STRICT COMPARISON HOLDS IN THE UNIFORM ROE ALGEBRA OF A DISCRETE AMENABLE GROUP

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Abstract. Let Γ be a countable discrete amenable group, and let $A = l^\infty(\Gamma) \rtimes \Gamma$ or $A = \mathrm{C}(M) \rtimes \Gamma$, where (M, Γ) is the universal minimal set of Γ. It is shown that if $a, b \in A \otimes \mathcal{K}$ are positive elements such that

$$\mathrm{d}_\tau(a) < \mathrm{d}_\tau(b), \quad \tau \in \mathrm{T}(A),$$

then a is Cuntz subequivalent to b.

1. Introduction

The uniform algebra of a discrete metric space was introduced by John Roe in [18] and [19] in connection with index theory. In the case the space is a discrete group Γ, the uniform Roe algebra is isomorphic to the reduced crossed product C*-algebra $l^\infty(\Gamma) \rtimes_{\mathrm{r}} \Gamma$ (see, for instance, Proposition 5.1.3 of [3]), where the action of Γ on $l^\infty(\Gamma)$ is induced by translation.

The C*-regularity properties of the uniform Roe algebra have been found to be closely related to the asymptotic dimension of the underlying space and dynamics (see [24], [11], [12], [23], [22], [7], etc.). In this paper, we shall focus on the group case, and so on the crossed product C*-algebra $l^\infty(\Gamma) \rtimes \Gamma$, where Γ is a countable discrete amenable group. We shall show that this C*-algebra (recall that the full and reduced crossed products are the same for an amenable group) has the strict comparison property:

Theorem 1.1 (Theorem 4.5). *Let Γ be a countable discrete amenable group, and let $A = l^\infty(\Gamma) \rtimes \Gamma$. If $a, b \in A \otimes \mathcal{K}$ are positive elements such that*

$$\mathrm{d}_\tau(a) < \mathrm{d}_\tau(b), \quad \tau \in \mathrm{T}(A),$$

then a is Cuntz subequivalent to b.

To show the strict comparison stated above, we consider the topological dynamical system $(\widehat{l^\infty(\Gamma)}, \Gamma)$, where $\widehat{l^\infty(\Gamma)}$ is the spectrum of the commutative C*-algebra $l^\infty(\Gamma)$ (which is homeomorphic to $\beta\Gamma$, the Stone-Čech compactification of Γ), and consider the uniform Rohklin property, URP, and the relative comparison property, (COS), which were studied in [14]. In general, these

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two properties together imply that the crossed-product C*-algebra can be weakly tracially approximated by the sub-C*-algebras from the Rohklin towers, which are homogeneous C*-algebras (see [14]). This pair of properties is known to hold for arbitrary free and minimal $\mathbb{Z}^d$-actions and extensions of a free actions of a group with sub-exponential growth on the Cantor set ([15], [14]).

We shall observe that the URP and property (COS) also hold for $(\beta\Gamma, \Gamma)$. Since Γ is amenable, by [5], Γ has (exact) tilings by Følner sets. This indeed implies that the dynamical system $(\beta\Gamma, \Gamma)$ has a strong version of the URP in the sense that the Rohklin towers form a partition of unity (Proposition 3.3). It then follows that the crossed-product C*-algebra $l^\infty(\Gamma) \rtimes \Gamma$ can be weakly tracially approximated by unital homogeneous C*-algebras (with zero-dimensional base spaces) (Proposition 3.8). This strong version of the URP also implies the property (COS) for the sub-C*-algebra $l^\infty(\Gamma)$ inside $l^\infty(\Gamma) \rtimes \Gamma$. Once the URP and property (COS) are established, strict comparison for $l^\infty(\Gamma) \rtimes \Gamma$ then follows from an argument similar to that for Theorem 4.8 of [14].

