REMARKS ON VILLADSEN ALGEBRAS

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ABSTRACT. It is shown that certain unital simple C*-algebras constructed by Villadsen in [9] are classified by the K_0-group together with radius of comparison.

1. Introduction

Villadsen algebras (of the first type) were constructed in [9] as simple unital AH C*-algebras which have perforation in their ordered K_0-group. This class of C*-algebras lies outside the scope of the current classification theorem, as Villadsen algebras do not absorb the Jiang-Su algebra $\mathcal{Z}$ tensorially. Indeed, a Villadsen type algebra has been constructed in [8] which has the same value of the Elliott invariant of an AI algebra, but itself is not isomorphic to that AI algebra.

Each Villadsen algebra is an inductive limit of homogeneous C*-algebras with connecting maps induced by coordinate projections together with a small portion of point evaluations (see Section 2). In this note, we first show that, with fixed connected seeds space and fixed numbers of coordinate projections, the Villadsen algebra is classifiable by its K_0-group together with radius of comparison (Theorem 6.1). Then, if the fixed seeds space is further assumed to be contractible, but the numbers of coordinated projections are allowed to vary, it is still shown to be classified by its K_0-group together with radius of comparison (Theorem 7.1 and Corollary 7.7). Note that this class of Villadsen algebras contains the example constructed in [8].

We hope this note could shed some light on the possible classification of more general non-$\mathcal{Z}$-stable C*-algebras.

2. The Villadsen algebra $A(X, (n_i), (k_i), E)$

Let X be a compact connected metric space, let (c_i) and (k_i) be two sequences of natural numbers, and let

$$\begin{cases} E_1 := \{x_{1,1}, ..., x_{1,k_1}\} \subseteq X, \\ E_2 := \{x_{2,1}, ..., x_{2,k_2}\} \subseteq X^{c_1}, \\ ... \\ E_i := \{x_{i,1}, ..., x_{i,k_i}\} \subseteq X^{c_{i-1}}, \\ ... \end{cases}$$

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be a sequence of finite subsets such that for each $i = 1, 2, ...,$ the set

$$\bigcup_{j=1}^{\infty} \bigcup_{s=1}^{c_{i+1}\cdots c_{i+j-1}} \pi_s(E_{i+j})$$

is dense in X^{c_i}, where π_s are the coordinate projections.

Construct the (generalized) Villadsen algebra

$$\mathrm{M}_{n_0}(\mathrm{C}(X)) \longrightarrow \mathrm{M}_{n_0(n_1+k_1)}(\mathrm{C}(X^{c_1})) \longrightarrow \mathrm{M}_{n_0(n_1+k_1)(n_2+k_2)}(\mathrm{C}(X^{c_1c_2})) \longrightarrow \cdots \longrightarrow A, \tag{2.1}$$

where the seed for the ith stage map,

$$\phi_i : \mathrm{C}(X^{c_1\cdots c_{i-1}}) \to \mathrm{M}_{n_i+k_i}(\mathrm{C}(X^{c_1\cdots c_{i-1}c_i})),$$

is defined by

$$\begin{aligned} f \quad \mapsto \quad & \mathrm{diag}\{\underbrace{\underbrace{f\circ\pi_1, ..., f\circ\pi_1}_{s_{i,1}}, ..., \underbrace{f\circ\pi_{c_i}, ..., f\circ\pi_{c_i}}_{s_{i,c_i}}}_{n_i}, \underbrace{f(x_{i,1}), ..., f(x_{i,k_i})}_{k_i}\} \\ & = \mathrm{diag}\{\underbrace{\underbrace{f\circ\pi_1, ..., f\circ\pi_1}_{s_{i,1}}, ..., \underbrace{f\circ\pi_{c_i}, ..., f\circ\pi_{c_i}}_{s_{i,c_i}}}_{n_i}, f(E_i)\}, \end{aligned}$$

where $s_{i,1}, ..., s_{i,d_i} \geq 1$ are natural numbers, and $n_i = \sum_{j=1}^{c_i} s_{i,j}$.

A direct calculation shows that the composed map

$$\phi_{i,i+j} : \mathrm{C}(X^{c_1\cdots c_{i-1}}) \to \mathrm{M}_{(n_i+k_i)\cdots(n_{i+j-1}+k_{i+j-1})}(\mathrm{C}(X^{c_1\cdots c_{i-1}\cdots c_{i+j-1}}))$$

is equal (up to a permutation) to

$$f \mapsto \mathrm{diag}\{\underbrace{f\circ\pi_1, ..., f\circ\pi_{c_i\cdots c_{i+j-1}}}_{n_i\cdots n_{i+j-1}}, \quad \underbrace{f(x_{i,1}), ..., f(x_{i,k_i})}_{k_i[(n_{i+1}+k_{i+1})\cdots(n_{i+j-1}+k_{i+j-1})]}, \quad \underbrace{f(\cdot), ..., f(\cdot)}_{\cdots}, ..., \},$$

i.e.,

$$\begin{aligned} f \mapsto \mathrm{diag}\{ & \underbrace{f\circ\pi_1, ..., f\circ\pi_{c_i\cdots c_{i+j-1}}}_{n_i\cdots n_{i+j-1}}, f(E_i)1_{(n_{i+1}+k_{i+1})\cdots(n_{i+j-1}+k_{i+j-1})}, \\ & (f(\pi_1(E_{i+1})), ..., f(\pi_{n_i}(E_{i+1})))1_{(n_{i+2}+k_{i+2})\cdots(n_{i+j-1}+k_{i+j-1})}, ..., \}. \end{aligned}$$

So, it can be described as

$$\mathrm{diag}(\underbrace{f\circ\pi_1, ..., f\circ\pi_{c_i\cdots c_{i+j-1}}}_{n_i\cdots n_{i+j-1}}, \text{point evaluations}).$$

We shall choose c_i, $s_{i,1}, ..., s_{i,c_i}$ (hence n_i) and k_i such that

$$\lim_{j\to\infty} \frac{n_i\cdots n_{i+j}}{(n_i+k_i)\cdots(n_{i+j}+k_{i+j})} = \lim_{j\to\infty} (\frac{n_i}{n_i+k_i})\cdots(\frac{n_{i+j}}{n_{i+j}+k_{i+j}}) \neq 0.$$

Hence

$$\lim_{i\to\infty}\lim_{j\to\infty} \frac{n_i\cdots n_{i+j}}{(n_i+k_i)\cdots(n_{i+j}+k_{i+j})} = \lim_{i\to\infty}\lim_{j\to\infty} (\frac{n_i}{n_i+k_i})\cdots(\frac{n_{i+j}}{n_{i+j}+k_{i+j}}) = 1. \tag{2.2}$$

Denote the limit algebra by

$$A(X, (n_i), (k_i), E).$$

In what follows, we shall show that this algebra is independent of the choice of the point evaluation set E.

Remark 2.1. If $c_i = 1$, $i = 1, 2, ...$, then $A(X, (n_i), (k_i), E)$ is the C*-algebra constructed by Goodearl in [4] which does not have real rank zero. On the other hand, if $s_{i,j} = 1$, $i = 1, 2, ...$, $j = 1, ..., c_i$, then $A(X, (n_i), (k_i), E)$ is the C*-algebra constructed by Villadsen in [9].

3. Mean dimension and radius of comparison

Let us first calculate the mean dimension of $A(X, (n_i), (k_i), E)$ (as formulated in [6]).

Theorem 3.1. *With* $A = A(X, (n_i), (k_i), E)$*, one has*

$$\mathrm{mdim}(A) \leq \frac{\dim(X)}{n_0} \cdot \lim_{i\to\infty} \frac{c_1 \cdots c_i}{(n_1 + k_1) \cdots (n_i + k_i)}, \tag{3.1}$$

where $\infty \cdot 0 = 0$.

Moreover, if X *is a CW complex, then equality holds in* (3.1)*, and the radius of comparison of* A*,* $\mathrm{rc}(A)$*, is equal to* $\frac{1}{2}\mathrm{mdim}(A)$*.*

Proof. Let us first prove (3.1). Consider the ith stage $\mathrm{M}_{m_i}(\mathrm{C}(X^{d_i}))$, where $m_i = n_0(n_1 + k_1) \cdots (n_i + k_i)$ and $d_i = c_1 \cdots c_i$, and let α be an open cover of X^{d_i}. Since the pull back of α by the constant map has degree zero, we have

$$\mathcal{D}(\phi_{i,j}(\alpha)) \leq c_i \cdots c_j \cdot \mathcal{D}(\alpha),$$

and then

$$\begin{aligned}
\lim_{j\to\infty} \frac{\mathcal{D}(\phi_{i,j}(\alpha))}{m_j} &\leq \frac{c_i \cdots c_j \cdot \mathcal{D}(\alpha)}{n_0(n_1 + k_1) \cdots (n_j + k_j)} \\
&= \frac{\mathcal{D}(\alpha)}{n_0} \cdot \lim_{j\to\infty} \frac{c_1 \cdots c_j}{(n_1 + k_1) \cdots (n_j + k_j)} \\
&\leq \frac{\dim(X)}{n_0} \cdot \lim_{j\to\infty} \frac{c_1 \cdots c_j}{(n_1 + k_1) \cdots (n_j + k_j)},
\end{aligned}$$

where $\infty \cdot 0 = 0$. With $i \to \infty$, this shows (3.1). In particular, we have

$$\mathrm{rc}(A) \leq \frac{1}{2}\mathrm{mdim}(A) \leq \frac{1}{2} \cdot \frac{\dim(X)}{n_0} \cdot \lim_{i\to\infty} \frac{c_1 \cdots c_i}{(n_1 + k_1) \cdots (n_i + k_i)}. \tag{3.2}$$

Now, assume that X is an CW-complex, and let us show that

$$\mathrm{rc}(A) \geq \frac{1}{2}\dim(X) \cdot \lim_{i\to\infty} \frac{c_1 \cdots c_i}{(n_1 + k_1) \cdots (n_i + k_i)}. \tag{3.3}$$

Together with (3.2), we have

$$\mathrm{rc}(A) = \frac{1}{2}\mathrm{mdim}(A) = \frac{1}{2} \cdot \frac{\dim(X)}{n_0} \cdot \lim_{i\to\infty} \frac{c_1 \cdots c_i}{(n_1 + k_1) \cdots (n_i + k_i)}.$$

Denote by

$$\gamma = \lim_{i\to\infty} \frac{c_1\cdots c_i}{(n_1+k_1)\cdots(n_i+k_i)}.$$

Since (3.3) holds trivially if $\gamma = 0$, let us assume that $\gamma \neq 0$ in the rest of the proof.

Assume that $\dim(X) < \infty$. Let $\varepsilon > 0$ be arbitrary for the time being. Choose i sufficiently large that

$$\frac{c_1\cdots c_i\cdot \dim(\mathrm{X})-2}{2n_0(n_1+k_1)\cdots(n_i+k_i)} > \frac{\gamma}{2}\cdot\frac{\dim(X)}{n_0}-\varepsilon$$

and

$$\frac{\dim(X)}{2n_0}\left(\frac{c_1\cdots c_i}{(n_1+k_1)\cdots(n_i+k_i)} - \frac{c_1\cdots c_j}{(n_1+k_1)\cdots(n_j+k_j)}\right) < \varepsilon, \quad j>i.$$

Note that $X^{c_1\cdots c_i}$ is an CW-complex with dimension $c_1\cdots c_i\dim(X)$. It contains a d-dimensional sphere S, where

$$c_1\cdots c_i\dim(X)-2 \leq d \leq c_1\cdots c_i\dim(X)-1$$

and d is even.

Pick a (complex) vector bundle E over S such that $\mathrm{rank}(E) = d/2$ and $e := \mathrm{c}_d(E) \in \mathrm{H}^d(S)$ is non-zero, where c_d is the dth Chern class. (One can pick a vector bundle with K-class $(d/2, 1) \in \mathbb{Z}\oplus\mathbb{Z} \cong \mathrm{K}_0(S)$. Since, together with trivial bundles, it generates $\mathrm{K}_0(S)$, its total Chern class must be non-zero at $\mathrm{H}^d(S)$.) Note that the total Chern class of E is $1+e$. Denote by p the projection in $\mathrm{M}_\infty(\mathrm{C}(S))$, and extend p to a positive element of $\mathrm{M}_\infty(\mathrm{C}(X^{d_i}))$ such that $\mathrm{rank}(p(x)) \geq d/2$, $x\in X^{d_i}$. Still denote this element by p.

Note that

$$\mathrm{d}_\tau(p) \geq \frac{d}{2n_0(n_1+k_1)\cdots(n_i+k_i)} \geq \frac{c_1\cdots c_i\cdot\dim(\mathrm{X})-2}{2n_0(n_1+k_1)\cdots(n_i+k_i)} > \frac{\gamma}{2}\cdot\frac{\dim(X)}{n_0}-\varepsilon.$$

Consider the element $\phi_{i,\infty}(p) \in A$. For each $j > i$, the restriction of $\phi_{i,j}(p) \in \mathrm{M}_{m_j}(\mathrm{C}(X^{d_j}))$ to $S\times\cdots\times S \subseteq X^{d_j}$ is a projection which corresponds to the vector bundle

$$E_j := (\bigoplus_{s_1}\pi_1^*(E)) \oplus\cdots\oplus (\bigoplus_{s_{c_i\cdots c_j}} \pi^*_{c_i\cdots c_j}(E)) \oplus \theta_j,$$

where θ_j is a trivial bundle. Then the total Chern class of E_j is

$$\begin{aligned} & \pi_1^*(1+c_d)^{s_1}\pi_2^*(1+c_d)^{s_2}\cdots\pi^*_{c_{i+1}\cdots c_j}(1+c_d)^{s_{c_{i+1}\cdots c_j}} \\ = \ & \pi_1^*(1+s_1e)\pi_2^*(1+s_2e)\cdots\pi^*_{c_{i+1}\cdots c_j}(1+s_{c_{i+1}\cdots c_j}e), \end{aligned}$$

and, by the Künneth Theorem, it is non-zero at degree $dc_{i+1}\cdots c_j$. Hence any trivial sub-bundle of E_j has rank at most (see Remark 3.2)

$$\begin{aligned}
& \mathrm{rank}(E_j) - \frac{1}{2}dc_{i+1}\cdots c_j \\
= \ & \mathrm{rank}(E)(n_{i+1}+k_{i+1})\cdots(n_j+k_j) - \frac{1}{2}dc_{i+1}\cdots c_j \\
= \ & \frac{d}{2}((n_{i+1}+k_{i+1})\cdots(n_j+k_j) - c_{i+1}\cdots c_j) \\
\leq \ & \frac{\dim(X)}{2}(c_1\cdots c_i(n_{i+1}+k_{i+1})\cdots(n_j+k_j) - c_1\cdots c_j) \\
= \ & \frac{\dim(X)}{2n_0}(\frac{c_1\cdots c_i}{(n_1+k_1)\cdots(n_i+k_i)} - \frac{c_1\cdots c_j}{n_0(n_1+k_1)\cdots(n_j+k_j)})(n_1+k_1)\cdots(n_j+k_j) \\
\leq \ & \varepsilon n_0(n_1+k_1)\cdots(n_j+k_j).
\end{aligned}$$

Let $r \in A$ be a trivial projection with $2\varepsilon < \mathrm{d}_\tau(r) < 3\varepsilon$. Then

$$\mathrm{d}_\tau(r) + (\frac{\gamma}{2}\cdot\frac{\dim(X)}{n_0} - 4\varepsilon) < \mathrm{d}_\tau(p), \quad \tau \in \mathrm{T}(A).$$

But the rank of the vector bundle of r at the stage j is at least

$$2\varepsilon n_0(n_1+k_1)\cdots(n_j+k_j) > \varepsilon n_0(n_1+k_1)\cdots(n_j+k_j),$$

which implies that r is not Cuntz subequivalent to p, and therefore

$$\mathrm{rc}(A) \geq \frac{\gamma}{2}\cdot\frac{\dim(X)}{n_0} - 4\varepsilon.$$

Since ε is arbitrary, this implies $\mathrm{rc}(A) \geq \frac{\gamma}{2}\dim(X)$.

