω.

The canonical normal conditional expectation is

$$\Psi\colon M^\omega \longrightarrow M, \qquad \Psi([(x_n)]) = \operatorname*{weak^*-lim}_{n\to\omega} x_n.$$

Its restriction to M_ω takes values in $Z(M)$, and $\psi_\omega = \psi \circ \Psi|_{M_\omega}$. Thus ψ_ω depends only on $\psi|_{Z(M)}$.

If M is finite and τ is a faithful normal tracial state, write $\|x\|_{2,\tau} = \tau(x^*x)^{1/2}$. Then

$$M^\omega = \ell^\infty(\mathbb{N}, M)/\{(x_n) : \lim_{n\to\omega} \|x_n\|_{2,\tau} = 0\}, \qquad M_\omega = M' \cap M^\omega.$$

Proof. The von Neumann algebra, normality, and faithfulness assertions are the ultrapower facts of [50, Section 5.1]. For centralizing representatives (x_n) and (y_n),

$$|\psi(x_n y_n - y_n x_n)| \le \|[x_n, \psi]\|\,\|y_n\| \longrightarrow_\omega 0.$$

Hence $\Psi(xy) = \Psi(yx)$ on M_ω, proving the trace assertion. Since Ψ is M-bimodular and $M_\omega \subseteq M' \cap M^\omega$, its image there lies in $Z(M)$. The formulas for ψ_ω follow from the definition. In the finite case $I_\omega(M)$ is the tracial 2-norm ideal of all of $\ell^\infty(\mathbb{N}, M)$, and Lemma 7.4 identifies M_ω with $M' \cap M^\omega$. □

7.3.2 Lifted actions and freeness

Definition 7.7. An automorphism θ of M induces automorphisms

$$\theta^\omega([(x_n)]) = [(\theta(x_n))], \qquad \theta_\omega = \theta^\omega|_{M_\omega}.$$

These automorphisms are called *liftable*. More generally, suppose that $\theta_n \in \mathrm{Aut}(M)$ converges to $\theta \in \mathrm{Aut}(M)$ along ω in the u-topology, that is,

$$\lim_{n\to\omega} \|\psi \circ \theta_n - \psi \circ \theta\| = 0 \qquad (\psi \in M_*).$$

Then $[(x_n)] \mapsto [(\theta_n(x_n))]$ defines an automorphism of M^ω leaving M_ω invariant [50, Section 5.2]. Such automorphisms, and their restrictions to M_ω, are called *semiliftable*.

For an action $\alpha\colon \Gamma \to \mathrm{Aut}(M)$, write α^ω and α_ω for the induced actions. The expectation is equivariant: $\Psi \circ \alpha_g^\omega = \alpha_g \circ \Psi$. In particular, if $\phi|_{Z(M)}$ is α-invariant, then ϕ_ω is an α_ω-invariant trace.

Definition 7.8. An automorphism of a von Neumann algebra is *properly outer* if its restriction to every nonzero invariant central corner is outer. An automorphism $\theta \in \mathrm{Aut}(M)$ is *centrally trivial* if $\theta_\omega = \mathrm{id}$. For separable predual this is equivalent to $\theta(x_n) - x_n \to 0$ σ-strong* for every ordinary centralizing sequence (x_n). It is *properly centrally nontrivial* if $\theta|_{pM}$ is not centrally trivial for every nonzero θ-invariant projection $p \in Z(M)$.

An action α on M is *free* if α_g is properly outer for every $g \neq e$, and is *centrally free* if α_g is properly centrally nontrivial for every $g \neq e$.

Definition 7.9 (Ocneanu's relative outerness)**.** An automorphism $\vartheta \in \mathrm{Aut}(M_\omega)$ is *strongly outer* if its restriction to $\mathcal{N}' \cap M_\omega$ is properly outer for every countable ϑ-invariant self-adjoint subset $\mathcal{N} \subseteq M_\omega$. Equivalently, one may use every invariant von Neumann subalgebra generated by a countable set. Self-adjointness ensures that the relative commutant is a von Neumann algebra.

An action β on M_ω is *strongly free* if each β_g, $g \neq e$, is strongly outer in this sense. An action α on M is strongly free if α_ω is strongly free.

Lemma 7.10 (Ocneanu)**.** *Let $\vartheta \in \mathrm{Aut}(M_\omega)$ be semiliftable, represented by automorphisms θ_n converging to θ along ω in the u-topology. If θ is properly centrally nontrivial, then ϑ is strongly outer. In particular, a centrally free action α on M induces a strongly free action α_ω on M_ω.*

Proof. The first assertion is the lemma in [50, Section 5.6]. Apply it to the constant sequence $\theta_n = \alpha_g$ for each $g \neq e$ to obtain the second assertion. □

7.3.3 Invariant traces and GNS actions

Let A be a unital separable C*-algebra, and let a countable discrete group Γ act on A. Write

$$\mathrm{T}^\alpha(A) = \{\tau \in \mathrm{T}(A) : \tau \circ \alpha_g = \tau \text{ for all } g \in \Gamma\}.$$

For $\tau \in \mathrm{T}^{\alpha}(A)$, put $M_\tau = \pi_\tau(A)''$. Trace invariance gives a normal action $\alpha_\tau \colon \Gamma \to \mathrm{Aut}(M_\tau)$ satisfying

$$(\alpha_\tau)_g(\pi_\tau(a)) = \pi_\tau(\alpha_g(a)).$$

It is implemented on the GNS space by $[a] \mapsto [\alpha_g(a)]$ and preserves the faithful normal tracial state $\bar{\tau}$ on M_τ. The GNS homomorphism π_τ need not be faithful on A.

Definition 7.11. The action α on A is W*-*centrally free* if α_τ is centrally free for every $\tau \in \mathrm{T}^{\alpha}(A)$. It is W*-*strongly free* if α_τ is strongly free, in the sense of Definition 7.9, for every $\tau \in \mathrm{T}^{\alpha}(A)$.

Lemma 7.10 shows that W*-central freeness implies W*-strong freeness. Consequently, the induced action on $(M_\tau)_\omega$ is properly outer on the relative commutant of every countable invariant self-adjoint set, as required by the small-tower lemma in the next section. When Γ is amenable and $\mathrm{T}(A) \neq \varnothing$, the invariant trace space is nonempty.

7.4 Exact Rokhlin towers in von Neumann algebras

We prove an exact tower theorem for prescribed tiling shapes and then lift these towers to C*-algebras with finitely many extreme invariant traces. The von Neumann algebra argument rests on Ocneanu's small-tower lemma.

Definition 7.12. An *exact tiling* of a discrete group Γ consists of finite nonempty shapes $D_1, \dots, D_S \subseteq \Gamma$, with $e \in D_s$, and center sets $C_s \subseteq \Gamma$, such that

$$\Gamma = \bigsqcup_{s=1}^{S} \bigsqcup_{c \in C_s} D_s c.$$

Thus the translated shapes $D_s c$ partition Γ. For a finite set $P \subseteq \Gamma$ containing e, put

$$\mathrm{Int}_P F = \{g \in F : Pg \subseteq F\}.$$

A finite nonempty set F is (P, δ)-invariant if $|F \setminus \mathrm{Int}_P F| < \delta |F|$.

Theorem 7.13 (Downarowicz–Huczek–Zhang)**.** *Let Γ be a countable discrete amenable group. For every finite $P \subseteq \Gamma$ containing e and every $\delta > 0$, there is an exact tiling all of whose shapes are (P, δ)-invariant.*

Proof. For infinite Γ, apply [19, Theorem 4.3] with $K = P$ and a tolerance smaller than $\min\{1, \delta/|P|\}$. Then [19, Lemma 2.6] converts its symmetric-difference invariance into the stated interior-boundary estimate. For finite Γ, use the single shape Γ and center set $\{e\}$. □

Definition 7.14. Let $\alpha \colon \Gamma \to \mathrm{Aut}(M)$ be an action on a von Neumann algebra with separable predual, and let ω be a free ultrafilter on $\mathbb{N}$. Write $\beta = \alpha_\omega$ for the induced action on M_ω. The action has the W*-*Rokhlin property* if, for every exact tiling with shapes $D_1, \dots, D_S$, there are

projections $p_1, \dots, p_S \in M_\omega$ such that the projections $\beta_k(p_s)$, $k \in D_s$, $1 \le s \le S$, are pairwise orthogonal and

$$\sum_{s=1}^{S} \sum_{k \in D_s} \beta_k(p_s) = 1.$$

For the next theorem and its proof, write $R = M_\omega$, $t = \phi_\omega$, and $\beta = \alpha_\omega$. Proposition 7.6 and equivariance of Ψ give a faithful normal β-invariant tracial state t on R.

Theorem 7.15. *Let Γ be a countable discrete amenable group, let M have separable predual, and let $\alpha\colon \Gamma \to \mathrm{Aut}(M)$ be centrally free. Suppose that ϕ is a faithful normal state on M and $\phi|_{Z(M)}$ is invariant under α. Then α has the* W**-Rokhlin property.*

More precisely, for every exact tiling with shapes $D_1, \dots, D_S$ and every countable subset $\mathcal{N} \subseteq M_\omega$, the projections in Definition 7.14 can be chosen in $\mathcal{N}' \cap M_\omega$.

The following form of Ocneanu's small-tower lemma applies. By Lemma 7.10, the induced action β in Theorem 7.15 is strongly free.

Lemma 7.16 (Ocneanu's small-tower lemma)**.** *Under the hypotheses of Theorem 7.15, let $K \subseteq \Gamma \setminus \{e\}$ be finite, let $\eta > 0$, and let $\mathcal{N} \subseteq R$ be a countable β-invariant subset. There is a finite partition of unity by projections $e_0, e_1, \dots, e_q \in \mathcal{N}' \cap R$ such that*

$$t(e_0) < \eta, \qquad e_j \beta_k(e_j) = 0 \quad (1 \le j \le q,\ k \in K).$$

Proof. Apply the small-tower lemma in [50, Section 6.3] in the relative commutant of the countably generated invariant von Neumann algebra generated by $\mathcal{N} \cup \mathcal{N}^*$, choosing the remainder 2-norm tolerance smaller than $\eta^{1/2}$. Strong freeness supplies proper outerness on this relative commutant, and t is the invariant trace used in that lemma. When $K = \varnothing$, take $e_0 = 0$ and $e_1 = 1$. □

Lemma 7.17. *Fix an exact tiling with shapes $D_1, \dots, D_S$. For every finite $P \subseteq \Gamma$ containing e and every $\delta > 0$, there is a finite nonempty (P, δ)-invariant set H which is a union of complete tiles of this fixed tiling.*

Proof. Put $L = \bigcup_s D_s D_s^{-1}$. Choose $0 < \eta < 1$ with $\eta/(1-\eta) < \delta$, and use amenability to choose a finite nonempty (LP, η)-invariant set F. Let H be the union of all tiles of the fixed tiling contained in F. If $g \in \mathrm{Int}_{LP} F$ and $p \in P$, then the unique tile containing pg is contained in $Lpg \subseteq F$. Thus $Pg \subseteq H$, and

$$\mathrm{Int}_{LP} F \subseteq \mathrm{Int}_P H, \qquad |H| \ge (1-\eta)|F| > 0.$$

Since $H \subseteq F$, it follows that

$$\frac{|H \setminus \mathrm{Int}_P H|}{|H|} \le \frac{|F \setminus \mathrm{Int}_{LP} F|}{(1-\eta)|F|} < \frac{\eta}{1-\eta} < \delta.$$

□

Lemma 7.18 (Increasing the tower mass)**.** *Assume the hypotheses of Theorem 7.15, and fix an exact tiling with shapes $D_1, \dots, D_S$ and a countable β-invariant subset $\mathcal{N} \subseteq R$. Suppose that $E_1, \dots, E_S \in \mathcal{N}' \cap R$ are projections whose levels $\beta_k(E_s)$, $k \in D_s$, are pairwise orthogonal. If*

$$b := t\left(\sum_{s=1}^{S} \sum_{k \in D_s} \beta_k(E_s)\right) < 1,$$

then there is another such family $E_1', \dots, E_S'$ with strictly larger total trace.