The C*-algebra $l^\infty(\Gamma) \rtimes \Gamma$ is not simple if Γ is an infinite amenable group. In fact, by [4], the dynamical system $(\beta\Gamma, \Gamma)$ has at least 2^c minimal (non-empty) closed invariant subsets. Moreover, by [8], all of these minimal subsets are universal (and hence are isomorphic), and therefore, all of the (non-zero) simple quotients of the C*-algebra $l^\infty(\Gamma) \rtimes \Gamma$ are isomorphic to $\mathrm{C}(M) \rtimes \Gamma$, where (M, Γ) is the universal minimal set of Γ. Note that the URP of $(\beta\Gamma, \Gamma)$ carries over to the sub-system (M, Γ), and hence $\mathrm{C}(M) \rtimes \Gamma$ also has strict comparison. This simple C*-algebra also can be shown to have stable rank one and real rank zero, and to be approximately divisible:

Theorem 1.2 (Theorem 4.9)**.** *Let Γ be a countable discrete amenable group, and denote by (M, Γ) the universal minimal set of Γ. Then, the C*-algebra $\mathrm{C}(M) \rtimes \Gamma$ has strict comparison. Moreover, it has stable rank one and real rank zero, and is approximately divisible.*

2. Traces and comparison

2.1. **Traces.** A tracial state of a C*-algebra A is a positive linear function $\tau : A \to \mathbb{C}$ such that $\|\tau\| = 1$ and

$$\tau(ab) = \tau(ba), \quad a, b \in A.$$

The set of all tracial states of A is denoted by $\mathrm{T}(A)$. It is a Choquet simplex if A is unital.

Let Γ be a discrete group. Then Γ acts on itself by (right) multiplication, and hence acts on the C*-algebra $l^\infty(\Gamma)$ (from the left). A (right) invariant mean of Γ is a positive linear functional $\rho : l^\infty(\Gamma) \to \mathbb{C}$ such that $\|\rho\| = 1$ and

$$\rho(\gamma\phi) = \rho(\phi), \quad \phi \in l^\infty(\Gamma),\ \gamma \in \Gamma.$$

The group Γ is said to be amenable if it has invariant means.

Note that the restriction of a tracial state of the reduced crossed product C*-algebra $l^\infty(\Gamma) \rtimes_{\mathrm{r}} \Gamma$ to $l^\infty(\Gamma)$ is an invariant mean of Γ. Also note that any invariant mean of Γ can be extended to a trace on $l^\infty(\Gamma) \rtimes_{\mathrm{r}} \Gamma$. Hence, the C*-algebra $l^\infty(\Gamma) \rtimes_{\mathrm{r}} \Gamma$ has tracial states if, and only if, Γ is amenable.

2.2. **Cuntz subequivalence and comparison of positive elements by traces.** Let A be a C*-algebra and let $a, b \in A$ be positive elements. Then a is said to be Cuntz subequivalent to b, denote it by $a \precsim b$, if there is a sequence $(x_n) \subseteq A$ such that

$$\lim_{n\to\infty} x_n^* b x_n = a.$$

For any $\varepsilon > 0$, consider the element $(a-\varepsilon)_+ = f(a)$, where $f(t) = \max\{t-\varepsilon, 0\}$, $t \in \mathbb{R}$. Note that $\|a - (a-\varepsilon)_+\| \leq \varepsilon$. The following lemma will be used freely.

Lemma 2.1 ([20])**.** *Let A be a C*-algebra, and let $a, b \in A$ be positive elements. Then $a \precsim b$ if and only if $(a-\varepsilon)_+ \precsim b$ for all $\varepsilon > 0$.*

Fix $\tau \in \mathrm{T}(A)$, and $a \in (A \otimes \mathcal{K})^+$, and define the rank of a with respect to τ as

$$\mathrm{d}_\tau(a) = \lim_{n\to\infty} \tau(a^{\frac{1}{n}}),$$

where τ is canonically extended to a lower semicontinuous trace of $A \otimes \mathcal{K}$. For any positive elements $a, b \in A \otimes \mathcal{K}$, if $a \precsim b$, then $\mathrm{d}_\tau(a) \leq \mathrm{d}_\tau(b)$ for all $\tau \in \mathrm{T}(A)$.

The reverse implication does not hold in general, even if $\mathrm{d}_\tau(a) < \mathrm{d}_\tau(b)$ is assumed. However, for a large class of C*-algebras, such as simple exact $\mathcal{Z}$-absorbing C*-algebras, strict Cuntz subequivalence is determined by traces (strict comparison) (see [21]). In this paper, we shall show that strict comparison also holds for the C*-algebra $l^\infty(\Gamma) \rtimes \Gamma$, where Γ is a countable discrete amenable group (Corollary 4.5).