If X is an infinite dimensional CW-complex (and $\lim_{i\to\infty}\frac{c_1\cdots c_i}{(n_1+k_1)\cdots(n_i+k_i)} \neq 0$), then the same argument as above (choose d arbitrarily large) shows that $\mathrm{rc}(A)$ is arbitrarily large, and hence $\mathrm{rc}(A) = \infty$. So, (3.3) always holds, as desired. □

Remark 3.2. Assume a (complex) vector bundle E over a compact metric space X has non-zero Chern class $\mathrm{c}_n(E) \in \mathrm{H}^n(X)$. Then the trivial sub-bundles of E have rank at most $\mathrm{rank}(E) - n/2$, as, if there is a trivial sub-bundle F of rank $r > \mathrm{rank}(E) - n/2$, then

$$\mathrm{c}(E) = \mathrm{c}(E' \oplus F) = \mathrm{c}(E')\mathrm{c}(F) = \mathrm{c}(E'),$$

but, since $\mathrm{rank}(E') = \mathrm{rank}(E) - \mathrm{rank}(F) < n/2$, we have $\mathrm{c}_n(E') = 0$, and hence $\mathrm{c}_n(E) = \mathrm{c}_n(E') = 0$, which contradicts the assumption.

4. The trace simplex

Let us choose two different evaluation sets

$$E_1, E_2, ..., E_i, ... \quad \text{and} \quad F_1, F_2, ..., F_i, ...$$

with sizes (k_i^E) and (k_i^F) respectively, and both satisfying Condition (2.2) (with respect to the same (n_i)), and assume that

$$\prod_{i=1}^{\infty}(n_i + k_i^{(E)}) = \prod_{i=1}^{\infty}(n_i + k_i^{(F)}) \tag{4.1}$$

as supernatural numbers, and

$$\lim_{i\to\infty} \frac{(n_1 + k_1^{(E)})\cdots(n_{i_1'-1} + k_i^{(E)})}{(n_1 + k_1^{(F)})\cdots(n_{i_1'-1} + k_i^{(F)})} = 1. \tag{4.2}$$

Definition 4.1. Let us also assume that $(k_i^{(E)})$ and $(k_i^{(F)})$ are sufficiently close in the sense that for any $\delta > 0$, there are arbitrarily large $i_1 > i_1'$ such that

$$1 - \prod_{j=0}^{\infty} \frac{n_{i_1'+j}}{n_{i_1'+j} + k_{i_1'+j}^{(E)}} < \delta,$$

$$\Pi_{i=1}^{i_1-1}(n_i + k_i^{(F)}) \text{ is divisible by } \Pi_{i=1}^{i_1'-1}(n_i + k_i^{(E)}),$$

and

$$\frac{(n_1 + k_1^{(F)})\cdots(n_{i_1'-1} + k_{i_1'-1}^{(F)})}{(n_1 + k_1^{(E)})\cdots(n_{i_1'-1} + k_{i_1'-1}^{(E)})} \cdot \frac{(n_{i_1'} + k_{i_1'}^{(F)})\cdots(n_{i_1-1} + k_{i_1-1}^{(F)})}{n_{i_1'}\cdots n_{i_1-1}} > 1, \tag{4.3}$$

and, furthermore, there are arbitrarily large $i_2 > i_2'$ such that

$$1 - \prod_{j=0}^{\infty} \frac{n_{i_2'+j}}{n_{i_2'+j} + k_{i_1'+j}^{(F)}} < \delta,$$

$$\Pi_{i=1}^{i_2-1}(n_i + k_i^{(E)}) \text{ is divided by } \Pi_{i=1}^{i_2'-1}(n_i + k_i^{(F)}),$$

and

$$\frac{(n_1 + k_1^{(E)})\cdots(n_{i_1'-1} + k_{i_2'-1}^{(E)})}{(n_1 + k_1^{(F)})\cdots(n_{i_2'-1} + k_{i_2'-1}^{(F)})} \cdot \frac{(n_{i_2'} + k_{i_2'}^{(E)})\cdots(n_{i_2-1} + k_{i_2-1}^{(E)})}{n_{i_2'}\cdots n_{i_2-1}} > 1. \tag{4.4}$$

Lemma 4.2. *If*

$$\lim_{i\to\infty} \frac{(n_1 + k_1^{(E)})\cdots(n_{i_1'-1} + k_i^{(E)})}{(n_1 + k_1^{(F)})\cdots(n_{i_1'-1} + k_i^{(F)})} = 1, \tag{4.5}$$

then (4.3) *and* (4.4) *automatically hold.*

Proof. Indeed, for the given $\delta > 0$, choose i_1' sufficiently large that

$$1 - \prod_{j=0}^{\infty} \frac{n_{i_1'+j}}{n_{i_1'+j} + k_{i_1'+j}^{(E)}} < \delta.$$

Then, with sufficiently large i_1, by (4.5), we have

$$\begin{aligned}
&\frac{(n_1+k_1^{(F)})\cdots(n_{i_1'-1}+k_{i_1'-1}^{(F)})}{(n_1+k_1^{(E)})\cdots(n_{i_1'-1}+k_{i_1'-1}^{(E)})}\cdot\frac{(n_{i_1'}+k_{i_1'}^{(F)})\cdots(n_{i_1-1}+k_{i_1-1}^{(F)})}{n_{i_1'}\cdots n_{i_1-1}}\\
=\ &\frac{(n_1+k_1^{(F)})\cdots(n_{i_1-1}+k_{i_1-1}^{(F)})}{(n_1+k_1^{(E)})\cdots(n_{i_1-1}+k_{i_1-1}^{(E)})}\cdot\frac{(n_{i_1'}+k_{i_1'}^{(E)})\cdots(n_{i_1-1}+k_{i_1-1}^{(E)})}{n_{i_1'}\cdots n_{i_1-1}}\\
>\ &\frac{(n_1+k_1^{(F)})\cdots(n_{i_1-1}+k_{i_1-1}^{(F)})}{(n_1+k_1^{(E)})\cdots(n_{i_1-1}+k_{i_1-1}^{(E)})}\cdot\frac{n_{i_1'}+k_{i_1'}^{(E)}}{n_{i_1'}}>1.
\end{aligned}$$

So (4.3) holds. The same argument also shows that (4.4) holds. □

Let

$$\mathcal{H}_1^{(E)},\mathcal{H}_1^{(F)}\subseteq \mathrm{M}_{n_0}(\mathrm{C}(X)),\quad \mathcal{H}_2^{(E)},\subseteq \mathrm{M}_{n_1+k_1^{(E)}}\mathrm{C}(X^{c_1}),\quad \mathcal{H}_2^{(F)}\subseteq \mathrm{M}_{n_1+k_1^{(F)}}\mathrm{C}(X^{c_1}),\quad ...$$

be finite subsets such that

$$\overline{\bigcup_{i=1}^{\infty}\mathcal{H}_i^{(E)}}=A_E\quad\text{and}\quad\overline{\bigcup_{i=1}^{\infty}\mathcal{H}_i^{(F)}}=A_F,$$

where A_E and A_F denote the C*-algebras $A(X,(n_i),(k_i),E)$ and $A(X,(n_i),(k_i),F)$ respectively. Let $\delta_1,\delta_2,...$ be a decreasing sequence of strictly positive numbers with

$$\sum_{n=1}^{\infty}\delta_n<1.$$

Consider the inductive sequences

$$\mathrm{M}_{n_0}(\mathrm{C}(X))\xrightarrow{\phi_1^{(E)}}\mathrm{M}_{n_0(n_1+k_1^{(E)})}(\mathrm{C}(X^{c_1}))\xrightarrow{\phi_2^{(E)}}\mathrm{M}_{n_0(n_1+k_1^{(E)})(n_2+k_2^{(E)})}(\mathrm{C}(X^{c_1c_2}))\longrightarrow\cdots\longrightarrow A_E,$$

$$\mathrm{M}_{n_0}(\mathrm{C}(X))\xrightarrow{\phi_1^{(F)}}\mathrm{M}_{n_0(n_1+k_1^{(F)})}(\mathrm{C}(X^{c_1}))\xrightarrow{\phi_2^{(F)}}\mathrm{M}_{n_0(n_1+k_1^{(F)})(n_2+k_2^{(F)})}(\mathrm{C}(X^{c_1c_2}))\longrightarrow\cdots\longrightarrow A_F.$$

Consider the subset $\mathcal{H}_1^{(E)}$ of $\mathrm{M}_{n_0}(\mathrm{C}(X))$. Since the sequences $(k_i^{(E)})$ and $(k_i^{(F)})$ are sufficiently close, there are $i_1'<i_1$ such that

$$1-\prod_{j=0}^{\infty}\frac{n_{i_1'+j}}{n_{i_1'+j}+k_{i_1'+j}^{(E)}}<\delta_1, \tag{4.6}$$

$$\Pi_{i=1}^{i_1-1}(n_i+k_i^{(F)})\text{ is divisible by }\Pi_{i=1}^{i_1'-1}(n_i+k_i^{(E)}), \tag{4.7}$$

and

$$\frac{(n_1+k_1^{(F)})\cdots(n_{i_1'-1}+k_{i_1'-1}^{(F)})}{(n_1+k_1^{(E)})\cdots(n_{i_1'-1}+k_{i_1'-1}^{(E)})}\cdot\frac{(n_{i_1'}+k_{i_1'}^{(F)})\cdots(n_{i_1-1}+k_{i_1-1}^{(F)})}{n_{i_1'}\cdots n_{i_1-1}}>1. \tag{4.8}$$

Then consider the diagram

$$\begin{tikzcd}
\mathrm{M}_{n_0}(\mathrm{C}(X)) \arrow[r, "\phi^{(E)}_{1,i'_1}"] & \mathrm{M}_{m^{(E)}_{i'_1}}(\mathrm{C}(X^{d_{i'_1}})) \arrow[r, "\phi^{(E)}_{i'_1,i_1}"] \arrow[dr, "\tilde{\phi}^{(E,F)}_{i'_1,i_1}"] & \mathrm{M}_{m^{(E)}_{i_1}}(\mathrm{C}(X^{d_{i_1}})) \arrow[r] & \cdots \arrow[r] & A_E \\
\mathrm{M}_{n_0}(\mathrm{C}(X)) \arrow[r, "\phi^{(F)}_{1,i'_1}"] & \mathrm{M}_{m^{(F)}_{i'_1}}(\mathrm{C}(X^{d_{i'_1}})) \arrow[r, "\phi^{(F)}_{i'_1,i_1}"] & \mathrm{M}_{m^{(F)}_{i_1}}(\mathrm{C}(X^{d_{i_1}})) \arrow[r] & \cdots \arrow[r] & A_F,
\end{tikzcd}$$

where

(4.9) $$m_i := n_0(n_1+k_1)\cdots(n_{i-1}+k_{i-1}), \quad d_i := c_1\cdots c_{i-1},$$

and $\tilde{\phi}^{(E,F)}_{i'_1,i_1} : \mathrm{M}_{m^{(E)}_{i'_1}}(\mathrm{C}(X^{d_{i'_1}})) \to \mathrm{M}_{m^{(F)}_{i_1}}(\mathrm{C}(X^{d_{i_1}}))$ is a map of the form

$$f \mapsto \mathrm{diag}(\underbrace{f\circ\pi_1, ..., f\circ\pi_{c_{i'_1}\cdots c_{i_1-1}}}_{n_{i'_1}\cdots n_{i_1-1}}, \text{point evaluations}),$$

where the point evaluations are arbitrarily chosen (the map $\tilde{\phi}^{(E,F)}_{i'_1,i_1}$ exists by (4.7) and (4.8)).

Write

$$\phi^{(E,F)}_{1,i_1} = \tilde{\phi}^{(E,F)}_{i'_1,i_1} \circ \phi^{(E)}_{1,i'_1},$$

and compress the diagram above as

$$\begin{tikzcd}
\mathrm{M}_{n_0}(\mathrm{C}(X)) \arrow[r, "\phi^{(E)}_{1,i_1}"] \arrow[dr, "\phi^{(E,F)}_{1,i_1}"] & \mathrm{M}_{m^{(E)}_{i_1}}(\mathrm{C}(X^{d_{i_1}})) \arrow[r, "\phi^{(E)}_{i_1}"] & \mathrm{M}_{m^{(E)}_{i_1+1}}(\mathrm{C}(X^{d_{i_1+1}})) \arrow[r] & \cdots \arrow[r] & A_E \\
\mathrm{M}_{n_0}(\mathrm{C}(X)) \arrow[r, "\phi^{(F)}_{1,i_1}"] & \mathrm{M}_{m^{(F)}_{i_1}}(\mathrm{C}(X^{d_{i_1}})) \arrow[r, "\phi^{(F)}_{i_1}"] & \mathrm{M}_{m^{(F)}_{i_1+1}}(\mathrm{C}(X^{d_{i_1+1}})) \arrow[r] & \cdots \arrow[r] & A_F.
\end{tikzcd}$$

Consider $\mathcal{H}^{(F)}_{i_1}$. There are $i'_2 < i_2$ such that

$$1 - \prod_{j=0}^{\infty} \frac{n_{i'_2+j}}{n_{i'_2+j} + k^{(F)}_{i'_2+j}} < \delta_2,$$

$$\Pi_{i=1}^{i_2-1}(n_i + k_i^{(E)}) \text{ is divided by } \Pi_{i=1}^{i'_2-1}(n_i + k_i^{(F)})$$

and

(4.10) $$\frac{(n_1+k^{(E)}_1)\cdots(n_{i'_2-1}+k^{(E)}_{i'_2-1})}{(n_1+k^{(F)}_1)\cdots(n_{i'_2-1}+k^{(F)}_{i'_2-1})} \cdot \frac{(n_{i'_2}+k^{(E)}_{i'_2})\cdots(n_{i_2-1}+k^{(E)}_{i_2-1})}{n_{i'_2}\cdots n_{i_2-1}} > 1.$$