Proof. Write $q = \sum_{s,k \in D_s} \beta_k(E_s)$, and choose $0 < \varepsilon < \min\{1/16, 1 - b\}$. Put

$$U = \bigcup_s D_s, \quad L = \bigcup_s D_s D_s^{-1}, \quad P = U^{-1}U, \quad m = \sum_s |D_s|, \quad \delta = \frac{\varepsilon}{16m}.$$

By Lemma 7.17, choose a $(PL \cup U, \delta)$-invariant union of complete tiles

$$H = \bigsqcup_s \bigsqcup_{c \in L_s} D_s c, \qquad L_s \subseteq C_s.$$

Retain the centers

$$L_s' = \{c \in L_s : D_s c \subseteq \operatorname{Int}_P H\}, \qquad J = \bigsqcup_s D_s L_s'.$$

If $g \in \operatorname{Int}_{PL} H$, every point of its tile belongs to Lg and hence to $\operatorname{Int}_P H$. Therefore this tile is retained, and

$$|H \setminus J| < \delta |H|. \tag{7.8}$$

Choose $\eta > 0$ such that $b < (1 - \varepsilon)(1 - \eta)$. Apply Lemma 7.16 with $K = H^{-1}H \setminus \{e\}$, enlarging $\mathcal{N}$ by all translates of the E_s. The resulting partition $e_0, \dots, e_r$ commutes with every $\beta_g(E_s)$, and the projections $\beta_h(e_j)$, $h \in H$, are pairwise orthogonal for $j \geq 1$. Set

$$x = \frac{1}{|H|} \sum_{h \in H} \beta_{h^{-1}}(q).$$

Then $0 \leq x \leq 1$, $t(x) = b$, and x commutes with each e_j. There is a nonzero $f = e_j$, $j \geq 1$, such that $t(fx) \leq (1 - \varepsilon) t(f)$. Otherwise,

$$b \geq t((1 - e_0)x) > (1 - \varepsilon)(1 - t(e_0)) > (1 - \varepsilon)(1 - \eta),$$

contrary to the choice of η. Define

$$f' = \sum_{h \in H} \beta_h(f), \qquad \rho = t(f') = |H| t(f) > 0.$$

This is a projection commuting with every translate of every E_s, and invariance of t gives

$$t(f'q) = |H| t(fx) \leq (1 - \varepsilon)\rho. \tag{7.9}$$

For each s, define a base projection from the retained centers:

$$S_s = \sum_{c \in L'_s} \beta_c(f).$$

The levels $\beta_k(S_s)$, $k \in D_s$, are pairwise orthogonal, since they are distinct levels indexed by the retained tiles in H. For every $k \in D_s$ and $l \in U$, we also have

$$\beta_k(S_s) \le \beta_l(f'). \tag{7.10}$$

Indeed, $kc \in \operatorname{Int}_P H$ and $l^{-1} \in P$ imply $l^{-1}kc \in H$, for each retained center c. In particular, $S_s \le f'$.

Set

$$E'_s = (1 - f')E_s + S_s.$$

Both summands are orthogonal projections in $\mathcal{N}' \cap R$. Their translated levels are pairwise orthogonal. The existing levels remain orthogonal after multiplication by the commuting projections, the new levels are orthogonal as above, and the mixed products vanish by (7.10).

To compare the total traces, equation (7.8) gives

$$A_0 := \sum_{s,k \in D_s} t(\beta_k(S_s)) = |J| t(f) > (1 - \delta)\rho.$$

For $k \in U$, invariance of H gives $|kH \triangle H| < 2\delta |H|$. Cancellation of the common summands, followed by the triangle inequality in $\| \cdot \|_{1,t}$, yields

$$\|\beta_k(f') - f'\|_{1,t} \le |kH \triangle H| t(f) < 2\delta\rho.$$

Consequently the removed mass satisfies

$$\begin{aligned} B_0 &:= \sum_{s,k \in D_s} t(\beta_k(f')\beta_k(E_s)) \\ &\le t(f'q) + 2m\delta\rho \le (1 - \varepsilon + 2m\delta)\rho. \end{aligned}$$

The mass of the new family is therefore

$$b' = b - B_0 + A_0 > b + (\varepsilon - \delta - 2m\delta)\rho > b + \frac{\varepsilon}{2}\rho > b,$$

which proves the assertion. □

Proof of Theorem 7.15. Adjoin adjoints to $\mathcal{N}$ and then take its countable β-orbit. Fix the prescribed tiling, and let b_* be the supremum of the total trace of orthogonal tower families with these shapes and bases in $\mathcal{N}' \cap R$. The zero family is admissible, and every admissible family has total trace at most one.

The supremum is attained. Indeed, choose admissible families $(E_s^{(r)})_{s=1}^S$ with masses tending to b_*, and regard $z_s = (E_s^{(r)})_r$ as elements of $\ell^\infty(R)$. Let C be the separable unital β-invariant

C*-algebra generated by the z_s, their translates, and the constant sequences from $\mathcal{N}$. The index selection lemma [50, Section 5.5] gives an equivariant homomorphism $\Theta\colon C \to R$ which fixes these constant sequences and satisfies

$$t(\Theta((x_r)_r)) = \lim_{r\to\omega} t(x_r) \quad ((x_r)_r \in C).$$

The lemma applies because M_* is separable, Γ is countable, and β is liftable. The projection, orthogonality, and relative-commutation relations hold coordinatewise in C and are preserved by Θ. Thus the projections $\Theta(z_s)$ form an admissible family of mass b_*.

If $b_* < 1$, Lemma 7.18 increases the mass of this maximizing family, a contradiction. Hence $b_* = 1$. Its sum of levels is a projection q with $t(q) = 1$. Faithfulness of t implies $q = 1$, proving the theorem. □

Theorem 7.19. *Let A be a unital separable* C*-*algebra with* $\mathrm{T}(A) \neq \varnothing$*, and let a countable discrete amenable group Γ act on A. Suppose that the action is* W*-*centrally free and that* $\mathcal{S} = \mathrm{T}^{\alpha}(A)$ *has finitely many extreme points. Then (A, Γ) has WRP.*

Proof. Amenability ensures that $\mathcal{S}$ is nonempty. Write $\partial_{\mathrm{e}}\mathcal{S} = \{\tau_1, \dots, \tau_n\}$ and put $\tau = n^{-1}\sum_i \tau_i$. Every $\sigma \in \mathcal{S}$ is a convex combination of the τ_i, so, for $a \in A$,

$$\|a\|_{2,\tau}^2 \leq \sup_{\sigma\in\mathcal{S}} \|a\|_{2,\sigma}^2 \leq n\|a\|_{2,\tau}^2. \tag{7.11}$$

In the norm ultrapower $A_\omega = \ell^\infty(A)/c_\omega(A)$, this proves $J_\tau = J_{\mathcal{S}}$.

Let $N = \pi_\tau(A)''$, with its faithful normal tracial state $\bar{\tau}$, and let N^ω be its tracial ultrapower. The GNS homomorphism induces an equivariant homomorphism

$$\Phi\colon A_\omega \longrightarrow N^\omega.$$

Kaplansky density, applied coordinatewise, and bounded positive lifting through $A \to \pi_\tau(A)$ show that Φ is surjective. Its kernel is J_τ. This construction applies even when τ is not faithful on A, since it uses the GNS homomorphism and requires no inclusion $A \subset N$.

The induced action on N is centrally free by hypothesis and preserves $\bar{\tau}$. Given an exact tiling with shapes $D_1, \dots, D_S$, Theorem 7.15 gives projections $p_s \in N' \cap N^\omega$ with pairwise orthogonal translated levels and

$$\sum_{s=1}^{S} \sum_{k\in D_s} \alpha_k(p_s) = 1.$$

Here $N_\omega = N' \cap N^\omega$, since N is finite.

To lift the towers centrally, choose sequences of positive contractions $b^{(s)} \in \ell^\infty(A)$ mapping to p_s, and set

$$C = \mathrm{C}^*(A, \alpha_g(b^{(s)}) : g \in \Gamma,\ 1 \leq s \leq S), \qquad I = C \cap \{(x_j) : \lim_{j\to\omega} \|x_j\|_{2,\tau} = 0\}.$$

The constants identify A with a subalgebra of $\ell^\infty(A)$; C is separable, and I is an invariant ideal. Centrality and orthogonality in N^ω imply

$$[b^{(s)}, a] \in I, \qquad \alpha_k(b^{(s)})\alpha_l(b^{(r)}) \in I$$

for $a \in A$ and distinct levels $(s,k) \neq (r,l)$. Choose a quasicentral approximate identity $(e^{(m)})_m$ for I in C. Every translate of this approximate identity is again quasicentral for I in C. Thus the cutoffs

$$(1 - e^{(m)})b^{(s)}(1 - e^{(m)})$$

have centrality defects tending to zero in norm, and the products of distinct translated cutoffs also tend to zero in norm. The images of the cutoffs in N^ω remain p_s.