We shall use the following results on the comparison properties of a homogeneous C*-algebra $A = \mathrm{M}_n(\mathrm{C}(\Omega))$, where Ω is a compact Hausdorff space. In general, it does not have strict comparison. But its diagonal subalgebra always has a certain relative comparison property, regardless of the topological dimension of Ω, and it has strict comparison if the topological dimension of Ω is zero. These statements are known for separable C*-algebras, and the general statements can be reduced as follows to the separable case.

Lemma 2.2 (cf. Theorem 7.8 of [14])**.** *Consider the C*-algebra $A = \mathrm{M}_n(\mathrm{C}(\Omega))$, where $n \in \mathbb{N}$ and Ω is a compact Hausdorff space. Then for any positive contractions $f = \mathrm{diag}\{f_1, ..., f_n\}$ and $g = \mathrm{diag}\{g_1, ..., g_n\}$ such that*

$$\mathrm{rank}(f(x)) < \frac{1}{4}\mathrm{rank}(g(x)) \quad \text{and} \quad 4 < \mathrm{rank}(g(x)), \quad x \in \Omega,$$

it follows that $f \precsim g$ in A.

Proof. By Theorem 7.8 of [14], the statement holds if A is separable. The general case can be reduced to the separable case: Pick a separable sub-C*-algebra $C \subseteq \mathrm{C}(\Omega)$ which contains $f_1, ..., f_n, g_1, ..., g_n$, and consider the separable sub-C*-algebra $\mathrm{M}_n(C) \subseteq \mathrm{M}_n(\mathrm{C}(\Omega))$. Write $C = \mathrm{C}(\Omega')$, and note that on the quotient Ω' of Ω,

$$\mathrm{rank}(f(x')) < \frac{1}{4}\mathrm{rank}(g(x')), \quad x' \in \Omega'.$$

Then, by Theorem 7.8 of [14], $f \precsim g$ in $\mathrm{M}_n(C) \subseteq \mathrm{M}_n(\mathrm{C}(\Omega))$, as desired. ∎

In the case that the topological covering dimension of Ω is zero (in particular, this includes the case $\Omega = \widehat{l^\infty(\Gamma)}$, since the C*-algebra $l^\infty(\Gamma)$ has real rank zero), one actually has strict comparison:

Lemma 2.3. *Consider the C*-algebra $A = \mathrm{M}_n(\mathrm{C}(\Omega))$, where $n \in \mathbb{N}$ and Ω is a compact Hausdorff space with zero covering dimension. Let $f, g \in A$ be positive elements such that*

$$\mathrm{rank}(f(x)) < \mathrm{rank}(g(x)), \quad x \in \Omega.$$

Then $f \precsim g$ in A.

As a trivial special case, if f and g are also diagonal projections, then there is a unitary $u \in A$ such that $u(x)$ is a permutation unitary for all $x \in \Omega$, $u^* f u$ is a diagonal projection, and $u^* f u \leq g$.

Proof. Note that A is a locally AF algebra. Since A has real rank zero, the Cuntz classes of both f and g are sups of increasing sequences of classes of projections. By compactness of $\mathrm{T}(A)$, each of the projection classes of f, since it is strictly less on $\mathrm{T}(A)$ than the sup of the projection classes for g, is eventually strictly less than on $\mathrm{T}(A)$ than the individual projection classes for g. Since strict comparability for projections clearly holds in A, taking sups in $\mathrm{Cu}(A)$ we find that f is Cuntz majorized by g.

