

# A CLASSIFICATION OF TRACIALLY APPROXIMATE SPLITTING INTERVAL ALGEBRAS. I. THE BUILDING BLOCKS AND THE LIMIT ALGEBRAS.

ZHUANG NIU

**ABSTRACT.** Motivated by Huaxin Lin’s axiomatization of AH-algebras, the class of TASI-algebras is introduced as the class of unital  $C^*$ -algebras which can be tracially approximated by splitting interval algebras—certain sub- $C^*$ -algebras of interval algebras. It is shown that the class of simple separable nuclear TASI-algebras satisfying the UCT is classified by the Elliott invariant. As a consequence, any such TASI-algebra is then isomorphic to an inductive limit of splitting interval algebras together with certain homogeneous  $C^*$ -algebras—so it is an ASH-algebra.

**Résumé.** Une classe de  $C^*$ -algèbres qui généralisent la classe bien connue TAI de Lin est considérée—basées sur, au lieu de l’intervalle, ce qui pourrait s’appeler l’intervalle fendu (“splitting interval”), de sorte que l’on les appelle la classe TASI. On montre que la classe de  $C^*$ -algèbres TASI qui sont simples, nucléaires, et à élément unité, qui vérifient le théorème à coefficients universels (UCT), peuvent se classer d’après l’invariant d’Elliott.

**1. Introduction** The Elliott programme for the classification of nuclear  $C^*$ -algebras has been developed rapidly in the last twenty years, and several classes of  $C^*$ -algebras have been covered in this programme. For instance, the class of separable simple nuclear purely infinite  $C^*$ -algebras which satisfy the Universal Coefficient Theorem (UCT) was shown by Kirchberg and Phillips to be classified by the  $K$ -functor. On the other hand, for finite  $C^*$ -algebras (a unital  $C^*$ -algebra is called finite if any left invertible element is also right invertible), the first step was made by Elliott in [8], where the class of separable approximately finite-dimensional  $C^*$ -algebras (AF-algebras for short) was classified using the order-unit  $K_0$ -functor. (In fact, the classification of AF-algebras is also regarded as the starting point of the whole classification programme.)

After a fifteen-year hiatus, the classification of AF-algebras was generalized to the class of simple approximate interval algebras (AI-algebras for short) and the class of simple approximate circle algebras (AT-algebras for short) with real rank zero. See [10], [11], and [13]. Both classes contain the class of simple AF-algebras as a subclass, but they are substantially larger: for AI-algebras, they might have an abundance of tracial states, and for AT-algebras, they might have non-trivial  $K_1$ -groups. So, for these classes of  $C^*$ -algebras, the classifying functor should

---

Received by the editors on June 26, 2012; revised March 26, 2013.

AMS Subject Classification: Primary: 46L35; secondary: 19K14.

Keywords: Classification of simple  $C^*$ -algebras, inductive limits of sub-homogeneous  $C^*$ -algebras, tracially approximate splitting interval algebras..

© Royal Society of Canada 2014.

not only contain the order-unit  $K_0$ -group, but also the simplex of tracial states (of course with the pairing with the  $K_0$ -group) and the  $K_1$ -group. This functor is called the Elliott invariant.

Using the Elliott invariant, in [14], [17], and [15], Elliott, G. Gong and L. Li were able to generalize the classification results above to a significantly large class of  $C^*$ -algebras, namely, the class of simple  $C^*$ -algebras arising as inductive limits of  $pM_n(C(X))p$ , where  $X$  is a compact metrizable space, and  $p$  is a projection in  $M_n(C(X))$ . Let us call such a  $C^*$ -algebra an AH-algebra. It is clear that AF-algebras, AI-algebras, and AT-algebras are all AH-algebras. Under the assumption of very slow dimension growth—that is, the dimension of  $X$  is sufficiently small compared to  $n$  as the building blocks go to infinity—the class of AH-algebras is classified by the Elliott invariant. (The restriction on the dimension growth turns out to be a crucial condition, which guarantees the regularity of the  $K$ -theoretical invariant, including the Cuntz semigroup ([36], [40], etc.). Without this restriction, plenty of exotic AH-algebras have been constructed. See, for instance, [42], [43], [39], etc.)

One important step in the classification of AH-algebras is the factorization theorem of [17], which states that the maps in the inductive sequence of an AH-algebra with very slow dimension growth can be approximately decomposed into the direct sum of a map factoring through an interval algebra and a map which is small at the level of traces.

Motivated by this factorization theorem, H. Lin introduced an axiomatization of AH-algebras in [23], [21], [22], [24], [25], etc. Instead of considering inductive limit decomposition, he considered tracial approximation by interval algebras in the sense of Egorov. More precisely, a simple  $C^*$ -algebra  $A$  is called a tracially approximate interval algebra (TAI-algebra for short) if for any  $\varepsilon > 0$ , any finite subset  $\mathcal{F} \subseteq A$  and any  $0 \neq a \in A^+$ , there is a non-zero projection  $p \in A$  and a sub- $C^*$ -algebra  $I \subseteq A$  with  $1_I = p$  and  $I \cong C([0, 1]) \otimes F$  for some finite-dimensional  $C^*$ -algebra  $F$  such that for all  $x \in \mathcal{F}$ ,

- $\|xp - px\| < \varepsilon$ ,
- $pxp \in_\varepsilon I$ , and
- $1 - p$  is Murray-von Neumann equivalent to a projection in  $\overline{aAa}$ .

It follows from the factorization theorem that any AH-algebra with very slow dimension growth is a TAI-algebra.

Lin introduced and studied TAI-algebras, and showed that the class of nuclear TAI-algebras which satisfy the UCT is classified by the Elliott invariant. Since AH-algebras with very slow dimension growth are TAI-algebras, and these AH-algebras exhaust all the possible invariants (see [41]), this class of TAI-algebras coincides with the class of AH-algebras with very slow dimension growth, and thus provides an axiomatization for AH-algebras.

The axiomatization of AH-algebras plays a crucial role in the recent progress in the classification programme. First, the condition of being tracially approximated by interval algebras is much easier to verify than the condition of being an inductive limit of certain building blocks. Hence, it provides a powerful tool

to verify whether some naturally arising  $C^*$ -algebras fall inside the class of AH-algebras; see, for example, [33] and [7].

More importantly, Lin's axiomatization of AH-algebras is the key ingredient of the further development of the classification on  $C^*$ -algebras which lack projections. Using the picture of tracial approximation by interval algebras, Lin proved a Basic Homotopy Lemma for TAI-algebras (see [5], [27] [28]). With this lemma, an asymptotical uniqueness theorem, among many other results, is obtained for TAI-algebras ([26], [29]). This enables one to use the method of Winter in [45] to push the classification theorem of TAI-algebras to the class of  $\mathcal{Z}$ -stable  $C^*$ -algebras which are TAI-algebras after tensoring with any UHF-algebra (the so-called rationally AH-algebras). See [30], [31] and [29] for more details. Note that the class of rationally AH-algebras contains the Jiang-Su algebra  $\mathcal{Z}$  itself, which has no non-trivial projections.

The range of the invariant for rationally AH-algebras is characterized in [32] as the  $K_0$ -group exhausting all simple order-unit groups with Riesz decomposition property after tensoring with  $\mathbb{Q}$  (the so-called rationally Riesz groups), and the canonical pairings with tracial simplexes preserving extreme points. In particular, the convex hull of the states on the  $K_0$ -group of any rationally AH-algebra must be a simplex.

In general, the convex of the states on the  $K_0$ -group of a stably finite  $C^*$ -algebra is not necessarily a simplex, and the canonical pairing does not necessarily preserve extreme points. Splitting interval algebras and their inductive limits, which were introduced by H. Su in [38], provide such examples. (More general examples are constructed by Elliott in [12] to exhaust all possible value of the invariants with weakly unperforated  $K_0$ -groups.)

Recall that splitting interval algebras are finite direct sums of  $C^*$ -algebras defined by

$$\{f \in M_n(C([0, 1])); f(0) \text{ and } f(1) \text{ are in block diagonal form}\}.$$

A typical example is the universal unital  $C^*$ -algebra generated by two projections, which is the splitting interval algebra with  $n = 2$ , and  $f(0)$  and  $f(1)$  required to be diagonal matrices.

Jiang and Su classified simple inductive limits of splitting interval algebras in [20] using the Elliott invariant. It was also shown by Elliott and Thomsen that there exists a simple inductive limit of splitting interval algebras with the convex hull of the states on the  $K_0$ -group being a square, and hence the  $K_0$ -group does not satisfy the Riesz decomposition property, even after tensoring with the group of rational numbers. Hence this  $C^*$ -algebra is not in the scope of AH-algebras intrinsically.

However, there are some limitations on the invariants of splitting interval algebras themselves, as they always have trivial  $K_1$ -group, and do not have any torsion in the  $K_0$ -group. In order to have torsion in the  $K_0$ -group, one needs to bring building blocks with two-dimensional base space into the inductive limit approximation; and in order to have non-trivial  $K_1$ -group, one needs to consider building blocks with circle and three-dimensional space as base space.

In this paper, instead considering the inductive limit of splitting interval algebras, we shall consider the C\*-algebras which are tracially approximated by splitting interval algebras (TASI-algebras). More precisely,

DEFINITION . A unital simple C\*-algebra  $A$  is called a tracially approximate splitting interval algebra (TASI-algebra for short) if for any  $\varepsilon > 0$ , any finite subset  $\mathcal{F} \subseteq A$  and any  $0 \neq a \in A^+$ , there is a projection  $p \in A$  and a sub-C\*-algebra  $S \subseteq A$  which is a splitting interval algebra with  $1_S = p \neq 0$  such that for all  $x \in \mathcal{F}$ ,

- $\|xp - px\| < \varepsilon$ ,
- $pxp \in_\varepsilon S$ , and
- $1 - p$  is Murray-von Neumann equivalent to a projection in  $\overline{aAa}$ .

This class of C\*-algebras contains the TAI-algebras and also the simple inductive limits of splitting interval algebras. The main theorem of the paper is that the class of simple separable amenable TASI-algebras satisfying the UCT is classified by the Elliott invariant.

THEOREM. *Let  $A$  and  $B$  be two simple separable nuclear TASI-algebra which satisfies the UCT. Then  $A \cong B$  if and only if*

$$\begin{aligned} & ((K_0(A), K_0(A)^+, [1_A]_0), K_1(A), T(A), r_A) \\ & \cong ((K_0(B), K_0(B)^+, [1_B]_0), K_1(B), T(B), r_B). \end{aligned}$$

*Moreover, the \*-isomorphism between the C\*-algebras can be chosen to induce the given isomorphism between their invariants.*

This is the part I of the paper. We will introduce the building blocks and study the basic properties of the C\*-algebras which are tracially approximated by those building blocks. This is a preparation for the uniqueness theorem, the existence theorem, and the classification theorem in the part II and part III of the paper.

## 2. Preliminaries, notation and basic building blocks

2.1. *Preliminaries and notation* Let  $A$  be a unital C\*-algebra. We shall use  $T(A)$  to denote the simplex of tracial states on  $A$ . Let  $(G, G^+, u)$  be an order-unit group. We shall use  $S_u(G)$  to denote the convex of the states on  $G$ , i.e.,

$$S_u(G) = \{\rho : G \rightarrow \mathbb{R}; \rho(G^+) \subseteq \mathbb{R}^+, \rho(u) = 1\}.$$

If the C\*-algebra  $A$  is stably finite (in particular, if  $A$  is simple and has tracial state), the triple  $(K_0(A), K_0(A)^+, [1]_0)$  is always an order-unit group. (See, for example, Proposition 6.6.3 of [1].) For any  $\tau \in T(A)$ , the restriction of  $\tau$  to projections in matrix algebras over  $A$  induces a positive homomorphism

$$r(\tau) : K_0(A) \rightarrow \mathbb{R},$$

which in fact is a state on  $K_0(A)$ . Thus  $r$  induces an affine map from  $T(A) \rightarrow S_u(K_0(A))$ . Combining the results in [4] and [19], one has that the map  $r$  is always surjective when  $A$  is exact.

DEFINITION 2.1. Let  $A$  be a simple unital stably finite  $C^*$ -algebras. Then the Elliott invariant of  $A$  is defined by

$$\text{Ell}(A) = (((K_0(A), K_0^+(A), [1_A]), T(A), r_A), K_1(A)).$$

REMARK 2.2. A simple ordered group  $G$  is said to be weakly unperforated if  $na > 0$  for some  $n \in \mathbb{N}$  and  $a \in G$  implies  $a > 0$ . Then, for any weakly unperforated simple order-unit group  $(G_0, G_0^+, u)$ , any Choquet simplex  $\Delta$ , any surjective map  $r : \Delta \rightarrow S_u(G)$ , and any abelian group  $G_1$ , it is shown in [12] that there is a unital simple  $C^*$ -algebra  $A$  such that

$$\text{Ell}(A) = (((G_0, G_0^+, u), \Delta, r), G_1).$$

DEFINITION 2.3. An ordered group is said to have the Riesz decomposition property if for any  $a \leq b + c$  with  $a, b, c \in G^+$ , there are  $b' \leq b$  and  $c' \leq c$  such that  $a = b' + c'$ . An ordered group is said to be a dimension group with torsion (see [9]) if it is a weakly unperforated Riesz group.

Note that if an order-unit group  $(G, G^+, u)$  has the Riesz decomposition property, then the convex  $S_u(G)$  is a simplex. The range of the invariant of AH-algebras was given by Villadsen.

THEOREM 2.4 ([41]). *Let  $G_0$  be a simple order-unit group. Let  $G_1$  be an abelian group. Let  $\Delta$  be a Choquet simplex, and let  $r : \Delta \rightarrow S_u(G_0)$  be a surjective map. Then there exists a simple AH-algebra  $A$  with very slow dimension growth such that*

$$(((G_0, G_0^+, u), \Delta, r), G_1) \cong \text{Ell}(A)$$

*if and only if  $G_0$  is a dimension group with torsion, and the map  $r$  preserves extreme points.*

DEFINITION 2.5. Let  $A$  and  $B$  be  $C^*$ -algebras. Let  $\mathcal{F} \subseteq A$  be a finite subset, and let  $\delta > 0$ . A linear map  $\phi : A \rightarrow B$  is said to be  $\mathcal{F}$ - $\delta$ -multiplicative if

$$\|\phi(ab) - \phi(a)\phi(b)\| < \delta \quad \text{and} \quad \|\phi^*(a) - \phi(a^*)\| < \delta$$

for any  $a, b \in \mathcal{F}$ .

