REMARKS ON VILLADSEN ALGEBRAS, II: A GENERALIZED CONSTRUCTION AND THE COMPARISON RADIUS FUNCTION

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Abstract. The authors' recent classification of Jesper Villadsen's remarkable generalization (based on a self-reproducing seed space) of Glimm's infinite tensor product (UHF) C*-algebras, by means of the Cuntz semigroup (in the case of a fixed, well-behaved, seed space), is extended to the analogous generalization of Bratteli's approximately finite-dimensional (AF) C*-algebras. Some progress is made in the direction of distinguishing between algebras based on different seed spaces.

1. Introduction

In [4], a beginning was made on classifying simple C*-algebras beyond what might now be called the classical classifiable class (often just called "classifiable") in which K-theory and traces suffice (see, for instance, in the unital finite case under consideration, [3]), by using new information contained in the Cuntz semigroup. (In particular, what was used in [4] was the Toms radius of comparison.)

In [4], what might be called Villadsen algebras of the first kind (introduced in [11], and quite different from the algebras studied later by Villadsen in [12]), or UHF-Villadsen algebras (as they reduce to Glimm's infinite tensor product algebras when the seed space is a single point), with a fixed well-behaved seed space (for instance a cube), were classified. In the present paper, this is extended to the analogous class of AF-Villadsen algebras.

Examples of this class of algebras were constructed by Hirshberg and Phillips in [5], to show that the particular Cuntz semigroup information used in [4] (the radius of comparison) was no longer sufficient. In the present paper, we consider more detailed information, replacing the radius of comparison, a single number (or as suggested in [5], a single number for each projection), by a function on the tracial simplex which we call the comparison radius function (with supremum the Toms invariant—see Corollary 3.6 and Theorem 5.13):

Theorem 1.1 (Theorem 4.3)**.** *Let X be a K-contractible solid space such that $0 < \dim(X) < \infty$, and let $A(X, G, \mathcal{E})$ and $B(X, H, \mathcal{F})$ be AF-Villadsen algebras with seed space X with rapid dimension growth (see (2.2)), where G and H are Bratteli diagrams and $\mathcal{E}$ and $\mathcal{F}$ are point evaluation sets. Then $A \cong B$ if, and only if, $(\mathrm{Cu}(A), [1_A]) \cong (\mathrm{Cu}(B), [1_B])$. Indeed, $A \cong B$ if, and only if,*

$$((\mathrm{K}_0(A), \mathrm{K}_0^+(A), [1_A]_0), r_\infty^{(0)}(A)) \cong ((\mathrm{K}_0(B), \mathrm{K}_0^+(B), [1_B]_0), r_\infty^{(0)}(B)),$$

where $r_\infty^{(0)}(A)$ and $r_\infty^{(0)}(B)$ are the comparison radius functions of A and B respectively.

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The comparison radius function $r_\infty^{(0)}$ considered in the theorem above factors through the map $\mathrm{T}(A) \to \mathrm{S}_u(\mathrm{K}_0(A))$. A more general comparison radius function, denoted by r_∞, which does not necessarily factor through $\mathrm{T}(A) \to \mathrm{S}_u(\mathrm{K}_0(A))$, is also constructed in the UHF-Villadsen algebra case:

Theorem 1.2 (Theorem 5.4)**.** *Let A be a UHF-Villadsen algebra with seed space a (finite) simplicial complex. There is a upper semicontinuous positive valued affine function r_∞ on $\mathrm{T}^+(A)$ with the following properties:*

(1) *If $h \in \mathrm{Aff}(\mathrm{T}^+(A))$ (continuous affine functions, 0 at 0) and $r_\infty \leq h$, then, h has the property that for any $a, b \in (A \otimes \mathcal{K})^+$,*

$$\mathrm{d}_\tau(a) + h(\tau) < \mathrm{d}_\tau(b),\ \ \tau \in \mathrm{T}^+(A) \quad \Rightarrow \quad a \precsim b.$$

(2) *If $h \in \mathrm{Aff}(\mathrm{T}^+(A))$ and $h(\tau_0) < r_\infty(\tau_0)$ for some $\tau_0 \in \mathrm{T}^+(A)$, then, there are $a, b \in (A \otimes \mathcal{K})^+$ such that*

$$\mathrm{d}_\tau(a) + h(\tau) < \mathrm{d}_\tau(b), \quad \tau \in \mathrm{T}^+(A),$$

but a is not Cuntz-subequivalent to b.

The general comparison radius function r_∞ in fact sometimes distinguishes between UHF-Villadsen algebras with different seed spaces (Corollary 5.7), and it also can be used to show that the action of $\mathrm{Aut}(A)$ on the extreme points of $\mathrm{T}(A)$, which is the Poulsen simplex (see [4]), is not transitive for certain UHF-Villadsen algebras A (Corollary 5.11).

The question of what the structure of the Cuntz semigroup for Villadsen algebras actually is clearly of considerable interest. Since the algebras are of stable rank one (as proved by Villadsen), the recent results of Thiel et al. ([9], [1]) are very much pertinent.

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2. AF-Villadsen algebras and the function $r_\infty^{(0)}$

2.1. **Growth along a Bratteli diagram.** Fix a metrizable compact space X as the seed space. Consider an inductive sequence (G_n, ϕ_n), where $G_n = \mathbb{Z}^{s_n}$ with order unit $u_n = (u_{n,1}, ..., u_{n,s_n})$. Consider the following inductive sequence of C*-algebras:

Set

$$\bigoplus_{j=1}^{s_n} \mathrm{M}_{u_{n,j}}(\mathrm{C}(X^{u_{n,j}})) = A_n.$$

For each $1 \leq j \leq s_{n+1}$, note that the map $\phi_n : G_n \to G_{n+1}$ is induced by a multiplicity matrix $(m_{i,j}^{(n)})$, $1 \leq i \leq s_n$, $1 \leq j \leq s_{n+1}$. Then, for each $1 \leq j \leq s_{n+1}$, choose a partition

$$P_1 \sqcup \cdots \sqcup P_{s_n} = \{1, 2, ..., u_{n+1,j}\}$$

such that

$$|P_i| = m_{i,j}^{(n)} u_{n,i}.$$

Inside each P_i, choose another partition

$$P_i = P_{i,1} \sqcup \cdots \sqcup P_{i,m_{i,j}^{(n)}}$$

such that

$$|P_{i,1}| = \cdots = \left|P_{i,m_{i,j}^{(n)}}\right|.$$

For each $k \in \{1, ..., m_{i,j}^{(n)}\}$, define π_k to be the projection of $X^{u_{i+1,j}}$ onto the coordinate subset $P_{i,k}$. In this case, we also write $\pi_k \in P_i$. Then, define a map $A_n \to A_{n+1}$:

$$\phi_n : A_n \ni (f_1, ..., f_{s_n}) \mapsto \bigoplus_{j=1}^{s_{n+1}} \bigoplus_{i=1}^{s_n} \bigoplus_{\pi_k \in P_i} f_i \circ \pi_k \in A_{n+1}.$$

Let us call P_i the supporting coordinates of f_i, and define the shape of $(\varphi_n)_j$ as

$$((u_{n,1}, m_{1,j}^{(n)}), ..., (u_{n,s_n}, m_{s_n,j}^{(n)})).$$

Denote the (non-simple) limit algebra by $A(X, (G_n, \phi_n), \mathcal{P})$, where $\mathcal{P}$ denotes the choice of partitions.

Remark 2.1. The reason to include the partition is that, unlike the UHF case, even if we define ϕ_n and ϕ_{n+1} so that their partitions are standard, the partition associated to the composition $\phi_{n+1} \circ \phi_n$ is not standard. Therefore, there is no canonical choice for the supporting coordinates of the functions f_i, $i = 1, ..., s_n$. However, we shall now show that, up to isomorphism, the limit algebra $A(X, (G_n, \phi_n), \mathcal{P})$ is independent of the choice of supporting coordinates. Therefore, we can omit $\mathcal{P}$, and just denote the algebra by $A(X, (G_n, \phi_n))$.

Proposition 2.2. *In the setting above, one has* $A(X, (G_n, \phi_n), \mathcal{P}) \cong B(X, (G_n, \phi_n), \mathcal{Q})$ *for any two partition systems* $\mathcal{P}$ *and* $\mathcal{Q}$.

Proof. It is enough to construct isomorphisms $\sigma_n : A_n \to B_n$, $n = 1, 2, ...$, such that

$$\phi_n^{(B)} \circ \sigma_n = \sigma_{n+1} \circ \phi_n^{(A)}, \quad n = 1, 2, ... \tag{2.1}$$

Set $\sigma_1 = \mathrm{id}$. Assume that σ_n is defined, and that it is induced by homeomorphisms $X^{u_{n,i}} \to X^{u_{n,i}}$, $i = 1, ..., s_n$, by the coordinate permutations:

$$(x_1, ..., x_{u_{n,i}}) \mapsto (x_{\sigma_i(1)}, ..., x_{\sigma_i(u_{n,i})}), \quad i = 1, ..., s_n.$$

A calculation shows that, up to a permutation, for each $j = 1, ..., s_{n+1}$,

$$\begin{aligned} & (\phi_n^{(A)}(f_1, ..., f_{s_n}))_j(x_1, ..., x_{u_{n+1,j}}) \\ = \ & \mathrm{diag}\{\underbrace{f_1(x_1, ..., x_{u_{n,1}}), ..., f_1(x_{(m_{1,1}-1)u_{n,1}+1}, ..., x_{m_{1,j}u_{n,s_n}})}_{m_{1,j}}, ..., \\ & \underbrace{f_{s_n}(x_{u_{n+1,j}-m_{s_n,j}u_{n,s_n}+1}, ..., x_{u_{n+1,j}-(m_{s_n,j})u_{n,s_n}+u_{n,s_n}}), ..., f_{s_n}(x_{u_{n+1,j}-u_{n,s_n}+1}, ..., x_{u_{n+1,j}})}_{m_{s_n,j}}\}. \end{aligned}$$

(As defined above, $(\phi_n^{(A)})_j$ has the shape $((u_{n,1}, m_{1,j}^{(n)}), ..., (u_{n,s_n}, m_{s_n,j}^{(n)}))$.)

On the other hand,

$$\begin{aligned}
&(\phi_n^{(B)}(\sigma_n(f_1(y_1,...,y_{u_{n,1}}),...,f_{s_n}(y_1,...,y_{u_{n,s_n}}))))_j(x_1,...,x_{u_{n+1,j}})\\
=\;&(\phi_n^{(B)}(f_1(y_{\sigma_1(1)},...,y_{\sigma_1(u_{n,1})}),...,f_{s_n}(y_{\sigma_{s_n}(1)},...,y_{\sigma_{s_n}(u_{n,s_n})}))))_j(x_1,...,x_{u_{n+1,j}})\\
=\;&\mathrm{diag}\{\underbrace{f_1(\sigma_1(x_1,...,x_{u_{n,1}})),...,f_1(\sigma_1(x_{(m_{1,1}-1)u_{n,1}+1},...,x_{m_{1,j}u_{n,s_n}}))}_{m_{1,j}},...,\\
&\underbrace{f_{s_n}(\sigma_{s_n}(x_{u_{n+1,j}-m_{s_n,j}u_{n,s_n}+1},...,x_{u_{n+1,j}-(m_{s_n,j})u_{n,s_n}+u_{n,s_n}})),...,f_{s_n}(\sigma_{s_n}(x_{u_{n+1,j}-u_{n,s_n}+1},...,x_{u_{n+1,j}}))}_{m_{s_n,j}}\}.
\end{aligned}$$

So, $(\phi_n^{(B)})_j$ has the same shape $((u_{n,1},m_{1,j}^{(n)}),...,(u_{n,s_n},m_{s_n,j}^{(n)}))$. Therefore, there are permutation homeomorphisms $X^{u_{n+1,j}}\to X^{u_{n+1,j}}$, $j=1,...,s_{n+1}$, which induce an isomorphism $\sigma_{n+1}:A_{n+1}\to A_{n+1}$ satisfying (2.1), as desired. □

2.2. **Adding point evaluations and the function $r_\infty^{(0)}$.** Let us assume X is K-contractible (i.e., $\mathrm{K}_*(\mathrm{C}(X))\cong\mathrm{K}_*(\mathbb{C})$, $*=0,1$) and

$$0<\dim(X)<\infty$$

in the rest of the paper.

Let a finite set $E_{i,j}^{(n)}\subseteq X^{u_{n,i}}$ be given for each n, each $1\le i\le s_n$, and each $1\le j\le s_{n+1}$. Set

$$\bigoplus_{i=1}^{s_n}\mathrm{M}_{\tilde u_{n,i}}(\mathrm{C}(X^{u_{n,i}}))=A_n,$$

where $\tilde u_{n,i}$ is defined recursively by

$$\tilde u_{n,i}=\sum_{i'=1}^{s_{n-1}}(m_{i',i}^{(n-1)}+\left|E_{i',i}^{(n-1)}\right|)\tilde u_{n-1,i'},\quad 1\le i\le s_n,$$

for $n>1$, and

$$\tilde u_{1,i}=u_{1,i},\quad i=1,...,s_1.$$

Define a map $\varphi_n:A_n\to A_{n+1}$ by

$$(f_1,...,f_{s_n})\mapsto\bigoplus_{j=1}^{s_{n+1}}\mathrm{diag}\{\underbrace{f_1\circ\pi_1^{(1)},...,f_1\circ\pi_{m_{1,j}^{(n)}}^{(1)}}_{m_{1,j}^{(n)}},f_1(E_{1,j}^{(n)}),...,\underbrace{f_{s_n}\circ\pi_1^{(s_n)},...,f_{s_n}\circ\pi_{m_{s_n,j}^{(n)}}^{(s_n)}}_{m_{s_n,j}^{(n)}},f_{s_n}(E_{s_n,j}^{(n)})\}.$$

Define the $\frac{1}{2}$-dimension ratios at stage n of the sequence $(A_i\to A_{i+1})$ to be

$$r_{n,j}:=\frac{\dim(X)}{2}\cdot\frac{u_{n,j}}{\tilde u_{n,j}}=\frac{\dim(X)}{2}\cdot\frac{\sum_{i=1}^{s_{n-1}}m_{i,j}^{(n-1)}u_{n-1,i}}{\sum_{i=1}^{s_{n-1}}(m_{i,j}^{(n-1)}+\left|E_{i,j}^{(n-1)}\right|)\tilde u_{n-1,i}},\quad j=1,...,s_n.$$

Note that $\mathrm{K}_0(A_n)\cong\mathbb{Z}^{s_n}$. Then, denote by

$$r_n^{(0)}:=(r_{n,1},...,r_{n,s_n})$$

the corresponding continuous affine function on $\mathrm{S}_u(\mathrm{K}_0(A_n))$, and then regard $r_n^{(0)}$ as an element of $\mathrm{Aff}(\mathrm{S}_u(\mathrm{K}_0(A)))$.

For each $i = 1, 2, ...$ and $j > i$, denote the (coordinate) multiplicity matrices of the partial maps $\phi_{i,j}$ and $\varphi_{i,j}$ by $[\phi_{i,j}]$ and $[\varphi_{i,j}]$ respectively, and denote by $[E_{i,j}]$ the multiplicity matrix of the point evaluation maps between A_i and A_j. Note that

$$[\varphi_{i,j}] = [\phi_{i,j}] + [E_{i,j}], \quad i = 1, 2, ..., \ i < j,$$

$$u_i = [\phi_{1,i}](u_1) \quad \text{and} \quad \tilde{u}_i = [\varphi_{1,i}](u_1), \quad i = 2, 3, ...,$$

and

$$\tilde{u}_1 = u_1.$$

Then

$$\begin{aligned} r_i^{(0)} &= \frac{\dim(X)}{2}(\frac{u_{i,j}}{\tilde{u}_{i,j}})_{1\leq j\leq s_i} = \frac{\dim(X)}{2}\frac{[\phi_{1,i}](u_1)}{[\varphi_{1,i}](u_1)} \\ &= \frac{\dim(X)}{2}\frac{[\phi_{1,i}](u_1)}{([\phi_{1,i}] + [E_{1,i}])(u_1)} \\ &= \frac{\dim(X)}{2}\frac{[\phi_{i-1,i}](\cdots[\phi_{1,2}](u_1)\cdots)}{([\phi_{i-1,i}] + [E_{i-1,i}])(\cdots([\phi_{1,2}] + [E_{1,2}])(u_1)\cdots)}, \end{aligned}$$

where the ratio of two vectors means the vector of individual fractions. So, it is clear from the last expression that r_i, $i = 1, 2, ...$, regarded as a sequence in $\mathrm{Aff}(\mathrm{S}_u(\mathrm{K}_0(A)))$, is decreasing.

Now, let us ensure that the point evaluation sets $E_{i,j}^{(n)}$, $i = 1, ..., s_{n-1}$, $j = 1, ..., s_n$, are sufficiently small that

$$(r_i^{(0)}) \text{ converges uniformly to a strictly positive function } r_\infty^{(0)} \in \mathrm{Aff}(\mathrm{S}_u(\mathrm{K}_0(A))). \tag{2.2}$$

(Recall that $\mathrm{Aff}(\mathrm{S}_u(\mathrm{K}_0(A))) = \varinjlim \mathrm{Aff}(\mathrm{S}_u(\mathrm{K}_0(A_n)))$.)