If f and g are diagonal projections, then, since Ω is totally disconnected and compact, there is a partition $\Omega = \Omega_1 \sqcup \Omega_2 \sqcup \cdots \sqcup \Omega_n$ such that Ω_i, $i = 1, ..., n$, are clopen sets, and the restrictions of f and g to each Ω_i are constant functions. Then, on each Ω_i, since $\mathrm{rank}(f(x)) < \mathrm{rank}(g(x))$, there is a permutation unitary u_i such that $u_i^* f u_i \leq g$ on Ω_i, and the desired unitary u can be obtained as u_i on Ω_i. ∎

3. Tilings, (URP), and (COS)

3.1. **Følner towers and tilings.** Let (Ω, Γ) be a topological dynamical system, where Ω is a compact Hausdorff space and Γ is a countable discrete group. A tower is a pair (Z, Γ_0), where $Z \subseteq \Omega$ is a subset and $\Gamma_0 \subseteq \Gamma$ is finite, such that the subsets

$$Z\gamma, \quad \gamma \in \Gamma_0,$$

are mutually disjoint.

A tower is said to be open (closed) if Z is open (closed). Clopen towers (Z, Γ_0) correspond one-to-one to pairs (p, Γ_0) where $p \in \mathrm{C}(\Omega)$ is a projection such that the projections

$$\gamma(p), \quad \gamma \in \Gamma_0,$$

are mutually orthogonal.

Using (exact) tilings of amenable groups, we are able to construct a partition of unity using towers for the (C*-)dynamical system $(l^\infty(\Gamma), \Gamma)$:

Definition 3.1. A tiling of a countable discrete group Γ consists of finite sets (shapes)

$$\Gamma_1, ..., \Gamma_n \subseteq \Gamma$$

and centres

$$\underbrace{c_1^{(1)}, c_2^{(1)}, ...,}_{\text{for } \Gamma_1} \ \underbrace{c_1^{(2)}, c_2^{(2)}, ...,}_{\text{for } \Gamma_2} \ ..., \underbrace{c_1^{(n)}, c_2^{(n)}, ...}_{\text{for } \Gamma_n} \in \Gamma,$$

such that the subsets

$$\Gamma_i c_j^{(i)}, \quad i = 1, ..., n, \ \ j = 1, 2, ...,$$

are mutually disjoint, and

$$\bigcup_{i=1}^{n} \bigcup_{j=1}^{\infty} \Gamma_i c_j^{(i)} = \Gamma.$$

The finite sets Γ_i, $i = 1, ..., n$, are called the shapes, and the elements $c_j^{(i)}$, $i = 1, ..., n$, $j = 1, 2, ...,$ are called the centres (of $\Gamma_i c_j^{(i)}$).

Let $K \subseteq \Gamma$ be a finite set and let $\varepsilon > 0$. A finite set $E \subseteq \Gamma$ is said to be (K, ε)-invariant if

$$\frac{|EK \Delta E|}{|E|} < \varepsilon.$$

Recall that the group Γ is amenable if, and only if, there is a sequence (Γ_n) of finite subsets of Γ such that for any (K, ε), Γ_n is (K, ε)-invariant if n is sufficiently large. The sequence (Γ_n) is called a Følner sequence, and the finite sets Γ_n, $n = 1, 2, ...,$ are called Følner sets.

Tilings of a discrete amenable group by Følner sets always exist:

Theorem 3.2 (Theorem 4.3 of [5])**.** *Let Γ be a countable discrete amenable group. For any (K, ε), where $K \subseteq \Gamma$ is a finite set and $\varepsilon > 0$, there exist (K, ε)-invariant finite sets $\Gamma_1, ..., \Gamma_S \subseteq \Gamma$ which tile the group Γ.*

A tiling of Γ naturally induces a partition of unity of $l^\infty(\Gamma)$ by clopen towers: Let $\{\Gamma_i, c_j^{(i)} : i = 1, ..., n, \ j = 1, 2, ...\}$ be a tiling of Γ. Without loss of generality, one may assume that $e \in \Gamma_i$, $i = 1, ..., n$. Define

$$p_i = \chi_{\{c_1^{(i)}, \ c_2^{(i)}, \ ...\}} \in l^\infty(\Gamma), \quad i = 1, ..., n.$$

Then p_i is a projection and

$$(p_i, \Gamma_i)$$

is a tower. Note that

$$\sum_{\gamma \in \Gamma_i} \gamma(p_i) = \chi_{\bigcup_{j=1}^{\infty} \Gamma_i c_j^{(i)}}.$$