By the same argument above, there is a unital homomorphism

$$\tilde{\phi}^{(F,E)}_{i'_2,i_2} : \mathrm{M}_{m^{(F)}_{i'_2}}(\mathrm{C}(X^{d_{i'_2}})) \to \mathrm{M}_{m^{(E)}_{i_2}}(\mathrm{C}(X^{d_{i_2}}))$$

of the form

$$f \mapsto \mathrm{diag}(\underbrace{f\circ\pi_1, ..., f\circ\pi_{c_i\cdots c_{i_2-1}}}_{n_{i_2'}\cdots n_{i_2-1}}, \text{point evaluations}).$$

Define

$$\phi^{(F,E)}_{i_1,i_2} = \tilde{\phi}^{(F,E)}_{i_2',i_2}\circ\phi^{(F)}_{i_1,i_2'}$$

and consider the augmented diagram

$$\begin{tikzcd}
\mathrm{M}_{n_0}(\mathrm{C}(X)) \arrow[r, "\phi^{(E)}_{1,i_1}"] \arrow[dr, "\phi^{(E,F)}_{1,i_1}"'] & \mathrm{M}_{m^{(E)}_{i_1}}(\mathrm{C}(X^{d_{i_1}})) \arrow[r, "\phi^{(E)}_{i_1,i_2}"] & \mathrm{M}_{m^{(E)}_{i_2}}(\mathrm{C}(X^{d_{i_2}})) \arrow[r] & \cdots \arrow[r] & A_E \\
\mathrm{M}_{n_0}(\mathrm{C}(X)) \arrow[r, "\phi^{(F)}_{1,i_1}"] & \mathrm{M}_{m^{(F)}_{i_1}}(\mathrm{C}(X^{d_{i_1}})) \arrow[r, "\phi^{(F)}_{i_1,i_2}"] \arrow[ur, "\phi^{(F,E)}_{i_1,i_2}"] & \mathrm{M}_{m^{(F)}_{i_2}}(\mathrm{C}(X^{d_{i_2}})) \arrow[r] & \cdots \arrow[r] & A_F.
\end{tikzcd}$$

Note that, by (4.6),

$$\left|\tau(\phi^{(F,E)}_{i_1,i_2}\circ\phi^{(E,F)}_{1,i_1}(h) - \phi^{(E)}_{i_1,i_2}\circ\phi^{(E)}_{1,i_1}(h))\right| < \delta_1, \quad h\in\mathrm{C}(X),$$

and $\phi^{(F,E)}_{i_1,i_2}\circ\phi^{(E,F)}_{1,i_1}$ has the form of

$$f \mapsto \mathrm{diag}(\underbrace{f\circ\pi_1, ..., f\circ\pi_{c_i\cdots c_{i_2-1}}}_{n_1\cdots n_{i_2-1}}, \text{point evaluations}).$$

Repeating this process, we have $i_1 < i_2 < \cdots$ with

$$1-\prod_{j=0}^{\infty}\frac{n_{i_s+j}}{n_{i_s+j}+k^{(E)}_{i_s+j}} < \delta_s \quad\text{and}\quad 1-\prod_{j=0}^{\infty}\frac{n_{i_s+j}}{n_{i_s+j}+k^{(F)}_{i_s+j}} < \delta_s, \quad s=1,2,..., \tag{4.11}$$

and the infinite intertwining diagram

$$\begin{tikzcd}
\mathrm{M}_{n_0}(\mathrm{C}(X)) \arrow[r, "\phi^{(E)}_{1,i_1}"] \arrow[dr, "\phi^{(E,F)}_{1,i_1}"'] & \mathrm{M}_{m^{(E)}_{i_1}}(\mathrm{C}(X^{d_{i_1}})) \arrow[r, "\phi^{(E)}_{i_1,i_2}"] & \mathrm{M}_{m^{(E)}_{i_2}}(\mathrm{C}(X^{d_{i_2}})) \arrow[r] \arrow[dr, "\phi^{(E,F)}_{i_2,i_3}"] & \cdots \arrow[r] & A_E \\
\mathrm{M}_{n_0}(\mathrm{C}(X)) \arrow[r, "\phi^{(F)}_{1,i_1}"] & \mathrm{M}_{m^{(F)}_{i_1}}(\mathrm{C}(X^{d_{i_1}})) \arrow[r, "\phi^{(F)}_{i_1,i_2}"] \arrow[ur, "\phi^{(F,E)}_{i_1,i_2}"] & \mathrm{M}_{m^{(F)}_{i_2}}(\mathrm{C}(X^{d_{i_2}})) \arrow[r] & \cdots \arrow[r] & A_F.
\end{tikzcd} \tag{4.12}$$

The diagram (7.24) is not commutative. But, by (4.11), we have

$$\left|\tau(\phi^{(F,E)}_{i_{s+1},i_{s+2}}\circ\phi^{(E,F)}_{i_s,i_{s+1}}(h) - \phi^{(E)}_{i_{s+1},i_{s+2}}\circ\phi^{(E)}_{i_s,i_{s+1}}(h))\right| < \delta_s$$

for any $s=0,2,...$, any $h\in\mathrm{M}_{m^{(E)}_{i_s}}(\mathrm{C}(X^{d_{i_s}}))$, and any $\tau\in\mathrm{T}(\mathrm{M}_{m^{(E)}_{i_{s+2}}}(\mathrm{C}(X^{d_{i_{s+2}}})))$; and

$$\left|\tau(\phi^{(E,F)}_{i_{s+1},i_{s+2}}\circ\phi^{(F,E)}_{i_s,i_{s+1}}(h) - \phi^{(F)}_{i_{s+1},i_{s+2}}\circ\phi^{(F)}_{i_s,i_{s+1}}(h))\right| < \delta_s$$

for any $s=1,3,...$, any $h\in\mathrm{M}_{m^{(F)}_{i_s}}(\mathrm{C}(X^{d_{i_s}}))$, and any $\tau\in\mathrm{T}(\mathrm{M}_{m^{(F)}_{i_{s+2}}}(\mathrm{C}(X^{d_{i_{s+2}}})))$. That is, the diagram (7.24) is approximately commutative on the level of traces. Note that this implies $\mathrm{T}(A_E)$

and $\mathrm{T}(A_F)$ are homeomorphic. Moreover, the maps $\phi^{(F,E)}_{i_{s+1},i_{s+2}} \circ \phi^{(E,F)}_{i_s,i_{s+1}}$ and $\phi^{(E)}_{i_{s+1},i_{s+2}} \circ \phi^{(E)}_{i_s,i_{s+1}}$ share the same coordinate projection part, and so are the maps $\phi^{(E,F)}_{i_{s+1},i_{s+2}} \circ \phi^{(F,E)}_{i_s,i_{s+1}}$ and $\phi^{(F)}_{i_{s+1},i_{s+2}} \circ \phi^{(F)}_{i_s,i_{s+1}}$.

Remark 4.3. In the case of a Goodearl algebra (i.e., $c_i = 1$, $i = 1, 2, ...$), the trace simplex is homeomorphic to the Bauer simplex with extreme boundary X, while in the case of the Villadsen algebra (i.e., $s_{i,j} = 1$, $i = 1, 2, ...,$ $j = 1, ..., c_i$), the trace simplex is homeomorphic to the Bauer simplex with extreme boundary X^∞. (Maybe it is always a Bauer simplex in the general case?)

5. A uniqueness theorem

Theorem 5.1. *Let X be a compact metrizable connected space, and let $\Delta : \mathrm{C}_1^+(X) \to (0, 1]$ be an order preserving map. Then, for any finite set $\mathcal{F} \subseteq \mathrm{C}(X)$ and any $\varepsilon > 0$, there exist finite sets $\mathcal{H}_0, \mathcal{H}_1 \subseteq \mathrm{C}^+(X)$ and $\delta > 0$ such that for any homomorphisms $\phi_0, \phi_1 : \mathrm{C}(X) \to \mathrm{M}_{n+k}(\mathrm{C}(X^d))$ with*

$$\phi_0(f) = \mathrm{diag}\{f \circ \pi_1, ..., f \circ \pi_n, f(x_1), ..., f(x_k)\}$$

and

$$\phi_1(f) = \mathrm{diag}\{f \circ \pi_1, ..., f \circ \pi_n, f(y_1), ..., f(y_k)\},$$

where $x_1, ..., x_k$ and $y_1, ..., y_k$ are points of X, and $\pi_1, ..., \pi_n$ are coordinate projections (with possible multiplicities), if

$$\tau(\phi_0(h)), \tau(\phi_1(h)) > \Delta(h), \quad h \in \mathcal{H}_0,$$

and

$$|\tau(\phi_0(h) - \phi_1(h))| < \delta, \quad h \in \mathcal{H}_1,\ \tau \in \mathrm{T}(\mathrm{M}_{n+k}(\mathrm{C}(X^d))),$$

then there is a unitary $u \in \mathrm{M}_{n+k}(\mathrm{C}(X^d))$ such that

$$\|\phi_0(f) - u^*\phi_1(f)u\| < \varepsilon, \quad f \in \mathcal{F}.$$

Proof. Fix a metric for X. Since X is compact, there is $\eta > 0$ such that for any $x, y \in X$ with $\mathrm{dist}(x, y) < 3\eta$, one has

$$|f(x) - f(y)| < \varepsilon, \quad f \in \mathcal{F}.$$

Choose an open cover

$$\mathcal{U} = \{U_1, U_2, ..., U_{|\mathcal{U}|}\}$$

with each U_i of diameter at most η. Let

$$\mathcal{O} = \{O_1, O_2, ..., O_S\}$$

denote the set of all finite unions of $U_1, U_2, ..., U_{|\mathcal{U}|}$. For each $O \in \mathcal{O}$, define

$$h_O(x) = \max\{1 - \mathrm{dist}(x, O)/\eta, 0\}, \quad x \in X.$$

Also, for each $O \in \mathcal{O}$ with $O_\eta \neq X$, where O_η denotes the η-neighborhood of O (hence $O_{2\eta} \setminus O_\eta \neq \varnothing$, as otherwise O_η is a clopen set and X is assumed to be connected), choose a non-zero positive function $g_O \in \mathrm{C}(X)$ such that $g_O \leq 1$ and

$$\mathrm{supp}(g_O) \subseteq O_{2\eta} \setminus O_\eta.$$

Then

$$\mathcal{H}_0 := \{g_O : O \in \mathcal{O},\ O_\eta \neq X\}, \quad \mathcal{H}_1 := \{h_O : O \in \mathcal{O}\}, \quad \text{and} \quad \delta := \min\{\Delta(g_O) : O \in \mathcal{O}\}$$

possess the property of the lemma.

Let ϕ_0 and ϕ_1 be given as in the statement of the lemma. Let $\tilde{X} \subseteq \{x_1, x_2, ..., x_k\}$ be an arbitrary subset. Let $U_{i_1}, U_{i_2}, ..., U_{i_l} \in \mathcal{U}$ be such that $U_{i_j} \cap \tilde{X} \neq \varnothing$, and consider the union

$$O = U_{i_1} \cup \cdots \cup U_{i_l} \in \mathcal{O}.$$

Assume $O_{2\eta} \neq X$ (hence $O_\eta \neq X$), and choose

$$x'_O \in X \setminus O_{2\eta},$$

and then choose $x_O \in X^d$ (e.g., pick $x_O = (x'_O, ..., x'_O)$) such that

$$\pi_1(x_O), ..., \pi_n(x_O) \in X \setminus O_{2\eta}.$$

Then

$$\begin{aligned}
\left|\tilde{X}\right| &\leq (n+k)\mathrm{tr}_{x_O}(\phi_0(h_O)) \\
&\leq (n+k)\mathrm{tr}_{x_O}(\phi_1(h_O)) + (n+k)\delta \\
&\leq |O_\eta \cap \{y_1, y_2, ..., y_k\}| + (n+k)\delta \\
&\leq |O_\eta \cap \{y_1, y_2, ..., y_k\}| + (n+k)\Delta(g_O) \\
&\leq |O_\eta \cap \{y_1, y_2, ..., y_k\}| + (n+k)\mathrm{tr}_{x_O}(g_O) \\
&\leq |O_\eta \cap \{y_1, y_2, ..., y_k\}| + |O_{\eta,2\eta} \cap \{y_1, y_2, ..., y_k\}| \\
&\leq |O_{2\eta} \cap \{y_1, y_2, ..., y_k\}| \\
&\leq \left|\tilde{X}_{3\eta} \cap \{y_1, y_2, ..., y_k\}\right| \qquad (O_{2\eta} \subseteq \tilde{X}_{3\eta}).
\end{aligned}$$

If $O_{2\eta} = X$, then $\tilde{X}_{3\eta} = X$. In particular, we still have

$$\left|\tilde{X}\right| \leq k = \left|\tilde{X}_{3\eta} \cap \{y_1, y_2, ..., y_k\}\right|.$$

That is, we always have

$$\left|\tilde{X}\right| \leq \left|\tilde{X}_{3\eta} \cap \{y_1, y_2, ..., y_k\}\right|. \tag{5.1}$$

The same calculation shows that for any subset $\tilde{Y} \subseteq \{y_1, y_2, ..., y_k\}$, we have

$$\left|\tilde{Y}\right| \leq \left|\tilde{Y}_{3\eta} \cap \{x_1, x_2, ..., x_k\}\right|. \tag{5.2}$$

Thus, by the Marriage Lemma ([5]), there is a one-to-one correspondence

$$\sigma : \{x_1, x_2, ..., x_k\} \to \{y_1, y_2, ..., y_k\}$$

such that

$$\mathrm{dist}(x_i, \sigma(x_i)) < 3\eta, \quad i = 1, 2, ..., k.$$

Denote by $w \in \mathrm{M}_k(\mathbb{C})$ the permutation unitary that induces σ. Then

$$u = \mathrm{diag}(1_n, w)$$

is the desired unitary. □

6. An isomorphism theorem

Theorem 6.1. *Assume X is a connected finite dimensional CW-complex, and assume that that the radii of comparison of $A_E := A(X, (n_i), (k_i^{(E)}), E)$ and $A_F := A(X, (n_i), (k_i^{(F)}), F)$ are non-zero. Then $A_E \cong A_F$ if, and only if,*

$$\rho(K_0(A_E)) = \rho(K_0(A_F)) \quad \text{and} \quad \mathrm{rc}(A_E) = \mathrm{rc}(A_F),$$

where ρ_{A_E} and ρ_{A_F} are the unique state of the $K_0(A_E)$ and $K_0(A_F)$, respectively.

Proof. If $\mathrm{rc}(A_E) = \mathrm{rc}(A_F) = 0$, then A_E and A_F are tracially AF algebras. Since X is connected, we have

$$((\mathrm{K}_0(A_E), \mathrm{K}_0^+(A_E), [1]_0), \mathrm{K}_1(A_E)) \cong ((\mathrm{K}_0(A_F), \mathrm{K}_0^+(A_F), [1]_0), \mathrm{K}_1(A_F)),$$

and hence $A_E \cong A_F$.