Since the sequence $(r_i^{(0)})$ converges uniformly, the function $r_\infty^{(0)}$ is continuous. By compactness of $\mathrm{S}_u(\mathrm{K}_0(A))$, there is $\delta > 0$ such that

$$r_i^{(0)}(\tau) \geq r_\infty^{(0)}(\tau) \geq \delta, \quad \tau \in \mathrm{S}_u(\mathrm{K}_0(A)), \ i = 1, 2, ...$$

This translates to

$$\frac{\dim(X)}{2}\frac{[\phi_{1,i}](u_1)}{([\phi_{1,i}] + [E_{1,i}])(u_1)} \geq \delta, \quad i = 1, 2, ..., \tag{2.3}$$

where "$\geq \delta$" means each entry of the vector is larger than δ.

Note that

$$\begin{aligned} r_i^{(0)} - r_{i+k}^{(0)} &= \frac{\dim(X)}{2}(\varphi_{i,i+k}^*(\frac{[\phi_{1,i}](u_1)}{([\phi_{1,i}] + [E_{1,i}])(u_1)}) - \frac{[\phi_{1,i+k}](u_1)}{([\phi_{1,i+k}] + [E_{1,i+k}])(u_1)}) \\ &= \frac{\dim(X)}{2}(\frac{([\phi_{i,i+k}] + [E_{i,i+k}])\circ[\phi_{1,i}](u_1)}{([\phi_{i,i+k}] + [E_{i,i+k}])\circ([\phi_{1,i}] + [E_{1,i}])(u_1)} \\ &\quad -\frac{[\phi_{i,i+k}]\circ[\phi_{1,i}](u_1)}{([\phi_{i,i+k}] + [E_{i,i+k}])\circ([\phi_{1,i}] + [E_{1,i}])(u_1)}) \\ &= \frac{\dim(X)}{2}\frac{[E_{i,i+k}]\circ[\phi_{1,i}](u_1)}{([\phi_{i,i+k}] + [E_{i,i+k}])\circ([\phi_{1,i}] + [E_{1,i}])(u_1)} \in \mathrm{Aff}(\mathrm{S}_u(\mathrm{K}_0(A_{i+k}))). \end{aligned}$$

By the Cauchy criterion, the uniform convergence of $(r_i^{(0)})$ translates to the condition

$$\lim_{i\to\infty}\sup_k \left\| \frac{[E_{i,i+k}]\circ[\phi_{1,i}](u_1)}{([\phi_{i,i+k}]+[E_{i,i+k}])\circ([\phi_{1,i}]+[E_{1,i}])(u_1)} \right\|_\infty = 0. \tag{2.4}$$

Lemma 2.3. *With the condition 2.2, one has*

$$\lim_{i\to\infty}\sup_k \left\| \frac{[E_{i,i+k}]\circ([\phi_{1,i}]+[E_{1,i}])(u_1)}{([\phi_{i,i+k}]+[E_{i,i+k}])\circ([\phi_{1,i}]+[E_{1,i}])(u_1)} \right\|_\infty = 0, \tag{2.5}$$

which, by definition, may be written as

$$\lim_{i\to\infty}\sup_k \max\{\frac{([E_{i,i+k}](\tilde{u}_i))_j}{([\varphi_{i,i+k}](\tilde{u}_i))_j} : j=1,...,s_{i+k}\} = 0. \tag{2.6}$$

Proof. By (2.3), one has

$$([\phi_{1,i}]+[E_{1,i}])(u_1) \leq \frac{\dim(X)}{2}\frac{1}{\delta}[\phi_{1,i}](u_1), \quad i=1,2,...,$$

and therefore, for each $i,k=1,2,...$,

$$\frac{[E_{i,i+k}]\circ([\phi_{1,i}]+[E_{1,i}])(u_1)}{([\phi_{i,i+k}]+[E_{i,i+k}])\circ([\phi_{1,i}]+[E_{1,i}])(u_1)} \leq \frac{\dim(X)}{2}\frac{1}{\delta}\frac{[E_{i,i+k}]\circ[\phi_{1,i}](u_1)}{([\phi_{i,i+k}]+[E_{i,i+k}])\circ([\phi_{1,i}]+[E_{1,i}])(u_1)}.$$

Thus, by (2.4),

$$\begin{aligned} & \lim_{i\to\infty}\sup_k \left\| \frac{[E_{i,i+k}]\circ([\phi_{1,i}]+[E_{1,i}])(u_1)}{([\phi_{i,i+k}]+[E_{i,i+k}])\circ([\phi_{1,i}]+[E_{1,i}])(u_1)} \right\|_\infty \\ \leq\ & \frac{\dim(X)}{2}\frac{1}{\delta}\lim_{i\to\infty}\sup_k \left\| \frac{[E_{i,i+k}]\circ[\phi_{1,i}](u_1)}{([\phi_{i,i+k}]+[E_{i,i+k}])\circ([\phi_{1,i}]+[E_{1,i}])(u_1)} \right\|_\infty \\ =\ & 0. \end{aligned}$$

which is (2.5). □

Note that

$$0 < r_\infty^{(0)}(\tau) < \frac{\dim(X)}{2}, \quad \tau \in \mathrm{S}_u(\mathrm{K}_0(A)).$$

Remark 2.4. The function $r_\infty^{(0)}$ also can be regarded as an affine function on $\mathrm{T}^+(\mathrm{K}_0(A))$, the cone of all positive homomorphisms $\mathrm{K}_0(A)\to\mathbb{R}$ (the simplex $\mathrm{S}_u(G)$ is a base for $\mathrm{T}^+(\mathrm{K}_0(A))$).

Example 2.5 ([5]). Let X be K-contractible. Consider two UHF-Villadsen algebras $A^{(1)} := A(X,(n_i^{(1)}),(k_i^{(1)}))$ and $A^{(2)} := A(X,(n_i^{(2)}),(k_i^{(2)}))$. Following [5], introduce $k_i^{(1,2)}$ point evaluations from $A_i^{(1)}$ to $A_{i+1}^{(2)}$ and $k_i^{(2,1)}$ point evaluations from $A_i^{(2)}$ to $A_{i+1}^{(1)}$. Thus, the multiplicity matrices for the connecting maps ϕ_i and φ_i, before and after adding point evaluations, are given by

$$[\phi_i] = \begin{pmatrix} n_i^{(1)} & \\ & n_i^{(2)} \end{pmatrix} \quad\text{and}\quad [\varphi_i] = \begin{pmatrix} n_i^{(1)} & \\ & n_i^{(2)} \end{pmatrix} + \begin{pmatrix} k_i^{(1)} & k_i^{(2,1)} \\ k_i^{(1,2)} & k_i^{(2)} \end{pmatrix}.$$

Hence,

$$\begin{pmatrix} u_{i,1} \\ u_{i,2} \end{pmatrix} = \begin{pmatrix} n_{i-1}^{(1)} & \\ & n_{i-1}^{(2)} \end{pmatrix} \cdots \begin{pmatrix} n_1^{(1)} & \\ & n_1^{(2)} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

and

$$\begin{pmatrix} \tilde{u}_{i,1} \\ \tilde{u}_{i,2} \end{pmatrix} = \begin{pmatrix} n_{i-1}^{(1)} + k_{i-1}^{(1)} & k_{i-1}^{(2,1)} \\ k_{i-1}^{(1,2)} & n_{i-1}^{(2)} + k_{i-1}^{(2)} \end{pmatrix} \cdots \begin{pmatrix} n_1^{(1)} + k_1^{(1)} & k_1^{(2,1)} \\ k_1^{(1,2)} & n_1^{(2)} + k_1^{(2)} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Note that the affine map

$$[\varphi_i]_0^* : [0,1] \cong \mathrm{S}_u(\mathrm{K}_0(A_i)) \leftarrow \mathrm{S}_u(\mathrm{K}_0(A_{i+1})) \cong [0,1]$$

is determined by the extreme point assignments

$$[0,1] \ni \frac{k_i^{(2,1)}\tilde{u}_{i,2}}{(n_i^{(1)} + k_i^{(1)})\tilde{u}_{i,1} + k_i^{(2,1)}\tilde{u}_{i,2}} \mapsfrom 0 \quad \text{and} \quad [0,1] \ni \frac{(n_i^{(2)} + k_i^{(2)})\tilde{u}_{i,2}}{k_i^{(1,2)}\tilde{u}_{i,1} + (n_i^{(2)} + k_i^{(2)})\tilde{u}_{i,2}} \mapsfrom 1.$$

Define the compression coefficient

$$c_i = \frac{(n_i^{(2)} + k_i^{(2)})\tilde{u}_{i,2}}{k_i^{(1,2)}\tilde{u}_{i,1} + (n_i^{(2)} + k_i^{(2)})\tilde{u}_{i,2}} - \frac{k_i^{(2,1)}\tilde{u}_{i,2}}{(n_i^{(1)} + k_i^{(1)})\tilde{u}_{i,1} + k_i^{(2,1)}\tilde{u}_{i,2}}.$$

Choose $k_i^{(2,1)}$, $k_i^{(1,2)}$, $i = 1, 2, ...$, sufficiently small that

$$c_1 c_2 \cdots > 0,$$

which implies that

$$\lim_{i\to\infty}(c_i c_{i+1} \cdots) = 1. \tag{2.7}$$

(This ensures that the simplex $\mathrm{S}_u(\mathrm{K}_0(A))$ does not collapse to a single point, and hence $\mathrm{S}_u(\mathrm{K}_0(A)) \cong [0,1]$.)

Write the extreme points of $\mathrm{S}_u(\mathrm{K}_0(A)) \cong \varprojlim([0,1], \varphi_i^*)$ as

$$\tau_1 = (s_1, s_2, ...) \in \prod_{i=1}^{\infty}[0,1] \quad \text{and} \quad \tau_2 = (t_1, t_2, ...) \in \prod_{i=1}^{\infty}[0,1],$$

where $s_1 < t_1,\ s_2 < t_2, ...$ Then

$$t_i - s_i = c_i c_{i+1} \cdots, \quad i = 1, 2, ...,$$

and, by (2.7),

$$\lim_{i\to\infty} s_i = 0 \quad \text{and} \quad \lim_{i\to\infty} t_i = 1.$$

Let us calculate the function $r_\infty^{(0)}$. For each $i = 1, 2, ...$,

$$r_i^{(0)}(\tau_1) = \frac{1}{2}\dim(X) \cdot (\frac{u_{i,1}}{\tilde{u}_{i,1}}(1 - s_i) + \frac{u_{i,2}}{\tilde{u}_{i,2}} \cdot s_i) \quad \text{and} \quad r_i^{(0)}(\tau_2) = \frac{1}{2}\dim(X) \cdot (\frac{u_{i,1}}{\tilde{u}_{i,1}}(1 - t_i) + \frac{u_{i,2}}{\tilde{u}_{i,2}} \cdot t_i).$$

Hence,

$$r_\infty^{(0)}(\tau_1) = \lim_{i\to\infty} r_i^{(0)}(\tau_1) = \frac{1}{2}\dim(X) \cdot \lim_{i\to\infty} \frac{u_{i,1}}{\tilde{u}_{i,1}} \quad \text{and} \quad r_\infty^{(0)}(\tau_2) = \lim_{i\to\infty} r_i^{(0)}(\tau_2) = \frac{1}{2}\dim(X) \cdot \lim_{i\to\infty} \frac{u_{i,2}}{\tilde{u}_{i,2}}.$$

3. The comparison property of $r_\infty^{(0)}$

In this section, let us study the comparison properties of the function $r_\infty^{(0)}$. It turns out that the function $r_\infty^{(0)}$ is the smallest continuous affine function on the state space of the order-unit K_0-group which guarantees comparison. Thus, the function $r_\infty^{(0)}$ can be recovered from the Cuntz semigroup of the limit algebra A:

Theorem 3.1. *Let A be an AF-Villadsen algebra which satisfies* (2.6)*. The function $r_\infty^{(0)}$, regarded as a function on $\mathrm{T}(A)$, has the following property:*

$$\mathrm{d}_\tau(a) + r_\infty^{(0)}(\tau) < \mathrm{d}_\tau(b), \quad \tau \in \mathrm{T}(A) \quad \Rightarrow \quad a \precsim b, \quad a, b \in A \otimes \mathcal{K}. \tag{3.1}$$

On the other hand, assume that X is K-contractible (i.e., $\mathrm{K}_*(\mathrm{C}(X)) = \mathrm{K}_*(\mathbb{C})$, $* = 0,1$) and is a finite dimensional solid space (i.e., it contains a Euclidean ball of dimension $\dim(X)$). If $h \in \mathrm{Aff}(\mathrm{S}_u(\mathrm{K}_0(A)))$ and if there is $\tau \in \mathrm{T}(A)$ such that

$$h(\tau) < r_\infty^{(0)}(\tau),$$

then there are $a, b \in (A \otimes \mathcal{K})^+$ such that

$$\mathrm{d}_\tau(a) + h(\tau) < \mathrm{d}_\tau(b), \quad \tau \in \mathrm{T}(A),$$

but a is not Cuntz subequivalent to b. Hence $r_\infty^{(0)}$ is the (unique) minimum among the continuous affine functions which satisfy (3.1) *and factor through $\mathrm{T}(A) \to \mathrm{S}_u(\mathrm{K}_0(A))$.*

Proof. Let $a, b \in (A \otimes \mathcal{K})^+$ be positive elements satisfying

$$\mathrm{d}_\tau(a) + r_\infty^{(0)}(\tau) < \mathrm{d}_\tau(b), \quad \tau \in \mathrm{T}(A).$$

Fix an arbitrary $\varepsilon > 0$ for the time being. Since the function r_∞ is continuous and strictly positive, by the compactness of $\mathrm{T}(A)$, there is $\delta > 0$ such that

$$\mathrm{d}_\tau((a-\varepsilon)_+) + r_\infty^{(0)}(\tau) + \delta < \mathrm{d}_\tau(b), \quad \tau \in \mathrm{T}(A).$$

Since (r_k) converges to $r_\infty^{(0)}$ uniformly, there is $k \in \mathbb{N}$ large enough that

$$\mathrm{d}_\tau((a-\varepsilon)_+) + r_k^{(0)}(\tau) + \frac{\delta}{2} < \mathrm{d}_\tau(b), \quad \tau \in \mathrm{T}(A),$$

and,

$$0 < r_k^{(0)}(\tau) - r_n^{(0)}(\tau) < \frac{\delta}{8}, \quad \tau \in \mathrm{T}(A_n),\ n > k, \tag{3.2}$$

where $r_k^{(0)}$ and $r_n^{(0)}$ are regarded as elements of $\mathrm{Aff}(\mathrm{S}_u(\mathrm{K}_0(A_n)))$.

Note that there is a (trivial) projection $q \in A_n$, for a sufficiently large n, such that

$$r_k^{(0)}(\tau) + \frac{\delta}{8} < \tau(q) < r_k^{(0)}(\tau) + \frac{\delta}{2}, \quad \tau \in \mathrm{T}(A_n), \tag{3.3}$$

and then

$$\mathrm{d}_\tau((a-\varepsilon)_+ \oplus q) \leq \mathrm{d}_\tau((a-\varepsilon)_+) + r_k^{(0)}(\tau) + \frac{\delta}{2} < \mathrm{d}_\tau(b), \quad \tau \in \mathrm{T}(A).$$

Since A is simple, this implies that there is $N \in \mathbb{N}$ such that

$$(N+1)([(a-\varepsilon)_+] + [q]) < N[b],$$

where $[\cdot]$ denotes the Cuntz class. (See the proof of Proposition 3.2 of [8].) Then, by Lemma 5.6 of [6], there are $a_n, b_n, \in A_n$, for n sufficiently large, such that

$$\|a_n - (a-\varepsilon)_+\| < \varepsilon, \quad \|b_n - b\| < \varepsilon, \quad b_n \precsim b,$$

and

$$(N+1)([a_n]+[q]) < N[b_n],$$

which implies

$$\mathrm{tr}(a_n(x)) + \mathrm{tr}(q(x)) < \mathrm{tr}(b_n(x)), \quad x \in X^{u_{n,j}}, \quad j = 1, ..., s_n, \tag{3.4}$$

where tr denotes the normalized trace of a matrix algebra.

Note that, by (3.2) and (3.3),

$$r_n^{(0)}(\tau) \approx_{\delta/8} r_k^{(0)}(\tau) < \tau(q) - \frac{\delta}{8}, \quad \tau \in \mathrm{T}(A_n),$$

and hence

$$r_n^{(0)}(\tau) < \tau(q), \quad \tau \in \mathrm{T}(A_n).$$

By (3.4),

$$\mathrm{tr}(a_n(x)) + r_n(\mathrm{tr}_x) < \mathrm{tr}(a_n(x)) + \mathrm{tr}(q(x)) < \mathrm{tr}(b_n(x)), \quad x \in X^{u_{n,j}},\ j = 1, ..., s_n.$$

Since

$$r_{n,j}^{(0)} = \frac{\dim(X)}{2} \cdot \frac{u_{n,j}}{\tilde{u}_{n,j}}, \quad j = 1, ..., s_n,$$

this implies

$$\mathrm{rank}(a_n(x)) + \frac{1}{2}\dim(X^{u_{n,j}}) < \mathrm{rank}(b_n(x)), \quad x \in X^{u_{n,j}},\ j = 1, ..., s_n. \tag{3.5}$$

By Theorem 4.6 of [10], one has $a_n \precsim b_n$, and hence

$$(a-2\varepsilon)_+ \precsim ((a-\varepsilon)_+ - \varepsilon)_+ \precsim a_n \precsim b_n \precsim b.$$

Since ε is arbitrary, one has $a \precsim b$. This shows (3.1).