Apply Kirchberg's epsilon test [40, Lemma 3.1], as in the proof of Lemma 3.33, including the coordinate 2-norm of the cutoff variable among the conditions. It gives a sequence $d = (d_j)$ of positive contractions with $\lim_{j\to\omega} \|d_j\|_{2,\tau} = 0$, such that

$$c_s = [(1 - d)b^{(s)}(1 - d)] \in A' \cap A_\omega$$

are positive contractions, $\Phi(c_s) = p_s$, and their translated levels are pairwise orthogonal. The argument uses the trace-kernel ideal pulled back to $\ell^\infty(A)$ and therefore applies even when τ is nonfaithful on A. Finally,

$$\Phi\left(1 - \sum_{s=1}^{S} \sum_{k\in D_s} \alpha_k(c_s)\right) = 0.$$

The complement therefore belongs to $J_\tau = J_{\mathcal{S}}$, proving WRP. □

Theorem 7.20. *Let A be a unital separable C^*-algebra with $\mathrm{T}(A) \neq \varnothing$, and let a countable discrete amenable group Γ act on A. If the action has a unique invariant tracial state and is W^*-centrally free, then (A, Γ) has WRP.*

Proof. Apply Theorem 7.19 to the singleton invariant trace space. □

Corollary 7.21. *Let A be a separable simple unital infinite-dimensional $\mathcal{Z}$-absorbing C^*-algebra with $\mathrm{T}(A) \neq \varnothing$, and let a countable discrete amenable group Γ act on A. Suppose the action is W^*-centrally free and has finitely many extreme invariant traces. Then $B = A \rtimes_{\alpha,\mathrm{r}} \Gamma$ is simple and weakly tracially $\mathcal{Z}$-absorbing. If B also has strict comparison, then it is tracially $\mathcal{Z}$-absorbing; if, in addition, A is nuclear, then $B \cong B \otimes \mathcal{Z}$.*

Proof. Theorem 7.19 gives WRP, and Corollary 3.32 gives simplicity of B, since A is simple. $\mathcal{Z}$-absorption supplies asymptotically central order zero maps from matrix algebras whose defects have uniformly small trace. Hence A is weakly tracially $\mathcal{Z}$-absorbing. Corollary 5.8 gives the conclusion. If A is nuclear, amenability of Γ makes B nuclear, giving the final assertion. □

Corollary 7.22. *Let (A, Γ) be a minimal C^*-dynamical system, where A is unital and separable, $\mathrm{T}(A) \neq \varnothing$, and Γ is countably infinite, discrete, and amenable. If the action is W^*-centrally free, has finitely many extreme invariant traces, and has (λ, m)-COS for some $\lambda \in (0, 1]$ and $m \in \mathbb{N}$, then*

$$\mathrm{tsr}(A \rtimes_{\alpha,\mathrm{r}} \Gamma) = 1.$$

Proof. Combine Theorem 7.19 with Theorem 6.17. □

7.5 Trace-kernel ideals and crossed-product traces

In the section, let us prepare notations and lemmas for the next section.

7.5.1 Trace-kernel ideals and Rokhlin towers

Definition 7.23. Let A be a unital C*-algebra with $\mathrm{T}(A) \neq \emptyset$, and let ω be a free ultrafilter on $\mathbb{N}$. Put

$$c_\omega(A) = \{(a_n) \in \ell^\infty(\mathbb{N}, A) : \lim_{n\to\omega} \|a_n\| = 0\}, \qquad A_\omega = \ell^\infty(\mathbb{N}, A)/c_\omega(A),$$

and identify A with the constant sequences in A_ω. For a nonempty subset $S \subseteq \mathrm{T}(A)$, define

$$\|a\|_{2,S} = \sup_{\tau\in S} \tau(a^*a)^{1/2} \quad (a \in A), \qquad \|[(a_n)]\|_{2,S} = \lim_{n\to\omega} \|a_n\|_{2,S}.$$

The latter seminorm is well defined on A_ω. The *trace-kernel ideal* associated to S is

$$J_S = \{x \in A_\omega : \|x\|_{2,S} = 0\}.$$

Write $q_S \colon A_\omega \to A_\omega/J_S$ for the quotient map.

The trace property gives $\|xy\|_{2,S} \le \|x\|\|y\|_{2,S}$ and $\|yx\|_{2,S} \le \|x\|\|y\|_{2,S}$. Consequently J_S is a norm-closed two-sided ideal. The supremum over S is taken before the ultralimit; vanishing against each fixed trace alone does not, in general, imply membership in J_S.

Definition 7.24 (σ-ideals)**.** An ideal $I \lhd B$ is a *σ-ideal* if, for every separable C*-subalgebra $C \subseteq B$, there is a positive contraction $e \in I \cap C'$ such that $eb = b$ for all $b \in I \cap C$.

Proposition 7.25 (Trace-kernel lifting)**.** *The ideal J_S is a σ-ideal of A_ω. If A is separable, then $J_S \cap (A_\omega \cap A')$ is a σ-ideal of $A_\omega \cap A'$, and*

$$q_S(A_\omega \cap A') = q_S(A)' \cap (A_\omega/J_S).$$

Proof. The first assertion is [40, Remark 4.7], with the constant family $S_n = S$; the relative assertion follows from [40, Proposition 4.5(i)]. For the displayed equality, lift $y \in q_S(A)' \cap (A_\omega/J_S)$ to $b \in A_\omega$ and apply the σ-ideal property to $C = \mathrm{C}^*(A, b)$. Choose $e \in J_S \cap C'$ absorbing $J_S \cap C$. Since $[b, a] \in J_S \cap C$ for $a \in A$, the element $(1-e)b(1-e)$ commutes with A and still maps to y. □

Lemma 7.26 (Orthogonal lifting of translated levels)**.** *Suppose that A is separable, a countable discrete group Γ acts on A, and $S \subseteq \mathrm{T}(A)$ is nonempty and α-invariant. Let $p_1, \dots, p_R$ be positive contractions in $q_S(A)' \cap (A_\omega/J_S)$, and let $K_1, \dots, K_R \subseteq \Gamma$ be finite. If the translated levels*

$$\{\alpha_g(p_s) : 1 \le s \le R,\ g \in K_s\}$$

are pairwise orthogonal, then there are positive contractions $c_s \in A_\omega \cap A'$ with $q_S(c_s) = p_s$ whose corresponding translated levels are pairwise orthogonal.

Proof. Invariance of S makes J_S invariant, so the action descends to the quotient. Choose positive contraction lifts $b_s \in A_\omega$ and put

$$C = \mathrm{C}^*(A, \alpha_g(b_s) : g \in \Gamma,\ 1 \le s \le R), \qquad I = J_S \cap C.$$

The algebra C is separable and invariant. Choose a positive contraction $e \in J_S \cap C'$ absorbing I, and set $c_s = (1-e)b_s(1-e)$. These elements have the prescribed images. Since $[b_s, a] \in I$ for $a \in A$, they commute with A.

Each $\alpha_g(e)$ commutes with C and absorbs I. For two distinct levels, put $b_{s,g} = \alpha_g(b_s)$ and $f_g = (1-\alpha_g(e))^2$. Their product in the quotient is zero, so $b_{s,g}b_{t,h} \in I$. Therefore

$$\alpha_g(c_s)\alpha_h(c_t) = f_g b_{s,g} f_h b_{t,h} = f_g b_{s,g} b_{t,h} f_h = 0.$$

Thus the translated levels are orthogonal, even when e is not invariant. □

Corollary 7.27 (WRP in the trace quotient)**.** *Let A be unital and separable with $\mathrm{T}(A) \neq \emptyset$, let Γ be countable, discrete, and amenable, and put $S = \mathrm{T}^\alpha(A) = \mathcal{M}_1(A, \Gamma)$. The action has WRP in the sense of Definition 3.4 if and only if, for every exact tiling with shapes $K_1, \dots, K_R$, there are projections $p_s \in q_S(A)' \cap (A_\omega/J_S)$ such that the levels $\alpha_g(p_s)$, $g \in K_s$, are pairwise orthogonal and*

$$\sum_{s=1}^{R} \sum_{g \in K_s} \alpha_g(p_s) = 1.$$

Proof. The images of WRP contractions are orthogonal positive elements summing to 1, hence projections. Conversely, lift the projections by Lemma 7.26. The complement of the sum of the lifted levels belongs to J_S, as in Definition 3.4. □

7.5.2 GNS crossed products and conditional expectations

For the remainder of this subsection, let Γ be countable and discrete, let A be unital, and assume $S = \mathrm{T}^\alpha(A) \neq \emptyset$. Write

$$B = A \rtimes_{\alpha,\mathrm{r}} \Gamma, \qquad E \colon B \to A$$

for the reduced crossed product and its canonical expectation. When Γ is amenable, the full and reduced crossed products coincide.

Lemma 7.28. *For $\tau \in S$, let*

$$M_\tau = \pi_\tau(A)'', \qquad N_\tau = \pi_{\tau\circ E}(B)''.$$

Let $\bar{\tau}$ be the faithful normal tracial state on M_τ, and let $(\alpha_\tau)_g$ be the induced normal automorphisms. The following assertions hold.

1. *The map*
$$[au_g] \longmapsto [a] \otimes \delta_g$$
extends to a unitary $H_{\tau\circ E} \to H_\tau \otimes \ell^2(\Gamma)$. *In this representation,*
$$\pi_{\tau\circ E}(a) = \pi_\tau(a) \otimes 1, \qquad \pi_{\tau\circ E}(u_g) = U_{\tau,g} \otimes \lambda_g,$$
where $U_{\tau,g}[a] = [\alpha_g(a)]$ *and* λ *is the left regular representation. In particular,* $\pi_{\tau\circ E}(A)''$ *is canonically normally isomorphic to* M_τ.

2. *Under this identification,*
$$N_\tau = M_\tau \rtimes_{\alpha_\tau} \Gamma.$$
There is a faithful normal conditional expectation $E_\tau \colon N_\tau \to M_\tau$ *extending* E, *in the sense that*
$$E_\tau(\pi_{\tau\circ E}(b)) = \pi_\tau(E(b)) \quad (b \in B).$$
The trace $\bar\tau \circ E_\tau$ *is faithful and normal. Writing* $v_{\tau,g} = \pi_{\tau\circ E}(u_g)$, *the Fourier coefficients*
$$x_g = E_\tau(x v_{\tau,g}^*) \quad (x \in N_\tau)$$
uniquely determine x. *The finite Fourier sums converge to* x *in the* 2*-norm of* $\bar\tau \circ E_\tau$ *as the finite subsets of* Γ *increase to* Γ.

3. *Put*
$$\pi = \bigoplus_{\tau\in S} \pi_\tau, \qquad \pi_o = \bigoplus_{\tau\in S} \pi_{\tau\circ E}, \qquad M = \pi(A)'', \quad M_o = \pi_o(A)'', \quad N = \pi_o(B)''.$$
There is a canonical normal isomorphism $M \cong M_o$. *After identifying these algebras, the action extends normally to an action* β *on* M, *and*
$$N = M \rtimes_\beta \Gamma.$$
The canonical faithful normal expectation $E_o \colon N \to M$ *extends* E. *Restriction to each direct summand gives surjective normal homomorphisms*
$$\rho_\tau^M \colon M \to M_\tau, \qquad \rho_\tau^N \colon N \to N_\tau.$$
Each family is jointly faithful, and
$$E_\tau \circ \rho_\tau^N = \rho_\tau^M \circ E_o.$$

Proof. Trace invariance gives
$$\langle [au_g], [bu_h] \rangle = \delta_{g,h} \tau(b^* a).$$
Thus the map in (1) is an isometry with dense range. Left multiplication by a and u_g gives the two displayed representation formulas. Conjugating further by the unitary
$$\xi \otimes \delta_h \longmapsto U_{\tau,h}^* \xi \otimes \delta_h$$

gives the usual regular representation: the coefficient algebra acts on the hth summand by $\pi_\tau(\alpha_{h^{-1}}(a))$, and the group acts by $1 \otimes \lambda$. Taking weak closures proves the crossed-product identification.

Compression to the identity summand gives the normal expectation E_τ; its faithfulness follows from positivity and the corresponding diagonal compressions on all summands. The invariant faithful trace on M_τ extends to $\bar{\tau} \circ E_\tau$. In its GNS Hilbert space the subspaces $L^2(M_\tau, \bar{\tau})v_{\tau,g}$ are mutually orthogonal and span a dense subspace. Orthogonal projection onto these subspaces gives the stated Fourier coefficients and their 2-norm convergence.