Consider  $K_0(A)$  and consider any finitely generated subgroup  $G \subset K_0(A)$ . Let us assume that  $G$  is generated by  $\mathcal{F} = \{[p_1], \dots, [p_n]\}$ , where  $p_i$  are projections in a matrix algebra over  $A$ . Since  $G$  is finitely generated as an abelian group, it is generated by  $\mathcal{F}$  with finitely many relations. Therefore, there is a sufficiently small  $\delta > 0$  such that for any  $\mathcal{F}$ - $\delta$ -multiplicative map  $\phi : A \rightarrow B$ , the element

$[\phi(p_i)]$  is well defined for any  $1 \leq i \leq n$ , and all the relations get preserved. Therefore, it induces a well-defined group homomorphism  $[\phi]|_G : G \rightarrow K_0(B)$ . The same statement holds for  $K_1$ -group and also  $K$ -groups with coefficients. In the rest part of the paper, we shall use them freely.

Finally, a  $C^*$ -algebra is said to have (SP) property if for any non-zero  $a \in A^+$ , the sub- $C^*$ -algebra  $\overline{aAa}$  contains a non-zero projection.

*2.2. Splitting tree algebras* The class of splitting tree algebras was introduced by Su in [38]. A subclass—splitting interval algebras—also arises naturally in the study of certain automorphisms of  $AT$ -algebras (for instance, see [6], [44]) and as universal  $C^*$ -algebras generated by certain projections (see [34]).

**DEFINITION 2.6.** Let  $T$  be a finite path-connected tree (as a 1-dimensional topological space). Denote its vertices by  $\{v_i\}_{i=1}^n$ , and its edges by  $\{e_i\}_{i=1}^m$ . Then, a splitting tree algebra is the  $*$ -algebra of continuous matrix-valued functions over  $T$  which take the prescribed block-diagonal form on each vertices of  $T$ . That is, there is an integer  $k > 0$  and  $\bar{k}_1, \dots, \bar{k}_n$ , which are partitions of  $\{1, 2, \dots, k\}$ , such that

$$S(\bar{k}_1, \dots, \bar{k}_n; T) := \{f \in M_k(C(T)); f(v_i) \in \bigoplus_{j \in \bar{k}_i} M_j(\mathbb{C}), i = 1, \dots, n\}.$$

The vertices of  $T$  are referred as the singular points of the splitting tree algebra  $S$ . The splitting points at  $v_i$  are labeled by  $v_i^1, \dots, v_i^{|\bar{k}_i|}$  respectively.

**REMARK 2.7.** From the definition above, the splitting tree algebra  $A$  is a unital sub- $C^*$ -algebra of the homogeneous  $C^*$ -algebra  $M_n(C(T))$ . It is a subhomogeneous  $C^*$ -algebra. The spectrum of  $S$  is almost the tree  $T$ , but is a non-Hausdorff tree—with the same edges as  $T$ , and each vertex  $v_i$  splits into  $|\bar{k}_i|$  points, according to the block-diagonal form at  $v_i$ ; the neighbourhood of each splitting point  $v_i^j$  includes its neighbourhood in the adjoint edges, but not the other splitting points at  $v_i$ .

**REMARK 2.8.** A routine argument shows that any splitting tree algebra  $S$  has stably rank one, i.e., the invertible elements are dense in  $S$ .

**REMARK 2.9.** If the tree  $T$  is the interval  $[0, 1]$  with vertices the two endpoints, the splitting tree algebra is referred to as a splitting interval algebra. In [20], the authors studied the unital simple inductive limits of splitting interval algebras, and showed that this class of  $C^*$ -algebras is classified by their Elliott invariants. The main purpose of this paper is to classify the  $C^*$ -algebras which are tracially approximate splitting interval algebras.

*2.3.  $K$ -theory of splitting tree algebras* Let us consider the  $K$ -theory of splitting tree algebras  $S(\bar{k}_1, \dots, \bar{k}_n; T)$ . For any splitting point  $v_i^j$ , consider the evaluation map

$$\text{ev}_i^j : S \rightarrow M_{\bar{k}_i(j)}(\mathbb{C}), \quad f \mapsto f(v_i^j).$$

Set  $\pi = \bigoplus_{i,j} \text{ev}_i^j$  the direct sum of all the evaluation maps at splitting points. Then the kernel of  $\pi$  is the ideal

$$I = \{f \in M_k(\mathbb{C}(T)); f(v_i) = 0\} \subseteq S,$$

and one has the following exact sequence:

$$0 \longrightarrow I \longrightarrow S \xrightarrow{\pi} F \longrightarrow 0,$$

where  $F$  is isomorphic to the finite-dimensional  $C^*$ -algebra  $\bigoplus_{i=1}^n \bigoplus_{i \in \bar{k}_i} M_i(\mathbb{C})$ .

A calculation shows that the index map of this short exact sequence is zero (see [38] for details on the calculation), and hence one has

LEMMA 2.10 ([38]). *The map  $[\pi]_0$  is injective. Hence, the  $K_0$ -group of the splitting tree algebra  $S$  is*

$$\left\{ (m_1, \dots, m_n) \in \bigoplus_{i=1}^n \mathbb{Z}^{|\bar{k}_i|}; \sum_j m_1^{(j)} = \dots = \sum_j m_n^{(j)} \right\}$$

with the ordered structure induced by the standard order on  $\bigoplus_{i=1}^n \mathbb{Z}^{|\bar{k}_i|}$ , and  $[1_S] = (\bar{k}_1, \dots, \bar{k}_n)$ .

Moreover,  $S$  always has trivial  $K_1$ -group.

Note that splitting tree algebra  $S$  has stable rank one, hence the Murray-von Neumann equivalence for projections is in fact induced by unitaries. One then has the follows.

COROLLARY 2.11. *Let  $S$  be a splitting tree algebra. For any  $(m_1, \dots, m_n) \in \bigoplus_{i=1}^n \mathbb{Z}^{|\bar{k}_i|}$  with  $\sum_i m_1^{(i)} = \dots = \sum_i m_n^{(i)}$ , and  $m_i \leq \bar{k}_i$ ,  $1 \leq i \leq n$ , there is a projection  $p$  in  $S$  such that*

$$[p] = (m_1, \dots, m_n).$$

Moreover, let  $p_1$  and  $p_2$  be two projections in  $S$ . If

$$\text{rank}(\text{ev}_i^j(p_1)) \leq \text{rank}(\text{ev}_i^j(p_2))$$

for all splitting point  $v_i^j$ , then  $p_1$  is unitarily equivalent to a subprojection of  $p_2$ .

PROOF. The first part of the statement follows directly from the description of positive cone of  $K_0(S)$ .

Note that  $[p_1] \leq [p_2]$ . Since  $S$  has stable rank one, it has cancelation on projections. Therefore,  $p_1$  is Murray-von Neumann equivalent to a subprojection of  $p_2$ .

Again, since  $S$  has stable rank one, any two Murray-von Neumann equivalent projections are in fact unitarily equivalent, and thus  $p_1$  is unitarily equivalent to a subprojection of  $p_2$ .  $\square$

From above, one has that the ordered  $K_0$ -group of  $S$  is completely determined by the evaluation map  $[e_i^j]_0 : K_0(S) \rightarrow \mathbb{Z}$ . Let us refer to these K-theory maps as the standard evaluation maps (on  $K_0$ -group). Since each of these maps is induced by a Dirac trace concentrating at a splitting point, we also use  $\text{Tr}$  to denote it.

*2.4. Finite presentation of  $K_0$ -groups of splitting interval algebras* In this part, let us consider the ordered  $K_0$ -groups of splitting interval algebras. We will show that the the positive cone of a such ordered group is generated by finitely many elements with finitely many relations. Moreover, we will list the generators and relations precisely.

Let us consider splitting interval algebra

$$(2.1) \quad S = \{f \in M_k(\mathbb{C}([0, 1])); f(0) \in \bigoplus_{j=1}^{h_0} M_{l_j}(\mathbb{C}), f(1) \in \bigoplus_{j=1}^{h_1} M_{l_j}(\mathbb{C})\}.$$

By Lemma 2.10, one has

$$K_0(S) = \{((m_1^0, \dots, m_{h_0}^0), (m_1^1, \dots, m_{h_1}^1)); \sum_{j=1}^{h_0} m_j^0 = \sum_{j=1}^{h_1} m_j^1\}$$

with the standard order.

Hence, as an abelian group,  $K_0(S)$  has rank  $h_0 + h_1 - 1$ . In particular, it is finitely generated. Furthermore, the positive cone of  $K_0(S)$  is generated by the equivalent classes of rank-one projections in  $S$ . In the following, we show that a set of free generators of  $K_0(S)$  (as an abelian group) can be chosen from the equivalent classes of rank-one projections.

For any  $1 \leq j \leq h_0$ , denote by  $s_0(j) = \sum_{k < j} l_k$ , define the rank-one projection

$$p_j(t) = \text{diag}\{\underbrace{0, \dots, 0}_{s_0(j)}, 1, 0, \dots\} \in S.$$

For any  $1 < j \leq h_1$ , denote by  $s_1(j) = \sum_{k < j} l_k$  denote by

$$q_j(t) = \begin{pmatrix} 1-t & 0 & \cdots & 0 & \sqrt{t(1-t)} & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ \sqrt{t(1-t)} & 0 & \cdots & 0 & t & 0 & \cdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \in S.$$

**LEMMA 2.12.** *The set of positive elements  $\mathcal{G} = \{[p_{j_0}], [q_{j_1}]; 1 \leq j_0 \leq h_0, 1 < j_1 \leq h_1\}$  is a set of free generators for  $K_0(S)$  as an abelian group.*



PROOF. It is enough to show that any rank-one  $K_0$ -element

$$m = (\underbrace{(0, \dots, 0, 1, 0, \dots)}_{m_0}, \underbrace{(0, \dots, 0, 1, 0, \dots)}_{m_1})$$

can be generated by  $\mathcal{G}$ . For each  $m_0$ , denote by  $m'_0$  the index such that  $\text{ev}_1^{m'_0}([p_{m_0}]) = 1$ . Then one has

$$m = [p_{m_0}] + [q_{m_1}] - [q_{m'_0}],$$

as desired.  $\square$

From the proof of the lemma above, it is clear that the positive elements  $\{[p_i], [q_j]; 1 \leq i \leq h_0, 1 < j \leq h_1\}$  satisfy the following relation  $[p_i] + [q_j] > [q_{i'}]$  for any  $i$ , where  $i'$  is the index such that  $\text{ev}_1^{i'}([p_i]) = 1$ . In fact, this relation characterizes the ordered group  $K_0(S)$ .

LEMMA 2.13. *Use the same notation as above. Let  $G$  be any ordered group, and let  $a_i, 1 \leq i \leq h_0$  and  $b_j, 1 < j \leq h_1$  be positive elements in  $G$  satisfying  $a_i + b_j > b_{i'}$  for any  $i$  and  $j$ , then the map  $[p_i] \mapsto a_i, [q_j] \mapsto b_j$  extends to a positive map  $\kappa : K_0(S) \rightarrow G$ .*

PROOF. Since  $\{[p_i], [q_j]; 1 \leq i \leq h_0, 1 < j \leq h_1\}$  is a set of free generators of  $K_0(S)$  as an abelian group, the map extends uniquely to a homomorphism  $\kappa : K_0(S) \rightarrow G$ . We only have to show that  $\kappa$  is positive.

Since the positive cone of  $K_0(S)$  is generated by rank-one elements, it is enough to show that  $\kappa(m)$  is positive for any rank-one element

$$m = (\underbrace{(0, \dots, 0, 1, 0, \dots)}_{m_0}, \underbrace{(0, \dots, 0, 1, 0, \dots)}_{m_1}).$$

Since  $m = [p_{m_0}] + [q_{m_1}] - [q_{m'_0}]$ , one has that

$$\begin{aligned} \kappa(m) &= \kappa([p_{m_0}]) + \kappa([q_{m_1}]) - \kappa([q_{m'_0}]) \\ &= a_{m_0} + b_{m_1} - b_{m'_0} > 0. \end{aligned}$$

Hence  $\kappa$  is positive.  $\square$

2.5. *Approximately halving an element of  $K_0(S)$*  Consider a splitting tree algebra  $S$ . In general, for any positive element  $a$  in  $K_0(S)$ , there might not exist positive elements  $b$  and  $c$  such that  $a = b + c$ , for instance, when  $a$  comes from a rank-one projection. But such a decomposition is always possible when  $a$  has large multiplicity. Moreover, in this part, we shall show that when  $a$  has sufficiently large multiplicity, one always has a decomposition of form  $a = 2c + 3d$  with  $2c$  and  $3d$  approximately have the same size.

Consider the group  $G$

$$\left\{ ((m_1^{(1)}, \dots, m_1^{(l_1)}), \dots, (m_n^{(1)}, \dots, m_n^{(l_n)})) \in \bigoplus_{i=1}^n \mathbb{Z}^{l_i}; \sum_i m_1^{(i)} = \dots = \sum_i m_n^{(i)} \right\}$$

with the natural order, and recall that a standard evaluation of  $G$  is a positive homomorphism  $\text{Tr} : G \rightarrow \mathbb{Z}$  in the form of

$$((m_1^{(1)}, \dots, m_1^{(l_1)}), \dots, (m_n^{(1)}, \dots, m_n^{(l_n)})) \mapsto m_j^{(i)}$$

for some  $i, j$ . Note that an element  $a$  in  $G$  is positive if and only if  $\text{Tr}(a)$  is positive for any standard evaluation map  $\text{Tr}$ .

LEMMA 2.14. *Let  $a = ((a_1^{(1)}, \dots, a_1^{(l_1)}), \dots, (a_n^{(1)}, \dots, a_n^{(l_n)}))$  be a positive element in  $G$  with  $a_j^{(i)} > 10$ . Then, there exist*

$$c = ((c_1^{(1)}, \dots, c_1^{(l_1)}), \dots, (c_n^{(1)}, \dots, c_n^{(l_n)})) \in G^+$$

and

$$d = ((d_1^{(1)}, \dots, d_1^{(l_1)}), \dots, (d_n^{(1)}, \dots, d_n^{(l_n)})) \in G^+$$

such that  $a = 2c + 3d$ . Moreover, one has that

$$|2\text{Tr}(2c) - \text{Tr}(a)| < 12$$

for any standard evaluation map  $\text{Tr}$  on  $G$ .