Now, let us show that r_∞ is the smallest continuous affine function which satisfies (3.1).

Let $h \in \mathrm{Aff}(\mathrm{S}_u(\mathrm{K}_0(A)))$ be such that $h([\tau]) < r_\infty^{(0)}([\tau])$ for some $\tau \in \mathrm{T}(A)$. Set

$$M = \max\{h(\tau) : \tau \in \mathrm{S}_u(\mathrm{K}_0(A))\}$$

and

$$\delta = \sup\{r_\infty^{(0)}(\tau) - h(\tau) : \tau \in \mathrm{S}_u(\mathrm{K}_0(A))\} > 0. \tag{3.6}$$

Since r_∞ and h are continuous, and $\mathrm{S}_u(\mathrm{K}_0(A))$ is compact, there is $\tau_0 \in \mathrm{S}_u(\mathrm{K}_0(A))$ such that

$$r_\infty^{(0)}(\tau_0) - h(\tau_0) = \delta. \tag{3.7}$$

Recall that (since X is K-contractible) $\mathrm{Aff}(\mathrm{S}_u(\mathrm{K}_0(A)))$ has a standard inductive limit decomposition:

$$(\mathbb{R}^{s_1}, \|\cdot\|_\infty) \longrightarrow (\mathbb{R}^{s_2}, \|\cdot\|_\infty) \longrightarrow \cdots \longrightarrow \mathrm{Aff}(\mathrm{S}_u(\mathrm{K}_0(A))),$$

where the connecting map $\mathbb{R}^{s_n} \to \mathbb{R}^{s_{n+1}}$ is given by

$$(t_1, ..., t_{s_n}) \mapsto \left(\frac{1}{\tilde{u}_{n+1,1}} \sum_{i=1}^{s_n} (m_{i,1}^{(n)} + \left|E_{i,1}^{(n)}\right|)(\tilde{u}_{n,i} t_i), ..., \frac{1}{\tilde{u}_{n+1,s_{n+1}}} \sum_{i=1}^{s_n} (m_{i,s_{n+1}}^{(n)} + \left|E_{i,s_{n+1}}^{(n)}\right|)(\tilde{u}_{n,i} t_i)\right). \tag{3.8}$$

Choose $\varepsilon > 0$ sufficiently small that

$$\frac{2\varepsilon}{\delta + \frac{3}{4}\varepsilon} < \frac{\delta}{64(M+1)}. \tag{3.9}$$

By (3.6) and (3.7), a compactness argument shows that if n is sufficiently large, there is $h_n \in \mathbb{R}^{s_n}$ such that

$$r_{n,j_n}^{(0)} - h_{n,j_n} \approx_{\varepsilon/3} \delta$$

for some $j_n \in \{1, ..., s_n\}$,

$$\left\|\varphi_{n,\infty}^*(r_n^{(0)}) - r_\infty^{(0)}\right\|_\infty < \varepsilon/3 \quad \text{and} \quad \left\|\varphi_{n,\infty}^*(h_n) - h\right\|_\infty < \varepsilon/3, \tag{3.10}$$

and

$$r_{n,j}^{(0)} - h_{n,j} < \delta + \varepsilon, \quad j = 1, ..., s_n. \tag{3.11}$$

Also assume n is sufficiently large that

$$h_{n,j} + \frac{\delta}{4} < (M+1) \tag{3.12}$$

and, furthermore, using (2.6), for any $k > n$,

$$\frac{1}{\tilde{u}_{k,j}} \sum_{i=1}^{s_n} [E_{n,k}]_{i,j} \tilde{u}_{n,i} < \frac{\delta}{64(M+1)}, \quad j = 1, ..., s_k. \tag{3.13}$$

By simplicity, one may also assume that $\tilde{u}_{n,j}$, $j = 1, ..., s_n$, are sufficiently large that for any positive real number t, there is $d \in \mathbb{N}$ such that

$$\frac{\delta}{8} < \frac{d}{\tilde{u}_{n,j}} - t < \frac{\delta}{4}. \tag{3.14}$$

For any $k > n$, consider $h_k := \phi_{n,k}^*(h_n)$ and $r_k = \phi_{n,k}^*(r_n)$. Note that there is j_k such that

$$r_{k,j_k}^{(0)} - h_{k,j_k} \approx_\varepsilon \delta.$$

Then, using (3.8), (3.11), and the definition of S below, one has

$$\begin{aligned}
\delta &\approx_\varepsilon r^{(0)}_{k,j_k} - h_{k,j_k} \\
&= \frac{1}{\tilde{u}_{k,j_k}} \sum_{i=1}^{s_n} (m_{i,j_k} + |E_{i,j_k}|)(\tilde{u}_{n,i})(r^{(0)}_{n,i} - h_{n,i}) \\
&= \frac{1}{\tilde{u}_{k,j_k}} \sum_{i\notin S} (m_{i,j_k} + |E_{i,j_k}|)(\tilde{u}_{n,i})(r^{(0)}_{n,i} - h_{n,i}) + \frac{1}{\tilde{u}_{k,j_k}} \sum_{i\in S} (m_{i,j_k} + |E_{i,j_k}|)(\tilde{u}_{n,i})(r^{(0)}_{n,i} - h_{n,i}) \\
&\leq (\frac{1}{\tilde{u}_{k,j_k}} \sum_{i\notin S} (m_{i,j_k} + |E_{i,j_k}|)(\tilde{u}_{n,i}))(\delta+\varepsilon) + (\frac{1}{\tilde{u}_{k,j_k}} \sum_{i\in S} (m_{i,j_k} + |E_{i,j_k}|)(\tilde{u}_{n,i}))\frac{\delta}{4} \\
&= (1-\gamma)(\delta+\varepsilon) + \gamma\frac{\delta}{4},
\end{aligned}$$

where

$$S := \{i = 1, ..., s_n : r^{(0)}_{n,i} - h_{n,i} \leq \frac{\delta}{4}\}$$

and

$$\gamma := \frac{1}{\tilde{u}_{k,j_k}} \sum_{i\in S} (m_{i,j_k} + |E_{i,j_k}|)(\tilde{u}_{n,i}).$$

Hence,

$$(1-\gamma)(\delta+\varepsilon) + \gamma\frac{\delta}{4} > \delta - \varepsilon,$$

which, together with (3.9), implies

$$\gamma < \frac{2\varepsilon}{\delta + \frac{3}{4}\varepsilon} < \frac{\delta}{64(M+1)},$$

or

$$\frac{1}{\tilde{u}_{k,j_k}} \sum_{i\in S} (m_{i,j_k} + |E_{i,j_k}|)(\tilde{u}_{n,i}) < \frac{\delta}{64(M+1)}. \tag{3.15}$$

By the choice of $\tilde{u}_{n,j}$, $j = 1, ..., s_n$ (see (3.14)), there are natural numbers d_j, $j = 1, ..., s_n$, such that

$$\frac{\delta}{8} < \frac{d_j}{\tilde{u}_{n,j}} - h_{n,j} < \frac{\delta}{4}. \tag{3.16}$$

Note that, together with (3.12), one has

$$d_i < (M+1)\tilde{u}_{n,j}, \quad j = 1, ..., s_n. \tag{3.17}$$

If $j \notin S$ (so that $h_{n,j} < r_{n,j} - \delta/4$), one has

$$\frac{d_j}{\tilde{u}_{n,j}} < h_{n,j} + \frac{\delta}{4} < r^{(0)}_{n,j} = \frac{1}{2}\mathrm{dim}(X)\frac{u_{n,j}}{\tilde{u}_{n,j}},$$

and so

$$2d_j < u_{n,j}\mathrm{dim}(X).$$

Since X is solid, there is a $(2d_j+1)$-dimensional Euclidean ball $B_j \subseteq X^{u_{n,j}}$, and there is a complex vector bundle E_j over ∂B_j ($\cong S^{2d_j}$) such that

$$\mathrm{rank}(E_j) = d_j \quad \text{and} \quad c_{d_j}(E_j) \in \mathrm{H}^{2d_j}(S^{2d_j}) \setminus \{0\}.$$

(Such a vector bundle exists, as, otherwise, the d_j-th Chern class of every vector bundle would be trivial, and then the Chern character would not induce a rational isomorphism between the K-group and the cohomology group of the sphere S^{2d_j}.)

Denote by $p_j \in \mathrm{C}(S^{2d_j}) \otimes \mathcal{K}$ the projection corresponding to E_j, and extended to a positive element of $\mathrm{C}(X^{u_{n,j}}) \otimes \mathcal{K}$ with rank at least d_j.

If $j \in S$, just choose p_j to be a constant function with rank d_j. Set

$$p = \bigoplus_{j=1}^{s_n} p_j.$$

Then, by (3.16),

$$\mathrm{d}_{\mathrm{tr}_x}(p) = \frac{\mathrm{rank}(p_j)}{\tilde{u}_{n,j}} \geq \frac{d_j}{\tilde{u}_{n,j}} > h_{n,j} + \frac{\delta}{8}, \quad x \in X_{n,j},\ j = 1, ..., s_n. \tag{3.18}$$

Choose $q \in A_n$ to be a trivial projection such that

$$\frac{\delta}{32} < \tau(q) < \frac{\delta}{16}, \quad \tau \in \mathrm{T}(A_n).$$

Then it follows from (3.10) and (3.18) that

$$\mathrm{d}_\tau(q) + h(\tau) < h(\tau) + \frac{\delta}{16} < h_n(\tau) + \frac{\delta}{8} < \mathrm{d}_\tau(p), \quad \tau \in \mathrm{T}(A).$$

Let us show that q actually is not Cuntz subequivalent to p. For any $k > n$, consider the direct summand A_{k,j_k}, and consider the closed subset

$$D := \underbrace{S^{2d_1} \times \cdots \times S^{2d_1}}_{m_{1,j_k}} \times \cdots \times \underbrace{S^{2d_1} \times \cdots \times S^{2d_1}}_{m_{1,j_k}} \subseteq X^{u_{k,j_k}}.$$

Then the restriction of p to this closed subset is equivalent to the projection onto a vector bundle with total Chern class non-zero at degree

$$2 \sum_{i \notin S} m_{i,j_k} d_i,$$

and by Remark 3.2 of [4], this implies that any trivial sub-bundle must have rank at most

$$\sum_{i=1}^{s_n} (m_{i,j_k} + |E_{i,j_k}|) d_i - \sum_{i \notin S} m_{i,j_k} d_i.$$

Then, by (3.17), (3.15), and (3.13), the (normalized) trace of the projection associated to this trivial sub-bundle is at most

$$\begin{aligned}
& \frac{1}{\tilde{u}_{k,j_k}}(\sum_{i=1}^{s_n}(m_{i,j_k}+|E_{i,j_k}|)d_i - \sum_{i\notin S} m_{i,j_k}d_i) \\
= \ & \frac{1}{\tilde{u}_{k,j_k}}(\sum_{i\in S}(m_{i,j_k}+|E_{i,j_k}|)d_i + \sum_{i\notin S}(m_{i,j_k}+|E_{i,j_k}|)d_i - \sum_{i\notin S} m_{i,j_k}d_i) \\
= \ & \frac{1}{\tilde{u}_{k,j_k}}(\sum_{i\in S}(m_{i,j_k}+|E_{i,j_k}|)d_i + \sum_{i\notin S}|E_{i,j_k}|\, d_i) \\
< \ & \frac{M+1}{\tilde{u}_{k,j_k}}(\sum_{i\in S}(m_{i,j_k}+|E_{i,j_k}|)\tilde{u}_{n,i} + \sum_{i=1}^{s_n}|E_{i,j_k}|\, \tilde{u}_{n,i}) \\
< \ & (M+1)(\frac{\delta}{64(M+1)} + \frac{\delta}{64(M+1)}) = \frac{\delta}{32}.
\end{aligned}$$

Since the trace of the restriction of q to D is larger than $\delta/32$, this implies that q is not Cuntz subequivalent to p, as asserted. □

Since the trace simplex $\mathrm{T}(A)$ is a base for the cone $\mathrm{T}^+(A)$, Theorem 3.1 can be reformulated in terms of $\mathrm{T}^+(A)$ and its affine functions:

Theorem 3.2. *Let A be an AF-Villadsen algebra which satisfies* (2.6)*. The function $r_\infty^{(0)}$, regarded as a continuous affine function on $\mathrm{T}^+(A)$, has the following property:*

$$\mathrm{d}_\tau(a) + r_\infty^{(0)}(\tau) < \mathrm{d}_\tau(b), \quad \tau \in \mathrm{T}^+(A)\setminus\{0\} \quad \Rightarrow \quad a \precsim b. \tag{3.19}$$

On the other hand, assume that X is a solid space. If $h \in \mathrm{Aff}(\mathrm{T}^+(\mathrm{K}_0(A)))$, where $\mathrm{T}^+(\mathrm{K}_0(A))$ denotes the cone $\mathbb{R}^+\mathrm{S}_u(\mathrm{K}_0(A))$, and if there is $\tau \in \mathrm{T}^+(A)$ such that

$$h([\tau]_0) < r_\infty^{(0)}([\tau]_0),$$

then there are $a, b \in (A\otimes\mathcal{K})^+$ such that

$$\mathrm{d}_\tau(a) + h([\tau]_0) < \mathrm{d}_\tau(b), \quad \tau \in \mathrm{T}^+(A)\setminus\{0\},$$

but a is not Cuntz subequivalent to b. Hence $r_\infty^{(0)}$ is the (unique) minimum among the continuous affine functions which factor through $\mathrm{T}^+(A) \to \mathrm{T}^+(\mathrm{K}_0(A))$ and satisfy (3.19)*.*

Remark 3.3. In Section 5, it will be shown that the function $r_\infty^{(0)}$ is not necessarily the minimum (if this exists) of all continuous affine functions on $\mathrm{T}^+(A)$ which satisfy (3.19), but not necessarily factoring through $\mathrm{T}^+(\mathrm{K}_0(A))$.

Corollary 3.4. *Let $A(X, G, \mathcal{E})$ be an AF-Villadsen algebra, and let $\sigma \in \mathrm{Aut}(A)$. Then*

$$r_\infty^{(0)}((([\sigma]_0)^*(\tau))) = r_\infty^{(0)}(\tau), \quad \tau \in \mathrm{S}_u(\mathrm{K}_0(A)).$$

Proof. By Theorem 3.1, the function $r_\infty^{(0)}$ is the minimum function which has the comparison property (3.1), and therefore the function $r_\infty^{(0)} \circ ([\sigma]_0)^*$ is also the minimum function which has the comparison property (3.1). Therefore $r_\infty^{(0)} \circ ([\sigma]_0)^* = r_\infty^{(0)}$. □

It is straightforward, as we shall now show, that

$$\mathrm{rc}(A) = \max\{r_\infty^{(0)}(\tau) : \tau \in \mathrm{T}(A)\}.$$

In fact, the radius of comparison of any unital hereditary sub-C*-algebra of $A \otimes \mathcal{K}$ can be recovered in a similar way.

Theorem 3.5. *Let A be a simple unital C*-algebra, and let $r \in \mathrm{Aff}(\mathrm{T}^+(A))$ have the following three properties:*

1. *The function r factors though $\mathrm{T}^+(A) \to \mathrm{T}^+(\mathrm{K}_0(A))$,*
2. *The function r has the property that for any $a, b \in (A \otimes \mathcal{K})^+$,*
$$\mathrm{d}_\tau(a) + r(\tau) < \mathrm{d}_\tau(b), \quad \tau \in \mathrm{T}^+(A) \quad \Longrightarrow \quad a \precsim b.$$
3. *r is the smallest element of $\mathrm{Aff}(\mathrm{T}^+(A))$ factoring though $\mathrm{T}^+(A) \to \mathrm{T}^+(\mathrm{K}_0(A))$ and having the comparison property (2), in the following sense: if $h \in \mathrm{Aff}(\mathrm{T}^+(A))$, h factors through $\mathrm{T}^+(A) \to \mathrm{T}^+(\mathrm{K}_0(A))$ and h has the comparison property (2), then*
$$r(\tau) \leq h(\tau), \quad \tau \in \mathrm{T}^+(A).$$

Then, for any projection $p \in A \otimes \mathcal{K}$, one has

$$\mathrm{rc}(p(A \otimes \mathcal{K})p) = \max\{r(\tau) : \tau(p) = 1,\ \tau \in \mathrm{T}^+(A)\}.$$

Proof. Since p is full, the set

$$C_p = \{\tau \in \mathrm{T}^+(A) : \tau(p) = 1\} \cong \mathrm{T}(p(A \otimes \mathcal{K})p)$$

is a base for $\mathrm{T}^+(A)$, and the set

$$C'_p = \{\tau \in \mathrm{T}^+(\mathrm{K}_0(A)) : \tau([p]) = 1\} \cong \mathrm{S}_{[p]}(\mathrm{K}_0(p(A \otimes \mathcal{K})p))$$

is a base for $\mathrm{T}^+(\mathrm{K}_0(A))$.