For (3), take the direct sum of the unitaries in (1) and use

$$\bigoplus_{\tau \in S} (H_\tau \otimes \ell^2(\Gamma)) \cong \left(\bigoplus_{\tau \in S} H_\tau \right) \otimes \ell^2(\Gamma).$$

Here the coefficient algebra is $M \otimes 1$, and the group unitaries are $(\bigoplus_\tau U_{\tau,g}) \otimes \lambda_g$. This proves the normal isomorphism and the regular crossed-product description, including the expectation. The projections onto the direct summands commute with the represented algebras. Their restrictions therefore define the normal coordinate homomorphisms, which are onto by taking weak closures and jointly faithful by the direct sum representation. The expectation identity holds on crossed-product polynomials and hence, by normality, on all of N. □

7.5.3 Proper outerness, trace extensions, and centers

Definition 7.29 (W*-proper outerness)**.** We call α W*-*properly outer* if $(\alpha_\tau)_g$ is properly outer for every $\tau \in S$ and every $g \neq e$, using Definition 7.8.

W*-central freeness implies W*-proper outerness: an inner automorphism is centrally trivial, also on any invariant central corner.

The following intertwiner criterion will be used in the trace and center calculations.

Lemma 7.30. *Let θ be a properly outer automorphism of a von Neumann algebra P. If $b \in P$ satisfies*

$$bm = \theta(m)b \quad (m \in P),$$

then $b = 0$.

Proof. If $b \neq 0$, write $b = v|b|$. The intertwining relation shows that b^*b and bb^* are central, so the initial and final projections $p = v^*v$ and $q = vv^*$ are nonzero central projections. Equivalent central projections are equal, hence $p = q$. For $z \in Z(P)$, the intertwining relation gives $zp = \theta(z)p$. Taking $z = p$ yields $p \leq \theta(p)$, whereas $z = \theta^{-1}(p)$ yields $p \leq \theta^{-1}(p)$. Thus $\theta(p) = p$. Cancelling $|b|$ on its support gives $\theta(m)p = vmpv^*$ for $m \in P$. Thus $\theta|_{pP}$ is inner, contradicting proper outerness. □

Proposition 7.31. *If α is* W*-*properly outer, then*

$$S \longrightarrow \mathrm{T}(B), \qquad \tau \longmapsto \tau \circ E$$

is an affine homeomorphism. Its inverse is restriction to A. Amenability of Γ is not required for this assertion for the reduced crossed product.

Proof. For $\tau \in S$, invariance and the trace identity on crossed-product polynomials show that $\tau \circ E$ is a trace. Conversely, let $\sigma \in \mathrm{T}(B)$ and put $\tau = \sigma|_A \in S$. In $Q = \pi_\sigma(B)''$, the induced trace $\bar{\sigma}$ is faithful and normal. The coefficient algebra $P = \pi_\sigma(A)''$ is canonically normally isomorphic to M_τ: compression to $\overline{\pi_\sigma(A)\xi_\sigma}$ is its GNS representation, and faithfulness of $\bar{\sigma}$ makes this compression faithful. There is a $\bar{\sigma}$-preserving normal conditional expectation $F\colon Q \to P$, obtained by orthogonal projection in $L^2(Q, \bar{\sigma})$.

Put $V_g = \pi_\sigma(u_g)$. Bimodularity gives

$$F(V_g)m = (\alpha_\tau)_g(m)F(V_g) \quad (m \in P).$$

For $g \neq e$, proper outerness and Lemma 7.30 imply $F(V_g) = 0$. Consequently $\sigma(au_g) = 0$ for $a \in A$ and $g \neq e$. It follows on polynomials, and then by continuity, that $\sigma = \tau \circ E$. The displayed map and restriction are affine and weak* continuous, proving the claim. □

Lemma 7.32. *Suppose that α is W^*-properly outer. In the notation of Lemma 7.28,*

$$M' \cap N = Z(M), \qquad Z(N) = Z(M)^\beta.$$

In particular $Z(N) \subseteq Z(M)$, and these conclusions apply to W^*-centrally free actions.

Proof. Let $x \in M' \cap N$. For each $\tau \in S$, surjectivity of ρ_τ^M implies $y = \rho_\tau^N(x) \in M_\tau' \cap N_\tau$. Write $y_g = E_\tau(yv_{\tau,g}^*)$. For $m \in M_\tau$, bimodularity and covariance give

$$my_g = E_\tau(ymv_{\tau,g}^*) = y_g(\alpha_\tau)_g(m).$$

Taking adjoints and applying Lemma 7.30 shows that $y_g = 0$ for $g \neq e$. Uniqueness of Fourier coefficients therefore gives $y = E_\tau(y)$. The expectation identity in Lemma 7.28 and joint faithfulness of the coordinate maps imply $x = E_o(x) \in M$. Since x commutes with M, it lies in $Z(M)$. The reverse inclusion in $M' \cap N = Z(M)$ is immediate. Finally, an element of $Z(M)$ is central in N precisely when it commutes with every implementing group unitary, equivalently when it is fixed by β. □

7.6 Weak Rokhlin property from invariant traces

Let us prove Theorem 7.48 in this section. We first construct a W^*-bundle over the extreme invariant traces and identify its ultrapower with the uniform trace quotient. Then, we use invariant slicings to glue fiber Rokhlin towers into bundle towers.

7.6.1 The invariant-trace bundle

We construct a continuous W^*-bundle over the extreme boundary of the invariant trace space. Since this space need not be a face of $\mathrm{T}(A)$, we apply the tracial completion theorem to the crossed product. Its conditional expectation then determines the coefficient bundle.

7.6.1.1 Uniform tracial completions and fibers

Definition 7.33. Let K be a compact Hausdorff space. A *continuous tracial* W^**-bundle* over K consists of a unital C^*-algebra $\mathcal{M}$, a specified unital inclusion $\mathrm{C}(K) \subseteq Z(\mathcal{M})$, and a faithful tracial conditional expectation

$$E_{\mathcal{M}} \colon \mathcal{M} \longrightarrow \mathrm{C}(K)$$

such that the closed unit ball $\mathcal{M}_1$ is complete for

$$\|x\|_{2,u} = \|E_{\mathcal{M}}(x^*x)\|^{1/2}.$$

Here traciality means $E_{\mathcal{M}}(xy) = E_{\mathcal{M}}(yx)$, and faithfulness means that $E_{\mathcal{M}}(x^*x) = 0$ implies $x = 0$. The bundle is *strictly separable* if $\mathcal{M}_1$ is separable for this metric.

This is the convention of [9, Definition 3.1]. For $\lambda \in K$, put

$$\tau_\lambda = \mathrm{ev}_\lambda \circ E_{\mathcal{M}}, \qquad \mathcal{M}_\lambda = \pi_{\tau_\lambda}(\mathcal{M}), \qquad \|x\|_{2,\lambda} = \tau_\lambda(x^*x)^{1/2}.$$

The fiber $\mathcal{M}_\lambda$ is already a finite von Neumann algebra, with its faithful normal GNS trace [53, Theorem 11]. The proof of that theorem also applies to nonmetrizable K; see [9, Section 3.1]. In particular, factoriality is not part of the bundle definition. On norm-bounded subsets, the strict topology from the Hilbert $\mathrm{C}(K)$-module $L^2(\mathcal{M}, E_{\mathcal{M}})$ agrees with the uniform 2-norm topology.

For a unital C^*-algebra D and a nonempty closed face $F \subseteq \mathrm{T}(D)$, write

$$\pi_F = \bigoplus_{\tau \in F} \pi_\tau, \qquad N_F = \pi_F(D)''.$$

Each $\tau \in F$ has a normal extension $\bar{\tau}$ to N_F. The *uniform tracial completion* $\overline{D}_F^{\,u}$ is realized inside N_F by adjoining limits of norm-bounded sequences from $\pi_F(D)$ which are Cauchy for

$$\|x\|_{2,F} = \sup_{\tau \in F} \bar{\tau}(x^*x)^{1/2}.$$

Thus completion is taken on bounded sets.

Theorem 7.34 (Ozawa)**.** *Let D be unital and separable, and let F be a nonempty closed face of $\mathrm{T}(D)$ with compact extreme boundary $L = \partial_e F$. There is a canonical unital isomorphism*

$$\theta_F \colon \mathrm{C}(L) \longrightarrow \overline{D}_F^{\,u} \cap Z(N_F)$$

such that

$$E_F = \theta_F^{-1} \circ \mathrm{ctr}_{N_F}|_{\overline{D}_F^{\,u}}$$

makes $\overline{D}_F^{\,u}$ a continuous W^*-bundle over L. Its expectation and uniform 2-norm satisfy

$$E_F(\pi_F(d))(\tau) = \tau(d), \qquad \|\pi_F(d)\|_{2,u} = \|d\|_{2,F} \quad (d \in D,\ \tau \in L).$$

Moreover, $\pi_F(D_1)$ is uniformly 2-norm dense in $(\overline{D}_F^{\,u})_1$, and the fiber at $\tau \in L$ is canonically $\pi_\tau(D)''$.

Proof. These are the Bauer-face conclusions of [53, Theorem 3]. The unit-ball density follows from the bounded convex approximation in its proof. □

7.6.1.2 Bundle actions and ultrapowers

Definition 7.35. For a discrete group Γ, an action $\beta\colon \Gamma \to \mathrm{Aut}(\mathcal{M})$ on a W*-bundle is a *bundle action* if it preserves the specified copy of $\mathrm{C}(K)$ and

$$E_{\mathcal{M}} \circ \beta_g = (\beta_g|_{\mathrm{C}(K)}) \circ E_{\mathcal{M}} \quad (g \in \Gamma).$$

It is *fiberwise* if $\beta_g|_{\mathrm{C}(K)} = \mathrm{id}$ for every g. In that case each τ_λ is invariant, and the induced normal fiber action is determined by

$$\beta_{\lambda,g}(\pi_{\tau_\lambda}(x)) = \pi_{\tau_\lambda}(\beta_g(x)) \quad (x \in \mathcal{M}).$$

Definition 7.36. Let ω be a free ultrafilter on $\mathbb{N}$. The W*-bundle ultrapower is

$$\mathcal{M}^\omega = \ell^\infty(\mathbb{N}, \mathcal{M})/c_\omega(\mathcal{M}), \qquad c_\omega(\mathcal{M}) = \{(x_n) \in \ell^\infty(\mathbb{N}, \mathcal{M}) : \lim_{n\to\omega} \|x_n\|_{2,u} = 0\}.$$

Here $c_\omega(\mathcal{M})$ is the uniform 2-norm ideal. The base K^ω is the ultracopower specified by

$$\mathrm{C}(K^\omega) = \mathrm{C}(K)_\omega,$$

where the right hand side is the norm ultrapower. The central inclusion of $\mathrm{C}(K^\omega)$ and the expectation are induced coordinatewise:

$$E^\omega_{\mathcal{M}}([(x_n)]) = [(E_{\mathcal{M}}(x_n))], \qquad \|[(x_n)]\|_{2,u} = \lim_{n\to\omega} \|x_n\|_{2,u}.$$

The resulting triple $(\mathcal{M}^\omega, K^\omega, E^\omega_{\mathcal{M}})$ is again a continuous W*-bundle [9, Proposition 3.9]. The base K^ω need not be metrizable. The set-theoretic ultrapower K_ω, formed by identifying (λ_n) and (μ_n) when $\{n : \lambda_n = \mu_n\} \in \omega$, embeds densely into K^ω by the characters

$$[(f_n)] \longmapsto \lim_{n\to\omega} f_n(\lambda_n).$$

A bundle action induces the coordinatewise action $\beta^\omega_g([(x_n)]) = [(\beta_g(x_n))]$; a fiberwise action remains fiberwise.