Now, assume $\mathrm{rc}(A_E) = \mathrm{rc}(A_F) \neq 0$. By Theorem 3.1, we have

$$\frac{c_1 \cdots c_i}{(n_1 + k_1^{(E)}) \cdots (n_i + k_i^{(E)})} = \frac{c_1 \cdots c_i}{(n_1 + k_1^{(F)}) \cdots (n_i + k_i^{(F)})} \neq 0,$$

and hence

$$\lim_{i \to \infty} \frac{(n_1 + k_1^{(E)}) \cdots (n_i + k_i^{(E)})}{(n_1 + k_1^{(F)}) \cdots (n_i + k_i^{(F)})} = 1.$$

By Lemma 4.2, together with the assumption $\mathrm{K}_0(A_E) \cong \mathrm{K}_0(A_F)$, we have that $(k_i^{(E)})$ and $(k_i^{(F)})$ are sufficiently close in the sense of Definition 4.1.

Consider the inductive limit constructions

$$\mathrm{M}_{n_0}(\mathrm{C}(X)) \xrightarrow{\phi_1^{(E)}} \mathrm{M}_{n_0(n_1+k_1^{(E)})}(\mathrm{C}(X^{c_1})) \xrightarrow{\phi_2^{(E)}} \mathrm{M}_{n_0(n_1+k_1^{(E)})(n_2+k_2^{(E)})}(\mathrm{C}(X^{c_1 c_2})) \longrightarrow \cdots \longrightarrow A_E$$

$$\mathrm{M}_{n_0}(\mathrm{C}(X)) \xrightarrow{\phi_1^{(F)}} \mathrm{M}_{n_0(n_1+k_1^{(F)})}(\mathrm{C}(X^{c_1})) \xrightarrow{\phi_2^{(F)}} \mathrm{M}_{n_0(n_1+k_1^{(F)})(n_2+k_2^{(F)})}(\mathrm{C}(X^{c_1 c_2})) \longrightarrow \cdots \longrightarrow A_F.$$

Choose finite subsets

$$\mathcal{F}_1^{(E)} \subseteq \mathrm{M}_{n_0}(C(X)),\ \mathcal{F}_2^{(E)} \subseteq \mathrm{M}_{n_0(n_1+k_1^{(E)})}(\mathrm{C}(X^{c_1})), ...$$

and

$$\mathcal{F}_1^{(F)} \subseteq \mathrm{M}_{n_0}(C(X)),\ \mathcal{F}_2^{(F)} \subseteq \mathrm{M}_{n_0(n_1+k_1^{(F)})}(\mathrm{C}(X^{c_1})), ...,$$

such that

$$\overline{\bigcup_{i=1}^{\infty} \mathcal{F}_i^{(E)}} = A_E \quad \text{and} \quad \overline{\bigcup_{i=1}^{\infty} \mathcal{F}_i^{(F)}} = A_F.$$

Also choose $\varepsilon_1 > \varepsilon_2 > \cdots > 0$ such that

$$\sum_{i=1}^{\infty} \varepsilon_i \leq 1.$$

Since A_E and A_F are simple, we have the non-zero density functions

$$\Delta_E(h) = \inf\{\tau(h) : \tau \in \mathrm{T}(A_E)\}, \quad h \in A_E^+,$$

and

$$\Delta_F(h) = \inf\{\tau(h) : \tau \in \mathrm{T}(A_F)\}, \quad h \in A_F^+.$$

Applying Theorem 5.1 to $(\mathcal{F}_i^{(E)}, \varepsilon_i)$, we obtain finite sets $\mathcal{H}_{i,0}^{(E)}, \mathcal{H}_{i,1}^{(E)} \subseteq \mathrm{M}_{m_i^{(E)}}(\mathrm{C}(X^{d_i}))$ and $\delta_i^{(E)} > 0$. Applying Theorem 5.1 to $(\mathcal{F}_i^{(F)}, \varepsilon_i)$, we obtain finite sets $\mathcal{H}_{i,0}^{(F)}, \mathcal{H}_{i,1}^{(F)} \subseteq \mathrm{M}_{m_i^{(F)}}(\mathrm{C}(X^{d_i}))$ and $\delta_i^{(F)} > 0$. Set $\delta_i = \min\{\delta_i^{(E)}, \delta_i^{(F)}\}$.

Since $(k_i^{(E)})$ and $(k_i^{(F)})$ are sufficiently close in the sense of Definition 4.1, by the construction of Section 4 (Diagram (7.24)), we have the diagram

$$\begin{tikzcd}[column sep=small]
\mathrm{M}_{n_0}(\mathrm{C}(X)) \arrow[r, "\phi_{1,i_1}^{(E)}"] \arrow[dr, "\phi_{1,i_1}^{(E,F)}"] & \mathrm{M}_{m_{i_1}^{(E)}}(\mathrm{C}(X^{d_{i_1}})) \arrow[r, "\phi_{i_1,i_2}^{(E)}"] & \mathrm{M}_{m_{i_2}^{(E)}}(\mathrm{C}(X^{d_{i_2}})) \arrow[r] \arrow[dr, "\phi_{i_2,i_3}^{(E,F)}"] & \cdots \arrow[r] & A_E \\
\mathrm{M}_{n_0}(\mathrm{C}(X)) \arrow[r, "\phi_{1,i_1}^{(F)}"] & \mathrm{M}_{m_{i_1}^{(F)}}(\mathrm{C}(X^{d_{i_1}})) \arrow[r, "\phi_{i_1,i_2}^{(F)}"] \arrow[ur, "\phi_{i_1,i_2}^{(F,E)}"] & \mathrm{M}_{m_{i_2}^{(F)}}(\mathrm{C}(X^{d_{i_2}})) \arrow[r] & \cdots \arrow[r] & A_F
\end{tikzcd} \tag{6.1}$$

with

$$\left|\tau(\phi_{i_{s+1},i_{s+2}}^{(F,E)} \circ \phi_{i_s,i_{s+1}}^{(E,F)}(h) - \phi_{i_{s+1},i_{s+2}}^{(E)} \circ \phi_{i_s,i_{s+1}}^{(E)}(h))\right| < \delta_s,$$

for any $s = 0, 2, ...$, any $h \in \mathrm{M}_{m_{i_s}^{(E)}}(\mathrm{C}(X^{d_{i_s}}))$, and any $\tau \in \mathrm{T}(\mathrm{M}_{m_{i_{s+2}}^{(E)}}(\mathrm{C}(X^{d_{i_{s+2}}})))$; and, furthermore,

$$\left|\tau(\phi_{i_{s+1},i_{s+2}}^{(E,F)} \circ \phi_{i_s,i_{s+1}}^{(F,E)}(h) - \phi_{i_{s+1},i_{s+2}}^{(F)} \circ \phi_{i_s,i_{s+1}}^{(F)}(h))\right| < \delta_s,$$

for any $s = 1, 3, ...$, any $h \in \mathrm{M}_{m_{i_s}^{(F)}}(\mathrm{C}(X^{d_{i_s}}))$, and any $\tau \in \mathrm{T}(\mathrm{M}_{m_{i_{s+2}}^{(F)}}(\mathrm{C}(X^{d_{i_{s+2}}})))$. Moreover, the maps $\phi_{i_{s+1},i_{s+2}}^{(F,E)} \circ \phi_{i_s,i_{s+1}}^{(E,F)}$ and $\phi_{i_{s+1},i_{s+2}}^{(E)} \circ \phi_{i_s,i_{s+1}}^{(E)}$ share the same coordinate projection part, and so do the maps $\phi_{i_{s+1},i_{s+2}}^{(E,F)} \circ \phi_{i_s,i_{s+1}}^{(F,E)}$ and $\phi_{i_{s+1},i_{s+2}}^{(F)} \circ \phi_{i_s,i_{s+1}}^{(F)}$.

Then, by Theorem 5.1, there are unitaries

$$u_2^{(E)} \in \mathrm{M}_{m_{i_2}}(\mathrm{C}(X^{d_{i_2}})), \quad u_4^{(E)} \in \mathrm{M}_{m_{i_4}}(\mathrm{C}(X^{d_{i_4}})), \ ...$$

and

$$u_3^{(E)} \in \mathrm{M}_{m_{i_3}}(\mathrm{C}(X^{d_{i_3}})), \quad u_5^{(E)} \in \mathrm{M}_{m_{i_5}}(\mathrm{C}(X^{d_{i_5}})), \ ...$$

such that

$$\left\|\phi_{i_{s+1},i_{s+2}}^{(F,E)} \circ \phi_{i_s,i_{s+1}}^{(E,F)}(f) - (u_{s+2}^{(E)})^*(\phi_{i_{s+1},i_{s+2}}^{(E)} \circ \phi_{i_s,i_{s+1}}^{(E)}(f))u_{s+2}^{(E)}\right\| < \varepsilon_s$$

for any $s = 0, 2, 4...$, any $f \in \mathcal{F}_{i_s}^{(E)} \subseteq \mathrm{M}_{m_{i_s}^{(E)}}(\mathrm{C}(X^{d_{i_s}}))$; and, furthermore,

$$\left\|\phi_{i_{s+1},i_{s+2}}^{(E,F)} \circ \phi_{i_s,i_{s+1}}^{(F,E)}(f) - (u_{s+2}^{(F)})^*\phi_{i_{s+1},i_{s+2}}^{(F)} \circ \phi_{i_s,i_{s+1}}^{(F)}(f)u_{s+2}^{(F)}\right\| < \varepsilon_s$$

for any $s = 1, 3, ...$, any $f \in \mathcal{F}_{i_s}^{(F)} \subseteq \mathrm{M}_{m_{i_s}^{(F)}}(\mathrm{C}(X^{d_{i_s}}))$.

That is, the ith triangle of the diagram

$$
\begin{tikzcd}
\mathrm{M}_{n_0}(\mathrm{C}(X)) \arrow[r, "\phi^{(E)}_{1,i_1}"] \arrow[dr, "\phi^{(E,F)}_{1,i_1}"'] & \mathrm{M}_{m^{(E)}_{i_1}}(\mathrm{C}(X^{d_{i_1}})) \arrow[r, "\mathrm{ad}(u_2^{(E)})\circ\phi^{(E)}_{i_1,i_2}"] & \mathrm{M}_{m^{(E)}_{i_2}}(\mathrm{C}(X^{d_{i_2}})) \arrow[r] \arrow[dr, "\phi^{(E,F)}_{i_2,i_3}"] & \cdots \arrow[r] & A_E \\
\mathrm{M}_{n_0}(\mathrm{C}(X)) \arrow[r, "\phi^{(F)}_{1,i_1}"'] & \mathrm{M}_{m^{(F)}_{i_1}}(\mathrm{C}(X^{d_{i_1}})) \arrow[r, "\phi^{(F)}_{i_1,i_2}"'] \arrow[ur, "\phi^{(F,E)}_{i_1,i_2}"] & \mathrm{M}_{m^{(F)}_{i_2}}(\mathrm{C}(X^{d_{i_2}})) \arrow[r, "\mathrm{ad}(u_3^{(F)})\circ\phi^{(F)}_{i_1,i_2}"'] & \cdots \arrow[r] & A_F
\end{tikzcd}
\tag{6.2}
$$

is approximately commutative up to $(\mathcal{F}_s^{(E)}, \varepsilon_s)$ or $(\mathcal{F}_s^{(F)}, \varepsilon_s)$. Then, by the approximate intertwining argument (Theorems 2.1 and 2.2 of [2]), we have

$$A_E \cong A_F,$$

as desired. □

Remark 6.2. In the case of Villadsen algebras, i.e., with coordinate projections of multiplicity one (and rapid dimension growth), both the trace simplex of $A(X, (n_i), (k_i))$ and the trace simplex of $A(X^2, (n_i), (k_i))$ are isomorphic to the simplex of probability Borel measures on X^∞. However, the algebras $A(X, (n_i), (k_i))$ and $A(X^2, (n_i), (k_i))$ are not isomorphic in general as their radii of comparison (see [7]) might be different. However, in the case that X is contractible, we will show below (Corollary 7.7) that these kind of C*-algebras are still classified by the K_0-group and the radius of comparison.

Remark 6.3. In the case of a Goodearl algebra ([4]), i.e., with only one coordinate projection, $\mathcal{Z}$-stability always holds, as the mean dimension in the sense of [6] is always zero.

7. Let (n_i) vary

Let us show the following theorem:

Theorem 7.1. *Let X be a contractible finite-dimensional CW-complex. Let*

$$A := A(X, (n_i^{(A)}), (k_i^{(A)}), E^{(A)}) \quad \text{and} \quad B := B(X, (n_i^{(B)}), (k_i^{(B)}), F^{(B)})$$

be Villadsen algebras (the multiplicities of the coordinate projections are 1*). Then*

$$A \cong B$$

if, and only if,

$$K_0(A) \cong K_0(B) \quad \text{and} \quad \mathrm{rc}(A) = \mathrm{rc}(B).$$

Remark 7.2. Since X is assumed to be contractible, we have

$$\mathrm{K}_0(A) \cong \mathbb{Z}[\frac{1}{n_0^{(A)}}, \frac{1}{n_1^{(A)} + k_1^{(A)}}, ...] \subseteq \mathbb{Q},$$

and

$$\mathrm{K}_0(B) \cong \mathbb{Z}[\frac{1}{n_0^{(B)}}, \frac{1}{n_1^{(B)} + k_1^{(B)}}, ...] \subseteq \mathbb{Q},$$

with the class of the unit being of course $1 \in \mathbb{Z}$.

7.1. **An intertwining diagram.**

Lemma 7.3. *Let*

$$A := A(X, (n_i^{(A)}), (k_i^{(A)}), E^{(A)}) \quad \text{and} \quad B := B(X, (n_i^{(B)}), (k_i^{(B)}), F^{(B)})$$

be Villadsen algebras (the multiplicities of the coordinate projections are 1*). Assume that*

$$n_0^{(A)} \prod_{i=1}^{\infty} (n_i^{(A)} + k_i^{(A)}) = n_0^{(B)} \prod_{i=1}^{\infty} (n_i^{(B)} + k_i^{(B)}), \tag{7.1}$$

as supernatural numbers, and

$$\frac{1}{n_0^{(A)}} \prod_{i=1}^{\infty} \frac{n_i^{(A)}}{n_i^{(A)} + k_i^{(A)}} = \frac{1}{n_0^{(B)}} \prod_{i=1}^{\infty} \frac{n_i^{(B)}}{n_i^{(B)} + k_i^{(B)}} \neq 0, \tag{7.2}$$

as real numbers.