Let

$$s < \max\{r(\tau) : \tau(p) = 1,\ \tau \in \mathrm{T}^+(A)\} = \max\{r(\tau) : \tau([p]_0) = 1,\ \tau \in \mathrm{T}^+(\mathrm{K}_0(A))\}.$$

Regard s as a constant affine function on C'_p. Since C'_p is a base for $\mathrm{T}^+(\mathrm{K}_0(A))$, the affine function s can be extended to a continuous affine function on $\mathrm{T}^+(\mathrm{K}_0(A))$, 0 at 0, and hence a continuous affine function on $\mathrm{T}^+(A)$ (factoring through $\mathrm{T}^+(A) \to \mathrm{T}^+(\mathrm{K}_0(A))$). Still denote it by s.

Then there is $\tau_0 \in \mathrm{T}^+(A)$ such that

$$s(\tau_0) < r(\tau_0).$$

By Condition (3), the affine function s does not have the comparison property (2), and hence there are positive elements $a, b \in A \otimes \mathcal{K}$ such that

$$\mathrm{d}_\tau(a) + s(\tau) < \mathrm{d}_\tau(b), \quad \tau \in \mathrm{T}^+(A),$$

but a is not Cuntz subequivalent to b. Restricting to C_p, one has

$$\mathrm{d}_\tau(a) + s < \mathrm{d}_\tau(b), \quad \tau \in C_p.$$

Since s is arbitrary,

$$\max\{r(\tau) : \tau(p) = 1,\ \tau \in \mathrm{T}^+(A)\} \leq \mathrm{rc}(p(A \otimes \mathcal{K})p).$$

Let us show the reverse inequality. Assume $a, b \in A \otimes \mathcal{K}$ are positive elements such that

$$\mathrm{d}_\tau(a) + \max\{r(\tau) : \tau(p) = 1,\ \tau \in \mathrm{T}^+(A)\} < \mathrm{d}_\tau(b), \quad \tau \in C_p.$$

Then

$$\mathrm{d}_\tau(a) + r(\tau) \leq \mathrm{d}_\tau(a) + \max\{r(\tau) : \tau(p) = 1,\ \tau \in \mathrm{T}^+(A)\} < \mathrm{d}_\tau(b), \quad \tau \in C_p.$$

Since C_p is a base for $\mathrm{T}^+(A)$, one has

$$\mathrm{d}_\tau(a) + r(\tau) < \mathrm{d}_\tau(b), \quad \tau \in \mathrm{T}^+(A).$$

By the comparison property of r, one has $a \precsim b$. Therefore,

$$\max\{r(\tau) : \tau(p) = 1,\ \tau \in \mathrm{T}^+(A)\} \geq \mathrm{rc}(p(A \otimes \mathcal{K})p),$$

as desired. □

Corollary 3.6. *Let A be an AF-Villadsen algebra which satisfies* (2.6)*. Then, for any projection $p \in A \otimes \mathcal{K}$, one has*

$$\mathrm{rc}(p(A \otimes \mathcal{K})p) = \max\{r_\infty^{(0)}(\tau) : \tau(p) = 1,\ \tau \in \mathrm{T}^+(A)\}.$$

Example 3.7 ([5]). Consider the Villadsen algebra of Example 2.5. Assume that

$$\lim_{i\to\infty} \frac{u_{i,1}}{\tilde{u}_{i,1}} \neq \lim_{i\to\infty} \frac{u_{i,2}}{\tilde{u}_{i,2}},$$

and hence that

$$r_\infty^{(0)}(\tau_1) \neq r_\infty^{(0)}(\tau_2),$$

where τ_1 and τ_2 are the extreme points of $\mathrm{S}_u(\mathrm{K}_0(A)) \cong [0,1]$ (see Example 2.5). Then, by Corollary 3.4, there is no automorphism $\sigma : A \to A$ such that $\tau_1 \circ ([\sigma]_0)^* = \tau_2$. That is, there is no automorphism which flips $\mathrm{S}_u(\mathrm{K}_0(A)) \cong [0,1]$.

Let $p \in A \otimes \mathcal{K}$ be a projection, and let us calculate $\mathrm{rc}(p(A \otimes \mathcal{K})p)$. Without loss of generality, one may assume that $p \in A_i$ for some $i \in \mathbb{N}$ (as, if p is unitarily equivalent to q, then the hereditary sub-C*-algebras generated by p and q are isomorphic).

Note that

$$\mathrm{T}^+(A) = \{\alpha\tau_1 + \beta\tau_2 : \alpha, \beta \in [0, +\infty)\},$$

and then consider the section

$$\{\tau \in \mathrm{T}^+(A) : \tau(p) = 1\}.$$

Write $\tau = \alpha\tau_1 + \beta\tau_2$, and then

$$1 = (\alpha\tau_1 + \beta\tau_2)(p) = \alpha\tau_1(p) + \beta\tau_2(p).$$

Since $\beta \geq 0$, a simple calculation shows that

$$0 \leq \alpha \leq \frac{1}{\tau_1(p)}.$$

Then, by Corollary 3.6,

$$\begin{aligned}
\mathrm{rc}(p(A\otimes\mathcal{K})p) &= \max\{r_\infty^{(0)}(\alpha\tau_1+\beta\tau_2) : (\alpha\tau_1+\beta\tau_2)(p)=1\}\\
&= \max\{\alpha r_\infty^{(0)}(\tau_1)+\frac{1}{\tau_2(p)}(1-\alpha\tau_1(p))r_\infty^{(0)}(\tau_2) : 0\leq\alpha\leq\frac{1}{\tau_1(p)}\}\\
&= \max\{\frac{r_\infty^{(0)}(\tau_2)}{\tau_2(p)}+(r_\infty^{(0)}(\tau_1)-\frac{\tau_1(p)}{\tau_2(p)}r_\infty^{(0)}(\tau_2))\alpha : 0\leq\alpha\leq\frac{1}{\tau_1(p)}\}\\
&= \max\{\frac{r_\infty^{(0)}(\tau_1)}{\tau_1(p)},\frac{r_\infty^{(0)}(\tau_2)}{\tau_2(p)}\}.
\end{aligned}$$

4. The isomorphism theorem

In this section, let us show that the AF-Villadsen algebras with a given (finite dimensional, K-contractible, and solid) seed space are classified by their K-groups together with the function $r_\infty^{(0)}$.

Proposition 4.1 (cf. Lemma 7.4 of [4]). *Let X be a K-contractible metrizable compact space such that $0<\dim(X)<\infty$, and let $A(X,G^{(A)},\mathcal{E}^{(A)})$ and $B(X,G^{(B)},\mathcal{E}^{(B)})$ be AF-Villadsen algebras with seed space X such that*

$$(\mathrm{K}_0(A),[1_A]_0,r_\infty^{(0)}(A))\cong(\mathrm{K}_0(B),[1_B]_0,r_\infty^{(0)}(B)).$$

Let $\delta_1>\delta_2>\cdots$ be a sequence of strictly positive numbers such that $\sum_{i=1}^\infty\delta_i<1$. Then, on telescoping, there is a diagram

$$\begin{tikzcd}
A_1 \arrow[r, "\phi_1"] \arrow[d, "\rho_1"'] & A_2 \arrow[r, "\phi_2"] \arrow[d, "\rho_2"'] & \cdots \\
B_1 \arrow[r, "\psi_1"] \arrow[ur, "\eta_1"] & B_2 \arrow[r, "\psi_1"] & \cdots
\end{tikzcd} \tag{4.1}$$

such that each of $\phi_i,\psi_i,\rho_i,\kappa_i$, $i=1,2,...$, consists of independent coordinate projections and point evaluations, i.e., restricted to each direct summand of the domain, it has the form

$$(f_1,f_2,...,f_s)\mapsto\mathrm{diag}\{f_1\circ P_1,...,f_s\circ P_s, \textit{point evaluations}\},$$

where $P_1,...,P_s$ are mutually disjoint sets of coordinate projections, and for each $i=1,2,...$, there are decompositions

$$\phi_i=\mathrm{diag}\{P_{A,i},R'_{A,i},\Theta_{A,i}\},\quad \psi_i=\mathrm{diag}\{P_{B,i},R'_{B,i},\Theta_{B,i}\},$$

and

$$\eta_i\circ\rho_i=\mathrm{diag}\{P_{A,i},R''_{A,i},\Theta_{A,i}\},\quad \rho_{i+1}\circ\eta_i=\mathrm{diag}\{P_{B,i},R''_{B,i},\Theta_{B,i}\},$$

where $P_{A,i}:A_i\to A_{i+1}$ and $P_{B,i}:B_i\to B_{i+1}$ consist of coordinate projections, and $\Theta_{A,i}:A_i\to A_{i+1}$ and $\Theta_{B,i}:B_i\to B_{i+1}$ consist of point evaluations, such that, for each $i=1,2,...$,

$$\frac{\mathrm{rank}_j(R'_{A,i}(1_{A_i}))}{\mathrm{rank}_j(\Theta_{A,i}(1_{A_i}))}=\frac{\mathrm{rank}_j(R''_{A,i}(1_{A_i}))}{\mathrm{rank}_j(\Theta_{A,i}(1_{A_i}))}<\delta_i,\quad j=1,...,s_{i+1}^{(A)},$$

and

$$\frac{\mathrm{rank}_j(R'_{B,i}(1_{B_i}))}{\mathrm{rank}_j(\Theta_{B,i}(1_{B_i}))} = \frac{\mathrm{rank}_j(R''_{B,i}(1_{B_i}))}{\mathrm{rank}_j(\Theta_{B,i}(1_{B_i}))} < \delta_i, \quad j = 1, ..., s_{i+1}^{(B)}.$$

In particular, $\mathrm{T}(A) \cong \mathrm{T}(B)$*, and in a way compatible with the isomorphism of* K_0*-groups.*

Proof. Since X is K-contractible, one has

$$(\mathrm{K}_0(G^{(A)}), \mathrm{K}_0^+(G^{(A)}), [\tilde{u}^{(A)}]) \cong (\mathrm{K}_0(A), \mathrm{K}_0^+(A), [1_A]_0)$$

and

$$(\mathrm{K}_0(G^{(B)}), \mathrm{K}_0^+(G^{(B)}), [\tilde{u}^{(B)}]) \cong (\mathrm{K}_0(B), \mathrm{K}_0^+(B), [1_B]_0).$$

Since $(\mathrm{K}_0(A), [1_A]_0) \cong (\mathrm{K}_0(B), [1_B]_0)$, there is an isomorphism

$$\kappa_\infty : (\mathrm{K}_0(G^{(A)}), \mathrm{K}_0^+(G^{(A)}), [\tilde{u}^{(A)}]) \cong (\mathrm{K}_0(G^{(B)}), \mathrm{K}_0^+(G^{(B)}), [\tilde{u}^{(B)}]).$$

Therefore, upon a telescoping, there is a commutative diagram

(4.2)
$$\begin{tikzcd}
G_1^{(A)} \arrow[r] \arrow[d, "\kappa_1^{[A,B]}"'] & G_2^{(A)} \arrow[r] \arrow[d, "\kappa_1^{[B,A]}"'] & \cdots \arrow[r] & \mathrm{K}_0(G^{(A)}) \arrow[d, leftrightarrow] \\
G_1^{(B)} \arrow[r] \arrow[ur] & G_2^{(B)} \arrow[r] \arrow[ur] & \cdots \arrow[r] & \mathrm{K}_0(G^{(B)})
\end{tikzcd}$$

which induces the isomorphism κ_∞.

Consider the inductive sequences

$$A_1 \longrightarrow A_2 \longrightarrow \cdots \longrightarrow A,$$

$$B_1 \longrightarrow B_2 \longrightarrow \cdots \longrightarrow B.$$

Write $r_i^{(0)}(A) \in \mathbb{R}^{s_i^{(A)}}$ and $r_i^{(0)}(B) \in \mathbb{R}^{s_i^{(B)}}$, $i = 1, 2, ...$, for the affine functions at stage i which converge (uniformly—see 2.2) to $r_\infty^{(0)}(A)$ and $r_\infty^{(0)}(B)$ respectively.

Consider δ_1 and consider A_1. It follows from the construction of A that there is $\Delta_1 > 0$ such that for each $i \geq 1$, if one decomposes $\phi_{1,i}$ as

$$\phi_{1,i} = P \oplus \Theta : A_1 \to A_i,$$

where $P : A_1 \to A_i$ consists of coordinate projections and $\Theta : A_1 \to A_i$ consists of point evaluations, then

$$\tau(\Theta(1_{A_1})) > \Delta_1, \quad \tau \in \mathrm{T}(A_i). \tag{4.3}$$

Choose δ'_1 sufficiently small that

$$\frac{2\delta'_1 + (1 - (1 - 2\delta'_1)^3)}{\Delta_1} < \delta_1.$$

Recall from (2.2) that $(r_i^{(0)}(A))$ and $(r_i^{(0)}(B))$ converge decreasingly and uniformly to $r_\infty^{(0)}(A)$ and $r_\infty^{(0)}(B)$ respectively. Since $r_\infty^{(0)}(A) = r_\infty^{(0)}(B)$ under the isomorphism induced by κ_∞, and since $r_\infty^{(0)}(A)$ and $r_\infty^{(0)}(B)$ are strictly positive, there is i'_1 such that the differences

$$\left\|r_i^{(0)}(A) - r_\infty^{(0)}(A)\right\|_\infty \quad \text{and} \quad \left\|r_i^{(0)}(B) - r_\infty^{(0)}(B)\right\|_\infty, \quad i \geq i'_1,$$

are small enough that

$$(1-\delta'_1)(\kappa_{i,j}^{A,B})^*(r_i^{(0)}(A)) < r_j^{(0)}(B), \quad j > i \geq i'_1. \tag{4.4}$$

By the assumption (2.6), i'_1 is large enough that

$$\sum_{i=1}^{s_{i'_1}^{(A)}}([\phi_{i'_1,k}]_{i,j} - (D[\phi_{i'_1,k}])_{i,j})\tilde{u}_{i'_1,i}^{(A)} < \delta'_1 \sum_{i=1}^{s_{i'_1}^{(A)}}[\phi_{i'_1,k}]_{i,j}\tilde{u}_{i'_1,i}^{(A)}, \quad j = 1, ..., s_k^{(A)},\ k > i'_1, \tag{4.5}$$

where $D(\cdot)$ denotes the multiplicity matrix of the coordinate projection component.

Fix i'_1. Since the ordered groups $\mathrm{K}_0(G^{(A)}) \cong \mathrm{K}_0(G^{(B)})$ are simple, if $i''_1 > i'_1$ is sufficiently large and if

$$(m_{i,j}), \quad i = 1, ..., s_{i'_1}^{(A)},\ j = 1, ..., s_{i''_1}^{(B)},$$

denotes the multiplicity matrix of $\kappa_{i'_1,i''_1}^{A,B}$, then there exist positive integers

$$\delta_{i,j}^{A,B}, \quad i = 1, ..., s_{i'_1}^{(A)},\ j = 1, ..., s_{i''_1}^{(B)},$$

satisfying

$$1 - 2\delta'_1 < \frac{\delta_{i,j}^{A,B}}{m_{i,j}^{A,B}} < 1 - \delta'_1, \quad i = 1, ..., s_{i'_1}^{(A)},\ j = 1, ..., s_{i''_1}^{(B)}. \tag{4.6}$$

Recall that

$$r_i^{(0)}(A) = \frac{1}{2}\dim(X)(\frac{u_{i,j}^{(A)}}{\tilde{u}_{i,j}^{(A)}})_j, \quad r_i^{(0)}(B) = \frac{1}{2}\dim(X)(\frac{u_{i,j}^{(B)}}{\tilde{u}_{i,j}^{(B)}})_j,$$

and $0<\dim(X)<\infty$. Then, by (4.4), one has, for each $j=1,...,s^{(B)}_{i''_1}$,

$$\begin{aligned}
\frac{1}{\tilde{u}^{(B)}_{i''_1,j}}\sum_{i=1}^{s^{(A)}_{i'_1}} u^{(A)}_{i'_1,i}\delta^{A,B}_{i,j} &= \sum_{i=1}^{s^{(A)}_{i'_1}}(\frac{u^{(A)}_{i'_1,i}}{\tilde{u}^{(A)}_{i'_1,i}})(\tilde{u}^{(A)}_{i'_1,i}m^{A,B}_{i,j}\frac{1}{\tilde{u}^{(B)}_{i''_1,j}})(\frac{\delta^{A,B}_{i,j}}{m^{A,B}_{i,j}}) \\
&< \sum_{i=1}^{s^{(A)}_{i'_1}}(\frac{u^{(A)}_{i'_1,i}}{\tilde{u}^{(A)}_{i'_1,i}})(\tilde{u}^{(A)}_{i'_1,i}m^{A,B}_{i,j}\frac{1}{\tilde{u}^{(B)}_{i''_1,j}})(1-\delta'_1) \\
&= (1-\delta'_1)\sum_{i=1}^{s^{(A)}_{i'_1}}(\frac{u^{(A)}_{i'_1,i}}{\tilde{u}^{(A)}_{i'_1,i}})(\tilde{u}^{(A)}_{i'_1,i}m^{A,B}_{i,j}\frac{1}{\tilde{u}^{(B)}_{i''_1,j}}) \\
&= (1-\delta'_1)\frac{2}{\dim(X)}((\kappa^{A,B}_{i'_1,i''_1})^*(r^{(A)}_{i''_1}))_j \\
&< \frac{2}{\dim(X)}(r^{(0)}_{i''_1}(B))_j = \frac{u^{(B)}_{i''_1,j}}{\tilde{u}^{(B)}_{i''_1,j}}.
\end{aligned}$$