PROOF. Denote by  $S = \sum_i m_1^{(i)}$ . Consider the first singular point ( $j = 1$ ). There are positive integers  $c_1^{(i)}$  and  $d_1^{(i)}$  such that

$$a_1^{(i)} = 2c_1^{(i)} + 3d_1^{(i)}.$$

Denoted by  $S_1 = \sum c_1^{(i)}$  and  $S_2 = \sum d_1^{(i)}$ . One can choose  $c_1^{(i)}$ 's and  $d_1^{(i)}$  such that

$$(2.2) \quad \left| 4c_1^{(i)} - a_1^{(i)} \right| \leq 12, \quad 1 \leq i \leq l_1$$

$$|2S_1 - S/2| \leq 6 \text{ and } |3S_2 - S/2| \leq 6.$$

Consider the second singular point ( $j = 2$ ). Let us choose positive integers  $c_2^{(i)}$  and  $d_2^{(i)}$  such that

$$a_2^{(i)} = 2c_2^{(i)} + 3d_2^{(i)}, \quad 1 \leq i \leq l_2 - 1,$$

$$(2.3) \quad \left| 4c_2^{(i)} - a_2^{(i)} \right| \leq 12, \quad 1 \leq i \leq l_2 - 1,$$

and

$$\left| \sum_{i=1}^k 2c_2^{(i)} - \sum_{i=1}^k 3d_2^{(i)} \right| \leq 5, \quad 1 \leq k \leq l_2 - 1,$$

as following: For  $k = 1$ , it is clear that there are  $c_2^{(1)}$  and  $d_2^{(1)}$  such that  $a_2^{(1)} = 2c_2^{(1)} + 3d_2^{(1)}$  with  $|2c_2^{(1)} - 3d_2^{(1)}| \leq 5$ . Assume that  $c_2^{(i)}$  and  $d_2^{(i)}$  are chosen for  $1 \leq i \leq k$ . Denote by

$$e = \sum_{i=1}^k 2c_2^{(i)} - \sum_{i=1}^k 3d_2^{(i)}.$$

Then  $|e| \leq 5$ . Assume  $e > 0$ . Write

$$a_2^{(k+1)} = 2c_2^{(k+1)} + 3d_2^{(k+1)}$$

with

$$\left| 2c_2^{(k+1)} - a_2^{(k+1)} \right| \leq 6,$$

and

$$e \leq 3d_2^{(k+1)} - 2c_2^{(k+1)} \leq e + 5.$$

We then have that

$$\left| \sum_{i=1}^{k+1} 2c_2^{(i)} - \sum_{i=1}^{k+1} 3d_2^{(i)} \right| \leq 5.$$

Thus, such  $\{c_2^{(i)}\}$  and  $\{d_2^{(i)}\}$  exist.

In particular

$$\left| \sum_{i=1}^{l_2-1} 2c_2^{(i)} - \sum_{i=1}^{l_2-1} a_2^{(i)}/2 \right| \leq 3,$$

and

$$\left| \sum_{i=1}^{l_2-1} 3d_2^{(i)} - \sum_{i=1}^{l_2-1} a_2^{(i)}/2 \right| \leq 3.$$

Then one has that

$$\sum_{i=1}^{l_2-1} 2c_2^{(i)} \leq \sum_{i=1}^{l_2-1} a_2^{(i)}/2 + 10 \leq S/2 - a_2^{(l_k)} + 10 \leq 2S_1 - a_2^{(l_k)} + 10,$$

and hence

$$S_1 - \sum_{i=1}^{l_2-1} c_2^{(i)} \geq a_2^{(l_k)}/2 - 10 > 0.$$

Same argument shows that

$$S_2 - \sum_{i=1}^{l_2-1} b_2^{(i)} > 0.$$

Define

$$c_2^{(l_k)} = S_1 - \sum_{i=1}^{l_2-1} c_2^{(i)}$$

and

$$d_2^{(l_k)} = S_2 - \sum_{i=1}^{l_2-1} d_2^{(i)}.$$

One then has that  $\sum_{i=1}^{l_1} c_1^{(i)} = S_1 = \sum_{i=1}^{l_2} c_2^{(i)}$  and  $\sum_{i=1}^{l_1} d_1^{(i)} = S_2 = \sum_{i=1}^{l_2} d_2^{(i)}$ .

The same argument shows that for any  $k$ , there are elements  $\{c_k^{(1)}, \dots, c_k^{(l_k)}\}$  and  $\{d_k^{(1)}, \dots, d_k^{(l_k)}\}$  such that

$$(2.4) \quad \left| 4c_k^{(i)} - a_k^{(i)} \right| \leq 12, \quad 1 \leq i \leq l_k - 1,$$

$S_1 = \sum_{i=1}^{l_k} c_k^{(i)}$  and  $S_2 = \sum_{i=1}^{l_k} d_k^{(i)}$ , and  $a_k^{(i)} = 2c_k^{(i)} + 3d_k^{(i)}$ . Denoted by

$$c = ((c_1^{(1)}, \dots, c_1^{(l_1)}), \dots, (c_n^{(1)}, \dots, c_n^{(l_n)})),$$

and

$$d = ((d_1^{(1)}, \dots, d_1^{(l_1)}), \dots, (d_n^{(1)}, \dots, d_n^{(l_n)})).$$

Then one has that  $\sum_i c_1^{(i)} = \dots = \sum_i c_n^{(i)}$ ,  $\sum_i d_1^{(i)} = \dots = \sum_i d_n^{(i)}$ . Hence the elements  $c$  and  $d$  are in  $G$  and  $a = 2c + 3d$ . It follows from (2.4) that

$$|2\text{Tr}(2c) - \text{Tr}(a)| < 12$$

for any standard evaluation map  $\text{Tr}$  on  $G$ , as desired.  $\square$

**COROLLARY 2.15.** *Let  $S$  be a splitting interval algebra, and let  $G$  be the ordered group*

$$\left\{ ((m_1^{(1)}, \dots, m_1^{(l_1)}), \dots, (m_n^{(1)}, \dots, m_n^{(l_n)})) \in \bigoplus_{i=1}^n \mathbb{Z}^{l_i}; \sum_i m_1^{(i)} = \dots = \sum_i m_n^{(i)} \right\}.$$

*Let  $\kappa : K_0(S) \rightarrow G$  be a strictly positive map. If  $\kappa(e)$  has multiplicity at least 36 for any minimal projection in  $S$ , then there exist positive maps  $\kappa_1, \kappa_2 : K_0(S) \rightarrow G$  such that  $\kappa = 2\kappa_1 + 3\kappa_2$ .*

**PROOF.** Denote by  $h_0$  and  $h_1$  the numbers of the splitting points of  $S$  at 0 and 1 respectively, and consider the generating set  $\{[p_i], [q_j]; 1 \leq i \leq h_0, 1 < j \leq h_1\}$  of  $K_0(S)$  specified in Lemma 2.13. By Lemma 2.14, there are positive elements  $\{a'_i, a''_i, b'_j, b''_j; 1 \leq i \leq h_0, 1 < j \leq h_1\}$  such that

$$\kappa([p_i]) = 2a'_i + 3a''_i \quad \text{and} \quad \kappa([q_j]) = 2b'_j + 3b''_j.$$

Consider the map  $\kappa_1$  and  $\kappa_2$  induced by sending  $[p_i]$  to  $a'_i$  and  $a''_i$ , and sending  $[q_j]$  to  $b'_j$  and  $b''_j$  respectively. In order to proof the corollary, it is enough to show that  $\kappa_1$  and  $\kappa_2$  are positive. By Lemma 2.13, it is enough to show that  $a'_i + b'_j > b'_j$ , and  $a''_i + b''_j > b''_j$ , for any  $i, j$  and  $i'$  with  $\text{ev}_1^{i'}([p_i]) = 1$ .

Consider  $\kappa([p_i]) + \kappa([q_j]) - \kappa([p_{i'}]) = \kappa([p_i] + [q_j] - [p_{i'}]) \in G^+$ . Since  $\kappa(e)$  has multiplicity at least 36 for any minimal projection in  $S$ , one has that

$$(2.5) \quad \text{Tr}(\kappa([p_i])) + \text{Tr}(\kappa([q_j])) - \text{Tr}(\kappa([p_{i'}])) > 36$$

for any standard trace  $\text{Tr}$  on  $G$ . By the second part of Lemma 2.14,

$$(2.6) \quad |4\text{Tr}(a'_i) - \text{Tr}(\kappa[p_i])| \leq 12 \quad \text{and} \quad |4\text{Tr}(b'_j) - \text{Tr}(\kappa[q_j])| \leq 12.$$

Therefore, by (2.5) and (2.6), one has that

$$\text{Tr}(a'_i + b'_j - a'_{i'}) > 0.$$

Since  $\text{Tr}$  is an arbitrary standard trace, one has  $a'_i + b'_j - b'_{i'}$  is positive. The same argument shows that  $a''_i + b''_j - b''_{i'}$  is positive. Hence by Lemma 2.13 the maps  $\kappa_1$  and  $\kappa_2$  are positive, as desired.  $\square$

*2.6. An existence theorem* In this part, let us show that any unit preserving positive map between the  $K_0$ -groups of splitting interval tree algebras is indeed induced by a  $*$ -homomorphism between splitting interval algebras.

As shown in [38], any tracial state of a splitting tree algebra is induced by a probability measure on the spectrum. More precisely, let  $S$  be a splitting tree algebra, and let  $\tau$  be a tracial state on  $S$ , then  $\tau$  comes from a convex combination of a probability measure concentrating on the tree and Dirac measures on the splitting points:

$$\tau(f) = \lambda_0 \int_0^1 \text{tr}(f)d\mu + \sum_{i,j} \lambda_i^j \text{tr}(f(v_i^j)), \quad \forall f \in S,$$

where  $\text{tr}$  is the normalized trace over a matrix algebra,  $\mu$  a probability measure on  $T$ ,  $\lambda_0, \lambda_i^j \geq 0$  and  $\lambda_0 + \sum_{i,j} \lambda_i^j = 1$ .

In general, this decomposition is not unique. But, if one assumes that at each singular point  $v_i$ , there exists at least one splitting point, say  $v_i^j$ , such that  $\lambda_i^j = 0$ , then the probability measure  $\mu$  and the coefficients  $\lambda_i^j$  are uniquely determined (in other words, the probability measure  $\mu$  is saturate). Let us assume this condition in the sequel.

From the description above, any extreme tracial state of  $S$  is represented by a Dirac measure on the spectrum of  $S$ , which is a non-Hausdorff splitting tree. The topology on these extreme tracial states agrees with the topology on the non-Hausdorff splitting tree. If an extreme tracial state  $\tau$  is induced by a Dirac measure on a splitting point, then the positive state  $r(\tau)$  is an extreme point in  $S_u(K_0(S))$ . However, if  $\tau$  is induced by a Dirac measure on a point in an edge of the tree, then the state  $r(\tau)$  is not extremal in general. In fact, it is then a linear combination of the standard evaluation maps on  $K_0(S)$ .

On the other hand, since  $S$  is of type I, it is an exact  $C^*$ -algebra, and hence any element in  $S_u(K_0(S))$  comes from a tracial state. Therefore, the convex  $S_u(K_0(S))$  is spanned by certain scalar multiples of the standard evaluation maps. Moreover, if one identifies the image of a state of  $K_0(S)$  to  $\mathbb{Z}$ , one has

LEMMA 2.16. *Let  $S$  be a splitting tree algebra. Then any positive homomorphism  $\alpha : K_0(S) \rightarrow \mathbb{Z}$  is a sum of standart evaluations.*

PROOF. Let  $S = S(\bar{k}_1, \dots, \bar{k}_n; T)$ . Then

$$K_0(S) = \{(m_1, \dots, m_n) \in \bigoplus_{i=1}^n \mathbb{Z}^{|\bar{k}_i|}; \sum_j m_1^{(j)} = \dots = \sum_j m_n^{(j)}\}.$$

Since  $\alpha$  comes from a trace (minght not be normalized), one has

$$\alpha((m_1, \dots, m_n)) = \lambda_0 \sum_j m_1(j) + \sum_{i,j} \lambda_i^j m_i(j)$$

where the coefficients  $\lambda$ 's are positive, and for any  $i$ , at least one of  $\{\lambda_i^j\}$  is 0. Therefore, there exists a rank-one projection  $p$  of  $S$  such that  $[\text{ev}_i^j]([p]) = 0$  whenever  $\lambda_i^j \neq 0$ . Apply  $\alpha$  to this element; we get  $\lambda_0 = \alpha([p]) \in \mathbb{Z}^+$ .

In the same way, for any  $i_0, j_0$  such that  $\lambda_{i_0}^{j_0} \neq 0$ , we can find a rank-one projection  $p$  in  $S$  such that  $[\text{ev}_{i_0}^{j_0}]([p]) = 1$  and  $[\text{ev}_i^j]([p]) = 0$  whenever  $\lambda_i^j \neq 0$  and  $i \neq i_0$ . Evaluating  $\alpha$  on  $p$ , we get  $\lambda_{i_0}^{j_0} = \alpha([p]) \in \mathbb{Z}^+$ .

Therefore, all coefficients are positive integers, and hence  $\alpha$  is a sum of standard evaluation maps on the  $K_0$ -group.  $\square$

Let us consider two splitting tree algebras and consider a unit-preserving positive homomorphism between them. Then, we have

PROPOSITION 2.17. *Let  $S_1, S_2$  be two splitting tree algebras, and  $\kappa : K_0(S_1) \rightarrow K_0(S_2)$  be a unit-preserving positive map. Then there exists a unital \*-homomorphism  $\varphi : S_1 \rightarrow S_2$  such that  $\varphi_* = \kappa$ .*

PROOF. Let us first assume that  $S_2 \cong M_n(\mathbb{C})$  for some  $n$ . Then  $\kappa$  is a positive map from  $K_0(S_1)$  to  $K_0(S_2) \cong \mathbb{Z}$ . Applying Lemma 2.16, there are splitting points  $s_1, \dots, s_k$  such that  $\kappa = \sum_{i=1}^k \text{ev}(s_i)$ , where  $\text{ev}(s_i)$  is the evaluation map at  $s_i$ . Since  $\kappa$  preserves the order unit, one has that  $\varphi := \bigoplus_{i=1}^k \text{ev}(s_i)$  is a map from  $S_1$  to  $M_n(\mathbb{C})$ , and  $[\varphi] = \kappa$ .

In the general setting, consider a homomorphism

$$\kappa : (K_0(S_1), K_0(S_1)^+, [1_{S_1}]_0) \rightarrow (K_0(S_2), K_0(S_2)^+, [1_{S_2}]_0)$$

for two splitting tree algebras  $S_1$  and  $S_2$ . Denote the singular points of  $S_1$  and  $S_2$  by  $\{w_i\}$  and  $\{v_i\}$  respectively. Let us consider the composition of  $\kappa$  with a evaluation map of  $K_0(S_2)$  at a splitting point, say  $v_i^j$ . Since the codomain of the map  $\text{ev}_i^j$  is  $\mathbb{Z}$ , the positive map  $\text{ev}_i^j \circ \kappa$  sends  $K_0(S_1)$  to  $\mathbb{Z}$ . By the argument same as that of the case when codomain is a matrix algebra, there is a \*-homomorphism from  $S_1$  to the block matrix at the splitting point  $v_i^j$ . Moreover, this \*-homomorphism is a sum of evaluation maps at certain splitting points  $S_1$ .