7.6.1.3 The coefficient bundle

Theorem 7.37. *Let A be unital and separable, and let $\alpha\colon \Gamma \to \mathrm{Aut}(A)$ be an action of a countable discrete group with $S = \mathrm{T}^\alpha(A) \neq \emptyset$. Suppose that α is W*-properly outer and $K = \partial_e S$ is compact. Then there are a strictly separable continuous W*-bundle $(\mathcal{M}, K, E_{\mathcal{M}})$, a fiberwise action β, and a unital equivariant homomorphism $\iota\colon A \to \mathcal{M}$ such that*

1. *$\iota(A_1)$ is uniformly 2-norm dense in $\mathcal{M}_1$.*
2. *For $a \in A$ and $\tau \in K$,*

$$E_{\mathcal{M}}(\iota(a))(\tau) = \tau(a), \qquad \|\iota(a)\|_{2,u} = \|a\|_{2,S}.$$

3. There is a canonical normal isomorphism

$$\mathcal{M}_\tau \cong M_\tau = \pi_\tau(A)'' \quad (\tau \in K)$$

intertwining β_τ with α_τ.

4. The kernel of ι is

$$\ker \iota = \{a \in A : \|a\|_{2,S} = 0\}.$$

If Γ is amenable, the nonemptiness assumption on S follows from $\mathrm{T}(A) \neq \emptyset$.

Proof. Put $B = A \rtimes_{\alpha,\mathrm{r}} \Gamma$ and let $E_\alpha \colon B \to A$ be the canonical expectation. Proposition 7.31 identifies $F = \mathrm{T}(B)$ affinely and homeomorphically with S by $\tau \mapsto \tau \circ E_\alpha$. Thus S is a Choquet simplex, and this map identifies its compact extreme boundary K with $\partial_e F$. In the representation of Lemma 7.28, write

$$\pi_o = \bigoplus_{\tau \in S} \pi_{\tau \circ E_\alpha}, \qquad M = \pi_o(A)'', \qquad N = \pi_o(B)''.$$

Let $E_o \colon N \to M$ be the normal extension of E_α, and let ctr_N be the center-valued trace of N.

By Theorem 7.34, the completion $\mathcal{B} = \overline{B}_F^u$ is a W*-bundle over K. We write

$$\theta \colon \mathrm{C}(K) \longrightarrow \mathcal{B} \cap Z(N), \qquad E_{\mathcal{B}} = \theta^{-1} \circ \mathrm{ctr}_N \,|_{\mathcal{B}}.$$

On N, put $\|x\|_{2,F} = \|\,\mathrm{ctr}_N(x^*x)\|^{1/2}$.

The expectations satisfy

$$\mathrm{ctr}_N \circ E_o = \mathrm{ctr}_N, \qquad E_o \circ \mathrm{ctr}_N = \mathrm{ctr}_N\,. \tag{7.12}$$

For a normal state f on $Z(N)$, the trace $f \circ \mathrm{ctr}_N$ restricts to a trace on B. Proposition 7.31 implies that this restriction equals its composition with E_α. Normality extends the equality to N. Since normal states separate $Z(N)$, this proves the first identity. The second follows from $Z(N) \subseteq Z(M) \subseteq M$, by Lemma 7.32.

Kadison's inequality and (7.12) give

$$\mathrm{ctr}_N(E_o(x)^* E_o(x)) \leq \mathrm{ctr}_N(E_o(x^*x)) = \mathrm{ctr}_N(x^*x).$$

Hence E_o is contractive for $\|\cdot\|_{2,F}$. For $x \in \mathcal{B}_1$, choose $b_n \in B_1$ such that $\|\pi_o(b_n) - x\|_{2,F} \to 0$. Then

$$\|\pi_o(E_\alpha(b_n)) - E_o(x)\|_{2,F} \longrightarrow 0, \qquad \|E_\alpha(b_n)\| \leq 1.$$

Thus $E_o(\mathcal{B}) \subseteq \mathcal{B} \cap M$. Set

$$\mathcal{M} = \mathcal{B} \cap M = E_o(\mathcal{B}), \qquad E_{\mathcal{M}} = E_{\mathcal{B}}|_{\mathcal{M}}, \qquad \iota(a) = \pi_o(a).$$

The unit ball of $\mathcal{M}$ is the fixed-point set of the uniformly 2-norm continuous contraction E_o on $\mathcal{B}_1$, and is therefore complete. Moreover,

$$\theta(\mathrm{C}(K)) = \mathcal{B} \cap Z(N) \subseteq \mathcal{M} \cap Z(\mathcal{M}).$$

Consequently $E_{\mathcal{M}}$ is a faithful tracial conditional expectation onto this specified central copy of $\mathrm{C}(K)$. Thus the construction defines a continuous W*-bundle.

Applying the preceding approximation to $x \in \mathcal{M}_1$ gives contractions $E_\alpha(b_n) \in A_1$ approximating x. This proves (1); separability of A also proves strict separability of the bundle. The expectation formula in (2) follows from the corresponding formula for $\mathcal{B}$. Since S is a Bauer simplex, an affine continuous function on S has the same supremum on S and on K. Applying this to $\tau \mapsto \tau(a^*a)$ gives the norm identity. Faithfulness of $E_{\mathcal{M}}$ then gives (4).

To identify the fibers, let $\rho_\tau^M \colon M \to M_\tau$ be the normal coordinate map of Lemma 7.28. For $\tau \in K$ and $x \in \mathcal{M}$,

$$(\mathrm{ev}_\tau \circ E_{\mathcal{M}})(x) = \bar{\tau}(\rho_\tau^M(x)).$$

Thus the bundle GNS representation is unitarily equivalent to $\rho_\tau^M|_{\mathcal{M}}$ on H_τ: its cyclic subspace contains $\pi_\tau(A)\xi_\tau$, which is dense in H_τ. Ozawa's fiber theorem shows that this image is a von Neumann algebra. It is contained in M_τ and contains $\pi_\tau(A)$, so it equals M_τ.

Finally, the normal action on M is implemented in N by the canonical group unitaries. Their inner automorphisms preserve $\mathcal{B}$, its uniform 2-norm, and M, so they restrict to an action β on $\mathcal{M}$. They fix $Z(N)$ pointwise, hence fix $\theta(\mathrm{C}(K))$ pointwise. The action is therefore fiberwise and ι is equivariant. The coordinate maps intertwine this action with α_τ, giving (3). □

The fibers in this theorem need not be factors: an extreme invariant trace need not be extreme in $\mathrm{T}(A)$. The map ι is injective whenever invariant traces separate A_+; in particular, this holds for a minimal action, since $\ker \iota$ is a proper invariant ideal.

7.6.2 From bundle towers to C*-algebra towers

The bundle ultrapower is canonically isomorphic to the uniform trace quotient. This identifies Rokhlin towers in the bundle with WRP towers in the C*-algebra. Throughout this subsection, ω is the fixed free ultrafilter on $\mathbb{N}$, A_ω is the norm ultrapower, and $\mathcal{M}^\omega$ is the W*-bundle ultrapower.

Definition 7.38. Let $\beta \colon \Gamma \to \mathrm{Aut}(\mathcal{M})$ be a bundle action of a countable discrete group. The action has the W*-*bundle Rokhlin property* if, for every exact tiling of Γ with shapes $K_1, \dots, K_R$, there are projections $p_1, \dots, p_R \in \mathcal{M}^\omega \cap \mathcal{M}'$ such that

$$\sum_{s=1}^{R} \sum_{g \in K_s} \beta_g^\omega(p_s) = 1.$$

The translated projections are automatically pairwise orthogonal, since their sum is 1.

The next lemma is the invariant-trace version of [9, Lemma 3.10]. Its proof uses the bounded density established in Theorem 7.37 and the trace-kernel lifting property from Section 7.5.

Lemma 7.39 (The uniform tracial quotient)**.** *Let $(\mathcal{M}, K, E_{\mathcal{M}})$, β, and $\iota \colon A \to \mathcal{M}$ be as in Theorem 7.37, and put $S = \mathrm{T}^\alpha(A)$. Then the coordinatewise map*

$$\Phi \colon A_\omega \longrightarrow \mathcal{M}^\omega, \qquad \Phi([(a_n)]) = [(\iota(a_n))]$$

is an equivariant surjective homomorphism satisfying

$$\|\Phi(x)\|_{2,u} = \|x\|_{2,S}, \qquad \ker \Phi = J_S.$$

Moreover, its restriction gives an equivariant surjection

$$\Phi\colon A_\omega \cap A' \longrightarrow \mathcal{M}^\omega \cap \mathcal{M}'.$$

In particular, there are canonical equivariant isomorphisms

$$A_\omega/J_S \cong \mathcal{M}^\omega, \qquad \frac{A_\omega \cap A'}{J_S \cap (A_\omega \cap A')} \cong \mathcal{M}^\omega \cap \mathcal{M}'.$$

Proof. The inequality $\|\iota(a)\|_{2,u} = \|a\|_{2,S} \leq \|a\|$ shows that the coordinatewise homomorphism annihilates norm-null sequences, so Φ is well defined. Equivariance and the displayed 2-norm identity follow from Theorem 7.37. Since the uniform 2-norm on $\mathcal{M}^\omega$ is faithful, this also identifies its kernel with J_S.

To prove surjectivity, represent $y \in \mathcal{M}^\omega$ by a bounded sequence (y_n) in $\mathcal{M}$, and choose $C \geq 1$ with $\|y_n\| \leq C$ for all n. The uniform 2-norm density of $\iota(A_1)$ in $\mathcal{M}_1$ gives $a_n \in A$ such that

$$\|a_n\| \leq C, \qquad \|\iota(a_n) - y_n\|_{2,u} < 1/n.$$

Thus $\Phi([(a_n)]) = y$. Let $\overline{\Phi}\colon A_\omega/J_S \to \mathcal{M}^\omega$ be the resulting equivariant isomorphism.