Therefore,

$$\sum_{i=1}^{s^{(A)}_{i'_1}} u^{(A)}_{i'_1,i}\delta^{A,B}_{i,j} < u^{(B)}_{i''_1,j},\quad j=1,...,s^{(B)}_{i''_1}. \tag{4.7}$$

By (2.6), i''_1 can be chosen even farther out so that

$$\sum_{i=1}^{s^{(B)}_{i'_1}}([\psi_{i''_1,k}]_{i,j}-(D[\psi_{i''_1,k}])_{i,j})\tilde{u}^{(B)}_{i'_2,i} < \delta'_1\sum_{i=1}^{s^{(B)}_{i'_1}}[\psi_{i''_1,k}]_{i,j}\tilde{u}^{(B)}_{i'_1,i},\quad j=1,...,s^{(B)}_k,\ k>i''_1,$$

which implies

$$\sum_{i=1}^{s^{B}_{i''_1}}(D[\psi_{i''_1,k}])_{i,j})\tilde{u}^{(B)}_{i'_2,i} > (1-\delta'_1)\sum_{i=1}^{s^{B}_{i''_1}}[\psi_{i''_1,k}]_{i,j}\tilde{u}^{(B)}_{i'_1,i},\quad j=1,...,s^{(B)}_k,\ k>i''_1. \tag{4.8}$$

Then, define the map

$$\rho'_1: A_{i'_1}\to B_{i''_1}$$

of the form

$$\begin{aligned}
\rho'_1(f_1,...,f_{s^{(A)}_{i'_1}}) \mapsto\ & (\mathrm{diag}\{f_1\circ P^{A,B}_{1,1},...,f_{s^{(A)}_{i''_1}}\circ P^{A,B}_{s^{(A)}_{i'_1},1},\text{point evaluations}\},..., \\
& \mathrm{diag}\{f_1\circ P^{A,B}_{1,s^{(B)}_{i''_1}},...,f_{s^{(A)}_{i'_1}}\circ P^{A,B}_{s^{(A)}_{i'_1},s^{(B)}_{i''_1}},\text{point evaluations}\}),
\end{aligned}$$

where each map

$$P^{A,B}_{i,j}: X^{u^{(B)}_{i''_1,j}}\to X^{u^{(A)}_{i'_1,i}},\quad i=1,...,s^{(A)}_{i'_1},\ j=1,...,s^{(B)}_{i''_1},$$

consists of $\delta^{A,B}_{i,j}$ disjoint coordinate projections, and the K_0-multiplicity of ρ_1' is $\kappa^{[A,B]}_{i_1',i_1''}$. Note that this is well defined by (4.7). Then, define

$$\rho_1=\rho_1'\circ\phi_{1,i_1'}:A_1\to B_{i_1''}.$$

The construction above can be illustrated by the following diagram:

$$\begin{tikzcd}
A_1 \arrow[r, "\phi_{1,i_1'}"] & A_{i_1'} \arrow[r] \arrow[dr, "\rho_1'"] & A_{i_1''} \arrow[r] & \cdots \arrow[r] & A \\
B_1 \arrow[r] & B_{i_1'} \arrow[r] & B_{i_1''} \arrow[r] & \cdots \arrow[r] & B.
\end{tikzcd}$$

Let us now construct the map η from B back to A. By the construction of B, there is $\Delta_2>0$ such that for each $i\geq i_1''$ and if one writes

$$\psi_{i_1'',i}=P\oplus\Theta:B_{i_1''}\to B_i,$$

where $P:B_{i_1''}\to B_i$ consists of coordinate projections and $\Theta:B_{i_1''}\to B_i$ consists of point evaluations, then

$$\mathrm{tr}(\Theta(1_{B_{i_1''}}))>\Delta_2,\quad \tau\in\mathrm{T}(B_i).$$

Then choose δ_2' sufficiently small that

$$\frac{2\delta_2'+(1-(1-2\delta_2')^3)}{\Delta_2}<\delta_2.$$

One also should ensure that $\delta_2'<\delta_1'$.

By the same argument as for ρ_1, there is a sufficiently large $i_2'>i_1''$ that

$$(1-\delta_2'')(\kappa^{B,A}_{i,j})^*(r^{(0)}_i(B))<r^{(0)}_j(A),\quad i,j\geq i_2', \tag{4.9}$$

and

$$\sum_{i=1}^{s^{(B)}_{i_2'}}([\psi_{i_2',k}]_{i,j}-(D[\psi_{i_2',k}])_{i,j})\tilde{u}^{(B)}_{i_1',i}<\delta_2'\sum_{i=1}^{s^{(B)}_{i_2'}}[\psi_{i_2',k}]_{i,j}\tilde{u}^{(B)}_{i_2',i},\quad j=1,...,s^{(B)}_k,\ k>i_2', \tag{4.10}$$

and then there is $i_2''>i_2'$ sufficiently large that

$$\sum_{i=1}^{s^{A}_{i_2''}}(D[\psi_{i_2'',k}])_{i,j})\tilde{u}^{(A)}_{i_2'',i}>(1-\delta_2')\sum_{i=1}^{s^{A}_{i_2''}}[\psi_{i_2'',k}]_{i,j}\tilde{u}^{(A)}_{i_2'',i},\quad j=1,...,s^{(A)}_k,\ k>i_2'', \tag{4.11}$$

and there are positive integers

$$\delta^{B,A}_{i,j},\quad i=1,...,s^{(B)}_{i_2'},\ j=1,...,s^{(A)}_{i_2''},$$

satisfying

$$1-2\delta_2'<\frac{\delta^{B,A}_{i,j}}{m^{B,A}_{i,j}}<1-\delta_2',\quad i=1,...,s^{(B)}_{i_2'},\ j=1,...,s^{(A)}_{i_2''}, \tag{4.12}$$

where $(m^{B,A}_{i,j})$, $i=1,...,s^{(B)}_{i'_2}$, $j=1,...,s^{(A)}_{i''_2}$, are the multiplicities of $\kappa^{B,A}_{i'_2,i''_2}$, and therefore

$$\sum_{i=1}^{s^{(B)}_{i'_2}} u^{(B)}_{i'_1,i}\delta^{B,A}_{i,j} < u^{(A)}_{i''_2,j}, \quad j=1,...,s^{(A)}_{i''_2}. \tag{4.13}$$

Thus, there is room to define a map $\eta'_1 : B_{i'_2} \to A_{i''_2}$ with the multiplicities of coordinate projections equal to $(\delta^{B,A}_{i,j})$. Then define

$$\eta_1 := \eta'_1 \circ \psi_{i''_1,i'_1}.$$

The construction can be illustrated by the following diagram:

$$\begin{tikzcd}
A_1 \arrow[r, "\phi_{1,i'_1}"] & A_{i'_1} \arrow[r] \arrow[dr, "\rho'_1"] & A_{i''_1} \arrow[r] & A_{i'_2} \arrow[r] & A_{i''_2} \arrow[r] & \cdots \arrow[r] & A \\
B_1 \arrow[r] & B_{i'_1} \arrow[r] & B_{i''_1} \arrow[r, "\psi_{i''_1,i'_2}"'] & B_{i'_2} \arrow[r] \arrow[ur, "\eta'_1"] & B_{i''_2} \arrow[r] & \cdots \arrow[r] & B.
\end{tikzcd}$$

Let us consider the composition $\eta_1 \circ \rho_1$, which is $(\eta'_1 \circ \psi_{i''_1,i'_2} \circ \rho'_1) \circ \phi_{1,i'_1}$, and compare it with the map ϕ_{1,i''_2}.

Note that the multiplicity matrix of coordinate projections of $\eta'_1 \circ \psi_{i''_1,i'_2} \circ \rho'_1$ is the product

$$(\delta^{B,A}_{i,j})(D[\psi_{i''_1,i'_2}])(\delta^{A,B}_{i,j}).$$

Then, using (4.6), (4.8), and (4.12) (note that $\delta'_2 < \delta'_1$), one has

$$\begin{aligned}
(\delta^{B,A}_{i,j})(D[\psi_{i''_1,i'_2}])(\delta^{A,B}_{i,j})(\tilde{u}^{(A)}_{i'_1,i}) &> (1-2\delta'_1)(m^{B,A}_{i,j})(D[\psi_{i''_1,i'_2}])(1-2\delta''_2)(m^{A,B}_{i,j})(\tilde{u}^{(A)}_{i'_1,i}) \\
&> (1-2\delta'_1)^2(m^{B,A}_{i,j})((D[\psi_{i''_1,i'_2}])(m^{A,B}_{i,j}))(\tilde{u}^{(A)}_{i'_1,i}) \\
&= (1-2\delta'_1)^2(m^{B,A}_{i,j})(D[\psi_{i''_1,i'_2}])(\tilde{u}^{(B)}_{i''_1,i}) \\
&> (1-2\delta'_1)^3(m^{B,A}_{i,j})[\psi_{i''_1,i'_2}](\tilde{u}^{(B)}_{i''_1,i}) \\
&= (1-2\delta'_1)^3(\kappa^{B,A}_{i'_2,i''_2} \circ [\psi_{i''_1,i'_2}] \circ \kappa^{A,B}_{i'_1,i''_1})(\tilde{u}^{(A)}_{i'_1,i}) \\
&= (1-2\delta'_1)^3[\phi_{i'_1,i''_2}](\tilde{u}^{(A)}_{i'_1,i}),
\end{aligned}$$

and therefore

$$(1-2\delta'_1)^3[\phi_{i'_1,i''_2}](\tilde{u}^{(A)}_{i'_1,i}) < D[\eta'_1 \circ \psi_{i''_1,i'_2} \circ \rho'_1](\tilde{u}^{(A)}_{i'_1,i}) \leq [\phi_{i'_1,i''_2}](\tilde{u}^{(A)}_{i'_1,i}).$$

That is,

$$\sum_{i=1}^{s^{(A)}_{i'_1}}([\phi_{i'_1,i''_2}]_{i,j}\tilde{u}^{(A)}_{i'_1,i} - D[\eta'_1 \circ \psi_{i''_1,i'_2} \circ \rho'_1]_{i,j}\tilde{u}^{(A)}_{i'_1,i}) < (1-(1-2\delta'_1)^3)\sum_{i=1}^{s^{(A)}_{i'_1}}[\phi_{i'_1,i''_2}]_{i,j}(\tilde{u}^{(A)}_{i'_1,i}), \tag{4.14}$$

for each $j=1,...,s^{(A)}_{i''_2}$.

Also note that, by (4.5),

$$\sum_{i=1}^{s_{i'_1}^{(A)}}([\phi_{i'_1,i''_2}]_{i,j}-(D[\phi_{i'_1,i''_2}])_{i,j})\tilde{u}_{i'_1,i}^{(A)}<\delta'_1\sum_{i=1}^{s_{i'_1}^{(A)}}[\phi_{i'_1,i''_2}]_{i,j}\tilde{u}_{i'_1,i}^{(A)}=\delta'_1\tilde{u}_{i''_2,j}^{(A)},\quad j=1,...,s_{i''_2}^{(A)}. \tag{4.15}$$

Therefore, for each $j=1,...,s_{i''_2}^{(A)}$, using the equation

$$\sum_{i=1}^{s_{i'_1}^{(A)}}[\phi_{i'_1,i''_2}]_{i,j}\tilde{u}_{i'_1,i}^{(A)}=u_{i''_2,j}^{(A)}$$

in the last step, one has

$$\begin{aligned}
&\sum_{i=1}^{s_{i'_1}^{(A)}}\left|(D[\eta'_1\circ\psi_{i''_1,i'_2}\circ\rho'_1])_{i,j}\tilde{u}_{i'_1,i}^{(A)}-(D[\phi_{i'_1,i''_2}])_{i,j}\tilde{u}_{i'_1,i}^{(A)}\right| \qquad (4.16)\\
\leq\ &\sum_{i=1}^{s_{i'_1}^{(A)}}(\left|(D[\eta'_1\circ\psi_{i''_1,i'_2}\circ\rho'_1])_{i,j}\tilde{u}_{i'_1,i}^{(A)}-[\phi_{i'_1,i''_2}]_{i,j}\tilde{u}_{i'_1,i}^{(A)}\right|+\left|[\phi_{i'_1,i''_2}]_{i,j}\tilde{u}_{i'_1,i}^{(A)}-(D[\phi_{i'_1,i''_2}])_{i,j}\tilde{u}_{i'_1,i}^{(A)}\right|)\\
<\ &\delta'_1(\sum_{i=1}^{s_{i'_1}^{(A)}}[\phi_{i'_1,i''_2}]_{i,j}\tilde{u}_{i'_1,i}^{(A)})+(1-(1-2\delta'_1)^3)(\sum_{i=1}^{s_{i'_1}^{(A)}}[\phi_{i'_1,i''_2}]_{i,j}\tilde{u}_{i'_1,i}^{(A)})\\
=\ &(\delta'_1+(1-(1-2\delta'_1)^3))\tilde{u}_{i''_2,j}^{(A)}.
\end{aligned}$$

Then, introducing the matrix $(c_{i,j})$ with

$$c_{i,j}:=\min\{(D[\eta'_1\circ\psi_{i''_1,i'_2}\circ\rho'_1])_{i,j},[\phi_{i'_1,i''_2}]_{i,j}\},i=1,...,s_{i'_1}^{(A)},\ j=1,...,s_{i''_2}^{(A)},$$

and defining $P:A_{i'_1}\to A_{i''_2}$ to be the diagonal map consisting of coordinate projections with multiplicities given by $(c_{i,j})$, one has the decompositions

$$\phi_{i'_1,i''_2}=P\oplus R_0,$$
$$\eta'_1\circ\psi_{i''_1,i'_2}\circ\rho'_1=P\oplus R_1,$$

where, by (4.16) and (4.15), for each $j=1,...,s_{i''_2}^{(A)}$,

$$\mathrm{rank}_j((R_0(1_{A_{i'_1}})))=\mathrm{rank}_j((R_1(1_{A_{i'_1}})))<((\delta'_1+(1-(1-2\delta'_1)^3))+\delta'_1)\tilde{u}_{i''_2,j}^{(A)}. \tag{4.17}$$

Write

$$\phi_{1,i'_1}=P_1\oplus\Theta,$$

where $P_1:A_1\to A_{i'_1}$ is a coordinate projection map and $\Theta:A_1\to A_{i'_1}$ is a point evaluation map; then

$$\begin{aligned}
\phi_{1,i''_2}&=\phi_{i'_1,i''_2}\circ\phi_{1,i'_1}\\
&=(P\oplus R_0)\circ(P_1\oplus\Theta)\\
&=(P\circ P_1)\oplus(R_0\circ P_1)\oplus((P\oplus R_0)\circ\Theta)
\end{aligned}$$

and

$$\begin{aligned}\eta_1\circ\rho_1 &= (\eta_1'\circ\psi_{i_1'',i_2'}\circ\rho_1')\circ\phi_{1,i_1'}\\ &= (P\oplus R_1)\circ(P_1\oplus\Theta)\\ &= (P\circ P_1)\oplus(R_1\circ P_1)\oplus((P\oplus R_1)\circ\Theta).\end{aligned}$$

Note that, since Θ is a point-evaluation map,

$$(P\oplus R_0)\circ\Theta=(P\oplus R_1)\circ\Theta. \tag{4.18}$$

By (4.17) and (4.3), for each $j=1,...,s^{(A)}_{i_2''}$, one has

$$\begin{aligned}\frac{\mathrm{rank}_j(R_0\circ P_1)(1_{A_1})}{\mathrm{rank}_j((P\oplus R_0)\circ\Theta)(1_{A_1})} &\leq \frac{\mathrm{rank}_j R_0(1_{A_{1_i'}})}{\mathrm{rank}_j(\phi_{i_1',i_2''}\circ\Theta)(1_{A_1})}\\ &\leq \frac{((\delta_1'+(1-(1-2\delta_1')^3))+\delta_1')\tilde{u}^{(A)}_{i_2'',j}}{\mathrm{rank}_j(\phi_{i_1',i_2''}\circ\Theta)(1_{A_1})}\\ &= \frac{(\delta_1'+(1-(1-2\delta_1')^3))+\delta_1'}{(\mathrm{tr}_j\circ\phi_{i_1',i_2''})(\Theta(1_{A_1}))}\\ &< \frac{(\delta_1'+(1-(1-2\delta_1')^3))+\delta_1'}{\Delta_1}<\delta_1,\end{aligned}$$

and, by the same argument,

$$\frac{\mathrm{rank}_j(R_1\circ P_1)(1_{A_1})}{\mathrm{rank}_j((P\oplus R_1)\circ\Theta)(1_{A_1})}<\delta_1,\quad j=1,...,s^{(A)}_{i_2''}.$$