Let

$$\mathcal{H}_1^{(A)} \subseteq \mathrm{M}_{n_0^{(A)}}(\mathrm{C}(X)), \quad \mathcal{H}_2^{(A)} \subseteq \mathrm{M}_{n_0^{(A)}(n_1^{(A)} + k_1^{(A)})} \mathrm{C}(X^{n_1}), \quad \ldots$$

and

$$\mathcal{H}_1^{(B)} \subseteq \mathrm{M}_{n_0^{(B)}}(\mathrm{C}(X)), \quad \mathcal{H}_2^{(B)} \subseteq \mathrm{M}_{n_0^{(B)}(n_1^{(B)} + k_1^{(B)})} \mathrm{C}(X^{n_1}), \quad \ldots$$

be finite subsets such that

$$\overline{\bigcup_{i=1}^{\infty} \mathcal{H}_i^{(A)}} = A \quad \text{and} \quad \overline{\bigcup_{i=1}^{\infty} \mathcal{H}_i^{(B)}} = B,$$

and let $\delta_1, \delta_2, \ldots$ be a decreasing sequence of strictly positive numbers with

$$\sum_{n=1}^{\infty} \delta_n < 1.$$

Then, there is a diagram

$$\begin{tikzcd}
\mathrm{M}_{n_0^{(A)}}(\mathrm{C}(X)) \arrow[r, "\phi_{1,i_1}^{(A)}"] \arrow[dr, "\phi_{1,i_1}^{(A,B)}"] & \mathrm{M}_{m_{i_1}^{(A)}}(\mathrm{C}(X^{d_{i_1}^{(A)}})) \arrow[r, "\phi_{i_1,i_2}^{(A)}"] & \mathrm{M}_{m_{i_2}^{(A)}}(\mathrm{C}(X^{d_{i_2}^{(A)}})) \arrow[r] \arrow[dr, "\phi_{i_2,i_3}^{(A,B)}"] & \cdots \arrow[r] & A \\
\mathrm{M}_{n_0^{(B)}}(\mathrm{C}(X)) \arrow[r, "\phi_{1,i_1}^{(B)}"] & \mathrm{M}_{m_{i_1}^{(B)}}(\mathrm{C}(X^{d_{i_1}^{(B)}})) \arrow[r, "\phi_{i_1,i_2}^{(B)}"] \arrow[ur, "\phi_{i_1,i_2}^{(B,A)}"] & \mathrm{M}_{m_{i_2}^{(F)}}(\mathrm{C}(X^{d_{i_2}^{(B)}})) \arrow[r] & \cdots \arrow[r] & B
\end{tikzcd} \tag{7.3}$$

such that

$$\left| \tau(\phi_{i_{s+1},i_{s+2}}^{(B,A)} \circ \phi_{i_s,i_{s+1}}^{(A,B)}(h) - \phi_{i_{s+1},i_{s+2}}^{(A)} \circ \phi_{i_s,i_{s+1}}^{(A)}(h)) \right| < \delta_{i_s}$$

for any $s = 0, 2, \ldots,$ any $h \in \mathrm{M}_{m_{i_s}^{(A)}}(\mathrm{C}(X^{d_{i_s}^{(A)}})),$ and any $\tau \in \mathrm{T}(\mathrm{M}_{m_{i_{s+2}}^{(A)}}(\mathrm{C}(X^{d_{i_{s+2}}^{(A)}})))$; and, furthermore,

$$\left| \tau(\phi_{i_{s+1},i_{s+2}}^{(A,B)} \circ \phi_{i_s,i_{s+1}}^{(B,A)}(h) - \phi_{i_{s+1},i_{s+2}}^{(B)} \circ \phi_{i_s,i_{s+1}}^{(B)}(h)) \right| < \delta_{i_s}$$

for any $s = 1, 3, \ldots,$ any $h \in \mathrm{M}_{m_{i_s}^{(B)}}(\mathrm{C}(X^{d_{i_s}^{(B)}})),$ and any $\tau \in \mathrm{T}(\mathrm{M}_{m_{i_{s+2}}^{(B)}}(\mathrm{C}(X^{d_{i_{s+2}}^{(B)}})))$.

Moreover, for each $s = 0, 2, ...,$

$$\phi^{(B,A)}_{i_{s+1},i_{s+2}} \circ \phi^{(A,B)}_{i_s,i_{s+1}} = \mathrm{diag}\{\pi_1^*, ..., \pi_{K_s}^*, \textit{point evaluations}\}$$

$$\phi^{(A)}_{i_{s+1},i_{s+2}} \circ \phi^{(A)}_{i_s,i_{s+1}} = \mathrm{diag}\{\pi_1^*, ..., \pi^*_{n^{(A)}_{i_s}\cdots n^{(A)}_{i_{s+1}}}, \textit{point evaluations}\}$$

with

$$\frac{n^{(A)}_{i_s}\cdots n^{(A)}_{i_{s+1}} - K_s}{(n^{(A)}_{i_s}+k^{(A)}_{i_s})\cdots(n^{(A)}_{i_{s+1}}+k^{(A)}_{i_{s+1}}) - n^{(A)}_{i_s}\cdots n^{(A)}_{i_{s+1}}} < \delta_{i_s};$$

and for each $s = 1, 3, ...,$

$$\phi^{(A,B)}_{i_{s+1},i_{s+2}} \circ \phi^{(B,A)}_{i_s,i_{s+1}} = \mathrm{diag}\{\pi_1^*, ..., \pi_{K_s}^*, \textit{point evaluations}\}$$

$$\phi^{(B)}_{i_{s+1},i_{s+2}} \circ \phi^{(B)}_{i_s,i_{s+1}} = \mathrm{diag}\{\pi_1^*, ..., \pi^*_{n^{(B)}_{i_s}\cdots n^{(B)}_{i_{s+1}}}, \textit{point evaluations}\}$$

with

$$\frac{n^{(B)}_{i_s}\cdots n^{(B)}_{i_{s+1}} - K_s}{(n^{(B)}_{i_s}+k^{(B)}_{i_s})\cdots(n^{(B)}_{i_{s+1}}+k^{(B)}_{i_{s+1}}) - n^{(B)}_{i_s}\cdots n^{(B)}_{i_{s+1}}} < \delta_{i_s}.$$

Proof. Consider the inductive sequences

$$\mathrm{M}_{n_0^{(A)}}(\mathrm{C}(X)) \xrightarrow{\phi_1^{(A)}} \mathrm{M}_{n_0^{(A)}(n_1^{(A)}+k_1^{(A)})}(\mathrm{C}(X^{n_1})) \xrightarrow{\phi_2^{(A)}} \mathrm{M}_{n_0^{(A)}(n_1^{(A)}+k_1^{(A)})(n_2^{(A)}+k_2^{(A)})}(\mathrm{C}(X^{n_1n_2})) \longrightarrow \cdots \longrightarrow A,$$

$$\mathrm{M}_{n_0^{(B)}}(\mathrm{C}(X)) \xrightarrow{\phi_1^{(B)}} \mathrm{M}_{n_0^{(B)}(n_1^{(B)}+k_1^{(B)})}(\mathrm{C}(X^{n_1})) \xrightarrow{\phi_2^{(B)}} \mathrm{M}_{n_0^{(B)}(n_1^{(B)}+k_1^{(B)})(n_2^{(B)}+k_2^{(B)})}(\mathrm{C}(X^{n_1n_2})) \longrightarrow \cdots \longrightarrow B.$$

Set (see (7.2))

$$\gamma := \lim_{i\to\infty} \frac{n_1^{(A)}\cdots n_i^{(A)}}{n_0^{(A)}(n_1^{(A)}+k_1^{(A)})\cdots(n_i^{(A)}+k_i^{(A)})} = \lim_{i\to\infty} \frac{n_1^{(B)}\cdots n_i^{(B)}}{n_0^{(B)}(n_1^{(B)}+k_1^{(B)})\cdots(n_i^{(B)}+k_i^{(B)})} \in (0,1).$$

Consider the set $\mathcal{H}_1^{(A)}$, and without loss of generality, let us assume

$$\delta_1 < \frac{k_1^{(A)}}{n_1^{(A)}+k_1^{(A)}}. \tag{7.4}$$

There is $i_1' > 0$ such that

$$1 - \prod_{j=0}^{\infty} \frac{n^{(A)}_{i_1'+j}}{n^{(A)}_{i_1'+j}+k^{(A)}_{i_1'+j}} < \delta_1. \tag{7.5}$$

Then, pick $\varepsilon' > 0$ such that

$$\frac{n_0^{(A)}(n_1^{(A)}+k_1^{(A)})\cdots(n^{(A)}_{i_1'-1}+k^{(A)}_{i_1'-1})}{n_1^{(A)}\cdots n^{(A)}_{i_1'-1}} < \frac{1}{\gamma} - \varepsilon', \tag{7.6}$$

and pick $\varepsilon'' > 0$ such that

$$(\frac{1}{\gamma}+\varepsilon')(\gamma+\varepsilon'') < 1.$$

By (7.1), there is $i_1 > i_1'$ such that

$$n_0^{(B)}\prod_{i=1}^{i_1-1}(n_i^{(B)}+k_i^{(B)}) \text{ is divisible by } n_0^{(A)}\prod_{i=1}^{i_1'-1}(n_i^{(A)}+k_i^{(A)}). \tag{7.7}$$

One may assume i_1 is sufficiently large that

$$\frac{n_1^{(B)}\cdots n_{i_1}^{(B)}}{n_0^{(B)}(n_1^{(B)}+k_1^{(B)})\cdots(n_{i_1}^{(B)}+k_{i_1}^{(B)})} < \gamma+\varepsilon'' \tag{7.8}$$

and

$$\frac{n_1^{(A)}\cdots n_{i_1'-1}^{(A)}}{n_1^{(B)}\cdots n_{i_1-1}^{(B)}} < \frac{\delta_1^2}{2}. \tag{7.9}$$

Pick $l_{i_1',i_1}\in\mathbb{N}$ such that

$$0\leq n_1^{(B)}\cdots n_{i_1}^{(B)} - (n_1^{(A)}\cdots n_{i_1'-1}^{(A)})l_{i_1',i_1} < n_1^{(A)}\cdots n_{i_1'-1}^{(A)}. \tag{7.10}$$

Then

$$\begin{aligned}
&\frac{n_0^{(A)}(n_1^{(A)}+k_1^{(A)})\cdots(n_{i_1'-1}^{(A)}+k_{i_1'-1}^{(A)})}{n_0^{(B)}(n_1^{(B)}+k_1^{(B)})\cdots(n_{i_1'-1}^{(B)}+k_{i_1'-1}^{(B)})}\cdot\frac{l_{i_1',i_1}}{(n_{i_1'}^{(B)}+k_{i_1'}^{(B)})\cdots(n_{i_1}^{(B)}+k_{i_1}^{(B)})} \\
\leq\ &\frac{n_0^{(A)}(n_1^{(A)}+k_1^{(A)})\cdots(n_{i_1'-1}^{(A)}+k_{i_1'-1}^{(A)})}{n_0^{(B)}(n_1^{(B)}+k_1^{(B)})\cdots(n_{i_1}^{(B)}+k_{i_1}^{(B)})}\cdot\frac{n_1^{(B)}\cdots n_{i_1}^{(B)}}{n_1^{(A)}\cdots n_{i_1'-1}^{(A)}} \qquad \text{(by (7.10))} \\
=\ &\frac{n_0^{(A)}(n_1^{(A)}+k_1^{(A)})\cdots(n_{i_1'-1}^{(A)}+k_{i_1'-1}^{(A)})}{n_1^{(A)}\cdots n_{i_1'-1}^{(A)}}\cdot\frac{n_1^{(B)}\cdots n_{i_1}^{(B)}}{n_0^{(B)}(n_1^{(B)}+k_1^{(B)})\cdots(n_{i_1}^{(B)}+k_{i_1}^{(B)})} \\
<\ &(\frac{1}{\gamma}-\varepsilon')(\gamma+\varepsilon'')<1 \qquad \text{(by (7.6), (7.8))}.
\end{aligned} \tag{7.11}$$

Then consider the diagram

$$\begin{tikzcd}
\mathrm{M}_{n_0^{(A)}}(\mathrm{C}(X)) \arrow[r, "\phi_{1,i_1'}^{(A)}"] & \mathrm{M}_{m_{i_1'}^{(A)}}(\mathrm{C}(X^{d_{i_1'}^{(A)}})) \arrow[r, "\phi_{i_1',i_1}^{(A)}"] \arrow[dr, "\tilde{\phi}_{i_1',i_1}^{(A,B)}"] & \mathrm{M}_{m_{i_1}^{(A)}}(\mathrm{C}(X^{d_{i_1}^{(A)}})) \arrow[r] & \cdots \arrow[r] & A \\
\mathrm{M}_{n_0^{(B)}}(\mathrm{C}(X)) \arrow[r, "\phi_{1,i_1'}^{(B)}"] & \mathrm{M}_{m_{i_1'}^{(B)}}(\mathrm{C}(X^{d_{i_1'}^{(B)}})) \arrow[r, "\phi_{i_1',i_1}^{(B)}"] & \mathrm{M}_{m_{i_1}^{(B)}}(\mathrm{C}(X^{d_{i_1}^{(B)}})) \arrow[r] & \cdots \arrow[r] & B,
\end{tikzcd}$$

where

$$m_i := n_0(n_1+k_1)\cdots(n_{i-1}+k_{i-1}), \quad d_i := n_1\cdots n_{i-1}, \tag{7.12}$$

and $\tilde{\phi}^{(A,B)}_{i'_1,i_1}: \mathrm{M}_{m^{(A)}_{i'_1}}(\mathrm{C}(X^{d^{(A)}_{i'_1}})) \to \mathrm{M}_{m^{(B)}_{i_1}}(\mathrm{C}(X^{d^{(B)}_{i_1}}))$ is a map of the form

$$f \mapsto \mathrm{diag}(\underbrace{f\circ\pi_1, ..., f\circ\pi_{l_{i'_1,i_1}}}_{l_{i'_1,i_1}}, \text{point evaluations}),$$

where the point evaluations are arbitrarily chosen (the map $\tilde{\phi}^{(A,B)}_{i'_1,i_1}$ exists by (7.7) and (7.11)).