Repeat this argument for all splitting points of  $S_2$ . In the following, we shall show that for any pair of singular points (vertices of the tree) of  $S_2$  which are connected by one edge, the sets of the point evaluations corresponding to them can be matched to each other.

Consider a singular point  $v$  of  $S_2$ , and consider the point evaluations at the splitting points of  $S_1$  which are induced by any splitting points at  $v$ . Then, for each splitting point  $w_i^j$  of  $S_1$ , denote by  $m_i^j$  the multiplicity of the point evaluations at  $w_i^j$  which appear in those associated to  $v$ . (So,  $m_i^j$  might be 0.) We then group these point evaluations of  $S_1$  in the following way: At each singular point  $w_i$  of  $S_1$ , set  $m_i = \min_j \{m_i^j\}$ . Then first group the point evaluations at this splitting point into two sets: one consists the point evaluations at each splitting point with multiplicity  $m_i$ , and the other one consists of all the remaining point evaluations at this singular point. In other words, we divide them into the point evaluations which are the same at the K-theory level as a point evaluation on a regular point with multiplicity  $m_i$  and the point evaluations which is not full at this singular point. In particular, the map  $[\text{ev}(v)] \circ \kappa$  can be realized as the  $K_0$ -map of a direct sum of a full point evaluation at singular points of  $S_1$  and some point evaluations at some splitting points of  $S_1$ .

Repeating this procedure at all singular points of  $S_1$ , we have the following decomposition

$$[\text{ev}(v)] \circ \kappa = m[\text{ev}(w)] + \sum_{i,j} m_{i,j} [\text{ev}(w_i^j)],$$

where  $w$  is a vertex in  $T$ , and for any singular point  $w_i$ , there is a splitting point  $w_i^j$  such that  $m_{i,j} = 0$ . Let us refer to  $\text{ev}(w)$  as the full point evaluation part, and refer to  $\sum_{i,j} m_{i,j} [\text{ev}(w_i^j)]$  as non-full point evaluation part. Note that the non-full part sends some rank-one projections to 0. Moreover, by the same argument in the proof of Lemma 2.16, the splitting points  $w_i^j$  in this decomposition is unique.

Decompose  $[\text{ev}(v)] \circ \kappa$  in such way for all singular points  $v$  of  $S_2$ . Then, for any pair of singular points  $v_1$  and  $v_2$  of  $S_2$  connected by a edge, one then can match the point evaluations associated to them in the following way. Since at each singular point  $v_i$  of  $S_2$ , the map  $[\text{ev}(v_i)] = [\sum_j \text{ev}(w_i^j)]$  is the same, and hence, by the uniqueness of the decomposition above, the full point evaluation parts of each singular point have the same multiplicity, and the non-full point evaluation parts are unique. Therefore, the non-full part of point evaluations can be glued together by constant maps. For the full evaluation part, one can glue them using a path in the spectrum of  $S_1$  connecting the corresponding evaluation points. Thus, the point evaluation maps from  $S_1$  to  $v_1$  and  $v_2$  extends to a \*-homomorphism from  $S_1$  to the splitting tree algebras restricted between  $v_1$  and  $v_2$ .

Repeating the argument above for any edge connecting to  $v_1$  or  $v_2$ , and after finite step, one has the desired homomorphism from  $S_1$  to  $S_2$ .  $\square$

*2.7. Homomorphisms between splitting interval algebras* In this part, we only focus on splitting interval algebras, and we shall show some results on the unique-

ness of the homomorphisms between these algebras.

The following Uniqueness Theorem was proved by Jiang-Su.

**THEOREM 2.18** (Theorem 3.1, [38]). *Let  $A, B$  be two splitting interval algebras. For any unital map  $\phi : A \rightarrow B$ , any finite subset  $\mathcal{F} \subseteq A$  and any  $\varepsilon > 0$ , there exists a unital map  $\phi' : A \rightarrow B$  such that:*

- $\|\phi(f) - \phi'(f)\| < \varepsilon$  for any  $f \in \mathcal{F}$ ;
- there exist continuous maps  $\{\lambda_j(\cdot)\}_{j=1}^p \subseteq C([0, 1]; \text{sp}(A))$  and a unitary  $W \in M_m(C[0, 1])$ , where  $M_m$  is the generic fibre of  $B$ , such that

$$(2.7) \quad \phi'(f)(y) = W^*(y) \text{diag}(f(\lambda_1(y)), \dots, f(\lambda_p(y)))W(y)$$

for any  $f \in A$ ,  $y \in [0, 1]$ .

**REMARK 2.19.** As in [38], a homomorphism in the form of (2.7) is said to have a standard form.

**REMARK 2.20.** The continuous maps  $\{\lambda_j\}_{j=1}^p$  are called the eigenvalue maps, and they have the following property: for any  $y \in [0, 1]$ , the image  $\lambda_j(y)$  is either a single splitting point, or a full point. Thus, the each eigenvalue map  $\lambda_j$  is either a constant map, or induced by a continuous map from  $[0, 1]$  to  $[0, 1]$ . Therefore, in the theorem above, one may assume that each eigenvalue map  $\lambda_j$  is either a constant map (with image in a splitting point of  $A$ ) or a map from  $[0, 1]$  to  $[0, 1]$  such that  $\lambda_j(y) \in (0, 1)$  for any  $y \in (0, 1)$ .

As in the case of interval algebras, if the eigenvalue maps are relatively dense, and most of them are constant, then the homomorphism approximately factors through a finite-dimensional  $C^*$ -algebra. (For the case of interval algebras, see [10].) More precisely, we have the following lemma.

**LEMMA 2.21.** *Let  $S$  be a splitting interval algebra. For any finite subset  $\mathcal{F}$  and any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for any  $0 < x_1 < \dots < x_n < 1$  with  $x_{i+1} - x_i < \delta$ ,  $0 \leq i \leq n$  (assume  $x_0 = 0$  and  $x_{n+1} = 1$ ), there is a homomorphism  $\Phi : S \rightarrow M_{nm+1}(S)$  factoring through finite dimension  $C^*$ -algebras such that*

$$\|\text{diag}\{f, f(x_1), \dots, f(x_n)\} - \Phi(f)\| < \varepsilon \quad \text{for any } f \in \mathcal{F},$$

where  $m$  is the dimension of generic fibres of  $S$  and each evaluation  $f(x_i)$  is regarded as in  $M_m(\mathbb{C}1_S) \subseteq M_m(S)$ .

**PROOF.** Take  $\delta > 0$  such that

$$\|f(x) - f(y)\| < \varepsilon/4, \quad \text{for any } f \in \mathcal{F}$$

and for any  $|x - y| < \delta$ .



Consider the homomorphism  $\Psi : S \rightarrow M_{n+1}(S)$  defined by

$$f \mapsto \text{diag}\{f, f(x_1), \dots, f(x_n)\}.$$

Then it is a map induced by the eigenvalue maps  $\lambda_0, \dots, \lambda_n : [0, 1] \rightarrow [0, 1]$  with  $\lambda_1(x) = x$  and  $\lambda_i(x) = x_i$ , where  $i = 1, \dots, n$ .

Denote by  $m$  the size of the generic fibre of  $S$ . We assert that there are unitary  $W \in M_{m(n+1)}(C([0, 1]))$  and continuous map  $\lambda'_0, \dots, \lambda'_n$  such that

$$\|\Psi(f) - W^* \text{diag}\{f \circ \lambda'_0, f \circ \lambda'_1, \dots, f \circ \lambda'_n\}W\| < \varepsilon/2$$

for any  $f \in \mathcal{F}$ , and moreover, the variation of each  $\lambda'_i$  is less than  $\delta$ . Then, define the map

$$\Phi : f \mapsto W^* \text{diag}\{f \circ \lambda'_0(0), f \circ \lambda'_1(0), \dots, f \circ \lambda'_n(0)\}W,$$

and the lemma follows.

By a small perturbation of  $\lambda_0$ , there is a continuous functions  $\tilde{\lambda}_0$  such that  $|\tilde{\lambda}_0 - \lambda_0| < \delta'$ , and for any  $i = 1, \dots, n$ ,

$$\tilde{\lambda}_0(x) = x_i \quad \text{for any } |x - x_i| < \delta'$$

for some  $\delta > \delta' > 0$  satisfying

$$\|f(x) - f(y)\| < \varepsilon/2 \quad \text{for any } |x - y| < \delta', f \in \mathcal{F}.$$

Note that for any  $f \in \mathcal{F}$ , one has that  $\text{diag}\{f \circ \tilde{\lambda}_0, f \circ \lambda_1, \dots, f \circ \lambda_n\} \in M_{n+1}(S)$ , and

$$\|\Phi(f) - \text{diag}\{f \circ \tilde{\lambda}_0, f \circ \lambda_1, \dots, f \circ \lambda_n\}\| < \varepsilon/2.$$

Consider the maps  $\tilde{\lambda}_0$  and  $\lambda_1$ . Set

$$\lambda'_0(x) := \min(\tilde{\lambda}_0, \lambda_1) \quad \text{and} \quad \tilde{\lambda}_1 := \max(\tilde{\lambda}_0, \lambda_1),$$

and set

$$U(x) = \begin{cases} 1 & \text{if } x \leq x_1 - \delta_1, \\ \text{diag}(Z((x - x_1 + \delta_1)/(2\delta_1)), 1) & \text{if } x_1 - \delta_1 \leq x \leq x_1 + \delta_1, \\ \text{diag}\left(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, 1\right) & \text{if } x \geq x_1 + \delta_1, \end{cases}$$

where  $Z(x)$  is a path of unitaries in  $M_2(\mathbb{C})$  with  $Z(0) = 1$  and  $Z(1) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ .

Then one has

$$\begin{aligned} & \text{diag}(f(\tilde{\lambda}_0(x)), f(\lambda_1(x)), \dots, f(\lambda_n(x))) \\ = & U_1^*(x) \text{diag}(f(\lambda'_0(x)), f(\tilde{\lambda}_1(x)), \dots, f(\lambda_n(x)))U_1(x), \end{aligned}$$

and therefore

$$\left\| \Phi(f) - U_1^*(x) \text{diag}(f(\lambda'_0(x)), f(\tilde{\lambda}_1(x)), \dots, f(\lambda_n(x))) U_1(x) \right\| < \varepsilon/2$$

for any  $f \in \mathcal{F}$ .

Note that the function  $\lambda'_1$  has variation less than  $x_1 < \delta$ . Then consider the maps  $\tilde{\lambda}_1$  and  $\lambda_2$ , and set

$$\lambda'_0(x) := \min(\tilde{\lambda}_1, \lambda_2) \quad \text{and} \quad \tilde{\lambda}_1 := \max(\tilde{\lambda}_1, \lambda_2).$$

A similar argument as above also shows that there is a unitary  $U_2$  such that

$$\begin{aligned} & \text{diag}(f(\tilde{\lambda}_0(x)), f(\lambda_1(x)), \dots, f(\lambda_n(x))) \\ &= U_2^*(x) U_1^*(x) \text{diag}(f(\lambda'_0(x)), f(\lambda'_1(x)), \dots, f(\lambda_n(x))) U_1(x) U_2(x), \end{aligned}$$

and hence

$$\left\| \Phi(f) - U_2^*(x) U_1^*(x) \text{diag}(f(\lambda'_0(x)), f(\tilde{\lambda}_1(x)), \dots, f(\lambda_n(x))) U_1(x) U_2(x) \right\| < \varepsilon/2$$

for any  $f \in \mathcal{F}$ . Note that the function  $\lambda'_1$  has variation at most  $\delta$ . Repeating this argument at each point  $x_i$ ,  $i = 1, \dots, n$ , one obtains the desired eigenvalue maps.  $\square$

Recall that a  $C^*$ -algebra  $A$  has cancellation on projections if for any projections  $p, q, r$  in a matrix algebra of  $A$ ,  $p \oplus r \preceq q \oplus r$  implies  $p \preceq q$ . (See, for example, 7.3 of [37].) For  $C^*$ -algebras with this cancellation property, one has

**LEMMA 2.22** (Lemma 7.3.2 of [37]). *Let  $A$  be a finite-dimensional  $C^*$ -algebra, and let  $B$  be a unital  $C^*$ -algebra with the cancellation of projections. Let  $\phi, \psi : A \rightarrow B$  be homomorphisms. Then  $[\phi]_0 = [\psi]_0$  if and only if  $\phi = \text{Adu} \circ \psi$  for some unitary  $u$  in  $B$ .*

With Lemma 2.21 and Lemma 2.22, one has the following stable uniqueness theorem for maps from a splitting interval algebra to any  $C^*$ -algebra with cancellation of projections.

**COROLLARY 2.23.** *Let  $S$  be a splitting interval algebra, and let  $A$  be a unital  $C^*$ -algebra with cancellation of projections. For any finite subset  $\mathcal{F} \subseteq S$  and any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that for any  $0 < x_1 < \dots < x_n < 1$  with  $x_{i+1} - x_i < \delta$ ,  $0 \leq i \leq n$  (assume  $x_0 = 0$  and  $x_{n+1} = 1$ ) and any homomorphisms  $\phi, \psi : S \rightarrow A$  with  $[\phi]_0 = [\psi]_0$ , there exists a unitary  $u \in M_{m_{n+1}}(A)$  such that*

$$\left\| \text{diag}\{\phi(f), f(x_1), \dots, f(x_n)\} - u^* \text{diag}\{\psi(f), f(x_1), \dots, f(x_n)\} u \right\| < \varepsilon$$

for any  $f \in \mathcal{F}$ , where  $m$  is the dimension of generic fibres of  $S$ , each evaluation  $f(x_i)$  is regarded as in  $M_m(\mathbb{C}1_A) \subseteq M_m(A)$

PROOF. By Lemma 2.21, there exists  $\delta > 0$  such that for any  $0 < x_1 < \dots < x_n < 1$  with  $x_{i+1} - x_i < \delta$ ,  $0 \leq i \leq n$  (assume  $x_0 = 0$  and  $x_{n+1} = 1$ ), there is a homomorphism  $\Theta : S \rightarrow M_{nm+1}(S)$  factoring through a finite-dimensional  $C^*$ -algebra such that

$$\|\text{diag}\{f, f(x_1), \dots, f(x_n)\} - \Theta(f)\| < \varepsilon/2 \quad \text{for any } f \in \mathcal{F}.$$

Denote by

$$\Phi = \phi \otimes 1, \Psi = \psi \otimes 1 : M_{nm+1}(S) \rightarrow M_{nm+1}(A),$$

and a direct calculation shows that for any  $f \in \mathcal{F}$ ,

$$\|\Phi \circ \Theta(f) - \text{diag}\{\phi(f), f(x_1), \dots, f(x_n)\}\| < \varepsilon/2.$$

and

$$\|\Psi \circ \Theta(f) - \text{diag}\{\psi(f), f(x_1), \dots, f(x_n)\}\| < \varepsilon/2.$$

Denote by  $F = \Theta(S)$ . It is a finite-dimensional  $C^*$ -algebra.