For the assertion concerning central sequences, observe that if $y \in \mathcal{M}^\omega$ commutes with $\iota(A)$, then, for $m \in \mathcal{M}_1$ and $a \in A_1$,

$$\|[y, m]\|_{2,u} \leq 2\|y\|\|m - \iota(a)\|_{2,u}.$$

Bounded density makes the right hand side arbitrarily small. Hence $[y, m] = 0$, and

$$\mathcal{M}^\omega \cap \iota(A)' = \mathcal{M}^\omega \cap \mathcal{M}'.$$

Proposition 7.25 gives

$$q_S(A_\omega \cap A') = q_S(A)' \cap (A_\omega/J_S).$$

Applying $\overline{\Phi}$ to this equality proves surjectivity on the central sequence algebras. The kernel of the restriction is $J_S \cap (A_\omega \cap A')$, as asserted. □

Lemma 7.40 (Transfer of Rokhlin towers)**.** *In the setting of Theorem 7.37, suppose also that* Γ *is amenable. Then* α *has WRP in the sense of Definition 3.4 if and only if* β *has the* W*-*bundle Rokhlin property of Definition 7.38.*

Proof. Fix an exact tiling with shapes $K_1, \ldots, K_R$. Suppose first that β has the W*-bundle Rokhlin property, and choose its projections p_s. Lemma 7.39 identifies these with projections in $q_S(A)' \cap (A_\omega/J_S)$. Their translated levels are pairwise orthogonal. Lemma 7.26 therefore gives

positive contractions $c_s \in A_\omega \cap A'$ satisfying $\Phi(c_s) = p_s$ whose translated levels are pairwise orthogonal in the norm ultrapower. Equivariance gives

$$\Phi\left(1 - \sum_{s=1}^{R} \sum_{g \in K_s} \alpha_g^\omega(c_s)\right) = 0.$$

Since $\ker \Phi = J_S$, the lifted towers satisfy WRP.

Conversely, let c_s be WRP contractions for this tiling. By Lemma 7.39, the positive contractions $p_s = \Phi(c_s)$ belong to $\mathcal{M}^\omega \cap \mathcal{M}'$. Their translated levels are pairwise orthogonal and sum to 1, because the WRP defect belongs to J_S. Orthogonal positive elements summing to 1 are projections: multiplying the sum by any one level shows that this level equals its square. Thus the translated levels, and hence the p_s, are projections. □

7.6.3 From fiber towers to bundle towers

The local-to-global argument combines exact Rokhlin towers in the fibers with invariant partitions which reproduce prescribed central trace weights.

Throughout the bundle arguments, $(\mathcal{M}, K, E)$ is a strictly separable continuous W*-bundle and $\beta\colon \Gamma \to \mathrm{Aut}(\mathcal{M})$ is a fiberwise action of a countable discrete group. Write $M_\lambda = \pi_\lambda(\mathcal{M})$ for the fiber, and $\bar{\tau}_\lambda$ for its faithful normal trace. Strict separability gives each fiber separable predual. The ultrapowers $(M_\lambda)^\omega$ are tracial, whereas $\mathcal{M}^\omega$ is the uniform bundle ultrapower. Put $E^\omega = E^\omega_{\mathcal{M}}$. Elements of $\mathrm{C}(K)$ and of $\mathcal{M}$ are identified with their constant images in the corresponding ultrapowers.

7.6.3.1 Fiber quotients and invariant slices

Lemma 7.41 (Lifting central sequences from a fiber)**.** *For every $\lambda \in K$, the coordinatewise fiber map gives equivariant surjections*

$$\Pi_\lambda\colon \mathcal{M}^\omega \longrightarrow (M_\lambda)^\omega, \qquad \Pi_\lambda\colon \mathcal{M}^\omega \cap \mathcal{M}' \longrightarrow (M_\lambda)^\omega \cap M_\lambda'.$$

Proof. The fiber map $\pi_\lambda\colon \mathcal{M} \to M_\lambda$ is surjective, and $\|\pi_\lambda(x)\|_{2,\bar{\tau}_\lambda} \le \|x\|_{2,u}$. Bounded lifting through this C*-quotient proves surjectivity of Π_λ; the inequality makes the map well defined. Fiberwise equivariance makes both maps equivariant.

Let $x = [(x_n)]$ belong to the fiber central sequence algebra. Choose lifts $a_n \in \mathcal{M}$ with uniformly bounded norms, and fix a uniformly 2-norm dense sequence (s_j) in $\mathcal{M}_1$. Centrality of x permits a choice of integers $k(n)$ tending to infinity along ω such that

$$\|[x_n, \pi_\lambda(s_j)]\|_{2,\bar{\tau}_\lambda} < 1/k(n) \quad (1 \le j \le k(n))$$

on an ω-large set. For these n, continuity of the fiber 2-norms gives an open neighborhood U_n of λ on which

$$\|[a_n, s_j]\|_{2,\mu} < 2/k(n) \quad (\mu \in U_n,\ 1 \le j \le k(n)).$$

Choose $f_n \in \mathrm{C}(K)$ with $0 \le f_n \le 1$, $f_n(\lambda) = 1$, and $\mathrm{supp}(f_n) \subseteq U_n$. On the remaining indices take $f_n = 1$. Centrality of f_n gives

$$\|[f_n a_n, s_j]\|_{2,u} \le 2/k(n) \quad (1 \le j \le k(n))$$

on the same ω-large set. Thus $[(f_n a_n)]$ commutes with every s_j. The uniform norm bound and 2-norm density extend commutation to all of $\mathcal{M}$, while $\Pi_\lambda([(f_n a_n)]) = x$. □

Definition 7.42. An *invariant n-slicing* consists of pairwise orthogonal projections

$$e_1, \dots, e_n \in (\mathcal{M}^\omega \cap \mathcal{M}')^{\beta\omega}$$

such that

$$\sum_{i=1}^{n} e_i = 1, \qquad E^\omega(e_i x) = \frac{1}{n} E(x) \quad (x \in \mathcal{M},\ 1 \le i \le n).$$

The expectation on the right is viewed as a constant element of $\mathrm{C}(K^\omega) = \mathrm{C}(K)_\omega$. Equivalently, the trace identity holds at every point of K^ω.

A unital homomorphism $\mathrm{M}_n \to (\mathcal{M}^\omega \cap \mathcal{M}')^{\beta\omega}$ gives an invariant n-slicing by taking its diagonal matrix units. Indeed, traciality of E^ω and commutation with x make $E^\omega(e_{ii}x)$ independent of i, and their sum is $E(x)$.

Lemma 7.43 (Relative slices)**.** *If an invariant n-slicing exists, then, for every countable subset $\mathcal{S} \subseteq \mathcal{M}^\omega$, there are pairwise orthogonal projections*

$$f_1, \dots, f_n \in (\mathcal{M}^\omega \cap \mathcal{M}' \cap \mathcal{S}')^{\beta\omega}$$

with

$$\sum_i f_i = 1, \qquad E^\omega(f_i x) = \frac{1}{n} E^\omega(x) \quad (x \in \mathcal{S}).$$

Proof. Adjoin to $\mathcal{S}$ the constant images of a countable uniformly 2-norm dense subset of $\mathcal{M}_1$, together with 1, adjoints, and all Γ-translates. Enumerate the resulting countable set as (x_j), and choose bounded representatives $(x_{j,m})_m$ in $\mathcal{M}$. Represent the given slicing projections by positive contraction sequences $(e_{i,k})_k$.

For each m, the finitely many $x_{j,m}$, $j \le m$, are elements of $\mathcal{M}$. The slicing identities and centrality give an index $k(m)$ such that

$$\|[e_{i,k(m)}, x_{j,m}]\|_{2,u} < 1/m, \qquad \left\| E(e_{i,k(m)} x_{j,m}) - \frac{1}{n} E(x_{j,m}) \right\| < 1/m \quad (i \le n,\ j \le m).$$

Choose this index so that the errors in the projection, orthogonality, and partition relations are also below $1/m$, as are the invariance errors for the first m elements of a fixed enumeration of Γ. Each condition holds on an ω-large set, so their finite intersection is nonempty. Then $f_i = [(e_{i,k(m)})_m]$ satisfies all the asserted relations. Commutation with the adjoined dense subset extends to $\mathcal{M}$ by boundedness. □

7.6.3.2 Partitions with prescribed central trace weights

Lemma 7.44. *Suppose that invariant n-slicings exist for every $n \geq 1$. Let $h_1, \ldots, h_l \in \mathrm{C}(K^\omega)_+$ satisfy $\sum_j h_j = 1$, and let $\mathcal{S} \subseteq \mathcal{M}^\omega$ be countable. Then there are pairwise orthogonal projections*

$$p_1, \ldots, p_l \in (\mathcal{M}^\omega \cap \mathcal{M}' \cap \mathcal{S}')^{\beta^\omega}$$

such that

$$\sum_j p_j = 1, \qquad E^\omega(p_j x) = h_j E^\omega(x) \quad (x \in \mathcal{S},\ 1 \leq j \leq l).$$

These identities extend to the uniform 2-norm closure of $\mathcal{S}$.

Proof. Adjoin 1 and a countable uniformly 2-norm dense subset of $\mathcal{M}_1$ to the given subset. For $N \geq 1$, Lemma 7.43 gives an invariant relative slicing $e_{1,N}, \ldots, e_{N,N}$ with

$$E^\omega(e_{r,N} x) = \frac{1}{N} E^\omega(x) \quad (x \in \mathcal{S}).$$

Put $s_0 = 0$ and $s_j = \sum_{i=1}^j h_i$, so $s_l = 1$. For $0 \leq t \leq 1$, define the continuous scalar functions

$$F_{r,N}(t) = \min\{1, \max\{0, Nt - r + 1\}\} \quad (1 \leq r \leq N).$$

They satisfy $\sum_{r=1}^N F_{r,N}(t) = Nt$. Use functional calculus in the specified central algebra to set

$$d_{j,r} = F_{r,N}(s_j) - F_{r,N}(s_{j-1}), \qquad a_{j,N} = \sum_{r=1}^N d_{j,r} e_{r,N}.$$

The coefficients $d_{j,r}$ are central positive contractions. Consequently the $a_{j,N}$ are positive contractions, commute with the specified elements, are invariant, and satisfy

$$\sum_j a_{j,N} = 1, \qquad E^\omega(a_{j,N} x) = h_j E^\omega(x) \quad (x \in \mathcal{S}).$$

We estimate their orthogonality in the uniform 2-norm. At a point $\lambda \in K^\omega$, the scalar $d_{j,r}(\lambda)$ is N times the length of the intersection of $[s_{j-1}(\lambda), s_j(\lambda)]$ with the rth interval of the equal N-partition of $[0,1]$. For $i \neq j$, both coefficients can be nonzero in at most one such interval. Since $E^\omega(e_{r,N}) = 1/N$, it follows that

$$\|a_{i,N} a_{j,N}\|_{2,u}^2 = \left\| \frac{1}{N} \sum_{r=1}^N (d_{i,r} d_{j,r})^2 \right\| \leq \frac{1}{N}. \tag{7.13}$$

A diagonal argument preserves the prescribed h_j and the specified elements, including their nonconstant ultrapower representatives. Fix representatives for them, and enumerate the countably many commutation, invariance, partition, and expectation relations. Choose $N_m > m^2$ and positive contraction representatives for a_{j,N_m}. There are nested sets $V_m \in \omega$, with $V_m \subseteq \{k : k \geq$

$m\}$, on which the errors in the first m relations and in all orthogonality relations are below $2/m$. For an index k, let

$$m(k) = \max\{m \le k : k \in V_m\},$$

using arbitrary contractions when this set is empty. Then $m(k) \to_\omega \infty$. At coordinate k, choose the kth coordinate of $a_{j,N_{m(k)}}$, retaining the fixed representatives of the h_j and the specified elements at that coordinate. The resulting p_j satisfy all the asserted relations.