Write

$$\phi^{(A,B)}_{1,i_1} = \tilde{\phi}^{(A,B)}_{i'_1,i_1} \circ \phi^{(A)}_{1,i'_1},$$

and compress the diagram above as

$$\begin{tikzcd}
\mathrm{M}_{n^{(A)}_0}(\mathrm{C}(X)) \arrow[r, "\phi^{(A)}_{1,i_1}"] \arrow[dr, "\phi^{(A,B)}_{1,i_1}"] & \mathrm{M}_{m^{(A)}_{i_1}}(\mathrm{C}(X^{d^{(A)}_{i_1}})) \arrow[r, "\phi^{(A)}_{i_1}"] & \mathrm{M}_{m^{(A)}_{i_1+1}}(\mathrm{C}(X^{d^{(A)}_{i_1+1}})) \arrow[r] & \cdots \arrow[r] & A \\
\mathrm{M}_{n^{(B)}_0}(\mathrm{C}(X)) \arrow[r, "\phi^{(B)}_{1,i_1}"] & \mathrm{M}_{m^{(B)}_{i_1}}(\mathrm{C}(X^{d^{(B)}_{i_1}})) \arrow[r, "\phi^{(B)}_{i_1}"] & \mathrm{M}_{m^{(B)}_{i_1+1}}(\mathrm{C}(X^{d^{(B)}_{i_1+1}})) \arrow[r] & \cdots \arrow[r] & B.
\end{tikzcd}$$

Note that $\phi^{(A,B)}_{i,i_1}$ has the form

$$f \mapsto \mathrm{diag}(\underbrace{f\circ\pi_1, ..., f\circ\pi_{(n^{(A)}_1\cdots n^{(A)}_{i'_1-1})l_{i'_1,i_1}}}_{(n^{(A)}_1\cdots n^{(A)}_{i'_1-1})l_{i'_1,i_1}}, \text{point evaluations}).$$

Consider the set $\mathcal{H}^{(B)}_{i_1}$, and, without loss of generality, let us assume that

$$\delta_2 < \frac{k^{(B)}_{i_1}}{n^{(B)}_{i_1} + k^{(B)}_{i_1}}.$$

The same argument as above shows that there are $i'_2 < i_2$ such that

$$1 - \prod_{j=0}^{\infty} \frac{n^{(B)}_{i'_2+j}}{n^{(B)}_{i'_2+j} + k^{(B)}_{i'_2+j}} < \delta_{i_1}, \tag{7.13}$$

$$n^{(A)}_0 \prod_{i=1}^{i_2-1}(n^{(A)}_i + k^{(A)}_i) \text{ is divided by } n^{(B)}_0 \prod_{i=1}^{i'_2-1}(n^{(B)}_i + k^{(B)}_i) \tag{7.14}$$

$$\frac{n^{(B)}_1\cdots n^{(B)}_{i'_2-1}}{n^{(A)}_1\cdots n^{(A)}_{i_2}} < \frac{n^{(B)}_{i_1}\cdots n^{(B)}_{i'_2-1}}{n^{(A)}_{i_1}\cdots n^{(A)}_{i_2-1}} < \frac{\delta^2_{i_1}}{2}. \tag{7.15}$$

and

$$\frac{n^{(B)}_0(n^{(B)}_1 + k^{(B)}_1)\cdots(n^{(B)}_{i'_2-1} + k^{(B)}_{i'_2-1})}{n^{(A)}_0(n^{(A)}_1 + k^{(A)}_1)\cdots(n^{(A)}_{i'_2-1} + k^{(A)}_{i'_2-1})} \cdot \frac{l_{i'_2,i_2}}{(n^{(A)}_{i'_1} + k^{(A)}_{i'_2})\cdots(n^{(A)}_{i_2} + k^{(A)}_{i_2})} < 1 \tag{7.16}$$

where

$$0 \leq n_1^{(A)} \cdots n_{i_2}^{(A)} - (n_1^{(B)} \cdots n_{i_2'-1}^{(B)}) l_{i_2',i_2} < n_1^{(B)} \cdots n_{i_2'-1}^{(B)}. \tag{7.17}$$

Consider the map $\tilde{\phi}_{i_2',i_2}^{(B,A)} : \mathrm{M}_{m_{i_2'}^{(B)}}(\mathrm{C}(X^{d_{i_2'}^{(B)}})) \to \mathrm{M}_{m_{i_2}^{(A)}}(\mathrm{C}(X^{d_{i_1}^{(A)}}))$,

$$f \mapsto \mathrm{diag}(\underbrace{f \circ \pi_1, ..., f \circ \pi_{l_{i_2',i_2}}}_{l_{i_2',i_2}}, \text{point evaluations}),$$

where the point evaluations are arbitrarily chosen (the map $\tilde{\phi}_{i_2',i_2}^{(B,A)}$ exists by (7.14) and (7.16)). Define

$$\phi_{i_1,i_2}^{(B,A)} = \tilde{\phi}_{i_2',i_2}^{(B,A)} \circ \phi_{i_1,i_2'}^{(B)}$$

and consider the augmented diagram

$$\begin{tikzcd}
\mathrm{M}_{n_0^{(A)}}\mathrm{C}(X) \arrow[r, "\phi_{1,i_1}^{(A)}"] \arrow[dr, "\phi_{1,i_1}^{(A,B)}"] & \mathrm{M}_{m_{i_1}^{(A)}}(\mathrm{C}(X^{d_{i_1}^{(A)}})) \arrow[r, "\phi_{i_1,i_2}^{(A)}"] & \mathrm{M}_{m_{i_2}^{(A)}}(\mathrm{C}(X^{d_{i_2}^{(A)}})) \arrow[r] & \cdots \arrow[r] & A \\
\mathrm{M}_{n_0^{(B)}}\mathrm{C}(X) \arrow[r, "\phi_{1,i_1}^{(B)}"] & \mathrm{M}_{m_{i_1}^{(B)}}(\mathrm{C}(X^{d_{i_1}^{(B)}})) \arrow[r, "\phi_{i_1,i_2}^{(B)}"] \arrow[ur, "\phi_{i_1,i_2}^{(B,A)}"] & \mathrm{M}_{m_{i_2}^{(B)}}(\mathrm{C}(X^{d_{i_2}^{(B)}})) \arrow[r] & \cdots \arrow[r] & B.
\end{tikzcd}$$

Note that, by (7.5),

$$\left| \tau(\phi_{i_1,i_2}^{(B,A)} \circ \phi_{1,i_1}^{(A,B)}(h) - \phi_{i_1,i_2}^{(A)} \circ \phi_{1,i_1}^{(A)}(h)) \right| < \delta_1, \quad h \in \mathrm{M}_{n_0^{(A)}}(\mathrm{C}(X)),$$

and $\phi_{i_1,i_2}^{(B,A)} \circ \phi_{1,i_1}^{(A,B)}$ has the form

$$f \mapsto \mathrm{diag}\{\underbrace{f \circ \pi_1, ..., f \circ \pi_{(n_1^{(A)} \cdots n_{i_1'}^{(A)}) l_{i_1',i_1} l_{i_2',i_2}}}_{(n_1^{(A)} \cdots n_{i_1'}^{(A)}) l_{i_1',i_1} (n_{i_1+1}^{(B)} \cdots n_{i_2'}^{(B)}) l_{i_2',i_2}}, \text{point evaluations}\},$$

and

$$\begin{aligned}
& n_1^{(A)}\cdots n_{i_2}^{(A)} - (n_1^{(A)}\cdots n_{i_1'-1}^{(A)})l_{i_1',i_1}(n_{i_1}^{(B)}\cdots n_{i_2'}^{(B)})l_{i_2',i_2} \\
= \ & n_1^{(A)}\cdots n_{i_2}^{(A)} - ((n_1^{(A)}\cdots n_{i_1'-1}^{(A)})l_{i_1',i_1})(n_{i_1}^{(B)}\cdots n_{i_2'}^{(B)})l_{i_2',i_2} \\
\leq \ & n_1^{(A)}\cdots n_{i_2}^{(A)} - (n_1^{(B)}\cdots n_{i_1-1}^{(B)})(n_{i_1}^{(B)}\cdots n_{i_2'}^{(B)})l_{i_2',i_2} + (n_1^{(A)}\cdots n_{i_1'-1}^{(A)})(n_{i_1}^{(B)}\cdots n_{i_2'}^{(B)})l_{i_2',i_2} \quad \text{(by (7.10))} \\
= \ & n_1^{(A)}\cdots n_{i_2}^{(A)} - (n_1^{(B)}\cdots n_{i_2'-1}^{(B)})l_{i_2',i_2} + (n_1^{(A)}\cdots n_{i_1'-1}^{(A)})(n_{i_1}^{(B)}\cdots n_{i_2'-1}^{(B)})l_{i_2',i_2} \\
\leq \ & n_1^{(B)}\cdots n_{i_2'-1}^{(B)} + (n_1^{(A)}\cdots n_{i_1'-1}^{(A)})(n_{i_1}^{(B)}\cdots n_{i_2'-1}^{(B)})l_{i_2',i_2} \qquad \text{(by (7.17))} \\
\leq \ & n_1^{(B)}\cdots n_{i_2'-1}^{(B)} + (n_1^{(A)}\cdots n_{i_1'-1}^{(A)})(n_{i_1}^{(B)}\cdots n_{i_2'-1}^{(B)})\frac{n_1^{(A)}\cdots n_{i_2}^{(A)}}{n_1^{(B)}\cdots n_{i_2'-1}^{(B)}} \qquad \text{(by (7.17))} \\
= \ & n_1^{(B)}\cdots n_{i_2'-1}^{(B)} + (n_1^{(A)}\cdots n_{i_1'-1}^{(A)})\frac{n_1^{(A)}\cdots n_{i_2}^{(A)}}{n_1^{(B)}\cdots n_{i_1-1}^{(B)}} \\
= \ & n_1^{(B)}\cdots n_{i_2'-1}^{(B)} + \frac{n_1^{(A)}\cdots n_{i_1'-1}^{(A)}}{n_1^{(B)}\cdots n_{i_1-1}^{(B)}}(n_1^{(A)}\cdots n_{i_2}^{(A)}) \\
\leq \ & \delta_1^2(n_1^{(A)}\cdots n_{i_2}^{(A)}) \qquad \text{(by (7.15), (7.9))}.
\end{aligned}$$

Note that

$$(n_1^{(A)}+k_1^{(A)})\cdots(n_{i_2}^{(A)}+k_{i_2}^{(A)}) - n_1^{(A)}\cdots n_{i_2}^{(A)}$$

is the number of the point evaluations appearing in $\phi_{i_1,i_2}^{(A)}\circ\phi_{1,i_1}^{(A)}$, which is is at least

$$\frac{k_1^{(A)}}{n_1^{(A)}+k_1^{(A)}}((n_1^{(A)}+k_1^{(A)})\cdots(n_{i_2}^{(A)}+k_{i_2}^{(A)})),$$

and hence, by (7.4), is at least

$$\delta_1(n_1^{(A)}\cdots n_{i_2}^{(A)}).$$

Therefore,

$$\begin{aligned}
& n_1^{(A)}\cdots n_{i_2}^{(A)} - (n_1^{(A)}\cdots n_{i_1'-1}^{(A)})l_{i_1',i_1}(n_{i_1}^{(B)}\cdots n_{i_2'}^{(B)})l_{i_2',i_2} \\
< \ & \delta_1^2(n_1^{(A)}\cdots n_{i_2}^{(A)}) \\
< \ & \delta_1((n_1^{(A)}+k_1^{(A)})\cdots(n_{i_2}^{(A)}+k_{i_2}^{(A)}) - n_1^{(A)}\cdots n_{i_2}^{(A)}).
\end{aligned}$$

That is,

$$\frac{n_1^{(A)}\cdots n_{i_2}^{(A)} - K_1}{((n_1^{(A)}+k_1^{(A)})\cdots(n_{i_2}^{(A)}+k_{i_2}^{(A)}) - n_1^{(A)}\cdots n_{i_2}^{(A)})} < \delta_1,$$

where $K_1 = (n_1^{(A)}\cdots n_{i_1'-1}^{(A)})l_{i_1',i_1}(n_{i_1}^{(B)}\cdots n_{i_2'}^{(B)})l_{i_2',i_2}$. Repeating this process, we have the intertwining diagram with the desired properties. $\square$

7.2. **The isomorphism theorem.** First, we have the following stable uniqueness theorem, which certainly is well known to experts. For the reader's convenience, we provide a proof.

Theorem 7.4. *Let X be a contractible compact metric space, and let $\Delta : \mathrm{C}(X) \to (0, +\infty)$ be a positive map. For any finite set $\mathcal{F} \subseteq \mathrm{C}(X)$ and any $\varepsilon > 0$, there exists a finite set $\mathcal{H} \subseteq \mathrm{C}(X)^+$ with $\mathrm{supp}(h) \neq X$ for each $h \in \mathcal{H}$ and there exists $M \in \mathbb{N}$ such that the following property holds: for any homomorphisms*

$$\varphi_1, \varphi_2 : \mathrm{C}(X) \to \mathrm{M}_n(\mathrm{C}(Y)) \quad \text{and} \quad \theta : \mathrm{C}(X) \to \mathrm{M}_m(\mathbb{C}) \subseteq \mathrm{M}_m(\mathrm{C}(Y))$$

where θ is a unital point-evaluation map with $nM < m$, and

$$\mathrm{tr}(\theta(h)) > \Delta(h), \quad h \in \mathcal{H},$$

there is a unitary $u \in \mathrm{M}_{n+m}(\mathrm{C}(Y))$ such that

$$\|\mathrm{diag}\{\phi_1(a), \theta(a)\} - u^* \mathrm{diag}\{\phi_2(a), \theta(a) u\}\| < \varepsilon, \quad a \in \mathcal{F}.$$

The theorem follows from the following two lemmas.

Lemma 7.5. *Let X be a contractible compact metric space, and let $\Delta : \mathrm{C}(X)^+ \to (0, +\infty)$ be a density function. For any finite set $\mathcal{F} \subseteq \mathrm{C}(X)$ and any $\varepsilon > 0$, there exists a finite set $\mathcal{H} \subseteq \mathrm{C}(X)^+$ with $\mathrm{supp}(h) \neq X$ for each $h \in \mathcal{H}$ and there exists $M \in \mathbb{N}$ such that the following property holds: for any homomorphisms*

$$\phi, \psi : \mathrm{C}(X) \to \mathrm{M}_n(\mathrm{C}(Y)) \quad \text{and} \quad \theta : \mathrm{C}(X) \to \mathrm{M}_n(\mathbb{C}) \subseteq \mathrm{M}_n(\mathrm{C}(Y))$$

where θ is a unital point-evaluation map with

$$\mathrm{tr}(\theta(h)) > \Delta(h), \quad h \in \mathcal{H},$$

there is a unitary $u \in \mathrm{M}_{(1+M)n}(\mathrm{C}(Y))$ such that

$$\left\| \mathrm{diag}\{\phi(a), \underbrace{\theta(a), ..., \theta(a)}_{M}\} - u^* \mathrm{diag}\{\psi(a), \underbrace{\theta(a), ..., \theta(a)}_{M}\} u \right\| < \varepsilon_0, \quad a \in \mathcal{F}_0.$$

Proof. Assume the statement were not true. Then there would be $(\mathcal{F}_0, \varepsilon_0)$ such that for any finite set $\mathcal{H} \subseteq \mathrm{C}(X)^+$ and any M, there are $\phi, \psi, \theta : \mathrm{C}(X) \to \mathrm{M}_n(\mathrm{C}(Y))$ for some Y and n with θ a unital point-evaluation map with

$$\mathrm{tr}(\theta(h)) > \Delta(h), \quad h \in \mathcal{H},$$

but

$$\left\| \mathrm{diag}\{\phi(a), \underbrace{\theta(a), ..., \theta(a)}_{M}\} - u^* \mathrm{diag}\{\psi(a), \underbrace{\theta(a), ..., \theta(a)}_{M}\} u \right\| \geq \varepsilon, \quad a \in \mathcal{F}.$$

In particular, let $\mathcal{H}_i$, $i = 1, 2, ...$, be an increasing sequence of finite sets with dense union in $\mathrm{C}(X)^+$ (and $\mathrm{supp}(h) \neq X$ for each $h \in \mathcal{H}_i$, $i = 1, 2, ...$). There are sequences of homomorphisms $\phi_i, \psi_i, \theta_i : \mathrm{C}(X) \to \mathrm{M}_{n_i}(\mathrm{C}(Y_i)) =: B_i$ for some Y_i and n_i with θ_i a unital point-evaluation map with

$$\mathrm{tr}(\theta_i(h)) > \Delta(h), \quad h \in \mathcal{H}_i, \tag{7.18}$$

but

$$\left\| \mathrm{diag}\{\phi_i(a), \underbrace{\theta_i(a), ..., \theta_i(a)}_{i}\} - u^* \mathrm{diag}\{\psi_i(a), \underbrace{\theta_i(a), ..., \theta_i(a)}_{i}\} u \right\| \geq \varepsilon_0, \quad a \in \mathcal{F}_0, \tag{7.19}$$

for all unitary $u \in \mathrm{M}_{1+i}(B_i)$.