Thus, in order to proof the Corollary, it is enough to show that the maps  $\Phi|_F$  and  $\Psi|_F$  are unitarily equivalent. Note that  $[\phi]_0 = [\psi]_0$ , and hence  $[\Phi]_0 = [\Psi]_0$ . Therefore,  $[\Phi|_F]_0 = [\Psi|_F]_0$ . Since  $A$  has cancellation of projections, so does  $M_{n+1}(A)$ . By Lemma 2.22, there exists a unitary  $u \in M_{n+1}(A)$  such that

$$u^* \Psi|_F u = \Phi|_F,$$

as desired. □

In general, homomorphisms between splitting interval algebras were studied by Jiang-Su in [20]. In the following, we collect several results from their paper.

DEFINITION 2.24. Let  $A$  and  $B$  be two unital stably finite  $C^*$ -algebras. Let  $\kappa : K_0(A) \rightarrow K_0(B)$  be a positive homomorphism, and let  $\theta : T(B) \rightarrow T(A)$  be a continuous affine maps. Then  $(\kappa; \theta)$  is called a compatible pair if

$$\tau(\kappa(p)) = \theta(\tau(p))$$

for any  $p \in K_0(A)$  and for any  $\tau \in T(B)$ . The pair  $(\kappa; \theta)$  is called  $\varepsilon$ -compatible for a given  $\varepsilon > 0$  if

$$|\tau(\kappa(p)) - \theta(\tau(p))| < \varepsilon$$

for any  $p \in K_0^+(A)$  with  $p \leq [1]_0$  and for any  $\tau \in T(B)$ .

If  $A$  and  $B$  are splitting interval algebras, then any approximately compatible pair can be perturbed into a compatible pair.

LEMMA 2.25 (Proposition 2.7 of [20]). *Let  $A$  be a finite direct sum of splitting interval algebras. Then, for any  $\varepsilon > 0$ , there is a constant  $\delta > 0$  such that for any  $\delta$ -compatible pair  $(\kappa; \theta^\delta)$  for  $(A; B)$ , where  $B$  is also a finite direct sum of splitting interval algebras, there exists a compatible pair  $(\kappa; \theta)$  for  $(A; B)$  such that*

$$\|\theta(\tau) - \theta^\delta(\tau)\| < \varepsilon$$

for all  $\tau \in \mathsf{T}(B)$ , where  $\|\cdot\|$  is the norm on  $A^*$ .

For any compatible pair on splitting interval algebras, there then exists a homomorphism between the  $C^*$ -algebras which induces the given pair approximately. More precisely,

THEOREM 2.26 (Theorem 3.7 of [20]). *Let  $A$  be a splitting interval algebra. Then for any finite set  $\mathcal{F} \subseteq A$  and any  $\varepsilon > 0$ , there is a constant  $N \in \mathbb{N}$ , such that for any compatible pair  $(\kappa; \theta)$  for  $(A; B)$ , where  $B$  is also a splitting interval algebra, there is a homomorphism  $\phi : A \rightarrow B$  of the standard form (cf. Remark 2.19) which induces  $\kappa$  and almost induces  $\theta$  in the sense that*

$$\|\phi^*(\tau)(f) - \theta(\tau)(f)\| < \varepsilon + \frac{2 + r_0 + r_1}{m} \cdot N \cdot \|f\|$$

for any  $f \in \mathcal{F}$ ,  $\tau \in \mathsf{T}(B)$ .

The following is a uniqueness theorem for the homomorphisms between splitting interval algebras.

THEOREM 2.27 (Theorem 4.2 of [20]). *Let  $A$  and  $B$  be two splitting interval algebras. For any finite subset  $\mathcal{F} \subseteq A$  and any  $\varepsilon > 0$ , there exist finite subsets  $\mathcal{G}_0 \subseteq A^+$ ,  $\mathcal{G}_1 \subset A$  such that for any homomorphisms  $\phi$  and  $\psi$  from  $A$  to  $B$  satisfying*

- $[\phi]_0 = [\psi]_0$ ,
- for some  $\delta > 0$ , one has that  $\tau(\phi(f)) > \delta$ ,  $\tau(\psi(f)) > \delta$  for any  $f \in \mathcal{G}_0$  and any  $\tau \in \mathsf{T}(B)$ ,
- for the same  $\delta > 0$ ,  $\|\phi^*(\tau)(f) - \psi^*(\tau)(f)\| < \delta$  for any  $\tau \in \mathsf{T}(B)$  and any  $f \in \mathcal{G}_1$ ,

there then exists a unitary  $u \in B$  such that

$$\|\phi(f) - u^* \psi(f) u\| < \varepsilon \quad \text{for any } f \in \mathcal{F}.$$

REMARK 2.28. The statement of the theorem above is different from Theorem 4.2 of [20]. In fact, once  $\mathcal{F}$  and  $\varepsilon > 0$  is given, one can choose  $n \in \mathbb{N}$  according to Condition (4) Theorem 4.2 of [20]. Then the finite subset  $\mathcal{G}_0$  can be chosen to be the central elements of  $A$  which is 0 on the complement of  $[r/n, (r+1)/n]$ , 1 at  $(2r+1)/2n$ , and linear in between; and the finite subset  $\mathcal{G}_1$  can be chosen to be the finite subset in Condition (3) of Theorem 4.2 of [20].

**3. Point-line algebras** The class of splitting tree algebras is a subclass of point-line algebras. They were introduced by Elliott in [12], where the author constructed a class of inductive limit of subhomogeneous C\*-algebras which exhausts all weakly unperforated invariants.

**DEFINITION 3.1.** A point-line algebra is a unital C\*-algebra  $A$  with an essential extension

$$(3.1) \quad 0 \longrightarrow I \longrightarrow A \longrightarrow F \longrightarrow 0,$$

where  $I$  is a homogeneous C\*-algebra with spectrum finitely many copies of  $\mathbb{R}$ , and  $F$  is a finite-dimensional C\*-algebra.

**REMARK 3.2.** The definition above is equivalent to the original definition of point-line algebras in [12]. In fact, any point-line algebra defined in [12] has an extension as above. On the other hand, consider any essential extension of the finite-dimensional C\*-algebra  $F$  by  $I \cong \bigoplus_{i=1}^l C_0(\mathbb{R}, M_{n_i})$ , and consider its Busby invariant

$$\tau : F \rightarrow \mathcal{M}(I)/I = \bigoplus_{i=1}^l (C_b(\mathbb{R}, M_{n_i})/C_0(\mathbb{R}, M_{n_i})).$$

Then, the weak isomorphism of the extensions as (3.1) is determined by the weak equivalence of the Busby maps, i.e., the unitarily equivalence of the map  $F \rightarrow \mathcal{M}(I)/I$ .

Note that the multiplier algebra  $\mathcal{M}(I) = \bigoplus_{i=1}^l C_b(\mathbb{R}, M_{n_i})$  has stable rank one. In fact, for any complex-valued bounded continuous function over  $\mathbb{R}$ , one can easily perturb it within arbitrary tolerance into a function with non-zero values. Therefore the algebra  $\mathcal{M}(I)/I$  also has stable rank one, and hence has cancelation of projections (see [35]).

Since  $F$  is a finite-dimensional C\*-algebra, the unitary equivalence of the maps  $F \rightarrow \mathcal{M}(I)/I$  is determined by the induced map on  $K_0$ -group (see, for instance Lemma 7.3.2 of [37]), which is the exponential map associated to (3.1). Thus, the weak isomorphism of such extensions, in particular, the isomorphism class of  $A$ , is determined by the exponential map associated to (3.1). Therefore,  $A$  must be isomorphic to a concrete model given in [12].

There is a natural way to visualize how the points in the spectrum of  $F$  are attached to the spectrum of  $I$ . Consider the ideals  $I_n = \bigoplus_{i=1}^l C_0((-n, n), M_{n_i})$ ,  $n \in \mathbb{N}$ . Then  $(I_n)$  is an increasing sequence with  $I = \bigcup_n I_n$ . Since  $F$  is semiprojective, one has that the map  $\tau$  has a lifting to  $\tau_n : F \rightarrow \mathcal{M}(I)/I_n$  for some  $n$ . Note that

$$\mathcal{M}(I)/I_n = \bigoplus_{i=1}^l C_b((-\infty, -n] \cup [n, +\infty), M_{n_i}),$$

which exact corresponds the points at infinity of  $I$ . Thus a point in the spectrum of  $F$ , which corresponds a simple direct sum of  $F$ , is attached to  $\bigcup_l \mathbb{R}$  at the infinity corresponding to its image under  $\tau_n$ .

REMARK 3.3. Any interval algebra and any finite-dimensional C\*-algebra have stable rank one. Since the index map associated to the extension above is automatically trivial, it follows that any point-line algebra has stable rank one.

From the short exact sequence, one has the following

$$0 \longrightarrow K_0(A) \longrightarrow \mathbb{Z}^p \longrightarrow \mathbb{Z}^l \longrightarrow K_1(A) \longrightarrow 0,$$

and the order on  $K_0(A)$  is induced by the order on  $\mathbb{Z}^p$ , where  $l$  is the number of lines in the spectrum of  $I$  and  $p$  is the number of points in the spectrum of  $F$ . The embedding  $K_0(A) \rightarrow \mathbb{Z}^p$  is referred to as the canonical embedding of  $K_0(A)$ .

DEFINITION 3.4. Let  $A$  and  $B$  be two point-line algebras. Consider the canonical embedding  $K_0(A) \subseteq \mathbb{Z}^p$ . A positive homomorphism  $\kappa : K_0(A) \rightarrow K_0(B)$  is called extendable if  $\kappa$  can be extended to a positive homomorphism  $\mathbb{Z}^p \rightarrow K_0(B)$ .

For extendable  $K_0$ -maps, one has the following

LEMMA 3.5 (5.2.1 of [12]). *Let  $A$  be a point-line algebra with trivial  $K_1$ -group, and let  $B$  be a splitting tree algebra. Let  $\kappa : K_0(A) \rightarrow K_0(B)$  be an extendable positive homomorphism preserving the order units. Then there exists a \*-homomorphism  $\phi : A \rightarrow B$  such that  $[\phi] = \kappa$ .*

PROOF. Consider the canonical embeddings  $K_0(A) \subseteq \mathbb{Z}^{p_1}$  and  $K_0(B) \subseteq \mathbb{Z}^{p_2}$ . Since  $\kappa$  is extendable, there is a positive homomorphism  $\mathbb{Z}^{p_1} \rightarrow \mathbb{Z}^{p_2}$  such that the restriction to  $K_0(A)$  is exactly the map  $\kappa$ . Then, using the point-line algebra  $B$  as the C\*-algebra  $B_0$  in the proof of 5.2.1 of [12], the argument of 5.2.1 of [12] produces a homomorphism  $\phi' : A \rightarrow M_n(B)$  for some  $n$  such that  $[\phi']_0 = \kappa$ . Consider the projection  $p := \phi'(1_A)$ . Then, one has that  $[p]_0 = \kappa([1_A]_0) = [1_B]$ . Since  $B$  has stable rank one, there is a unitary  $u \in M_n(B)$  such that  $1_B = u^* \phi'(1_A) u$ , and therefore, the map  $\phi = \text{ad}(u) \circ \phi'$  is the desired map.  $\square$

In order to verify whether a given map is extendable, it is enough to show that its composition with the point evaluation at any point at infinity is extendable. This is a positive homomorphism to  $\mathbb{Z}$ , and the following lemma, which is an analogue to Lemma 2.16, states that any such map is extendable if it has certain multiplicity.

LEMMA 3.6. *Let  $A$  be a point-line algebra with trivial  $K_1$ -group. Then, there exists  $M$  such that for any positive map  $\kappa : K_0(A) \rightarrow \mathbb{Z}$  with any element in  $\kappa(K_0(A))$  is divisible by  $M$ , there is a homomorphism  $\phi : A \rightarrow M_k(\mathbb{C})$  such that  $[\phi]_0 = \kappa$ . Moreover, this homomorphism can be chosen to be a direct sum of point evaluations at the points at infinity.*

PROOF. Tensoring  $K_0(A)$  with  $\mathbb{Q}$ , and consider the embedding of vector spaces  $K_0(A) \otimes \mathbb{Q} \subseteq \mathbb{Z}^p \otimes \mathbb{Q} \cong \mathbb{Q}^p$ . There is a positive linear map  $L : \mathbb{Q}^p \rightarrow K_0(A) \otimes \mathbb{Q}$  such that the restriction to  $K_0(A) \otimes \mathbb{Q}$  is identity. There then exists  $M$  such that the image  $L(\mathbb{Z}^p)$  is in  $K_0(A)/M$ . Then, this  $M$  is the desired constant. Indeed, consider the map  $\kappa \circ L : \mathbb{Q}^p \rightarrow \mathbb{Q}$ , and its restriction to  $\mathbb{Z}^p$ . If each element of  $\kappa(K_0(A))$  is in  $M\mathbb{Z}$ , one has that  $\kappa \circ L(\mathbb{Z}^p) \subseteq \mathbb{Z}$ , and hence  $\kappa$  can be extend to a positive map defined on  $\mathbb{Z}^p$ , which corresponds to the points at infinity.  $\square$

Using the lemma above, one has the following existence result.

PROPOSITION 3.7. *Let  $A$  be a point-line algebra with trivial  $K_1$ -group. Then, there exists  $M$  such that for any splitting tree algebra  $B$  with dimension of any irreducible representation divisible by  $M$ , any positive homomorphism  $\kappa : K_0(A) \rightarrow K_0(B)$  which preserve the order-unit, there exists a homomorphism  $\phi : A \rightarrow B$  such that  $[\phi] = \kappa$ .*

PROOF. By Lemma 3.6, there exists  $M$  such that if the size of a point at infinity of  $B$  is a multiple of  $M$ , then, the composition of the point evaluation at that point of  $B$  with  $\kappa$  is induced by a direct sum of point evaluations of  $A$  at some points at infinity. Thus, the map  $\kappa$  is extendable. By Lemma 3.5, the map  $\kappa$  can be lifted to a \*-homomorphism  $\phi : A \rightarrow B$ , as desired.  $\square$

If the domain algebra is a splitting interval algebra, and the codomain algebra is any point-line algebra, then, using the same argument as that of Proposition 2.17, one has

PROPOSITION 3.8. *Let  $A$  be a splitting interval algebra, and let  $B$  be a point-line algebra with  $K_0(B) = 0$ . Then, for any positive homomorphism  $\kappa : K_0(A) \rightarrow K_0(B)$ , there is a \*-homomorphism  $\phi : A \rightarrow B$  such that  $[\phi] = \kappa$ .*

**4. Tracially approximate splitting tree algebras** Motivated by Gong's decomposition theorem in the study of AH-algebras ([17]), tracial approximation for  $C^*$ -algebras was introduced by Lin as an axiomatization for AH-algebras (see [23], [25], etc.). Compare to inductive limit decompositions, it only requires locally approximation rather than an increasing sequence of sub- $C^*$ -algebras. Moreover, the basic building blocks in tracial approximation are usually semiprojective, and this make it easier to verify whether some naturally arising  $C^*$ -algebras are in a class of classified  $C^*$ -algebras or not. In this chapter, We shall introduces the class of  $C^*$ -algebra which can be tracially approximated by splitting tree algebras.