They are positive contractions with pairwise zero products and sum 1, so $p_j^2 = p_j$. Finally, commutators with a fixed bounded element and the maps $x \mapsto E^\omega(p_j x)$ and $x \mapsto h_j E^\omega(x)$ are uniformly 2-norm continuous in x. This proves the extension assertion. □

Theorem 7.45 (From fiber towers to bundle towers)**.** *Suppose that every fiber action β_λ has the* W*-*Rokhlin property of Definition 7.14, and that invariant n-slicings exist for all n. Then β has the* W*-*bundle Rokhlin property.*

Proof. Fix an exact tiling with shapes $K_1, \dots, K_R$. Let $\mathcal{F} \subseteq \mathcal{M}_1$ be finite and let $\varepsilon > 0$. At $\lambda \in K$, take fiber Rokhlin projections in $(M_\lambda)^\omega \cap M_\lambda'$. Spectral cutoffs give projection representatives in M_λ. Choose one coordinate and lift its base projections to positive contractions $f_{\lambda,s} \in \mathcal{M}$. The coordinate can be chosen so that, at λ,

$$\left\| 1 - \sum_{s=1}^{R} \sum_{g \in K_s} \beta_g(f_{\lambda,s}) \right\|_{2,\lambda} < \varepsilon/2, \qquad \|[f_{\lambda,s}, a]\|_{2,\lambda} < \varepsilon/2 \quad (a \in \mathcal{F}).$$

Also $\|f_{\lambda,s}^2 - f_{\lambda,s}\|_{2,\lambda} = 0$. Continuity gives an open neighborhood U_λ on which all three kinds of errors are below ε. Take a finite subcover $U_1, \dots, U_L$, corresponding lifts $f_{j,s}$, and a partition of unity $h_1, \dots, h_L \in \mathrm{C}(K)_+$ subordinate to this cover.

Apply Lemma 7.44 with a countable uniformly 2-norm dense subset of $\mathcal{M}_1$. Its extension assertion gives invariant orthogonal projections p_j summing to 1 and satisfying

$$p_j \in \mathcal{M}^\omega \cap \mathcal{M}', \qquad E^\omega(p_j x) = h_j E(x) \quad (x \in \mathcal{M}).$$

Define positive contractions

$$q_s = \sum_{j=1}^{L} p_j f_{j,s} \in \mathcal{M}^\omega.$$

For arbitrary $d_1, \dots, d_L \in \mathcal{M}$, centrality and orthogonality of the p_j give

$$\left\| \sum_j p_j d_j \right\|_{2,u}^2 = \left\| \sum_j h_j E(d_j^* d_j) \right\|. \tag{7.14}$$

In each application below, $\|d_j\|_{2,\mu} < \varepsilon$ on U_j. Since $\mathrm{supp}(h_j) \subseteq U_j$, the right hand side is at most ε^2. Apply (7.14) to

$$f_{j,s}^2 - f_{j,s}, \qquad 1 - \sum_{s,g \in K_s} \beta_g(f_{j,s}), \qquad [f_{j,s}, a] \ (a \in \mathcal{F}).$$

Invariance of the p_j is used in the middle expression. We obtain

$$\|q_s^2 - q_s\|_{2,u} \le \varepsilon, \qquad \left\| 1 - \sum_{s,g\in K_s} \beta_g^\omega(q_s) \right\|_{2,u} \le \varepsilon, \qquad \|[q_s, a]\|_{2,u} \le \varepsilon \quad (a \in \mathcal{F}).$$

Let $\mathcal{F}_m$ increase through a dense sequence in $\mathcal{M}_1$ and let $\varepsilon = 1/m$. For each resulting family, choose a coordinate of positive contraction representatives at which the finitely many displayed errors are below $2/m$. The sequence of these coordinates gives projections $e_s \in \mathcal{M}^\omega \cap \mathcal{M}'$ satisfying

$$\sum_{s=1}^{R} \sum_{g\in K_s} \beta_g^\omega(e_s) = 1.$$

All translated levels are projections, and their sum is 1, so they are pairwise orthogonal. Thus the bundle action has the W*-bundle Rokhlin property. □

7.6.3.3 Finite-dimensional gluing and the main conclusion

Lemma 7.46 (Invariant diffuse gluing)**.** *Suppose that the strictly separable bundle $(\mathcal{M}, K, E)$ has compact metrizable base of finite covering dimension, and that β is a fiberwise bundle action of a countable group, in the sense of Definition 7.35. Assume that, for every $\lambda \in K$,*

$$Z(M_\lambda)^{\beta_\lambda} = \mathbb{C}1$$

and there is a positive contraction

$$z_\lambda \in ((M_\lambda)^\omega \cap M_\lambda')^{\beta_\lambda^\omega}$$

with Lebesgue distribution on $[0,1]$ for the ultrapower trace. Then invariant n-slicings exist for every $n \ge 1$.

Proof. Put $c = \dim K + 1$. We first construct a commuting tuple in the bundle ultrapower from the fiber contractions. Fix finite sets $\mathcal{F}_m \subset \mathcal{M}_1$ increasing through a uniformly 2-norm dense sequence, finite sets $H_m \subset \Gamma$ increasing to Γ, and numbers $\delta_m \downarrow 0$.

Successive covering families. For each m, construct c finite families successively. To construct family ℓ, at each $\lambda \in K$ choose a positive contraction coordinate $t_\lambda \in M_\lambda$ of z_λ that is δ_m-central for the images of $\mathcal{F}_m$ and of every element in all earlier families, δ_m-invariant under H_m, and satisfies

$$\left| \bar{\tau}_\lambda(t_\lambda^p) - \frac{1}{p+1} \right| < \delta_m \quad (1 \le p \le m).$$

Lift t_λ to a positive contraction $z \in \mathcal{M}$. Choose the coordinate with a sufficiently small initial error. Continuity then gives a neighborhood U_λ on which all these estimates hold with error δ_m. Choose a smaller neighborhood V_λ with $\overline{V_\lambda} \subset U_\lambda$ and a central function $f_\lambda \in \mathrm{C}(K)_+$ satisfying

$$0 \le f_\lambda \le 1, \qquad f_\lambda|_{V_\lambda} = 1, \qquad \mathrm{supp}(f_\lambda) \subset U_\lambda.$$

Replace the lift by

$$b_\lambda = f_\lambda z + (1 - f_\lambda)\tfrac{1}{2}1.$$

Its commutators with the specified elements and its invariance errors are below δ_m in the uniform 2-norm: outside U_λ it is scalar, and the cutoff is central and fixed by β. Its moments satisfy the stated estimates on V_λ. Choose a finite subcover consisting of sets V_λ; the corresponding contractions b_λ form family ℓ. Every family covers all of K, and each later family uniformly almost commutes with every earlier family. This is the scalar version of the commuting covering-system construction in [31, Claim 8.4.1].

After constructing all c families, take a common refinement of their neighborhood covers. Finite covering dimension gives a subordinate finite partition of unity (h_j) with c colors such that $h_j h_k = 0$ for distinct indices of the same color. Such a partition is obtained from a refinement of order at most c by coloring the barycentric subdivision of its nerve according to simplex dimension. For each member of color ℓ, choose $b_{\ell,j}$ from family ℓ whose moment estimates hold throughout its support. Put

$$a_{\ell,m} = \sum_{j \text{ of color } \ell} h_j b_{\ell,j}, \qquad w_{\ell,m} = \sum_{j \text{ of color } \ell} h_j.$$

Then $0 \le a_{\ell,m} \le w_{\ell,m}$ and $\sum_\ell w_{\ell,m} = 1$. At each fiber, at most one summand of a given color is nonzero. Consequently all centrality, invariance, and cross-color commutator errors are bounded by δ_m, independently of the number of patches. Also, for $1 \le p \le m$,

$$\left\| E(a_{\ell,m}^p) - \frac{w_{\ell,m}^p}{p+1} \right\| < \delta_m.$$

Thus, writing $Y = K^\omega$ and $\mathrm{C}(Y) = \mathrm{C}(K)_\omega$, the classes $a_\ell = [(a_{\ell,m})]$ commute with each other, belong to $(\mathcal{M}^\omega \cap \mathcal{M}')^{\beta^\omega}$, and satisfy

$$0 \le a_\ell \le w_\ell, \qquad \sum_\ell w_\ell = 1, \qquad E^\omega(a_\ell^p) = \frac{w_\ell^p}{p+1} \quad (p \ge 1), \quad w_\ell = [(w_{\ell,m})] \in \mathrm{C}(Y). \tag{7.15}$$

These moment identities determine the distribution of $(a_\ell)_y$ at every $y \in Y$: it is uniform on $[0, w_\ell(y)]$, with the point mass at zero when $w_\ell(y) = 0$.

A uniform diffuse variable. The maximum

$$a = \max_{1 \le \ell \le c} a_\ell$$

is defined by joint continuous functional calculus in the commuting algebra generated by the a_ℓ and $\mathrm{C}(Y)$. In each fiber the commuting tuple is a tuple of random variables with the marginals in (7.15). The following estimate depends only on these marginal distributions. For an interval $[t, t+\delta] \subset [0,1]$, put $r = \sqrt{\delta}$. If $t \ge r$, a variable attaining the maximum in this interval has weight at least t. The union bound gives interval mass at most $c\delta/t \le c\sqrt{\delta}$. If $t < r$, choose a weight at least $1/c$; the event that the maximum lies in the interval forces the corresponding

variable, after division by its weight, below $c(r+\delta)$. This rescaled variable is uniform on $[0,1]$. In either case,

$$\bar{\tau}_y(1_{[t,t+\delta]}(a_y)) \le 2c\sqrt{\delta}. \tag{7.16}$$

Intervals meeting the endpoints are handled by truncation. In particular, all fiber distributions are atomless with a common modulus of continuity.

Continuous ramps at t are Cauchy in the uniform 2-norm by (7.16). Uniform 2-norm completeness of the ultrapower bundle therefore gives $P_t = 1_{[0,t]}(a) \in \mathcal{M}^\omega$. Set $H_t = E^\omega(P_t) \in \mathrm{C}(Y)$. The same estimate gives

$$\|H_t - H_s\| \le 2c\sqrt{|t-s|}, \qquad H_0 = 0, \quad H_1 = 1.$$

Hence $H(y,t) = H_t(y)$ belongs to $\mathrm{C}(Y \times [0,1])$. Joint continuous functional calculus with the central algebra $\mathrm{C}(Y)$ defines $u = H(a)$. In the fiber at y, this is the continuous distribution function of a_y evaluated at a_y; the probability integral transform gives

$$E^\omega(f(u)) = \left(\int_0^1 f(t)\,dt\right) 1 \quad (f \in \mathrm{C}([0,1])).$$

Thus $u \in (\mathcal{M}^\omega \cap \mathcal{M}')^{\beta^\omega}$ has uniform distribution in every fiber.