Consider $\Phi := (\phi_i), \Psi := (\psi_i), \Theta := (\theta_i) : \mathrm{C}(X) \to \prod B_i / \bigoplus B_i$. Since X is contractive, we have

$$[\Phi] = [\Psi] \quad \text{in } \mathrm{KK}(\mathrm{C}(X), \prod B_i / \bigoplus B_i). \tag{7.20}$$

By (7.18), the map Θ is a unital full embedding (see Definition 2.8 of [1], and we leave the verification to the reader), and then, by Theorem 2.22 together with Theorem 4.5 of [1], there exist $l \in \mathbb{N}$ and a unitary $u \in \mathrm{M}_{1+l}(\prod B_i / \bigoplus B_i)$ such that

$$\left\| \mathrm{diag}\{\Phi(a), \underbrace{\Theta(a), ..., \Theta(a)}_{l}\} - u^* \mathrm{diag}\{\Psi(a), \underbrace{\Theta(a), ..., \Theta(a)}_{l}\} u \right\| < \varepsilon_0, \quad a \in \mathcal{F}_0.$$

Lifting u to a unitary of $\prod B_i$, one has a contradiction to (7.19). □

Lemma 7.6. *Let X be a compact metric space, and let $\Delta : \mathrm{C}(X)^+ \to (0, +\infty)$ be a density function. For any finite set $\mathcal{F} \subseteq \mathrm{C}(X)$, any $\varepsilon > 0$, and any $M \in \mathbb{N}$, there exist a finite set $\mathcal{H} \subseteq \mathrm{C}(X)^+$ and $L \in \mathbb{N}$ with* $\mathrm{supp}(h) \neq X$ *for each $h \in \mathcal{H}$ such that if $\theta : \mathrm{C}(X) \to \mathrm{M}_n(\mathbb{C})$, where $n > L$, is a point-evaluation map with*

$$\mathrm{tr}(\theta(h)) > \Delta(h), \quad h \in \mathcal{H},$$

then there are homomorphisms $\theta_0 : \mathrm{C}(X) \to \mathrm{M}_{n_0}(\mathbb{C}), \theta_1 : \mathrm{C}(X) \to \mathrm{M}_{n_1}(\mathbb{C})$ for some n_0, n_1 with $n_0 + Mn_1 = n$ and a (permutation) unitary u such that

$$\left\| \theta(a) - u^*(\theta_0(h) \oplus \underbrace{\theta_1(a) \oplus \cdots \theta_1(a)}_{M}) u \right\| < \varepsilon, \quad a \in F,$$

and

$$n_0 \leq n_1.$$

Proof. Pick $\delta > 0$ such that $\mathrm{dist}(x, y) < \delta$ implies $|f(x) - f(y)| < \varepsilon$ for all $f \in \mathcal{F}$. Pick a finite set $\{y_1, ..., y_s\} \subseteq X$ which is δ-dense, and continuous functions $h_i : X \to [0, 1]$, $i = 1, ..., s$, such that

$$h_i(x) = \begin{cases} 1, & \mathrm{dist}(x, y_i) < \delta/3, \\ 0, & \mathrm{dist}(x, y_i) > \delta/2. \end{cases}$$

Set $\mathcal{H} = \{h_1, ..., h_s\}$, and pick an integer

$$L > \frac{M^2 + M}{\min\{\Delta(h_i) : i = 1, ..., s\}}.$$

Then $\mathcal{H}$ and L have the property of the lemma.

Indeed, let $\theta : \mathrm{C}(X) \to \mathrm{M}_n$ be a point-evaluation map satisfying

$$\mathrm{tr}(\theta(h)) > \Delta(h), \quad h \in \mathcal{H}. \tag{7.21}$$

Write $\{x_1, ..., x_n\}$ be the evaluation points of θ. Then, choose a map $\sigma : \{x_1, ..., x_n\} \to \{y_1, ..., y_s\}$ such that

$$\mathrm{dist}(x_i, \alpha(x_i)) < \delta, \quad i = 1, ..., L,$$

and for each $j = 1, ..., s$,

$$\alpha(x_i) = y_j \quad \text{if } \mathrm{dist}(x_i, y_j) < \delta/3.$$

Let $\theta' : \mathrm{C}(X) \to \mathrm{M}_n(\mathbb{C})$ denote the point-evaluation map on $\sigma(x_1), ..., \sigma(x_n)$. Then,

$$\|\theta(f) - \theta'(f)\| < \varepsilon, \quad f \in \mathcal{F}.$$

Also, up to a permutation,

$$\theta' = \mathrm{diag}\{\underbrace{\mathrm{ev}_{y_1}, ..., \mathrm{ev}_{y_1}}_{m_1}, ..., \underbrace{\mathrm{ev}_{y_1}, ..., \mathrm{ev}_{y_1}}_{m_s}\}.$$

By (7.21), we have

$$m_i \geq n\Delta(h_i) \geq L\Delta(h_i) > M^2 + M.$$

Then, write $m_i = Md_i + r_i$ with $0 \leq r_i \leq M - 1$, so that, in particular, $d_i > r_i$. Write

$$\theta_0 = \mathrm{diag}\{\underbrace{\overbrace{\mathrm{ev}_{y_1}, ..., \mathrm{ev}_{y_1}}^{r_1}, 0, ..., 0}_{m_1}, ..., \underbrace{\overbrace{\mathrm{ev}_{y_s}, ..., \mathrm{ev}_{y_s}}^{r_s}, 0, ..., 0}_{m_s}\}$$

and

$$\theta_1 = \mathrm{diag}\{\underbrace{\overbrace{0, ..., 0}^{r_1}, \overbrace{\mathrm{ev}_{y_1}, ..., \mathrm{ev}_{y_1}}^{d_1}, 0, ..., 0}_{m_1}, ..., \underbrace{\overbrace{0, ..., 0}^{r_s}, \overbrace{\mathrm{ev}_{y_s}, ..., \mathrm{ev}_{y_s}}^{d_s}, 0, ..., 0}_{m_s}\}.$$

A straightforward calculation shows that θ_0 and θ_1 are the desired maps. □

Proof of Theorem 7.4. Applying Lemma 7.5 to $(\mathcal{F}, \varepsilon/2)$ with respect to the density function $\Delta/4$, one obtains $\mathcal{H}_1 \subseteq \mathrm{C}(X)^+$ and M_1. Applying Lemma 7.6 to $(\mathcal{F} \cup \mathcal{H}_1, \min\{\varepsilon, \Delta(h)/4 : h \in \mathcal{H}_1\})$ and $2M_1$ with respect to the density function Δ, we obtain $\mathcal{H}_2$ and L. Then $\mathcal{H} := \mathcal{H}_1 \cup \mathcal{H}_2$ and $M := 2LM_1$ satisfy the condition of the theorem.

Let $\phi, \psi : \mathrm{C}(X) \to \mathrm{M}_n(\mathrm{C}(Y))$ and $\theta : \mathrm{C}(X) \to \mathrm{M}_m(\mathrm{C}(Y))$ be homomorphisms such that θ is a unital point-evaluation map with $n(2LM_1) < m$ (in particular, $L < m$ and $n < m/2M_1$), and

$$\mathrm{tr}(\theta(h)) > \Delta(h), \quad h \in \mathcal{H}.$$

By Lemma 7.6, there are $\theta_0 : \mathrm{C}(X) \to \mathrm{M}_{n_0}(\mathbb{C})$ and $\theta_1 : \mathrm{C}(X) \to \mathrm{M}_{n_1}(\mathbb{C})$ such that

$$\Big\|\theta(a) - u^*(\theta_0(h) \oplus \underbrace{\theta_1(a) \oplus \cdots \theta_1(a)}_{2M_1})u\Big\| < \min\{\varepsilon/2, \Delta(h)/4 : h \in \mathcal{H}_1\}, \quad a \in \mathcal{F} \cup \mathcal{H}_1,$$

and

$$n_0 \leq n_1.$$

Note that then

$$\mathrm{tr}(\theta_1(h)) > \frac{1}{4}\Delta(h), \quad h \in \mathcal{H}_1.$$

Now, consider the maps

$$(\phi \oplus \theta_0) \oplus (\underbrace{\theta_1 \oplus \cdots \oplus \theta_1}_{2M}) \quad \text{and} \quad (\psi \oplus \theta_0) \oplus (\underbrace{\theta_1 \oplus \cdots \oplus \theta_1}_{2M}).$$

Then the theorem follows from Lemma 7.5. □

Proof of Theorem 7.1. If $\mathrm{rc}(A) = \mathrm{rc}(B) = 0$, then A and B are UHF algebras, and hence $A \cong B$. So, let us assume $\mathrm{rc}(A) = \mathrm{rc}(B) \neq 0$.

Consider the inductive limit constructions

$$\mathrm{M}_{n_0^{(A)}}(\mathrm{C}(X)) \xrightarrow{\phi_1^{(A)}} \mathrm{M}_{n_0^{(A)}(n_1^{(A)}+k_1^{(A)})}(\mathrm{C}(X^{n_1^{(A)}})) \xrightarrow{\phi_2^{(A)}} \mathrm{M}_{n_0^{(A)}(n_1^{(A)}+k_1^{(A)})(n_2^{(A)}+k_2^{(A)})}(\mathrm{C}(X^{n_1^{(A)}n_2^{(A)}})) \longrightarrow \cdots \longrightarrow A,$$

$$\mathrm{M}_{n_0^{(B)}}(\mathrm{C}(X)) \xrightarrow{\phi_1^{(B)}} \mathrm{M}_{n_0^{(B)}(n_1^{(B)}+k_1^{(B)})}(\mathrm{C}(X^{n_1^{(B)}})) \xrightarrow{\phi_2^{(F)}} \mathrm{M}_{n_0^{(B)}(n_1^{(B)}+k_1^{(B)})(n_2^{(B)}+k_2^{(B)})}(\mathrm{C}(X^{n_1^{(B)}n_2^{(B)}})) \longrightarrow \cdots \longrightarrow B.$$

Choose finite subsets

$$\mathcal{F}_1^{(A)} \subseteq \mathrm{M}_{n_0^{(A)}}(C(X)), \mathcal{F}_2^{(A)} \subseteq \mathrm{M}_{n_0^{(A)}(n_1^{(A)}+k_1^{(A)})}(\mathrm{C}(X^{n_1^{(A)}})), ...,$$

and

$$\mathcal{F}_1^{(B)} \subseteq \mathrm{M}_{n_0^{(B)}}(C(X)), \mathcal{F}_2^{(B)} \subseteq \mathrm{M}_{n_0^{(B)}(n_1^{(B)}+k_1^{(B)})}(\mathrm{C}(X^{n_1^{(B)}})), ...,$$

such that

$$\overline{\bigcup_{i=1}^{\infty} \mathcal{F}_i^{(A)}} = A \quad \text{and} \quad \overline{\bigcup_{i=1}^{\infty} \mathcal{F}_i^{(B)}} = B.$$

Also choose $\varepsilon_1 > \varepsilon_2 > \cdots > 0$ such that

$$\sum_{i=1}^{\infty} \varepsilon_i \leq 1.$$

Since A and B are simple, we have the non-zero density functions

$$\Delta_A(h) = \inf\{\tau(h) : \tau \in \mathrm{T}(A)\}, \quad h \in A^+,$$

and

$$\Delta_B(h) = \inf\{\tau(h) : \tau \in \mathrm{T}(B)\}, \quad h \in B^+.$$

For $i = 0, 2, ...$, applying Theorem 5.1 to $(\mathcal{F}_i^{(A)}, \varepsilon_i/2)$ with respect to $\Delta_A/4$, we obtain finite sets $\mathcal{H}_{i,0}^{(A)}, \mathcal{H}_{i,1}^{(A)} \subseteq \mathrm{M}_{m_i^{(A)}}(\mathrm{C}(X^{d_i^{(A)}}))$ and $\delta_i^{(A)} > 0$. Applying Theorem 7.4 to $(\mathcal{F}_i^{(A)}, \varepsilon_i/2)$ with respect to $\Delta_A/2$, we obtain $M_i^{(A)} > 0$ and a finite set $\mathcal{H}_{i,2}^{(A)} \subseteq \mathrm{M}_{m_i^{(A)}}(\mathrm{C}(X^{d_i^{(A)}}))$.

For $i = 1, 3, ...$, applying Theorem 5.1 to $(\mathcal{F}_i^{(F)}, \varepsilon_i/2)$ with respect to $\Delta_B/4$, we obtain finite sets $\mathcal{H}_{i,0}^{(B)}, \mathcal{H}_{i,1}^{(B)} \subseteq \mathrm{M}_{m_i^{(B)}}(\mathrm{C}(X^{d_i^{(B)}}))$ and $\delta_i^{(B)} > 0$. Applying Theorem 7.4 to $(\mathcal{F}_i^{(B)}, \varepsilon_i/2)$ with respect to $\Delta_B/2$, we obtain $M_i^{(B)} > 0$ and a finite set $\mathcal{H}_{i,2}^{(B)} \subseteq \mathrm{M}_{m_i^{(B)}}(\mathrm{C}(X^{d_i^{(B)}}))$.