*4.1. Tracially approximate splitting tree algebras* Let us consider the class of splitting tree algebras as building blocks. Then, as considered in [16], Lin's tracial approximation with the class of splitting tree algebras can be described as follows.

DEFINITION 4.1. A unital simple  $C^*$ -algebra  $A$  is called a tracially approximate splitting tree algebra (TAS-algebra for short) if for any  $\varepsilon > 0$ , any finite subset  $\mathcal{F} \subseteq A$  and any  $0 \neq a \in A^+$ , there is a projection  $p \in A$  and a sub- $C^*$ -algebra  $S \subseteq A$  which is a splitting tree algebra with  $1_S = p \neq 0$  such that for all  $x \in \mathcal{F}$ ,

- $\|xp - px\| < \varepsilon$ ,
- $pxp \in_\varepsilon S$ , and
- $1 - p$  is Murray-von Neumann equivalent to a projection in  $\overline{aAa}$ .

If the sub- $C^*$ -algebra  $S$  can be chosen to be a splitting interval algebra, then let us say that  $A$  is a tracially approximate splitting interval algebra (TASI-algebra for short.)

REMARK 4.2. By Proposition 2.7 of [16], if any unital simple  $C^*$ -algebra  $A$  satisfies Condition (4.1) and Condition (4.1), then  $A$  has (SP) property, and thus, there always exist non-zero projections in  $\overline{aAa}$ .

It is known that any splitting tree algebra has stable rank one, and the strict order on projections are determined by traces. Thus, it follows from the results in [16] that

PROPOSITION 4.3 ([16]). *Any separable TAS-algebra has the following properties:*

- *has at least one tracial state, and thus is stably finite;*
- *has stable rank one;*
- *has the (SP) property;*
- *the strict order on projections are determined by traces, and thus has weakly unperforated  $K_0$ -group.*

Moreover, any separable TAS-algebra can be embedded into an asymptotical sequence algebra of finite-dimensional algebras, which is MF in sense of [2].

LEMMA 4.4. *Any separable simple  $C^*$ -algebra  $A$  which satisfies Conditions (4.1) and (4.1) of Definition 4.1 can be embedded into the algebra  $\prod_i M_{n_i}(\mathbb{C}) / \bigoplus_i M_{n_i}(\mathbb{C})$  for a sequence of positive integers  $(n_i)$ .*

PROOF. Let  $\mathcal{F} = \{a_1, a_2, \dots\}$  be a countable dense subset of the unit ball of  $A$  with  $a_1 = 1$ , and set  $\varepsilon_n = 1/2^n$ . Apply the tracial approximation property with  $\mathcal{F}_n = \{a_1, \dots, a_n\}$  and  $\varepsilon_n$ . Then there is a sub- $C^*$ -algebra  $S_n$  and a projection  $p_n$  such that  $p_n a_i p_n \in_\varepsilon S_n$  and  $\|p_n a_i - a_i p_n\| \leq \varepsilon_n$  for all  $a_i, 1 \leq i \leq n$ . Take a point evaluation map  $\pi_n$  of  $S_n$  to  $M_{n_i}$ . Then  $\pi_n$  can be extended to a positive linear contraction from  $p_n A p_n$  to the matrix algebra. Denote it by  $\pi_n$  as above. Then  $\Phi_n(x) = \pi_n(p_n x p_n)$  gives us a positive linear contraction from  $\mathcal{A}$  to the matrix algebra with  $\|\Phi_n(a_i a_j) - \Phi_n(a_i) \Phi_n(a_j)\| < \varepsilon_n$  and  $\|\Phi_n(a_i^*) - \Phi_n(a_i)^*\| < \varepsilon_n$  for any  $1 \leq i, j \leq n$ .

Applying this procedure for each  $n$ , we get a sequence of positive linear contractions  $\{\Phi_n\}$  with the approximation properties above. Therefore the map  $\Phi$



from  $A$  to the asymptotic sequence algebra  $\prod_i M_{n_i}(\mathbb{C}) / \bigoplus_i M_{n_i}(\mathbb{C})$  induced by  $(\Phi_1, \Phi_2, \dots, \Phi_n, \dots)$  is a  $*$ -homomorphism. Since  $\Phi(1) = 1$ , the map  $\Phi$  is a non-zero. By the simplicity of  $A$ , the  $*$ -homomorphism  $\Phi$  maps  $A$  onto-to-one into  $\prod_i M_{n_i}(\mathbb{C}) / \bigoplus_i M_{n_i}(\mathbb{C})$ , as desired.  $\square$

LEMMA 4.5. *Let  $A$  be a TAS-algebra. Let  $p \in A$  be a projection. Then, for any  $\varepsilon > 0$  and any  $N$ , there is a splitting tree algebra  $S \subseteq A$  and orthogonal projections  $p_1 \in A$  and  $p_2 \in S$  such that  $\|p - (p_1 + p_2)\| < \varepsilon$ , and  $\text{rank}(\text{ev}_i^j(p_2)) > N$  for the point evaluation at any splitting point.*

PROOF. Since  $A$  is simple, there exists  $\delta > 0$  such that  $\tau(p) > \delta$  for all  $\tau \in \text{T}(A)$ . If the lemma were not true, there exists a sequence of  $\mathcal{F} - \varepsilon$  multiplicative maps  $\Phi_i : A \rightarrow M_{n_i}$  which factor through point evaluations on some splitting tree algebras  $\{S_n\}$  such that  $\text{rank}(\Phi_i(p))$  is bounded. However,  $n_i$  converges to  $\infty$ , and hence  $\text{tr}(\Phi_i(p))$  converges to 0. A standard compactness argument shows that there exists a tracial state  $\tau \in \text{T}(A)$  such that  $\tau(p) = 0$ , which is a contradiction.  $\square$

Let  $G$  be an order-unit group. Recall that the canonical map  $\rho : G \rightarrow \text{Aff}(S(G))$  is defined by

$$g \mapsto (s \mapsto s(g)) \quad \text{for any } s \in S(G).$$

Let  $A$  be a simple TAS-algebra, and consider the order-unit group  $K_0(A)$ . The following lemma shows that the image of  $K_0(A)$  under  $\rho$  is dense in  $\text{Aff}(S(K_0(A)))$ .

LEMMA 4.6. *Let  $A$  be a simple TAS-algebra. The image of the canonical map  $\rho : K_0(A) \rightarrow \text{Aff}(S(K_0(A), K_0^+(A), [1]_0))$  is dense in  $\text{Aff}(S(K_0(A), K_0^+(A), [1]_0))$ .*

PROOF. Denote by  $E$  the linear subspace of  $\text{Aff}(S(K_0(A), K_0^+(A), [1]_0))$  spanned by the image of  $K_0(A)$ . Since  $E$  contains  $\rho(1_A)$ , the constant function 1,  $E$  contains all constant functions. Moreover, since  $K_0(A)$  separates the states,  $E$  separates the states. Thus the linear subspace  $E$  is dense in  $\text{Aff}(S(K_0(A), K_0^+(A), [1]_0))$  by Corollary 7.4 of [18].

To prove the lemma, we must show that  $\rho(K_0(A))$  is dense in  $E$ . It is enough to show that for any projection  $p \in A$  and any positive real number  $r < 1$ , the positive affine function  $r \cdot \rho([p])$  can be approximated by elements in  $\rho(K_0(A))$ .

For any  $\varepsilon > 0$ , there exist  $n, k \in \mathbb{N}$  such that

$$\frac{k-1}{n+1} \leq r \leq \frac{k+1}{n} \quad \text{and} \quad \left| \frac{k+1}{n} - \frac{k-1}{n-1} \right| \leq \varepsilon.$$

Since  $A$  is a TAS-algebra, one has that  $[p] = [p'] + [p'']$  with  $p'' \in S$  for some splitting tree algebra  $S$ , and  $\tau([p'])$  is uniformly small (less than  $\frac{1}{n}$ ) for all  $\tau \in S(K_0(A))$ . Since  $A$  is simple, by Lemma 4.5, we may assume that  $\text{rank}(\text{ev}_i^j(p'')) \geq n^2$ .

$$\text{rank}(\text{ev}_1^j(p'')) = na_{1,j} + r_{1,j}.$$

For each singular point  $i$ , there exists  $a_{i,j}, r_{i,j} \in \mathbb{N}$  and  $r_{i,j} < n$  such that

$$\text{rank}(\text{ev}_i^j(p'')) = na_{i,j} + r_{i,j},$$

for  $1 \leq j \leq |\bar{k}_i| - 1$ .

Consider  $R_i := \sum_{j=1}^{|\bar{k}_i|-1} r_{i,j}$ , and  $R' = \max\{R_i\}$ . Without loss of generality, we may assume that  $R_1 = R'$ . Then, there exists  $a_{1,|\bar{k}_1|}$  and  $r_{1,|\bar{k}_1|} < n$  such that

$$\text{rank}(\text{ev}_1^{|\bar{k}_1|}(p'')) = na_{1,|\bar{k}_1|} + r_{1,|\bar{k}_1|}.$$

Set  $R = R' + r_{1,|\bar{k}_1|}$ .

For any singular point  $i$ , note that  $R - R_i > 0$ . If  $R - R_i > \bar{k}_i(|\bar{k}_i|)$ , denote by  $d_i = R - R_i - \bar{k}_i(|\bar{k}_i|)$ , and consider  $l_i = \lceil d_i/n \rceil$ . Since  $R < \text{rank}(p'') - n$ , one has

$$d_i = R - R_i - \bar{k}_i(|\bar{k}_i|) < \text{rank}(p'') - R_i - \bar{k}_i(|\bar{k}_i|) - n = n \sum_{j=1}^{|\bar{k}_i|-1} a_{i,j} - n,$$

and hence  $l_i < \sum_{j=1}^{|\bar{k}_i|-1} a_{i,j}$ . Then there exists  $0 \leq b_{i,j} \leq a_{i,j}$  such that  $l_i = \sum_j b_{i,j}$ . Note that

$$\bar{k}_i(|\bar{k}_i|) > R - (R_i + n \sum_j b_{i,j}) = R - R_i - nl_i > \bar{k}_i(|\bar{k}_i|) - n > 0.$$

Replacing  $a_{i,j}$  by  $a_{i,j} - b_{i,j}$ , and still use the same notation for  $R_i$ , one still has  $R - R_i > 0$ , and moreover  $R - R_i < \bar{k}_i(|\bar{k}_i|)$ .

Note that

$$\begin{aligned} \bar{k}_i(|\bar{k}_i|) - (R - R_i) &= \bar{k}_i(|\bar{k}_i|) - (\text{rank}(p'') - n \sum_j a_{1,j} - \sum_{j=1}^{|\bar{k}_i|-1} (\bar{k}_i(j) - na_{i,j})) \\ &= n \left( \sum_{j=1}^{|\bar{k}_i|} a_{1,j} - \sum_{j=1}^{|\bar{k}_i|-1} a_{i,j} \right). \end{aligned}$$

Set  $a_{i,|\bar{k}_i|} = \sum_{j=1}^{|\bar{k}_i|} a_{1,j} - \sum_{j=1}^{|\bar{k}_i|-1} a_{i,j}$  and  $r_{i,|\bar{k}_i|} = R - R_i$ . One then has

$$\sum_{j=1}^{|\bar{k}_1|} r_{i,j} = \cdots = \sum_{j=1}^{|\bar{k}_n|} r_{n,j}.$$

By Corollary 2.11, there exists a projection  $q'' < p''$  such that  $\text{ev}_i^j(q'') = r_{i,j}$ . Moreover, there exists projection  $p''_1, \dots, p''_n$  with  $p''_i < p''$  in  $S$  such that

$$p'' = p''_1 + p''_2 + \cdots + p''_n + q'',$$

where  $p''_i \sim p''_j$  and  $q'' \preceq p''_1$ . Thus for any  $\tau \in \text{S}(\text{K}_0(A))$ ,

$$\frac{k}{n+1} \tau([p'']) \leq \tau([p''_1] + \cdots + [p''_k]) \leq \frac{k}{n} \tau([p'']).$$

Since  $\tau(p') \leq \frac{1}{n}$ , it follows that

$$\frac{k-1}{n+1} \tau([p]) \leq \tau([p''_1] + \cdots + [p''_k]) \leq \frac{k+1}{n} \tau([p])$$

holds for every  $\tau \in \text{S}(\text{K}_0(A))$ , which implies

$$|r \cdot \rho([p]) - \rho([p''_1] + \cdots + [p''_k])| \leq \varepsilon.$$

Therefore, the image  $\rho(\text{K}_0(A))$  is dense in  $E$ , and hence it is dense in the space  $\text{Aff}(\text{S}(\text{K}_0(A), \text{K}_0^+(A), [1]_0))$ , as desired.  $\square$

REMARK 4.7. It is worth pointing out that if one considers the canonical map  $\text{K}_0(A) \rightarrow \text{Aff}(\text{T}(A))$ , the image is not dense unless the projections of  $A$  separate traces. But in general, a TAS-algebra might not have this property.

4.2. *Tracially approximate divisibility of TASI-algebras* Approximate divisibility was introduced in [3]. At this moment, all classifiable stably finite  $C^*$ -algebras with (SP) property are approximately divisible. For instance, one can show directly that the class of simple inductive limits of interval algebras are approximately divisible, as well as the class of simple inductive limit of circle algebras. (Simple AH-algebras without slow dimension growth are also approximately divisible, however, one has to use the classification theorem to show this.)

For tracial approximation, Lin has introduced tracially approximate divisibility. In this section, let us show that any tracially approximate splitting interval algebra is tracially approximately divisible (Theorem 4.13). First, one has the following lemma:

LEMMA 4.8. *Let  $A$  be a splitting interval algebra. Then for any finite subset  $\mathcal{F} \subseteq A$  and any  $\varepsilon > 0$ , there exist a finite subset  $\mathcal{G}_0 \subset A^+$  and a positive integer valued function  $M(\delta)$ , where  $\delta$  is a positive number, such that for any unital homomorphism  $\phi : A \rightarrow B$ , where  $B$  is a splitting interval algebra, if for some  $\delta > 0$ ,  $\tau(\phi(g)) > \delta$  for any  $g \in \mathcal{G}_0$  any  $\tau \in \text{T}(B)$ , and the dimension of any irreducible representation of  $B$  is at least  $M(\delta)$ , then there exists a unital sub- $C^*$ -algebra  $C \cong \text{M}_2 \oplus \text{M}_3 \subseteq B$*

$$\|[f, y]\| < \varepsilon$$

for any  $f \in \mathcal{F}$  and  $y$  in the unit ball of  $C$ . Moreover, one has that  $|\tau(1_{\text{M}_2}) - 1| < \varepsilon$  for any  $\tau \in \text{T}(B)$ .