Then the maps ρ_1 and η_1 possess the properties of the proposition with

$$P_{A,1}=P\circ P_1,\quad R'_{A,1}=R_0\circ P_1,\quad R''_{A,1}=R_0\circ P_1,\quad\text{and}\quad \Theta_{A,1}=(P\oplus R_0)\circ\Theta.$$

Repeating this process, one has the maps ρ_i,η_i, $i=1,2,...,$ which have the desired property. $\square$

Recall the following stable uniqueness theorem:

Theorem 4.2 (Theorem 7.5 of [4]). *Let X be a K-contractible metrizable compact space (i.e., $\mathrm{K}_0(\mathrm{C}(X))=\mathbb{Z}$ and $\mathrm{K}_1(\mathrm{C}(X))=\{0\}$), and let $\Delta:\mathrm{C}(X)^+\to(0,+\infty)$ be a map. For any finite set $\mathcal{F}\subseteq\mathrm{C}(X)$ and any $\varepsilon>0$, there exists a finite set $\mathcal{H}\subseteq\mathrm{C}(X)^+$ with $\mathrm{supp}(h)\neq X$ for each $h\in\mathcal{H}$ and there exists $M\in\mathbb{N}$ such that the following property holds: for any unital homomorphisms*

$$\phi,\psi:\mathrm{C}(X)\to\mathrm{M}_n(\mathrm{C}(Y))\quad\text{and}\quad\theta:\mathrm{C}(X)\to\mathrm{M}_m(\mathbb{C})\subseteq\mathrm{M}_m(\mathrm{C}(Y)),$$

where θ is a unital point-evaluation map with $nM<m$, and such that

$$\mathrm{tr}(\theta(h))>\Delta(h),\quad h\in\mathcal{H},$$

there is a unitary $u\in\mathrm{M}_{n+m}(\mathrm{C}(Y))$ such that

$$\|\mathrm{diag}\{\phi(a),\theta(a)\}-u^*\mathrm{diag}\{\psi(a),\theta(a)\}u\|<\varepsilon,\quad a\in\mathcal{F}.$$

Theorem 4.3. *Let X be a K-contractible solid space such that $0 < \dim(X) < \infty$, and let $A(X, G, \mathcal{E})$ and $B(X, H, \mathcal{F})$ be AF-Villadsen algebras with seed space X satisfying* (2.2) *(and therefore* (2.6)*), where G and H are Bratteli diagrams and $\mathcal{E}$ and $\mathcal{F}$ are point evaluation sets. Then $A \cong B$ if, and only if, $(\mathrm{Cu}(A), [1_A]) \cong (\mathrm{Cu}(B), [1_B])$. Indeed, $A \cong B$ if, and only if,*

$$((\mathrm{K}_0(A), \mathrm{K}_0^+(A), [1_A]_0), r_\infty^{(0)}(A)) \cong ((\mathrm{K}_0(B), \mathrm{K}_0^+(B), [1_B]_0), r_\infty^{(0)}(B)).$$

Proof. Assume that

$$(\mathrm{Cu}(A), [1_A]) \cong (\mathrm{Cu}(B), [1_B]).$$

By Theorem 3.1, this implies that

$$((\mathrm{K}_0(A), \mathrm{K}_0^+(A), [1_A]_0), r_\infty^{(0)}(A)) \cong ((\mathrm{K}_0(B), \mathrm{K}_0^+(B), [1_B]_0), r_\infty^{(0)}(B)).$$

Let us prove the implication

$$((\mathrm{K}_0(A), \mathrm{K}_0^+(A), [1_A]_0), r_\infty^{(0)}(A)) \cong ((\mathrm{K}_0(B), \mathrm{K}_0^+(B), [1_B]_0), r_\infty^{(0)}(B)) \quad \Longrightarrow \quad A \cong B.$$

Choose finite subsets $\mathcal{F}_i^{(A)} \subseteq A_i$, $\mathcal{F}_i^{(B)} \subseteq A_i$, $i = 1, 2, ...$, such that

$$\mathcal{F}_1^{(A)} \subseteq \mathcal{F}_2^{(A)} \subseteq \cdots \quad \text{and} \quad \mathcal{F}_1^{(B)} \subseteq \mathcal{F}_2^{(B)} \subseteq \cdots,$$

and

$$\overline{\bigcup_{i=1}^{\infty} \mathcal{F}_i^{(A)}} = A \quad \text{and} \quad \overline{\bigcup_{i=1}^{\infty} \mathcal{F}_i^{(B)}} = B.$$

Choose $\varepsilon_1 > \varepsilon_2 > \cdots > 0$ such that

$$\sum_{i=1}^{\infty} \varepsilon_i < \infty.$$

For each A_i, $i = 1, 2, ...$, consider

$$\Delta^{(A)}(a) := \inf\{\tau(a) : \tau \in \mathrm{T}(A)\}, \quad a \in A_i^+,$$

and for each B_i, $i = 1, 2, ...$,

$$\Delta^{(B)}(b) := \inf\{\tau(b) : \tau \in \mathrm{T}(B)\}, \quad b \in B_i^+.$$

For each $(\mathcal{F}_i^{(A)}, \varepsilon_i)$, applying Theorem 4.2 with respect to $\frac{1}{2}\Delta^{(A)}$, one obtains a finite set of positive contractions $\mathcal{H}_i^{(A)} \subseteq A_i$ and $M_i^{(A)} \in \mathbb{N}$ with the property of Theorem 4.2. Similarly, for each $(\mathcal{F}_i^{(B)}, \varepsilon_i)$, applying Theorem 4.2 with respect to $\frac{1}{2}\Delta^{(B)}$, one obtains a finite set of positive contractions $\mathcal{H}_i^{(B)} \subseteq B_i$ and $M_i^{(B)} \in \mathbb{N}$ with the property of Theorem 4.2.

By Proposition 4.1, upon a telescoping, there is a diagram

$$\begin{tikzcd} A_1 \arrow[r, "\phi_1"] \arrow[d, "\rho_1"'] & A_2 \arrow[r, "\phi_2"] \arrow[d, "\rho_2"'] & \cdots \\ B_1 \arrow[r, "\psi_1"'] \arrow[ur, "\eta_1"] & B_2 \arrow[r, "\psi_1"'] & \cdots \end{tikzcd} \tag{4.19}$$

such that each of $\phi_i, \psi_i, \rho_i, \kappa_i$, $i = 1, 2, ...$, consists of independent coordinate projections and point evaluations, i.e., restricted to each direct summand of the domain, it has the form

$$(f_1, f_2, ..., f_s) \mapsto \mathrm{diag}\{f_1 \circ P_1, ..., f_s \circ P_s, \text{point evaluations}\},$$

where $P_1, ..., P_s$ are mutually disjoint sets of coordinate projections, and for each $i = 1, 2, ...,$ there are decompositions

$$\phi_i = \mathrm{diag}\{P_{A,i}, R'_{A,i}, \Theta_{A,i}\}, \quad \psi_i = \mathrm{diag}\{P_{B,i}, R'_{B,i}, \Theta_{B,i}\},$$

and

$$\eta_i \circ \rho_i = \mathrm{diag}\{P_{A,i}, R''_{A,i}, \Theta_{A,i}\}, \quad \rho_{i+1} \circ \eta_i = \mathrm{diag}\{P_{B,i}, R''_{B,i}, \Theta_{B,i}\},$$

where $P_{A,i} : A_i \to A_{i+1}$ and $P_{B,i} : B_i \to B_{i+1}$ consist of coordinate projections, and $\Theta_{A,i} : A_i \to A_{i+1}$ and $\Theta_{B,i} : B_i \to B_{i+1}$ consist of point evaluations, such that, for each $i = 1, 2, ...,$

$$\frac{\mathrm{rank}_j(R'_{A,i}(1_{A_i}))}{\mathrm{rank}_j(\Theta_{A,i}(1_{A_i}))} = \frac{\mathrm{rank}_j(R''_{A,i}(1_{A_i}))}{\mathrm{rank}_j(\Theta_{A,i}(1_{A_i}))} < \delta_i, \quad j = 1, ..., s^{(A)}_{i+1},$$

and

$$\frac{\mathrm{rank}_j(R'_{B,i}(1_{B_i}))}{\mathrm{rank}_j(\Theta_{B,i}(1_{B_i}))} = \frac{\mathrm{rank}_j(R''_{B,i}(1_{B_i}))}{\mathrm{rank}_j(\Theta_{B,i}(1_{B_i}))} < \delta_i, \quad j = 1, ..., s^{(B)}_{i+1},$$

where

$$\delta_i = \min\{\frac{1}{M^{(A)}_i},\ \frac{1}{M^{(B)}_i},\ \frac{1}{2}\Delta^{(A)}(h^{(A)}),\ \frac{1}{2}\Delta^{(B)}(h^{(B)}) : h^{(A)} \in \mathcal{H}^{(A)}_i,\ h^{(B)} \in \mathcal{H}^{(B)}_i\}.$$

Let us compare the maps

$$\phi_1 = \mathrm{diag}\{P_{A,1}, R'_{A,1}, \Theta_{A,1}\} \quad \text{and} \quad \eta_1 \circ \rho_1 = \mathrm{diag}\{P_{A,1}, R''_{A,1}, \Theta_{A,1}\}.$$

Since none of the elements of $\mathcal{H}^{(A)}_1$ has full support, for each $h \in \mathcal{H}^{(A)}_1$, there is $x_0 \in X_1$, where X_1 is the base space of A_1, such that $h(x_0) = 0$. Since the map $P_{A,1}$ consists of coordinate projections, there is $y_0 \in X_2$, where X_2 is the base space of A_2, such that $P_{A,1}(h)(y_0) = 0$, and therefore

$$\mathrm{tr}(\phi_1(h)(y_0)) = \mathrm{tr}(\mathrm{diag}\{P_{A,1}(h)(y_0), R'_{A,1}(h)(y_0), \Theta_{A,1}(h)(y_0)\}) > \Delta^{(A)}_1(h),$$

where tr is the normalized trace of the matrix algebra over y_0.

Hence

$$\frac{\mathrm{Tr}(\phi_1(h)(y_0))}{\mathrm{rank}(1_{R_{A,1}} + 1_{\Theta_{A,1}})} = \frac{\mathrm{Tr}(R'_{A,1}(h)(y_0)) + \mathrm{Tr}(\Theta_{A,1}(h)(y_0))}{\mathrm{rank}(R_{A,1}(1_{A_1}) + \Theta_{A,1}(1_{A_1}))} > \Delta^{(A)}_1(h),$$

where Tr is the unnormalized trace of the matrix algebra over y_0, and therefore (note that the image of $\Theta_{A,1}$ consists of constant functions),

$$\begin{aligned}
\frac{\mathrm{Tr}(\Theta_{A,1}(h))}{\mathrm{rank}(\Theta_{A,1}(1_{A_1}))} &> \Delta^{(A)}_1(h)(\frac{\mathrm{rank}(R_{A,1}(1_{A_1}))}{\mathrm{rank}(\Theta_{A,1}(1_{A_1}))} + 1) - \frac{\mathrm{Tr}(R'_{A,1}(h)(y_0))}{\mathrm{rank}(\Theta_{A,1}(1_{A_1}))} \\
&> \Delta^{(A)}_1(h)(\delta_1 + 1) - \frac{\mathrm{rank}(R'_{A,1}(1_{A_1}))}{\mathrm{rank}(\Theta_{A,1}(1_{A_1}))} \cdot \frac{\mathrm{Tr}(R'_{A,1}(h)(y_0))}{\mathrm{rank}(R'_{A,1}(1_{A_1}))} \\
&> \Delta^{(A)}_1(h) - \delta_1 \\
&> \frac{1}{2}\Delta^{(A)}_1(h).
\end{aligned}$$

Since

$$\frac{\mathrm{rank}_j(R'_{A,1}(1_{A_1}))}{\mathrm{rank}_j(\Theta_{A,1}(1_{A_1}))} = \frac{\mathrm{rank}_j(R''_{A,1}(1_{A_1}))}{\mathrm{rank}_j(\Theta_{A,1}(1_{A_1}))} < \delta_i < \frac{1}{M_1^{(A)}}, \quad j = 1, ..., s_2^{(A)},$$

it follows from Theorem 4.2 that there is a unitary $u_1 \in A_2$ such that

$$\|\phi_1(f) - u_1^*(\eta_1 \circ \rho_1)(f)u_1\| < \varepsilon_1, \quad f \in \mathcal{F}^{(A)}.$$

Replacing $\eta_1(\cdot)$ by $u_1^*\eta(\cdot)u_1$, and still denoting it by η_1, we have

$$\|\phi_1(f) - (\eta_1 \circ \rho_1)(f)\| < \varepsilon_1, \quad f \in \mathcal{F}^{(A)}.$$

Repeating this process, we have a diagram

$$\begin{tikzcd} A_1 \arrow[r, "\phi_1"] \arrow[d, "\rho_1"'] & A_2 \arrow[r, "\phi_2"] \arrow[d, "\rho_2"'] & \cdots \\ B_1 \arrow[r, "\psi_1"] \arrow[ur, "\eta_1"] & B_2 \arrow[r, "\psi_1"] & \cdots \end{tikzcd} \tag{4.20}$$

such that for each $i = 1, 2, ...,$

$$\|\phi_i(f) - (\eta_i \circ \rho_i)(f)\| < \varepsilon_i, \quad f \in \mathcal{F}_i^{(A)},$$

and

$$\|\psi_i(f) - (\rho_{i+1} \circ \eta_i)(f)\| < \varepsilon_i, \quad f \in \mathcal{F}_i^{(B)}.$$

By the approximate intertwining argument (Theorems 2.1 and 2.2 of [2]), we have $A \cong B$, as desired. $\square$

5. A generalized version of the comparison radius function for UHF-Villadsen algebras

Note that the function $r_\infty^{(0)}$ of Section 2 and Section 3 degenerates to the radius of comparison $\mathrm{rc}(A)$ if A is a Villadsen algebra of UHF type. In this section, let us introduce a more general function, denoted by r_∞, for Villadsen algebras of UHF type, which is not constant in general, but still has similar properties to the (numerical) radius of comparison (Theorem 5.4).

Definition 5.1. Let X be a compact Hausdorff space. For each $x \in X$, define

$$\mathrm{loc.dim}(x) = \min\{\dim(V) : V \text{ is a closed neighbourhood of } x\}.$$

Note that the function $x \mapsto \mathrm{loc.dim}(x)$ is upper semicontinuous, and, if X is a simplicial complex, then

$$\mathrm{loc.dim}((x_1, ..., x_n)) = \mathrm{loc.dim}(x_1) + \cdots + \mathrm{loc.dim}(x_n), \quad (x_1, ..., x_n) \in X^n.$$

Let X be a finite simplical complex, and let $A(X, (n_s), (k_s))$ be a Villadsen algebra with seed space X (see [4] and [11]). Let us briefly recall its construction [4]: $A(X, (n_s), (k_s))$ is the inductive limit of the sequence

$$\mathrm{C}(X) \longrightarrow \mathrm{M}_{(n_1+k_1)}(\mathrm{C}(X^{n_1})) \longrightarrow \mathrm{M}_{(n_1+k_1)(n_2+k_2)}(\mathrm{C}(X^{n_1 n_2})) \longrightarrow \cdots, \tag{5.1}$$

where the seed for the sth-stage map,

$$\phi_i : \mathrm{C}(X^{n_1\cdots n_{s-1}}) \to \mathrm{M}_{n_s+k_s}(\mathrm{C}(X^{n_1\cdots n_{s-1}n_s})),$$

is defined by

$$f \quad\mapsto\quad \mathrm{diag}\{f\circ\pi_1, ..., f\circ\pi_{n_s}, f(\theta_{s,1}), ..., f(\theta_{s,k_s})\}$$

where $\theta_{s,1}, ..., \theta_{s,k_s} \in X^{n_1\cdots n_{s-1}}$ are evaluation points. The evaluation points are chosen in such a way (dense enough) that the limit algebra is simple, and the growth sequences (n_s) and (k_s) are chosen so that

$$\lim_{s\to\infty}\lim_{t\to\infty}\frac{n_s\cdots n_t}{(n_s+k_s)\cdots(n_t+k_t)} = \lim_{i\to\infty}\lim_{j\to\infty}(\frac{n_s}{n_s+k_s})\cdots(\frac{n_t}{n_t+k_t}) = 1. \tag{5.2}$$

Note that, by Corollary 6.2 of [4], the simple limit algebra is independent of the evaluation points.

For each $s = 1, 2, ...,$ consider the function

$$r_s(x) = \frac{1}{2}\cdot\frac{\mathrm{loc.dim}(x)}{(n_1+k_1)\cdots(n_{s-1}+k_{s-1})},\quad x \in X^{n_1\cdots n_{s-1}}.$$

It is an upper semicontinuous function on $X^{n_1\cdots n_{s-1}} = \partial\mathrm{T}(A_s)$, and hence is an upper semicontinuous affine function on $\mathrm{T}(A_s)$, and hence on $\mathrm{T}(A)$.

On regarding $x \mapsto \mathrm{loc.dim}(x)$ as the upper left corner of A_s, there is a decreasing sequence of positive contractions $(f_{s,n}) \subseteq A_s$ such that

$$\lim_{n\to\infty}\tau(f_{s,n}) = r_s(\tau),\quad \tau\in\mathrm{T}(A_s). \tag{5.3}$$

(This will be used in the proof of Theorem 5.4(1) below.)