For each $i = 1, 2, ...,$ set

$$\mathcal{H}_i^{(A)} = \mathcal{H}_{i,0}^{(A)} \cup \mathcal{H}_{i,1}^{(A)} \cup \mathcal{H}_{i,2}^{(A)}, \quad \mathcal{H}_i^{(B)} = \mathcal{H}_{i,0}^{(B)} \cup \mathcal{H}_{i,1}^{(B)} \cup \mathcal{H}_{i,2}^{(B)},$$

and

$$\delta_i = \min\{\delta_i^{(A)}, \delta_i^{(B)}, \frac{1}{M_i^{(A)}}, \frac{1}{M_i^{(B)}}, \frac{1}{4}\Delta_A(h_A), \frac{1}{4}\Delta_B(h_B) : h_A \in \mathcal{H}_i^{(A)}, h_B \in \mathcal{H}_i^{(B)}\}.$$

After telescope, one may assume that

$$\tau(\phi_{i,i+1}^{(A)}(h)) > \Delta_A(h)/2, \quad h \in \mathcal{H}_i^{(A)},\ \tau \in \mathrm{T}(\mathrm{M}_{m_{i+1}^{(A)}}(\mathrm{C}(X^{d_{i+1}^{(A)}}))),\ i = 0, 1, ... \tag{7.22}$$

and

$$\tau(\phi_{i,i+1}^{(B)}(h)) > \Delta_B(h)/2, \quad h \in \mathcal{H}_i^{(B)},\ \tau \in \mathrm{T}(\mathrm{M}_{m_{i+1}^{(B)}}(\mathrm{C}(X^{d_{i+1}^{(B)}}))),\ i = 0, 1, ... \tag{7.23}$$

By Lemma 7.3, there is a diagram

$$\begin{tikzcd}
\mathrm{M}_{n_0^{(A)}}(\mathrm{C}(X)) \arrow[r, "\phi_{1,i_1}^{(A)}"] \arrow[dr, "\phi_{1,i_1}^{(A,B)}"] & \mathrm{M}_{m_{i_1}^{(A)}}(\mathrm{C}(X^{d_{i_1}^{(A)}})) \arrow[r, "\phi_{i_1,i_2}^{(A)}"] & \mathrm{M}_{m_{i_2}^{(A)}}(\mathrm{C}(X^{d_{i_2}^{(A)}})) \arrow[r] \arrow[dr, "\phi_{i_2,i_3}^{(A,B)}"] & \cdots \arrow[r] & A \\
\mathrm{M}_{n_0^{(B)}}(\mathrm{C}(X)) \arrow[r, "\phi_{1,i_1}^{(B)}"] & \mathrm{M}_{m_{i_1}^{(B)}}(\mathrm{C}(X^{d_{i_1}^{(B)}})) \arrow[r, "\phi_{i_1,i_2}^{(B)}"] \arrow[ur, "\phi_{i_1,i_2}^{(B,A)}"] & \mathrm{M}_{m_{i_2}^{(F)}}(\mathrm{C}(X^{d_{i_2}^{(B)}})) \arrow[r] & \cdots \arrow[r] & B.
\end{tikzcd} \tag{7.24}$$

such that

$$\left|\tau(\phi_{i_{s+1},i_{s+2}}^{(B,A)} \circ \phi_{i_s,i_{s+1}}^{(A,B)}(h) - \phi_{i_{s+1},i_{s+2}}^{(A)} \circ \phi_{i_s,i_{s+1}}^{(A)}(h))\right| < \delta_{i_s}$$

for any $s = 0, 2, ...,$ any $h \in \mathrm{M}_{m_{i_s}^{(A)}}(\mathrm{C}(X^{d_{i_s}^{(A)}}))$, and any $\tau \in \mathrm{T}(\mathrm{M}_{m_{i_{s+2}}^{(A)}}(\mathrm{C}(X^{d_{i_{s+2}}^{(A)}})))$; and

$$\left|\tau(\phi_{i_{s+1},i_{s+2}}^{(A,B)} \circ \phi_{i_s,i_{s+1}}^{(B,A)}(h) - \phi_{i_{s+1},i_{s+2}}^{(B)} \circ \phi_{i_s,i_{s+1}}^{(B)}(h))\right| < \delta_{i_s}$$

for any $s = 1, 3, ...,$ any $h \in \mathrm{M}_{m_{i_s}^{(B)}}(\mathrm{C}(X^{d_{i_s}^{(B)}}))$, and any $\tau \in \mathrm{T}(\mathrm{M}_{m_{i_{s+2}}^{(B)}}(\mathrm{C}(X^{d_{i_{s+2}}^{(B)}})))$.

Moreover, for each $s = 0, 2, ...,$

$$\phi_{i_{s+1},i_{s+2}}^{(B,A)} \circ \phi_{i_s,i_{s+1}}^{(A,B)} = \mathrm{diag}\{\pi_1^*, ..., \pi_{K_s}^*, \text{point evaluations}\},$$

and

$$\phi_{i_{s+1},i_{s+2}}^{(A)} \circ \phi_{i_s,i_{s+1}}^{(A)} = \mathrm{diag}\{(\pi_1^{(A)})^*, ..., (\pi_{n_{i_s}^{(A)} \cdots n_{i_{s+1}}^{(A)}})^*, \text{point evaluations}\},$$

with

$$\frac{n_{i_s}^{(A)} \cdots n_{i_{s+1}}^{(A)} - K_s}{(n_{i_s}^{(A)} + k_{i_s}^{(A)}) \cdots (n_{i_{s+1}}^{(A)} + k_{i_{s+1}}^{(A)}) - n_{i_s}^{(A)} \cdots n_{i_{s+1}}^{(A)}} < \delta_{i_s};$$

and for each $s = 1, 3, ...,$

$$\phi_{i_{s+1},i_{s+2}}^{(A,B)} \circ \phi_{i_s,i_{s+1}}^{(B,A)} = \mathrm{diag}\{\pi_1^*, ..., \pi_{K_s}^*, \text{point evaluations}\},$$

and

$$\phi_{i_{s+1},i_{s+2}}^{(B)} \circ \phi_{i_s,i_{s+1}}^{(B)} = \mathrm{diag}\{(\pi_1)^*, ..., (\pi_{n_{i_s}^{(B)} \cdots n_{i_{s+1}}^{(B)}})^*, \text{point evaluations}\},$$

with

$$\frac{n_{i_s}^{(B)} \cdots n_{i_{s+1}}^{(B)} - K_s}{(n_{i_s}^{(B)} + k_{i_s}^{(B)}) \cdots (n_{i_{s+1}}^{(B)} + k_{i_{s+1}}^{(B)}) - n_{i_s}^{(B)} \cdots n_{i_{s+1}}^{(B)}} < \delta_{i_s}.$$

Also note that, by (7.22) and (7.23),

$$\tau(\phi_{i_{s+1},i_{s+2}}^{(A)} \circ \phi_{i_s,i_{s+1}}^{(A)}(h)) > \Delta_A(h)/2, \quad h \in \mathcal{H}_{i_s}^{(A)},\ \tau \in \mathrm{T}(\mathrm{M}_{m_{i_{s+2}}^{(A)}}(\mathrm{C}(X^{d_{i_{s+2}}^{(A)}}))),\ s = 0, 2, ...,$$

and

$$\tau(\phi_{i_{s+1},i_{s+2}}^{(B)} \circ \phi_{i_s,i_{s+1}}^{(B)}(h)) > \Delta_B(h)/2, \quad h \in \mathcal{H}_{i_s}^{(B)},\ \tau \in \mathrm{T}(\mathrm{M}_{m_{i_{s+2}}^{(B)}}(\mathrm{C}(X^{d_{i_{s+2}}^{(B)}}))),\ s = 1, 3, ...$$

Write

$$\phi_{i_{s+1},i_{s+2}}^{(B,A)} \circ \phi_{i_s,i_{s+1}}^{(A,B)} = \mathrm{diag}\{\pi_1^*, ..., \pi_{K_s}^*, \Theta_0, \tilde{\Theta}_1\},$$

and

$$\phi_{i_{s+1},i_{s+2}}^{(A)} \circ \phi_{i_s,i_{s+1}}^{(A)} = \mathrm{diag}\{\pi_1^*, ..., \pi_{K_s}^*, P_0, \Theta_1\},$$

where $\tilde{\Theta}_1, \Theta_1$ are point evaluations with order

$$(n_{i_s}^{(A)} + k_{i_s}^{(A)}) \cdots (n_{i_{s+1}}^{(A)} + k_{i_{s+1}}^{(A)}) - n_{i_s}^{(A)} \cdots n_{i_{s+1}}^{(A)},$$

Θ_0 is a point evaluation with order $n_{i_s}^{(A)} \cdots n_{i_{s+1}}^{(A)} - K_s$, and P_0 is a coordinate projection map with order $n_{i_s}^{(A)} \cdots n_{i_{s+1}}^{(A)} - K_s$.

By Theorem 5.1, there are unitary u_s, $s = 2, 3, ...$, such that

$$\left\| u_s^* \mathrm{diag}\{\pi_1^*, ..., \pi_{K_s}^*, \tilde{\Theta}_1\} u_s - \mathrm{diag}\{\pi_1^*, ..., \pi_{K_s}^*, \Theta_1\} \right\| < \frac{\varepsilon_{i_s}}{2} \quad \text{on } \mathcal{F}_{i_s}^{(A)}. \tag{7.25}$$

Consider the maps

$$\mathrm{diag}\{\Theta_0, \Theta_1\} \quad \text{and} \quad \mathrm{diag}\{P_0, \Theta_1\}.$$

By Theorem 7.4, there is a unitary w_s such that

$$\| w_s^* \mathrm{diag}\{\Theta_0, \Theta_1\} w_s - \mathrm{diag}\{P_0, \Theta_1\} \| < \frac{\varepsilon_{i_s}}{2} \quad \text{on } \mathcal{F}_{i_s}^{(A)}. \tag{7.26}$$

Setting $v_s = u_s w_s$, we have

$$\left\| v_s^* \mathrm{diag}\{\pi_1^*, ..., \pi_{K_s}^*, \Theta_0, \tilde{\Theta}_1\} v_s - \mathrm{diag}\{\pi_1^*, ..., \pi_{K_s}^*, P_0, \Theta_1\} \right\| < \frac{\varepsilon_{i_s}}{2} \quad \text{on } \mathcal{F}_{i_s}^{(A)}.$$

That is,

$$\left\| v_s^* (\phi_{i_{s+1},i_{s+2}}^{(B,A)} \circ \phi_{i_s,i_{s+1}}^{(A,B)}) v_s - \phi_{i_{s+1},i_{s+2}}^{(A)} \circ \phi_{i_s,i_{s+1}}^{(A)} \right\| < \frac{\varepsilon_{i_s}}{2} \quad \text{on } \mathcal{F}_{i_s}^{(A)}.$$

The same argument shows that for $s = 1, 3, ...$, there are unitaries v_s such that

$$\left\| v_s^* (\phi_{i_{s+1},i_{s+2}}^{(A,B)} \circ \phi_{i_s,i_{s+1}}^{(B,A)}) v_s - \phi_{i_{s+1},i_{s+2}}^{(B)} \circ \phi_{i_s,i_{s+1}}^{(B)} \right\| < \frac{\varepsilon_{i_s}}{2} \quad \text{on } \mathcal{F}_{i_s}^{(B)}.$$

Therefore, in the diagram

$$\begin{tikzcd}
\mathrm{M}_{n_0^{(A)}}(\mathrm{C}(X)) \arrow[r, "\phi^{(A)}_{1,i_1}"] \arrow[dr, "\phi^{(A,B)}_{1,i_1}"] & \mathrm{M}_{m^{(A)}_{i_1}}(\mathrm{C}(X^{d^{(A)}_{i_1}})) \arrow[r, "\mathrm{ad}(v_2^{(A)})\circ\phi^{(A)}_{i_1,i_2}"] & \mathrm{M}_{m^{(A)}_{i_2}}(\mathrm{C}(X^{d^{(A)}_{i_2}})) \arrow[r] \arrow[dr, "\phi^{(A,B)}_{i_2,i_3}"] & \cdots \arrow[r] & A \\
\mathrm{M}_{n_0^{(B)}}(\mathrm{C}(X)) \arrow[r, "\phi^{(B)}_{1,i_1}"'] & \mathrm{M}_{m^{(B)}_{i_1}}(\mathrm{C}(X^{d^{(B)}_{i_1}})) \arrow[r, "\phi^{(B)}_{i_1,i_2}"'] \arrow[ur, "\phi^{(B,A)}_{i_1,i_2}"] & \mathrm{M}_{m^{(B)}_{i_2}}(\mathrm{C}(X^{d^{(B)}_{i_2}})) \arrow[r, "\mathrm{ad}(v_3^{(B)})\circ\phi^{(B)}_{i_1,i_2}"'] & \cdots \arrow[r] & B
\end{tikzcd} \tag{7.27}$$

the sth triangle is approximately commutative to within $(\mathcal{F}^{(A)}_{i_s}, \varepsilon_{i_s})$ or $(\mathcal{F}^{(B)}_{i_s}, \varepsilon_{i_s})$. By the approximate intertwining argument ([2]), we have

$$A \cong B,$$

as desired. $\square$

Corollary 7.7. *Let X be a contractible finite-dimensional CW-complex. Let*

$$A := A(X^p, (n_i^{(A)}), (k_i^{(A)}), E^{(A)}) \quad \text{and} \quad B := B(X^q, (n_i^{(B)}), (k_i^{(B)}), F^{(B)})$$

be Villadsen algebras (multiplicities of coordinate projections are 1). Then

$$A \cong B$$

if, and only if,

$$K_0(A) \cong K_0(B) \quad \text{and} \quad \mathrm{rc}(A) = \mathrm{rc}(B).$$

Proof. Consider the inductive limit construction

$$\mathrm{M}_{n_0^{(A)}}(\mathrm{C}(X^p)) \longrightarrow \mathrm{M}_{m_1^{(A)}}(\mathrm{C}(X^{pd_1^{(A)}})) \longrightarrow \mathrm{M}_{m_2^{(A)}}(\mathrm{C}(X^{pd_2^{(A)}})) \longrightarrow \cdots \longrightarrow A.$$

Tensor it with $\mathrm{M}_{pq}(\mathbb{C})$ and add the first map which is induced by coordinate projections with multiplicity one (but no point evaluation):

$$\mathrm{M}_{qn_0^{(A)}}(\mathrm{C}(X)) \longrightarrow \mathrm{M}_{pq}(\mathrm{M}_{n_0^{(A)}}(\mathrm{C}(X^p))) \longrightarrow \mathrm{M}_{pq}(\mathrm{M}_{m_1^{(A)}}(\mathrm{C}(X^{pd_1^{(A)}}))) \longrightarrow \cdots \longrightarrow \mathrm{M}_{pq}(A).$$

Similarly, also consider B, and consider the inductive limit construction

$$\mathrm{M}_{pn_0^{(B)}}(\mathrm{C}(X)) \longrightarrow \mathrm{M}_{pq}(\mathrm{M}_{n_0^{(B)}}(\mathrm{C}(X^q)) \longrightarrow \mathrm{M}_{pq}(\mathrm{M}_{m_1^{(B)}}(\mathrm{C}(X^{qd_1^{(B)}}))) \longrightarrow \cdots \longrightarrow \mathrm{M}_{pq}(B).$$

Since $\mathrm{rc}(A) = \mathrm{rc}(A)$, we have $\mathrm{rc}(\mathrm{M}_{pq}(A)) = \mathrm{rc}(\mathrm{M}_{pq}(A))$, and by Theorem 7.1, $\mathrm{M}_{pq}(A) \cong \mathrm{M}_{pq}(B)$. Since A and B have stable rank one ([3]), they have cancellation of projections, which implies $A \cong B$, as desired. $\square$

Remark 7.8. In fact, in the proof of Theorem 7.4 (and hence the proof of Theorem 7.1 and the corollary above), it is enough to just assume $\mathrm{K}_0(\mathrm{C}(X)) = \mathbb{Z}$ and $\mathrm{K}_1(\mathrm{C}(X)) = \{0\}$, as this is enough to get (7.20), which is the only place we use the assumption that X is contractive. (For a finite CW-complex, is it possible that being K-contractible is actually same as being contractible? On the other hand, to calculate the radius of comparison, we don't really need X to be a CW-complex, as long as X contains a ball of full dimension.)

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