PROOF. Without loss of generality, one may assume that  $\|f\| = 1$  for any  $f \in F$ .

Let  $\mathcal{G}_0 \subseteq A^+$  and  $\mathcal{G}_1 \subseteq A$  be the finite subsets respectively as in Theorem 2.27 with respect to  $\mathcal{F}$  and  $\varepsilon/2$ . Let  $N$  be the constant of Theorem 2.26 with respect to  $A$ ,  $\mathcal{G}_0 \cup \mathcal{G}_1 \subset A$  and  $\delta/4$ . Let  $\delta_0$  be the constant of Lemma 2.25 with respect to  $A$  and  $\delta/4$ .

Set  $M = \max\{4N/\delta, 48/\delta_0\}$ . Note that  $M$  only depends on  $\varepsilon, \delta, \mathcal{F}$ . Consider the compatible pair  $(\kappa; \theta)$ , where  $\kappa := [\phi]_0 : K_0(A) \rightarrow K_0(B)$  and  $\theta := \phi_* : T(B) \rightarrow T(A)$ . Therefore,  $\theta(\tau)(g) > \delta$  for any  $g \in \mathcal{G}_0$ .

By Corollary 2.15, there are maps  $\kappa_1, \kappa_2 : K_0(A) \rightarrow K_0(B)$  such that  $\kappa = 2\kappa_1 + 3\kappa_2$ . Since  $M \geq 48/\delta_0$ , one has,

$$|4\tau(\kappa_1([p])) - \tau(\kappa([p]))| < \delta_0/4.$$

for any projection  $p \in A$  and any  $\tau \in T(B)$ .

Let  $q_1$  and  $q_2$  be projections in  $B$  such that  $q_1 \perp q_2$  and  $[q_1] = \kappa_1([1_A])$  and  $[q_2] = \kappa_2([1_A])$ . Note that  $q_1$  and  $q_2$  are full projections in  $B$ , and there is an isomorphism  $\iota_1 : T(q_1 B q_1) \rightarrow T(B)$ . Note that since  $M \geq 48/\delta_0$ , one has that for any  $\tau \in T(B)$  and any projection  $q \in q_1 B q_1 \subseteq B$ , one has

$$|\iota_1^{-1}(\tau)(q) - 4\tau(q)| < \delta_0/4.$$

Consider the map  $\theta'_1 := \theta \circ \iota_1 : T(q_1 B q_1) \rightarrow T(A)$ , and we have that for any projection  $p \in A$ , and any  $\tau \in T(B)$ ,

$$\begin{aligned} |\iota_1^{-1}(\tau)(\kappa_1([p])) - \theta'_1(\iota_1^{-1}(\tau))(p)| &< |4\tau(\kappa_1([p])) - \theta(\tau)(p)| + \delta_0/4 \\ &< |\tau(\kappa([p])) - \theta(\tau)(p)| + 2\delta \\ &< \delta_0/2 < \delta_0. \end{aligned}$$

Therefore, by Lemma 2.25, there exists a map  $\theta_1 : T(q_1 B q_1) \rightarrow T(A)$  such that  $\theta_1$  is compatible to  $\kappa_1$  and

$$\|\theta_1(\tau) - \theta'_1(\tau)\| < \delta/4$$

for any  $\tau \in T(q_1 B q_1)$ .

By Theorem 2.26, there exists a homomorphism  $\psi_1 : A \rightarrow q_1 B q_1$  such that  $[\psi_1]_0 = \kappa_1$  and

$$|\psi_1^*(\tau)(f) - \theta_1(\tau)(f)| < \delta/4 + \delta/4 = \delta/2$$

for any  $f \in \mathcal{G}_0 \cup \mathcal{G}_1$  and any  $\tau \in T(q_1 B q_1)$ .

Note that for any  $g \in \mathcal{G}_0$  and any  $\tau \in T(q_1 B q_1)$ , one has

$$\theta'_1(\tau)(g) = \theta \circ \iota_1(\tau)(g) = \iota_1(\tau)(\phi(g)) < \delta$$

and hence

$$(4.1) \quad \psi_1^*(\tau)(g) > \theta \circ \iota_1(\tau)(g) - 3\delta/4 = \delta/4.$$

The same argument shows that there exists a homomorphism  $\psi_2 : A \rightarrow q_2 B q_2$  such that  $[\psi_2]_0 = \kappa_2$  and

$$|\psi_2^*(\tau)(f) - \theta_2(\tau)(f)| < \delta/2$$

for any  $f \in \mathcal{G}_0 \cup \mathcal{G}_1$  and any  $\tau \in \mathbb{T}(q_2 B q_2)$ .

Consider the homomorphism  $\psi := \psi_1 \oplus \psi_1 \oplus \psi_2 \oplus \psi_2 : A \rightarrow B$ . It is clear that

$$[\psi]_0 = 2\kappa_1 + 3\kappa_2 = \kappa = [\phi]_0.$$

Moreover, for any  $f \in \mathcal{G}_0 \cup \mathcal{G}_1$  and any  $\tau \in \mathbb{T}(B)$ ,

$$\begin{aligned} & |\psi^*(\tau)(f) - \theta(\tau)(f)| \\ & < \tau(p_1) |\psi_1^*(\iota_1^{-1}(\tau))(f) - \theta_1(\iota_1^{-1}(\tau))(f)| + \tau(p_2) |\psi_2^*(\iota_1^{-1}(\tau))(f) - \theta_2(\iota_2^{-1}(\tau))(f)| \\ & < \delta/2. \end{aligned}$$

In particular, one has that for any  $\tau \in \mathbb{T}(B)$ ,

- $\tau(\psi(f)) > \tau(\psi(f)) - \delta/2 > \delta/2$  for any  $f \in \mathcal{G}_0$ , and
- $|\psi^*(\tau)(f) - \phi^*(\tau)(f)| < \delta/2$  for any  $f \in \mathcal{G}_1$ .

Therefore, by Theorem 2.27, there exists a unitary  $w \in B$  such that

$$\|\phi(f) - w^* \psi(f) w\| < \varepsilon$$

for any  $f \in \mathcal{F}$ . Note that there exists a unital sub-C\*-algebra  $C' \cong M_2 \oplus M_3 \subseteq B$  such that

$$[\psi(A), C'] = 0.$$

Thus  $C := w^* C' w$  is the desired sub-C\*-algebra.  $\square$

**DEFINITION 4.9.** Let  $A$  and  $B$  be two unital C\*-algebras, and let  $\phi_1$  and  $\phi_2$  be homomorphisms (not necessarily unital) from  $A$  to  $B$ . We write  $\phi_1 \sim \phi_2$  if there exists a partial isometry  $v \in B$  such that  $\phi_1 = v^* \phi_2 v$ .

Using the lemma above, one immediately has the following corollary.

**COROLLARY 4.10.** *Let  $A$  be a splitting interval algebra. Then for any finite subset  $\mathcal{F} \subseteq A$  and any  $\varepsilon > 0$ , there exist a finite subset  $\mathcal{G}_0 \subset A^+$  and a positive integer valued function  $M(\delta)$ , where  $\delta$  is a positive number, such that for any unital homomorphism  $\phi : A \rightarrow B$ , where  $B$  is a splitting interval algebra, if for some  $\delta > 0$ ,  $\tau(\phi(g)) > \delta$  for any  $g \in \mathcal{G}_0$  any  $\tau \in \mathbb{T}(B)$ , and the dimension of any irreducible representation of  $B$  is at least  $M(\delta)$ , there are homomorphisms  $\phi_i : A \rightarrow B$ ,  $i = 1, \dots, 5$ , such that  $\phi_i(1_A)$  are mutually orthogonal with sum  $1_B$ ,  $\phi_1 \sim \phi_2$  and  $\phi_3 \sim \phi_4 \sim \phi_5$ , and*

$$\left\| \phi(f) - \bigoplus_{i=1}^5 \phi_i(f) \right\| < \varepsilon$$

for any  $f \in \mathcal{F}$ . Moreover, the sub- $C^*$ -algebra  $\phi_i(1_A)B\phi_i(1_A)$ , which is also a splitting interval algebra, has multiplicity at least  $1/6$  of the multiplicity of  $B$  for each  $i = 1, \dots, 5$ , and  $\tau(\phi_i(g)) > \delta/4$  for any  $\tau \in \mathbb{T}(\phi_i(1_A)B\phi_i(1_A))$ .

PROOF. The maps  $\phi_1$  can be chosen to be the maps  $\psi_1, \psi_2$  and their conjugates in the proof of Lemma 4.8. The estimation  $\tau(\phi_i(g)) > \delta/4$  follows from (4.1).  $\square$

Moreover, one can has the decomposition of  $\phi$  further if the codomain algebra  $B$  has sufficiently large multiplicity.

COROLLARY 4.11. *Let  $A$  be a splitting interval algebra. Then for any finite subset  $\mathcal{F} \subseteq A$ , any  $\varepsilon > 0$ , and any  $n \in \mathbb{N}$ , there exist a finite subset  $\mathcal{G}_0 \subset A^+$  and a positive integer valued function  $M(\delta)$ , where  $\delta$  is a positive number, such that for any unital homomorphism  $\phi : A \rightarrow B$ , where  $B$  is a splitting interval algebra, if for some  $\delta > 0$ ,  $\tau(\phi(g)) > \delta$  for any  $g \in \mathcal{G}_0$  any  $\tau \in \mathbb{T}(B)$ , and the dimension of any irreducible representation of  $B$  is at least  $M(\delta)$ , there are homomorphisms  $\phi_{i,j} : A \rightarrow B$ ,  $i = 1, \dots, m$ ,  $j = 1, \dots, m_i$ , such that  $\phi_{i,j}(1_A)$  are mutually orthogonal with sum  $1_B$ ,  $\phi_{i,j_1} \sim \phi_{i,j_2}$  for any  $1 \leq i \leq m$  and  $1 \leq j_1, j_2 < m_i$ , and*

$$\left\| \phi(f) - \bigoplus_{i,j} \phi_{i,j}(f) \right\| < \varepsilon$$

for any  $f \in \mathcal{F}$ . Moreover,  $m_i > n$  for any  $i = 1, \dots, m$ .

PROOF. Set  $k = \lceil \log_2 n \rceil + 1$ . Denote by  $\mathcal{G}_0$  and  $M'(\delta)$  the finite subset and the function in Corollary 4.10 with respect to  $\mathcal{F}$  and  $\varepsilon/k$ , respectively, and set  $M(\delta) = 6^n M'(\delta/4^k)$ .

By Corollary 4.10, there are homomorphisms  $\phi_i : A \rightarrow B$ ,  $i = 1, \dots, 5$ , such that  $\phi_1(1_A)$  are mutually orthogonal with sum  $1_B$ ,  $\phi_1 \sim \phi_2$  and  $\phi_3 \sim \phi_4 \sim \phi_5$ , and

$$\left\| \phi(f) - \bigoplus_{i=1}^5 \phi_i(f) \right\| < \varepsilon/n$$

for any  $f \in \mathcal{F}$ . Moreover the sub- $C^*$ -algebra  $\phi_i(1_A)B\phi_i(1_A)$  has multiplicity at least  $6^{n-1}M'(\delta)$  for each  $i = 1, \dots, 5$ .

Consider each homomorphism  $\phi_i : A \rightarrow \phi_i(1_A)B\phi_i(1_A)$ . One then has that  $\tau(\phi_i(g)) > \delta/4$  for any  $\tau \in \mathbb{T}(\phi_i(1_A)B\phi_i(1_A))$ . Therefore, one can apply Corollary 4.10 again to each  $\phi_i$ . Then resulting decomposition of  $\phi$  has multiplicity at least  $2^2 = 4$ . Applying Corollary 4.10  $k$  times, one then has the desired decomposition of  $\phi$ .  $\square$

COROLLARY 4.12. *Let  $A$  be a splitting interval algebra. Then for any finite subset  $\mathcal{F} \subseteq A$ , any  $\varepsilon > 0$ , and any  $n \in \mathbb{N}$ , there exist a finite subset  $\mathcal{G}_0 \subset A^+$  and a positive integer valued function  $M(\delta)$ , where  $\delta$  is a positive number, such that for any unital homomorphism  $\phi : A \rightarrow B$ , where  $B$  is a splitting interval algebra, if for some  $\delta > 0$ ,  $\tau(\phi(g)) > \delta$  for any  $g \in \mathcal{G}_0$  any  $\tau \in \mathbb{T}(B)$ , and*

the dimension of any irreducible representation of  $B$  is at least  $M(\delta)$ , there are homomorphisms  $L_i : A \rightarrow B$ ,  $i = 0, \dots, n$ , such that  $L_1 \sim L_2 \sim \dots \sim L_n$ ,

$$\|\phi(f) - L_0(f) \oplus (L_1(f) \oplus \dots \oplus L_n(f))\| < \varepsilon$$

for any  $f \in \mathcal{F}$ , and  $\tau(L_0(1_A)) < \varepsilon$  for any  $\tau \in \mathbb{T}(B)$ .

PROOF. Applying Corollary 4.11 to  $\mathcal{F}$ ,  $\varepsilon$  and  $n/\varepsilon$ , there exist  $\mathcal{G}_0$  and  $M(\delta)$  such that for any unital homomorphism  $\phi : A \rightarrow B$ , where  $B$  is a splitting interval algebra, if for some  $\delta > 0$ ,  $\tau(\phi(g)) > \delta$  for any  $g \in \mathcal{G}_0$  any  $\tau \in \mathbb{T}(B)$ , and the dimension of any irreducible representation of  $B$  is at least  $M(\delta)$ , there are homomorphisms  $\phi_{i,j} : A \rightarrow B$ ,  $i = 1, \dots, m$ ,  $j = 1, \dots, m_i$ , such that

$$\|\phi(f) - \text{diag}\{\{\phi_{1,1}, \phi_{1,2}, \dots, \phi_{1,m_1}\}, \dots, \{\phi_{m,1}, \phi_{m,2}, \dots, \phi_{m,m_m}\}\}\| < \varepsilon$$

for any  $f \in \mathcal{F}$ , where  $\phi_{i,j_1} \sim \phi_{i,j_2}$  for any  $1 \leq j_1, j_2 \leq m_i$ , and  $m_i > n/\varepsilon$  for any  $i = 1, \dots, m$ .