Mixed trace factorization. Strict separability gives countably many unitaries $v_j \in \mathcal{M}$ whose images generate every fiber as a von Neumann algebra: use exponentials of a countable uniformly 2-norm dense set of selfadjoint contractions. Let G be the countable group of automorphisms generated by $\mathrm{Ad}(v_j)$ and β_g, $g \in \Gamma$. Each automorphism fixes $\mathrm{C}(K)$ and preserves E. Choose a symmetric full-support probability μ on G with $\mu(1) \ge 1/2$, and set $T = \sum_{\theta \in G} \mu(\theta)\theta$. On each fiber L^2 space, T is a selfadjoint contraction. It is positive because the identity has weight at least $1/2$; full support of μ identifies its fixed vectors as

$$L^2(Z(M_\lambda))^{\beta_\lambda} = \mathbb{C}1.$$

For $x \in \mathcal{M}$, the continuous functions

$$\lambda \longmapsto E(|T^k x - E(x)|^2)(\lambda)$$

decrease pointwise to zero. Dini's theorem gives $\|T^k x - E(x)\|_{2,u} \to 0$. If $q \in (\mathcal{M}^\omega \cap \mathcal{M}')^{\beta^\omega}$, traciality and invariance imply $E^\omega(q\theta(x)) = E^\omega(qx)$ for every $\theta \in G$. Consequently

$$E^\omega(qx) = E^\omega(qT^k x) \longrightarrow E^\omega(q)E(x),$$

so $E^\omega(qx) = E^\omega(q)E(x)$.

Finally, take the spectral projections of u for the n equal subintervals of $[0,1]$. As above, uniform 2-norm limits of continuous ramps supply these projections inside $\mathcal{M}^\omega$. They are invariant and commute with $\mathcal{M}$, are mutually orthogonal, sum to 1, and have expectation $1/n$. The mixed trace factorization therefore makes them an invariant n-slicing. □

Corollary 7.47 (Tensoring with a McDuff tracial algebra)**.** *Let A be unital and separable with $\mathrm{T}(A) \neq \emptyset$, and let a countable discrete amenable group Γ act on A. Suppose that α is W^*-centrally free and $S = \mathrm{T}^\alpha(A)$ is Bauer. Let B be unital and separable with a unique tracial state ν, and suppose that $N = \pi_\nu(B)''$ is McDuff. Then*

$$\alpha \otimes \mathrm{id}_B \colon \Gamma \longrightarrow \mathrm{Aut}(A \otimes_{\min} B)$$

has WRP.

Proof. Every tracial state on $D = A \otimes_{\min} B$ is a product $\rho \otimes \nu$. Indeed, for $a \in A_+$ the positive trace $b \mapsto \sigma(a \otimes b)$ is a scalar multiple of ν; linearity gives the product formula. Hence the invariant trace space of $\gamma = \alpha \otimes \mathrm{id}_B$ is canonically S, and its GNS algebras are

$$\pi_{\rho\otimes\nu}(D)'' = M_\rho \mathbin{\bar{\otimes}} N.$$

Uniqueness of ν makes N a factor.

The action γ is W^*-centrally free. To see this, note that $Z(M_\rho \bar{\otimes} N) = Z(M_\rho) \otimes 1$. A centrally trivial restriction of $(\alpha_\rho)_g \otimes \mathrm{id}_N$ to a nonzero invariant central corner would act trivially on the embedded centralizing sequences from the corresponding corner of M_ρ. These sequences remain centralizing after tensoring with 1_N, as is seen from the product trace and density of elementary tensors. This would make the restriction of $(\alpha_\rho)_g$ centrally trivial, contradicting the hypothesis.

Construct the bundle $(\mathcal{N}, \partial_e S)$ for (D, γ) by Theorem 7.37. For each n, McDuffness of N gives a unital copy of M_n in $N^\omega \cap N'$. Choose bounded matrix-unit representatives and approximate them coordinatewise by elements $\pi_\nu(b_{ij,k})$, $b_{ij,k} \in B$, using Kaplansky density and bounded lifting through the GNS homomorphism. All matrix-unit and centrality errors tend to zero in $\|\cdot\|_{2,\nu}$. For $c \in B$,

$$\|1 \otimes c\|_{2,S_D} = \|c\|_{2,\nu}, \qquad S_D = \{\rho \otimes \nu : \rho \in S\}.$$

Moreover, for $a \in A$ and $b \in B$,

$$\|[1 \otimes b_{ij,k}, a \otimes b]\|_{2,S_D} \leq \|a\| \|[b_{ij,k}, b]\|_{2,\nu}.$$

Thus their images give a unital homomorphism

$$\mathrm{M}_n \longrightarrow (\mathcal{N}^\omega \cap \mathcal{N}')^{\gamma^\omega}.$$

Commutation with the whole bundle follows from its bounded uniform 2-norm density of the image of D. The image is invariant because γ fixes the second tensor factor.

These matrix algebras supply invariant slicings. The centrally free fiber actions have exact W^*-Rokhlin towers by Theorem 7.15. Apply Theorem 7.45 and then Lemma 7.40 to conclude. □

Theorem 7.48. *Let A be a unital separable nuclear C^*-algebra with $\mathrm{T}(A) \neq \emptyset$, and let $\alpha \colon \Gamma \to \mathrm{Aut}(A)$ be an action of a countable discrete amenable group. Suppose that α is W^*-centrally free and that $S = \mathrm{T}^\alpha(A)$ is a Bauer simplex with $\dim(\partial_e S) < \infty$. If Γ is infinite, then α has WRP. The same conclusion holds for finite Γ when A has no finite-dimensional quotients.*

Proof. W*-central freeness implies W*-proper outerness. Theorem 7.37 therefore gives a strictly separable bundle over $K = \partial_e S$, with a fiberwise action β and fibers

$$M_\tau = \pi_\tau(A)'' \quad (\tau \in K).$$

Write $\bar{\tau}$ for the faithful normal trace on M_τ. Extremality in the invariant trace simplex implies

$$Z(M_\tau)^{\alpha_\tau} = \mathbb{C}1. \tag{7.17}$$

Indeed, a nontrivial invariant central projection would split τ into two invariant tracial states. These are distinct: Kaplansky density distinguishes their normal extensions on the two central corners, contradicting extremality. The central supports of the type decomposition are invariant. Thus M_τ is either wholly of type II_1 or homogeneous of type I_r for one finite r.

In the type II_1 case, nuclearity gives hyperfiniteness. For every N, [31, Corollary 5.5] gives a unital homomorphism

$$\mathrm{M}_N \longrightarrow ((M_\tau)^\omega \cap M_\tau')^{\alpha_\tau^\omega}.$$

The cited result applies also to nonfactorial hyperfinite type II_1 algebras. The diagonal contraction with N equally weighted, equally spaced eigenvalues has moments converging to those of Lebesgue measure. Letting $N \to \infty$ and choosing coordinates diagonally for increasing finite centrality, invariance, and moment conditions gives a positive contraction in the displayed invariant central sequence algebra with exact Lebesgue distribution.

In the type I_r case, write

$$M_\tau = L^\infty(Y_\tau, \nu_\tau) \mathbin{\bar{\otimes}} \mathrm{M}_r.$$

Equation (7.17) says that the induced probability-measure-preserving action on Y_τ is ergodic. It is also essentially free: if a nonidentity g had a positive-measure fixed-point set, its characteristic projection would give an α_g-invariant central corner on which α_g fixes the center. An automorphism of a homogeneous finite type I algebra fixing its center is inner, hence centrally trivial, contrary to W*-central freeness.

Suppose first that Γ is infinite. Essential freeness implies that ν_τ is nonatomic, since an atom would have infinitely many disjoint translates of equal positive measure. By the hyperfiniteness theorem for amenable equivalence relations [14, Theorem 10], the orbit relation on Y_τ is, modulo null sets, the increasing union of finite probability-measure-preserving subrelations $\mathcal{R}_k$. Each finite Borel subrelation is smooth, so its quotient is a standard Borel space. The pushforward of ν_τ is nonatomic: the preimage of a quotient point is a finite class and has measure zero. For any N, split this quotient probability space into N sets of measure $1/N$ and pull them back to central projections $e_{1,k}, \dots, e_{N,k} \in M_\tau$. They are an exact partition of 1 and satisfy

$$\bar{\tau}(e_{i,k}) = \frac{1}{N}, \qquad \|\alpha_g(e_{i,k}) - e_{i,k}\|_{2,\bar{\tau}}^2 \le \nu_\tau\{y : (g^{-1}y, y) \notin \mathcal{R}_k\} \longrightarrow 0.$$

Their ultrapower classes are invariant central projections. The corresponding equally spaced diagonal contractions, with $N \to \infty$ and a diagonal choice of k, give a positive contraction with Lebesgue distribution in the invariant central sequence algebra of M_τ. If instead Γ is finite,

its ergodic action on Y_τ is a single finite orbit. Then M_τ is finite-dimensional, and the GNS homomorphism maps A onto M_τ. This contradicts the assumed absence of finite-dimensional quotients, excluding the type I case for a finite group.

Thus every fiber satisfies the hypotheses of Lemma 7.46, which gives invariant n-slicings for all n. The centrally free fiber actions have exact W*-Rokhlin towers by Theorem 7.15. Theorem 7.45 gives bundle towers, and Lemma 7.40 lifts them to WRP for α. □

Remark 7.49 (The finite-group restriction). The quotient hypothesis cannot be dropped for arbitrary finite groups. Let $\mathbb{Z}/2\mathbb{Z}$ act antipodally on the circle, and let $A = \mathrm{C}(S^1)$. The action is free, hence W*-centrally free, and its invariant-measure simplex is Bauer with extreme boundary $S^1/(\mathbb{Z}/2\mathbb{Z})$ of dimension one. Nevertheless,

$$\|f\|_{2,S} \geq \frac{1}{\sqrt{2}}\|f\| \quad (f \in \mathrm{C}(S^1)),$$

so $J_S = 0$ in the norm ultrapower. WRP for the single full-group tiling would give orthogonal positive contractions $p, \alpha(p)$ with $p + \alpha(p) = 1$; hence p is a projection. The norm ultrapower of $\mathrm{C}(S^1)$ has only the projections 0 and 1: uniform approximate projections on the connected circle approach scalar projections, and their scalar ultrapower is $\mathbb{C}$. Neither projection can satisfy the displayed partition.

Corollary 7.50 (Commutative coefficients)**.** *Let a countably infinite discrete amenable group act freely on a nonempty compact metrizable space X. If its invariant-measure simplex is Bauer with finite-dimensional extreme boundary, then the action has topological URP, and the induced action on* $\mathrm{C}(X)$ *has WRP.*

Proof. $\mathrm{C}(X)$ is separable and nuclear, and its invariant traces are exactly the invariant probability measures. Freeness implies that every invariant-measure GNS action on $L^\infty(X, \mu)$ is centrally free: in an abelian von Neumann algebra a centrally trivial automorphism is the identity, and a nonzero invariant central corner on which a nonidentity g acts identically would be a positive-measure fixed-point set. Apply Theorem 7.48 and Proposition 3.7. □

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