Lemma 5.2. *The sequence $r_1, r_2, ...,$ of upper semicontinuous positive real-valued affine functions on $\mathrm{T}(A)$ is decreasing, and for any $s < t$,*

$$\|r_s - r_t\|_\infty \leq \frac{1}{2}\mathrm{dim}(X)(\frac{n_1\cdots n_{s-1}}{(n_1+k_1)\cdots(n_{s-1}+k_{s-1})} - \frac{n_1\cdots n_{t-1}}{(n_1+k_1)\cdots(n_{t-1}+k_{t-1})}).$$

Proof. For each $x \in X^{(n_1\cdots n_{s-1})n_s}$, note that

$$\begin{aligned}
(\varphi_s)_*(r_s)((x_1,...,x_{n_s})) &= \frac{1}{n_s+k_s}(r_s(x_1)+\cdots+r_s(x_{n_s})+r_s(\theta_1)+\cdots+r_s(\theta_{k_s}))\\
&= \frac{1}{2}\cdot\frac{1}{(n_1+k_1)\cdots(n_s+k_s)}(\mathrm{loc.dim}(x_1)+\cdots+\mathrm{loc.dim}(x_{n_s}))+\\
&\quad \frac{1}{2}\cdot\frac{1}{(n_1+k_1)\cdots(n_s+k_s)}(\mathrm{loc.dim}(\theta_1)+\cdots+\mathrm{loc.dim}(\theta_{k_s}))\\
&= \frac{1}{2}\cdot\frac{1}{(n_1+k_1)\cdots(n_s+k_s)}\mathrm{loc.dim}(x_1,...,x_{n_s})+\\
&\quad \frac{1}{2}\cdot\frac{1}{(n_1+k_1)\cdots(n_s+k_s)}(\mathrm{loc.dim}(\theta_1)+\cdots+\mathrm{loc.dim}(\theta_{k_s})).
\end{aligned}$$

In particular, $r_s > r_{s+1}$ and

$$\begin{aligned}\|r_s - r_{s+1}\|_\infty &= \left\|\frac{1}{2}\cdot\frac{1}{(n_1+k_1)\cdots(n_s+k_s)}(\mathrm{loc.dim}(\theta_1)+\cdots+\mathrm{loc.dim}(\theta_{k_s}))\right\|_\infty \\ &\leq \frac{1}{2}\cdot\frac{k_s}{(n_1+k_1)\cdots(n_s+k_s)}(n_1\cdots n_{s-1})\mathrm{dim}(X).\end{aligned}$$

In general, the same argument shows that for any $s<t$,

$$\begin{aligned}\|r_s - r_t\|_\infty &\leq \frac{1}{2}\cdot\frac{(n_s+k_s)\cdots(n_{t-1}+k_{t-1})-n_s\cdots n_{t-1}}{(n_1+k_1)\cdots(n_{t-1}+k_{t-1})}(n_1\cdots n_{s-1})\mathrm{dim}(X) \\ &= \frac{1}{2}\cdot\frac{n_1\cdots n_{s-1}(n_s+k_s)\cdots(n_{t-1}+k_{t-1})-n_1\cdots n_{t-1}}{(n_1+k_1)\cdots(n_{t-1}+k_{t-1})}\mathrm{dim}(X) \\ &= \frac{1}{2}\cdot(\frac{n_1\cdots n_{s-1}}{(n_1+k_1)\cdots(n_{s-1}+k_{s-1})}-\frac{n_1\cdots n_{t-1}}{(n_1+k_1)\cdots(n_{t-1}+k_{t-1})})\mathrm{dim}(X).\end{aligned}$$

$\square$

Thus, by (5.2), the sequence (r_s) converges uniformly. Denote its limit by r_∞. By the construction, the function r_∞ is the pointwise limit of a decreasing sequence of the upper semicontinuous functions r_s, $s=1,2,...$, and hence r_∞ is also upper semicontinuous.

Definition 5.3. Let A be a simple C*-algebra. Define the set of (continuous) gap functions, G_A, to be the set of continuous positive real-valued affine functions $h:\mathrm{T}^+(A)\to[0,+\infty)$, 0 at 0, such that for any $a,b\in(A\otimes\mathcal{K})^+$,

$$\mathrm{d}_\tau(a)+h(\tau)<\mathrm{d}_\tau(b),\ \tau\in\mathrm{T}^+(A)\quad\Rightarrow\quad a\precsim b.$$

Theorem 5.4. *Let A be a UHF-Villadsen algebra with seed space a (finite) simplicial complex. Then the upper semicontinuous positive real-valued affine function r_∞ on $\mathrm{T}^+(A)$ has the property*

$$\{h\in\mathrm{Aff}(\mathrm{T}^+(A)):r_\infty\leq h\}=G_A. \tag{5.4}$$

Remark 5.5. Since r_∞ is upper semicontinuous and affine, one has

$$r_\infty=\inf\{h\in\mathrm{Aff}(\mathrm{T}^+(A)):r_\infty\leq h\}.$$

Together with (5.4), this implies

$$r_\infty=\inf G_A.$$

Proof. To prove (5.4), it is enough to show the following two properties:

(1) If h is a continuous positive real-valued affine function on $\mathrm{T}^+(A)$, 0 at 0, and $r_\infty\leq h$, then $h\in G_A$; that is, h has the property that for any $a,b\in(A\otimes\mathcal{K})^+$,
$$\mathrm{d}_\tau(a)+h(\tau)<\mathrm{d}_\tau(b),\ \tau\in\mathrm{T}^+(A)\quad\Rightarrow\quad a\precsim b.$$
(2) If h is a continuous positive real-valued affine function on $\mathrm{T}^+(A)$ and $h(\tau_0)<r_\infty(\tau_0)$ for some $\tau_0\in\mathrm{T}^+(A)$, then $h\notin G_A$; that is, there are $a,b\in(A\otimes\mathcal{K})^+$ such that
$$\mathrm{d}_\tau(a)+h(\tau)<\mathrm{d}_\tau(b),\quad\tau\in\mathrm{T}^+(A),$$
but a is not Cuntz subequivalent to b.

Proof of (1). Let $h \in \mathrm{Aff}(\mathrm{T}^+(A))$ be continuous and $r_\infty \leq h$, and let $a, b \in (A \otimes \mathcal{K})^+$ be such that

$$\mathrm{d}_\tau(a) + h(\tau) < \mathrm{d}_\tau(b), \quad \tau \in \mathrm{T}^+(A). \tag{5.5}$$

Let $\varepsilon > 0$ be arbitrary. There is $\delta > 0$ such that

$$\mathrm{d}_\tau((a-\varepsilon)_+) + h(\tau) + \delta < \mathrm{d}_\tau(b), \quad \tau \in \mathrm{T}^+(A). \tag{5.6}$$

Since $r_\infty \leq h$ and, by Lemma 5.2, (r_n) converges uniformly to r_∞, there is k such that

$$r_k(\tau) < h(\tau) + \frac{\delta}{8}, \quad \tau \in \mathrm{T}(A). \tag{5.7}$$

Choose a non-zero trivial projection $q \in A_i$ for some $i \in \mathbb{N}$ such that

$$\frac{3}{4}\delta < \tau(q) < \delta, \quad \tau \in A_i. \tag{5.8}$$

Since A has stable rank one, by Theorem 8.11 of [9] there is $c \in (A \otimes \mathcal{K})^+$ such that

$$\mathrm{d}_\tau(c) = h(\tau), \quad \tau \in \mathrm{T}(A).$$

Therefore, by (5.5) and (5.6),

$$\mathrm{d}_\tau(a \oplus c \oplus q) < \mathrm{d}_\tau(b), \quad \tau \in \mathrm{T}(A).$$

Note that, since $\tau \to \mathrm{d}_\tau(c) = h(\tau)$ is continuous, by Dini's theorem, there is $\delta' > 0$ such that

$$h(\tau) - \frac{\delta}{4} = \mathrm{d}_\tau(c) - \frac{\delta}{4} < \tau(f_{\delta'}(c)) \leq \mathrm{d}_\tau(c) = h(\tau), \quad \tau \in \mathrm{T}(A), \tag{5.9}$$

where $f_{\delta'} : \mathbb{R} \to [0,1]$ is the continuous function which is 0 on $(-\infty, \delta']$, 1 on $[2\delta', \infty)$, and linear in between. Fix δ'.

Since A is simple, by the proof of Proposition 3.2 of [8], there is $N \in \mathbb{N}$ such that

$$(a \oplus c \oplus q) \otimes 1_{N+1} \precsim b \otimes 1_N.$$

By Lemma 5.6 of [6] (and its proof), for any $\varepsilon' > 0$ (to be determined later), there is $i \in \mathbb{N}$ such that there are positive elements $\tilde{a}$, $\tilde{c}$, and $\tilde{b}$ in A_i (and $q \in A_i$) such that

$$\|a - \tilde{a}\| < \varepsilon', \quad \|c - \tilde{c}\| < \varepsilon', \quad \left\|b - \tilde{b}\right\| < \varepsilon',$$

$$((\tilde{a} - \varepsilon')_+ \oplus (\tilde{c} - \varepsilon')_+ \oplus q) \otimes 1_{N+1} \precsim \tilde{b} \otimes 1_N, \quad \text{and} \quad \tilde{b} \precsim b.$$

Hence

$$\mathrm{d}_\tau(\tilde{a} - \varepsilon')_+ + \mathrm{d}_\tau(\tilde{c} - \varepsilon')_+ + \tau(q) < \mathrm{d}_\tau(\tilde{b}), \quad \tau \in \mathrm{T}(A_i).$$

and, by (5.8),

$$\mathrm{d}_\tau(\tilde{a} - \varepsilon')_+ + \mathrm{d}_\tau(\tilde{c} - \varepsilon')_+ + \frac{3}{4}\delta < \mathrm{d}_\tau(\tilde{b}), \quad \tau \in \mathrm{T}(A_i). \tag{5.10}$$

Note that, with ε' sufficiently small, one has

$$\|f_{\delta'}(c) - f_{\delta'}(\tilde{c})\| < \frac{\delta}{4},$$

and hence

$$|\tau(f_{\delta'}(c)) - \tau(f_{\delta'}(\tilde{c}))| < \frac{\delta}{4}, \quad \tau \in \mathrm{T}(A). \tag{5.11}$$

Then, with $\varepsilon' < \delta'$, by (5.9) and (5.11),

$$\mathrm{d}_\tau(\tilde{c} - \varepsilon')_+ \geq \tau(f_{\delta'}(\tilde{c})) > \tau(f_{\delta'}(c)) - \frac{\delta}{4} > h(\tau) - \frac{\delta}{2}, \quad \tau \in \mathrm{T}(A). \tag{5.12}$$

By (5.7) one has one more step,

$$\mathrm{d}_\tau(\tilde{c} - \varepsilon')_+ \geq \tau(f_{\delta'}(\tilde{c})) > \tau(f_{\delta'}(c)) - \frac{\delta}{4} > h(\tau) - \frac{\delta}{2} > r_k(\tau) - \frac{3}{4}\delta, \quad \tau \in \mathrm{T}(A). \tag{5.13}$$

One should also assume that $\varepsilon' < \varepsilon$. Then, fix ε'. Since $\tau \mapsto r_k(\tau)$ is upper semicontinuous, a compactness argument (Dini's theorem) shows that there is $\delta'' > 0$ such that

$$\tau(f_{\delta''}((\tilde{c} - \varepsilon')_+)) > r_k(\tau) - \frac{3}{4}\delta, \quad \tau \in \mathrm{T}(A). \tag{5.14}$$

Thus, there is n such that

$$\tau(f_{\delta''}((\tilde{c} - \varepsilon')_+)) > \tau(f_{k,n}) - \frac{3}{4}\delta \geq r_k(\tau) - \frac{3}{4}\delta, \quad \tau \in \mathrm{T}(A), \tag{5.15}$$

where $(f_{k,n})_n$ is defined in (5.3), as, for any $\tau_0 \in \mathrm{T}(A)$, by (5.14), there is $N \in \mathbb{N}$ such that

$$\tau_0(f_{\delta''}((\tilde{c} - \varepsilon')_+)) > \tau_0(f_{k,n}) - \frac{3}{4}\delta > r_k(\tau_0) - \frac{3}{4}\delta, \quad n > N.$$

Since r_k is upper semicontinuous and $(f_{k,n})_n$ is decreasing, there is a neighbourhood $U \subseteq \mathrm{T}(A)$ of τ_0 such that

$$\tau(f_{\delta''}((\tilde{c} - \varepsilon')_+)) > \tau(f_{k,n}) - \frac{3}{4}\delta > r_k(\tau) - \frac{3}{4}\delta, \quad n > N, \quad \tau \in U.$$

Then, a compactness argument shows (5.15).

Then, by (5.15), there is $s > \max\{i, k\}$ such that

$$\tau(f_{\delta''}((\tilde{c} - \varepsilon')_+)) > \tau(f_{k,n}) - \frac{3}{4}\delta > r_k(\tau) - \frac{3}{4}\delta, \quad \tau \in \mathrm{T}(A_s). \tag{5.16}$$

As, otherwise, there is a sequence $\tau_{s_1} \in \mathrm{T}(A_{s_1}), \tau_{s_2} \in \mathrm{T}(A_{s_2}), ...$ such that

$$\tau_{s_i}(f_{\delta''}((\tilde{c} - \varepsilon')_+)) \leq \tau_{s_i}(f_{k,n}) - \frac{3}{4}\delta, \quad \tau \in \mathrm{T}(A_s).$$

Extend each τ_{s_i}, $i = 1, 2, ...$, to a state of A, and pick τ_∞ to be an accumulation point of $\{\tau_{s_i},\ i = 1, 2, ...\}$. Then $\tau_\infty \in \mathrm{T}(A)$ and it fails to satisfy (5.15).

Therefore,

$$\mathrm{d}_\tau((\tilde{c} - \varepsilon')_+) \geq \tau(f_{\delta''}((\tilde{c} - \varepsilon')_+)) > r_k(\tau) - \frac{3}{4}\delta, \quad \tau \in \mathrm{T}(A_s). \tag{5.17}$$

Then, by (5.17) and (5.10), for all $\tau \in \mathrm{T}(A_s)$, one has

$$\mathrm{d}_\tau((\tilde{a} - \varepsilon)_+) + r_s(\tau) \leq \mathrm{d}_\tau((\tilde{a} - \varepsilon)_+) + r_k(\tau) < \mathrm{d}_\tau((\tilde{a} - \varepsilon')_+) + \mathrm{d}_\tau((\tilde{c} - \varepsilon')_+) + \frac{3}{4}\delta < \mathrm{d}_\tau(\tilde{b}).$$

In particular,

$$\mathrm{rank}((\tilde{a}-\varepsilon)_+(x)) + \frac{1}{2}\cdot \mathrm{loc.dim}(x) < \mathrm{rank}(\tilde{b}(x)), \quad x \in X^{n_1\cdots n_{s-1}}. \tag{5.18}$$

Since X is a (finite) simplicial complex, writing

$$\mathrm{loc.dim}(X) = \{d_1, d_2, ..., d_l\},$$

where $d_1 > d_2 > \cdots > d_l$, and defining

$$X_{d_i} = \{x \in X^{n_1\cdots n_{s-1}} : \mathrm{loc.dim}(x) = d_i\},$$

one has

$$\dim(\overline{X_{d_i}}) = d_i, \quad i = 1, ..., l.$$

Note that there is a decomposition

$$X^{n_1\cdots n_{s-1}} = X_{d_1} \cup X_{d_2} \cup \cdots \cup X_{d_l},$$

and, since the function $\mathrm{loc.dim}(\cdot)$ is upper semicontinuous, the sets

$$Y_i := X_{d_1} \cup \cdots \cup X_{d_i}, \quad i = 1, ..., l,$$

are closed. This induces a recursive subhomogeneous decomposition (see [7])

$$A_s = \mathrm{M}_{(n_1+k_1)\cdots(n_{s-1}+k_{s-1})}(\mathrm{C}(X^{n_1\cdots n_{s-1}})) = (\cdots((A_1 \oplus_{A_2^{(0)}} A_2) \oplus_{A_3^{(0)}} A_3) \oplus \cdots) \oplus_{A_l^{(0)}} A_l,$$

where

$$A_i = \mathrm{M}_{(n_1+k_1)\cdots(n_{s-1}+k_{s-1})}(\mathrm{C}(\overline{X_{d_i}})) \quad \text{and} \quad A_i^{(0)} = \mathrm{M}_{(n_1+k_1)\cdots(n_{s-1}+k_{s-1})}(\mathrm{C}(Y_{i-1} \cap \overline{X_{d_i}})).$$

For each X_i, $i = 1, ..., l$, one has

$$\dim(\overline{X_i}) = d_i = \mathrm{loc.dim}(x), \quad x \in X_i.$$

Thus, by (5.18),

$$\mathrm{rank}((\tilde{a}-\varepsilon)_+(x)) + \frac{1}{2}\cdot \dim(\overline{X_i}) < \mathrm{rank}(\tilde{b}(x)), \quad x \in X_i,\ i = 1, ..., l.$$

By Theorem 4.6 of [10],

$$(\tilde{a}-\varepsilon)_+ \precsim \tilde{b} \precsim b.$$

Since

$$a \approx_{\varepsilon'} \tilde{a} \approx_{\varepsilon} (\tilde{a}-\varepsilon)_+,$$

one has

$$(a-2\varepsilon)_+ \precsim b.$$

Since ε is arbitrary, this implies $a \precsim b$. This shows (1).