For any group  $\{\phi_{i,1}, \phi_{i,2}, \dots, \phi_{i,m_i}\}$ , divide it further into

$$\underbrace{\underbrace{\{\phi_{i,1}, \dots, \phi_{i,n}\}}_n, \dots, \underbrace{\{\phi_{i,(k_i-1)n+1}, \dots, \phi_{i,k_i n}\}}_n, \underbrace{\{\phi_{i,k_i n+1}, \dots, \phi_{i,m_i}\}}_{l_i}}_{k_i}$$

where  $k_i = \lfloor m_i/n \rfloor$  and  $l_i = m_i - k_i n < n$ .

Define

$$L_0 = \text{diag}\{\underbrace{\{\phi_{1,k_1 n+1}, \dots, \phi_{1,m_1}\}}_{l_1}, \dots, \underbrace{\{\phi_{m,k_m n+1}, \dots, \phi_{m,m_m}\}}_{l_m}\},$$

and

$$L_k = \text{diag}\{\underbrace{\{\phi_{1,k}, \phi_{1,n+k}, \dots, \phi_{1,k_1 n-n+k}\}}_{k_1}, \dots, \underbrace{\{\phi_{1,k}, \phi_{1,n+k}, \dots, \phi_{1,k_m n-n+k}\}}_{k_m}\}$$

for any  $1 \leq k \leq n$ . Then, it is clear that  $L_1 \sim L_2 \sim \dots \sim L_n$  and,

$$\|\phi(f) - L_0(f) \oplus (L_1(f) \oplus \dots \oplus L_n(f))\| < \varepsilon$$

for any  $f \in \mathcal{F}$ . Moreover, for any  $\tau \in \mathbb{T}(B)$ , one has

$$\tau(L_0(1_A)) \leq \max\left\{\frac{l_i}{m_i}; 1 \leq i \leq m\right\} < \max\left\{\frac{n}{m_i}; 1 \leq i \leq m\right\} < \varepsilon,$$

as desired.  $\square$

**THEOREM 4.13.** *Let  $A$  be a TASI-algebra, and let  $\mathcal{F}$  be a finite subset of  $A$ . Then for any natural number  $n$ , any  $\varepsilon > 0$ , there exist mutually orthogonal projections  $q, p_1, \dots, p_n$  with  $q + p_1 + \dots + p_n = 1$ ,  $q \preceq p_1$  and  $p_i \sim p_1$ , a splitting interval sub-C\*-algebra  $S$  with  $1_S = p_1$  and two  $\mathcal{F} - \varepsilon$  multiplicative linear unital maps  $L_0 : A \rightarrow qAq$ ,  $L_1 : A \rightarrow S$ , such that*

$$\|x - L_0(x) \oplus \underbrace{(L_1(x) \oplus \dots \oplus L_1(x))}_{n \text{ copies}}\| < \varepsilon, \quad \forall x \in \mathcal{F},$$

and  $\tau(q) < \varepsilon$  for all tracial state on  $A$ .

**PROOF.** Without loss of generality, one may assume that  $\|x\| = 1$  for any  $x \in \mathcal{F}$  and  $\varepsilon < 1/(n+1)$ . Since  $A$  is a TASI-algebra, there exists a splitting interval algebra  $S_0$  inside  $A$  with  $p = 1_{S_0}$  such that

- $\|[x, p]\| \leq \varepsilon/8$  for any  $x \in \mathcal{F}$ ,
- $pxp \in_{\varepsilon/8} S_0$  for any  $x \in \mathcal{F}$ ,
- $\tau(1-p) < \varepsilon/4$  for any trace  $\tau$  of  $A$ .

Therefore, there are  $\mathcal{F} - \varepsilon/2$  multiplicative map  $\phi_0 : A \rightarrow (1-p)A(1-p)$  and  $\phi_1 : A \rightarrow S_0$  such that

$$\|x - \phi_0(x) \oplus \phi_1(x)\| < \varepsilon/2 \quad \text{for any } x \in \mathcal{F}.$$

Consider  $\mathcal{F}' = \phi_1(\mathcal{F}) \subset S_0$ . Denote by  $\mathcal{G}_0 \subset S^+$  and  $M(\delta)$  the finite subset and integer valued function of Corollary 4.12 with respect to  $\mathcal{F}' \subset S_0$ ,  $n$ , and  $\varepsilon/4$ .

Consider the C\*-algebra  $pAp$ . Since it is simple, there exists  $\delta > 0$  such that  $\tau(g) > \delta$  for any  $g \in \mathcal{G}_0$ . Noting that  $pAp$  is a simple TASI-algebra,  $S_0 \subset pAp$ , and  $S_0$  is generated by stable relations, there exist a splitting interval algebra  $S_1 \subseteq pAp$ , and two homomorphisms  $\psi_0 : S_0 \rightarrow (1-q)A(1-q)$  and  $\psi_1 : S_0 \rightarrow S_1$ , where  $q = 1_{S_1}$ , such that

- $\|x - \psi_0(x) \oplus \psi_1(x)\| < \varepsilon/4$  for any  $x \in \mathcal{F}' \cup \mathcal{G}_0$ ,
- $\tau(q) < \min\{\varepsilon/4, \delta/4\}$  for any  $\tau \in \mathbf{T}(A)$ .

Moreover, an asymptotical argument as that of Lemma 4.4 or Lemma 4.5 shows that the algebra  $S_1$  can be chosen such that

- $\tau(\psi_1(g)) > \delta/2$  for any  $\tau \in \mathbf{T}(S_1)$  and any  $g \in \mathcal{G}_0$ , and
- the dimension of any irreducible representation of  $S_1$  is at least  $M(\delta/2)$ .

Hence, by Corollary 4.12, there exist homomorphisms  $L'_i : S_0 \rightarrow S_1$ ,  $i = 0, \dots, n$ , such that  $L'_1 \sim L'_2 \sim \dots \sim L'_n$ ,

$$\|\psi_1(x') - L'_0(x') \oplus (L'_1(x') \oplus \dots \oplus L'_n(x'))\| < \varepsilon/4$$

for any  $x' \in \mathcal{F}'$ , and  $\tau(L'_0(1_S)) < \varepsilon/4$  for any  $\tau \in \mathbf{T}(B)$ .



Set

$$L_0 = \phi_0 \oplus \psi_0 \oplus L'_0 \quad \text{and} \quad L_i = L'_i, \quad i = 1, \dots, n.$$

It is clear that for any  $x \in \mathcal{F}$ ,

$$\begin{aligned} & \|x - L_0(x) \oplus (L_1(x) \oplus \cdots \oplus L_n(x))\| \\ & < \|\psi_1(x') - L'_0(f) \oplus (L'_1(f) \oplus \cdots \oplus L'_n(f))\| + 3\varepsilon/4 \\ & < \varepsilon, \end{aligned}$$

where  $x' = \phi_1(x) \in \mathcal{F}'$ .

Denote by  $q = L_0(1_A)$ ,  $p_i = L_i(1_A)$ , and  $S = p_1 S_1 p_1$ . It is clear that  $p_i \sim p_j$ ,  $1 \leq i, j \leq n$ . Note that  $\tau(q) < \varepsilon < 1/(n+1)$ , and hence  $\tau(q) < \tau(p_i)$  for any  $\tau \in T(A)$ . Since the projections of  $A$  are determined by traces, one has  $q \preceq p_i$ , as desired.  $\square$

#### REFERENCES

1. B. Blackadar. *K-theory for Operator Algebras*. Number 5 in Mathematical Sciences Research Institute Publications. Cambridge University Press, second edition, 1998.
2. B. Blackadar and E. Kirchberg. Generalized inductive limits of finite-dimensional  $C^*$ -algebras. *Math. Ann.*, 307(3):343–380, 1997.
3. B. Blackadar, A. Kumjian, and M. Rørdam. Approximately central matrix units and the structure of noncommutative tori. *K-Theory*, 6(3):267–284, 1992.
4. B. Blackadar and M. Rørdam. Extending states on preordered semigroups and existence of the quasitrace on  $C^*$ -algebras. *J. Algebra*, 152(1):240–247, 1992.
5. O. Bratteli, G. A. Elliott, D. E. Evans, and A. Kishimoto. Homotopy of a pair of approximately commuting unitaries in a simple  $C^*$ -algebra. *J. Funct. Anal.*, 160(2):466–523, 1998.
6. O. Bratteli, D. E. Evans, and A. Kishimoto. Crossed products of totally disconnected spaces by  $\mathbb{Z}_2 * \mathbb{Z}_2$ . *Ergodic Theory Dynam. Systems*, 13(3):445–484, 1993.
7. S. Echterhoff, W. Lück, N. C. Phillips, and S. Walters. The structure of crossed products of irrational rotation algebras by finite subgroups of  $SL_2(\mathbb{Z})$ . *J. Reine Angew. Math.*, 639:173–221, 2010.
8. G. A. Elliott. On the classification of inductive limits of sequences of semisimple finite-dimensional algebras. *J. Algebra*, 38(1):29–44, 1976.
9. G. A. Elliott. Dimension groups with torsion. *Internat. J. Math.*, 1(4):361–380, 1990.
10. G. A. Elliott. A classification of certain simple  $C^*$ -algebras. In *Quantum and non-commutative analysis*, number 16 in Math. Phys. Stud., pages 373–385, Kyoto, 1992, 1993. Kluwer Acad. Publ., Dordrecht.
11. G. A. Elliott. On the classification of  $C^*$ -algebras of real rank zero. *J. Reine Angew. Math.*, 443:179–219, 1993.
12. G. A. Elliott. An invariant for simple  $C^*$ -algebras. *Canadian Mathematical Society. 1945–1995*, 3:61–90, 1996.
13. G. A. Elliott. A classification of certain simple  $C^*$ -algebras. II. *J. Ramanujan Math. Soc.*, 12(1):97–134, 1997.
14. G. A. Elliott and G. Gong. On the classification of  $C^*$ -algebras of real rank zero. II. *Ann. of Math. (2)*, 144(3):497–610, 1996.
15. G. A. Elliott, G. Gong, and L. Li. On the classification of simple inductive limit  $C^*$ -algebras. II. The isomorphism theorem. *Invent. Math.*, 168(2):249–320, 2007.
16. G. A. Elliott and Z. Niu. On tracial approximation. *J. Funct. Anal.*, 254(2):396–440, 2008.
17. G. Gong. On the classification of simple inductive limit  $C^*$ -algebras. I. The reduction theorem. *Doc. Math.*, 7:255–461, 2002.

18. K. R. Goodearl. *Partially ordered abelian groups with interpolation*. Number 20 in Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1986.
19. U. Haagerup. Every quasi-trace on an exact  $C^*$ -algebra is a trace. *C. R. Math. Acad. Sci. Soc. R. Can.*, 35(1): ??-??, 2014.
20. X. Jiang and H. Su. A classification of simple limits of splitting interval algebras. *J. Funct. Anal.*, 151(1):50–76, 1997.
21. H. Lin. Classification of simple tracially AF  $C^*$ -algebras. *Canad. J. Math.*, 53(1):161–194, 2001.
22. H. Lin. The tracial topological rank of  $C^*$ -algebras. *Proc. London Math. Soc.*, 83(1):199–234, 2001.
23. H. Lin. Tracially AF  $C^*$ -algebras. *Trans. Amer. Math. Soc.*, 353(2):693–722, 2001.
24. H. Lin. Classification of simple  $C^*$ -algebras of tracial topological rank zero. *Duke Math. J.*, 125(1):91–119, 2004.
25. H. Lin. Simple nuclear  $C^*$ -algebras of tracial topological rank one. *J. Funct. Anal.*, 251(2):601–679, 2007.
26. H. Lin. Asymptotically unitary equivalence and asymptotically inner automorphisms. *Amer. J. Math.*, 131(6):1589–1677, 2009.
27. H. Lin. Approximate homotopy of homomorphisms from  $C(X)$  into a simple  $C^*$ -algebra. *Mem. Amer. Math. Soc.*, 205(963):vi+131, 2010.
28. H. Lin. Homotopy of unitaries in simple  $C^*$ -algebras with tracial rank one. *J. Funct. Anal.*, (6):1822–1882, 2010.
29. H. Lin. Asymptotic unitary equivalence and classification of simple amenable  $C^*$ -algebras. *Invent. Math.*, 183(2):385–450, 2011.
30. H. Lin. Localizing the Elliott conjecture at strongly self-absorbing  $C^*$ -algebras, II. *To appear in Crelle's Journal*, 2014.
31. H. Lin and Z. Niu. Lifting KK-elements, asymptotic unitary equivalence and classification of simple  $C^*$ -algebras. *Adv. Math.*, 219(5):1729–1769, 2008.
32. H. Lin and Z. Niu. The range of a class of classifiable separable simple amenable  $C^*$ -algebras. *J. Funct. Anal.*, 260(1):1–29, 2011.
33. H. Lin and N. C. Phillips. Crossed products by minimal homeomorphisms. *J. Reine Angew. Math.*, 641:95–122, 2010.
34. Z. Niu. Generators and relations for certain splitting tree algebras. *J. Operator Theory*, 61(2):331–345, 2009.
35. Marc A. Rieffel. Dimension and stable rank in the K-theory of  $C^*$ -algebras. *Proc. London Math. Soc. (3)*, 46(2):301–333, 1983.
36. M. Rørdam. The stable and the real rank of  $\mathcal{Z}$ -absorbing  $C^*$ -algebras. *Internat. J. Math.*, 15(10):1065–1084, 2004.
37. M. Rørdam, F. Larsen, and N. J. Laustsen. *An introduction to K-theory for  $C^*$ -algebras*. Number 49 in London Mathematical Society Student Texts. Cambridge University Press, 2000.
38. H. Su. On the classification of  $C^*$ -algebras of real rank zero: inductive limits of matrix algebras over non-Hausdorff graphs. *Mem. of Amer. Math. Soc.*, 114(547), 1995.
39. A. S. Toms. On the classification problem for nuclear  $C^*$ -algebras. *Ann. of Math. (2)*, 167(3):1029–1044, 2008.
40. A. S. Toms. K-theoretic rigidity and slow dimension growth. *Invent. Math.*, 183(2):225–244, 2011.
41. J. Villadsen. On the range of the Elliott invariant of the simple AH-algebras with slow dimension growth. *K-theory*, 15:1–12, 1998.
42. J. Villadsen. Simple  $C^*$ -algebras with perforation. *J. Funct. Anal.*, 154(1):110–116, 1998.
43. J. Villadsen. On the stable rank of simple  $C^*$ -algebras. *J. Amer. Math. Soc.*, 12(4):1091–1102, 1999.
44. S. G. Walters. Inductive limit automorphisms of the irrational rotation algebra. *Comm. Math. Phys.*, 171(2):365–381, 1995.
45. W. Winter. Localizing the Elliott conjecture at strongly self-absorbing  $C^*$ -algebras. *To appear in Crelle's Journal*, 2014.