Proof of (2). Let $h \in \mathrm{Aff}^+(\mathrm{T}^+(A))$ such that

$$h(\tau_0) < r_\infty(\tau_0)$$

for some $\tau_0 \in \mathrm{T}^+(A)$. Let us show that $h \notin G_A$.

Set

$$\delta = \max\{r_\infty(\tau) - h(\tau) : \tau \in \mathrm{T}^+(A),\ \tau(1_A) = 1\} > 0$$

and

$$M = \max\{h(\tau) : \tau \in \mathrm{T}^+(A),\ \tau(1_A) = 1\}.$$

By (5.1), one has the following inductive limit decomposition of the ordered Banach space $\mathrm{Aff}(\mathrm{T}(A))$:

$$\mathrm{C}_{\mathbb{R}}(X) \xrightarrow{\varphi_1^*} \mathrm{C}_{\mathbb{R}}(X^{n_1}) \xrightarrow{\varphi_2^*} \mathrm{C}_{\mathbb{R}}(X^{n_1 n_2}) \xrightarrow{\varphi_3^*} \cdots \longrightarrow \mathrm{Aff}(\mathrm{T}(A)).$$

Then there is $h_s \in \mathrm{C}_{\mathbb{R}^+}(X^{n_1 \cdots n_s})$ such that

$$\|h_s - h\|_\infty < \frac{\delta}{4}. \tag{5.19}$$

Hence, with s sufficiently large, there is $\tau_0 \in \mathrm{T}(A_s) = \mathcal{M}_1(X^{n_1\cdots n_s})$ such that

$$h_s(\tau_0) < r_s(\tau_0) - \frac{3}{4}\delta,$$

and this implies that there is $x_0 \in X^{n_1 \cdots n_s}$ such that

$$h_s(x_0) < r_s(x_0) - \frac{3}{4}\delta = \frac{1}{2} \cdot \frac{\mathrm{loc.dim}(x_0)}{(n_1+k_1)\cdots(n_s+k_s)} - \frac{3}{4}\delta.$$

Since $X^{n_1\cdots n_s}$ is a (finite) simplical complex, there is a $\mathrm{loc.dim}(x_0)$-dimensional ball $B_s \subseteq X^{n_1\cdots n_s}$ in any neighbourhood of x_0. Then, since h_s is continuous, there is a Euclidean ball $B_s \subseteq X^{n_1\cdots n_s}$ (in any neighbourhood of x_0) with dimension d_{x_0}, where

$$d_{x_0} = \begin{cases} \mathrm{loc.dim}(x_0), & \text{if } \mathrm{loc.dim}(x_0) \text{ is odd}, \\ \mathrm{loc.dim}(x_0) - 1, & \text{if } \mathrm{loc.dim}(x_0) \text{ is even}, \end{cases}$$

such that

$$h_s(x) < r_s(x) - \frac{3}{4}\delta = r_s(x_0) - \frac{3}{4}\delta, \quad x \in B_s. \tag{5.20}$$

One should also assume s is sufficiently large that

$$\left| \frac{\mathrm{loc.dim}(x) - 1}{(n_1+k_1)\cdots(n_s+k_s)} - r_s(x) \right| < \frac{\delta}{4} \tag{5.21}$$

and

$$1 - \left(\frac{n_{s+1}}{n_{s+1}+k_{s+1}}\right)\cdots\left(\frac{n_t}{n_t+k_t}\right) < \frac{\delta}{8}, \quad t > s. \tag{5.22}$$

Over ∂B_s, which is a $(d_{x_0}-1)$-dimensional sphere, there is a complex vector bundle E such that

$$\mathrm{rank}(E) = \frac{1}{2}(d_{x_0} - 1) \quad \text{and} \quad c_{\frac{d_{x_0}-1}{2}} \in H^{d_{x_0}-1}(S^{d_{x_0}-1}) \setminus \{0\}.$$

(Such a vector bundle exists, as, otherwise, the $\frac{d_{x_0}-1}{2}$-th Chern class of every vector bundle would be trivial, and then the Chern character would not induce a rational isomorphism between the K-group and the cohomology group of the sphere $S^{d_{x_0}-1}$.)

Denote by p the projection associated to E_i. Then,

$$\mathrm{tr}_x(p) = \frac{1}{2} \cdot \frac{d_{x_0} - 1}{(n_1+k_1)\cdots(n_s+k_s)}, \quad x \in \partial B_s, \tag{5.23}$$

where tr_x is the tracial state of A_s which is concentrated at x. Extend p to a positive element of $A_s \otimes \mathcal{K} = \mathrm{C}(X^{n_1 \cdots n_s}) \otimes \mathcal{K}$ and still denote it by p.

Choose a positive matrix e such that

$$\mathrm{rank}(e) \geq (M + 2\delta)(n_1 + k_1) \cdots (n_s + k_s),$$

and set

$$p_0 : X^{n_1 \cdots n_s} \ni x \mapsto \mathrm{dist}(x, \partial B_s) e \in \mathcal{K}.$$

Consider the element $p + p_0$, and still denote it by p. Then, together with (5.23), (5.21), and (5.20), one has

$$\tau(p) > h_s(\tau) + \frac{\delta}{2}, \quad \tau \in \mathrm{T}(A_s), \tag{5.24}$$

and the restriction of p to ∂B_s is a projection such that the corresponding vector bundle has non-zero total Chern class at degree $\mathrm{loc.dim}(x_0) - 1$.

Let $q \in A_s$ be a trivial projection with

$$\frac{\delta}{4} > \tau(q) > \frac{\delta}{8}, \quad \tau \in \mathrm{T}(A_s).$$

Then, by (5.19) and (5.24),

$$\tau(q) + h(\tau) < \frac{\delta}{4} + (h_s(\tau) + \frac{\delta}{4}) < \tau(p), \quad \tau \in \mathrm{T}(A).$$

To show the theorem, it is enough to show that q is not Cuntz sub-equivalent to p.

Let $t > s$ be arbitrary, and consider the building block A_t. Consider the closed subset

$$C_t := \underbrace{\partial B_s \times \cdots \times \partial B_s}_{n_{s+1} \cdots n_t} \subseteq \underbrace{X^{n_1 \cdots n_s} \times \cdots \times X^{n_1 \cdots n_s}}_{n_{s+1} \cdots n_t}.$$

Then

$$\phi_{s,t}(p)|_{C_t} = \mathrm{diag}\{p|_{\partial B_s} \circ \pi_1, ..., p|_{\partial B_s} \circ \pi_{n_{s+1} \cdots n_t}, c\},$$

where c is a constant positive matrix of rank at most

$$(n_1 + k_1) \cdots (n_t + k_t) - (n_1 + k_1) \cdots (n_s + k_s)(n_{s+1} \cdots n_t).$$

Hence the positive element $\phi_{s,t}(p)|_{C_t}$ is Cuntz equivalent to a projection of rank at most

$$R_t := \frac{1}{2}(d_{x_0} - 1)(n_{s+1} \cdots n_t) + (n_1 + k_1) \cdots (n_t + k_t) - (n_1 + k_1) \cdots (n_s + k_s)(n_{s+1} \cdots n_t)$$

and with non-zero total Chern class (by the Künneth Theorem) at

$$H^{(d_{x_0} - 1)(n_{s+1} \cdots n_t)}(C_t).$$

Thus, by Remark 3.2 of [4], the trivial subprojection of $[\phi_{s,t}(p)|_{C_t}]$ has rank at most

$$R_t - \frac{1}{2}(d_{x_0} - 1)(n_{s+1} \cdots n_t),$$

and hence has (normalized) trace at most

$$\begin{aligned}
& \frac{R_t - \frac{1}{2}(d_{x_0} - 1)(n_{s+1} \cdots n_t)}{(n_1 + k_1) \cdots (n_t + k_t)} \\
= \; & \frac{(n_1 + k_1) \cdots (n_t + k_t) - (n_1 + k_1) \cdots (n_s + k_s)(n_{s+1} \cdots n_t)}{(n_1 + k_1) \cdots (n_t + k_t)} \\
= \; & 1 - \frac{n_{s+1}}{n_{s+1} + k_{s+1}} \cdots \frac{n_t}{n_t + k_t} \\
< \; & \frac{\delta}{8}.
\end{aligned}$$

Since $\tau(\phi_{s,t}(q)) > \delta/8$ for all $\tau \in \mathrm{T}(A_t)$, this implies that q is not Cuntz subequivalent to p. This shows (2). □

Remark 5.6. If X has the property that loc.dim$(\cdot)$ is constant, then

$$r_\infty(\tau) = \mathrm{rc}(A)(\tau(1_A)), \quad \tau \in \mathrm{T}^+(A).$$

Corollary 5.7. *Let* $X_1 = [0,1]^2$ *and* $X_2 = [0,1] \vee [0,1]^2$, *and let* $A_1 = A(X_1, (n_i), (k_i))$ *and* $A_2 = (X_2, (n_i), (k_i))$ *be UHF-Villadsen algebras with seed spaces* X_1 *and* X_2 *respectively. Then*

$$\mathrm{rc}(A_1) = \mathrm{rc}(A_2) \quad \text{but} \quad A_1 \otimes \mathcal{K} \ncong A_2 \otimes \mathcal{K}.$$

Indeed,

$$\mathrm{Cu}(A_1) \ncong \mathrm{Cu}(A_2).$$

Proof. By Remark 5.6, the affine function $r_\infty(A_1)$ is constant on $\mathrm{T}(A_1)$, and so it factors through $\mathrm{T}^+(A_1) \to \mathrm{T}^+(\mathrm{K}_0(A_1)) \cong \mathbb{R}^+$. On the other hand, the function $r_\infty(A_2)$ is not constant on $\mathrm{T}(A_2)$, and so does not factor through $\mathrm{T}^+(A_2) \to \mathrm{T}^+(\mathrm{K}_0(A_2)) \cong \mathbb{R}^+$. By Theorem 5.4, the functions $r_\infty(A_1)$ and $r_\infty(A_2)$ are invariant (uniquely determined by the Cuntz semigroup). Hence, $\mathrm{Cu}(A_1) \ncong \mathrm{Cu}(A_2)$, as desired. □

Remark 5.8. Note that, since X_1 and X_2 are contractible, by Corollary 6.1 of [13],

$$X_1^\infty \cong X_2^\infty.$$

So, the Cuntz semigroup of a Villadsen algebra contains information which is finer than the infinite product of the seed space.

On the other hand, the Villadsen algebras $A([0,1], (n_i), (k_i))$ and $A([0,1]^2, (n_i), (k_i))$ are stably isomorphic (but not isomorphic). Therefore, their Cuntz semigroups are isomorphic.

Remark 5.9. Let $X_1 = [0,1] \vee [0,1]^2$ and $X_2 = [0,1] \vee [0,1]^2 \vee [0,1]$. It would be interesting to know if the Villadsen algebras $A_1 = A(X_1, (n_i), (k_i))$ and $A_2 = A(X_2, (n_i), (k_i))$ share the same Cuntz semigroup.

Corollary 5.10 (cf. Corollary 3.4)**.** *Let* $A(X, (n_i), (k_i))$ *be a UHF-Villadsen algebra, and let* $\sigma \in \mathrm{Aut}(A)$. *Then*

$$r_\infty((\sigma^*(\tau))) = r_\infty(\tau), \quad \tau \in \mathrm{T}(A).$$

Proof. Since σ is an automorphism, one has $\sigma_*(G_A) = G_A$, and therefore $\sigma_*(r_\infty)$ is also a lower enveloping function of G_A. By Lemma 5.5, such a lower enveloping function is unique, and hence $\sigma_*(r_\infty) = r_\infty$, as asserted. $\square$

Corollary 5.11 (cf. Example 3.7)**.** *Let $A = A(X, (n_i), (k_i))$ be a UHF-Villadsen algebra with seed space $X = [0,1] \vee [0,1]^2$. Then the action of* $\mathrm{Aut}(A)$ *on the extreme points of* $\mathrm{T}(A)$*, the Poulsen simplex (see* [4]*), is not transitive.*

Proof. The restriction of the function r_∞ to $\mathrm{T}(A)$ is not constant, and so there are $\tau_1, \tau_2 \in \partial\mathrm{T}(A)$ such that $r_\infty(\tau_1) \neq r_\infty(\tau_2)$. By the corollary above, there is no $\sigma \in \mathrm{Aut}(A)$ such that $\sigma^*(\tau_1) = \tau_2$, as desired. $\square$

Definition 5.12. Let A be a C*-algebra. An upper semicontinuous extended positive real valued affine function r_∞ on $\mathrm{T}^+(A)$ will be called the comparison radius function if it has the following property:

$$\{h \in \mathrm{Aff}(\mathrm{T}^+(A)) : r_\infty \leq h\} = G_A.$$

Note that, by Remark 5.5, the comparison radius function, if it exists, satisfies

$$r_\infty = \inf G_A,$$

and hence is unique.

The radius of comparison can be recovered from the comparison radius function r_∞ (cf. Remark 5.6).

Theorem 5.13. *Let A be a C*-algebra such that the comparison radius function r_∞ exists (e.g., A is a UHF-Villadsen algebra with seed space a (finite) simplicial complex). Then, for any non-zero projection $p \in A \otimes \mathcal{K}$, one has*

$$\mathrm{rc}(p(A \otimes \mathcal{K})p) = \sup\{r_\infty(\tau) : \tau(p) = 1,\ \tau \in \mathrm{T}^+(A)\}.$$

Proof. Let $s > \sup\{r_\infty(\tau) : \tau(p) = 1,\ \tau \in \mathrm{T}^+(A)\}$ be a real number. Then, regarding s as a constant (continuous) affine function on the section $\{\tau \in \mathrm{T}^+(A) : \tau(p) = 1\} = \mathrm{T}(p(A \otimes \mathcal{K})p)$, and extending s to $\mathrm{T}^+(p(A \otimes \mathcal{K})p) = \mathrm{T}^+(A)$, one has

$$r_\infty(\tau) < s(\tau), \quad \tau \in \mathrm{T}^+(A).$$

Therefore $s \in G_A$ (see (1) of Theorem 5.4), and so s has the property

$$\mathrm{d}_\tau(a) + s < \mathrm{d}_\tau(b),\ \tau \in \mathrm{T}(p(A \otimes \mathcal{K})p) \quad \Longrightarrow \quad a \precsim b, \quad a, b \in (A \otimes \mathcal{K})^+,$$

and hence $\mathrm{rc}(p(A \otimes \mathcal{K})p) \leq s$. This shows that

$$\mathrm{rc}(p(A \otimes \mathcal{K})p) \leq \sup\{r_\infty(\tau) : \tau(p) = 1,\ \tau \in \mathrm{T}^+(A)\}.$$

Now, let $s \leq \sup\{r_\infty(\tau) : \tau(p) = 1,\ \tau \in \mathrm{T}^+(A)\}$ be a real number. Then, for an arbitrary $\varepsilon > 0$, there is $\tau_0 \in \{\tau \in \mathrm{T}^+(A) : \tau(p) = 1\}$ such that $s - \varepsilon < r(\tau_0)$. Regarding $s - \varepsilon$ as a continuous affine function on $\mathrm{T}^+(A)$ as above (constant equal to the number $s - \varepsilon$ on $\mathrm{T}(p(A \otimes \mathcal{K})p)$), one has $s - \varepsilon \notin G_A$ (by (2) of Theorem 5.4); that is, there are $a, b \in (A \otimes \mathcal{K})^+$ such that

$$\mathrm{d}_\tau(a) + (s - \varepsilon) < \mathrm{d}_\tau(b),\ \tau \in \mathrm{T}(p(A \otimes \mathcal{K})p),$$

but a is not Cuntz subequivalent to b. Therefore,

$$s - \varepsilon \leq \mathrm{rc}(p(A \otimes \mathcal{K})p).$$

Since ε is arbitrary, this implies $s \leq \mathrm{rc}(p(A \otimes \mathcal{K})p)$. This shows that

$$\sup\{r_\infty(\tau) : \tau(p) = 1,\ \tau \in \mathrm{T}^+(A)\} \leq \mathrm{rc}(p(A \otimes \mathcal{K})p).$$

Together with the opposite inequality proved above, one has

$$\mathrm{rc}(p(A \otimes \mathcal{K})p) = \sup\{r_\infty(\tau) : \tau(p) = 1,\ \tau \in \mathrm{T}^+(A)\},$$

as asserted. □

Remark 5.14. Since r_∞ is upper semicontinuous, one has

$$\sup\{r_\infty(\tau) : \tau(p) = 1,\ \tau \in \mathrm{T}^+(A)\} = \max\{r_\infty(\tau) : \tau(p) = 1,\ \tau \in \mathrm{T}^+(A)\}.$$

Remark 5.15. Does the comparison radius function r_∞ exist for every simple C*-algebra? At least for simple C*-algebras of stable rank one?

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