Since $\{\Gamma_i, c_j^{(i)} : i = 1, ..., n, \ j = 1, 2, ...\}$ is a tiling of Γ, one has

$$\sum_{i=1}^{n} \sum_{\gamma \in \Gamma_i} \gamma(p_i) = \chi_\Gamma = \mathbf{1}_{l^\infty(\Gamma)}.$$

3.2. **Rokhlin Partitions.** Recall ([14]) that a dynamical system (X, Γ) has the uniform Rokhlin property if for any finite set $K \subseteq \Gamma$ and any $\varepsilon > 0$, there are mutually disjoint open towers

$$(Z_s, \Gamma_s), \quad s = 1, ..., S,$$

such that

(1) each Γ_s, $s = 1, ..., S$, is (K, ε)-invariant, and
(2) $\mu(X \setminus \bigcup_{s=1}^{S} \bigcup_{\gamma \in \Gamma_s} Z_s\gamma) < \varepsilon$ for every invariant probability Borel measure μ.

Consider the dynamical system $(l^\infty(\Gamma), \Gamma)$. Then, by Theorem 3.2 and the discussion at the beginning of this section (Subsection 3.1), it actually has the following stronger version of the Uniform Rokhlin Property (URP), namely, the Rokhlin towers form a partition of unity:

Proposition 3.3. *Consider the dynamical system $(l^\infty(\Gamma), \Gamma)$. For any finite set $K \subseteq \Gamma$ and any $\varepsilon > 0$, there are mutually disjoint (clopen) towers*

$$(p_s, \Gamma_s), \quad s = 1, ..., S,$$

such that

(1) *each Γ_s is (K, ε)-invariant, and*
(2) $\sum_{s=1}^{S} \sum_{\gamma \in \Gamma_s} \gamma(p_s) = \mathbf{1}_{l^\infty(\Gamma)}$.

Definition 3.4. A dynamical system (X, Γ) is said to have the strong URP if it satisfies the conclusion of the proposition above, i.e., there are Rokhlin towers which are arbitrarily close to being invariant and form an exact partition of unity.

Remark 3.5. Let (X, Γ) and (Y, Γ) be topological dynamical systems. If (X, Γ) is a quotient of (Y, Γ), and if (X, Γ) has the strong URP above, then (Y, Γ) also has the strong URP.

By [8], any minimal closed subset $M \subseteq \beta\Gamma$ is universal in the sense that (M, Γ) is minimal and if (X, Γ) is a minimal dynamical system where X is compact, then it is a quotient of (M, Γ). Since the strong URP above passes to subsystems, the universal minimal system of Γ also has the strong URP.

Corollary 3.6. *Let Γ be a discrete countable amenable group. Then, its universal minimal set (M, Γ) has the strong URP.*

Proof. Let $M \subseteq \beta\Gamma$ be a minimal compact invariant subset. By [8], (M, Γ) provides a realization of the universal minimal system of Γ. Let $K \subseteq \Gamma$ be a finite set, and let $\varepsilon > 0$. By Proposition 3.3, there are mutually disjoint (clopen) towers (Z_s, Γ_s), $s = 1, ..., S$, such that

(1) each Γ_s is (K, ε)-invariant, and
(2) $\bigcup_{s=1}^{S} \bigcup_{\gamma \in \Gamma_s} Z_s\gamma = \beta\Gamma$.

Then

$$(Z_s \cap M, \Gamma_s), \quad s = 1, ..., S,$$

are the desired towers for the universal minimal system (M, Γ). □

Remark 3.7. This strong URP should be compared to the Bratteli-Vershik models studied in [17] and [9]. It also should be compared to the dynamical quasitiling considered in Definition 6.1 of [6]. Lower semicontinuous analogues were considered in Section 8 of [14].

As in [14], this strong version of the URP implies that the C*-algebra $l^\infty(\Gamma)\rtimes\Gamma$ can be tracially approximated by unital sub-C*-algebras from the Rokhlin towers, which are homogeneous C*-algebras (but the cutting element in general is still not a projection). The following proposition also can be compared it to [11] in which the C*-algebra $l^\infty(\Gamma)\rtimes\Gamma$ is shown to be (non-separable) AF if Γ is locally finite.

Proposition 3.8. *Let (Ω, Γ) be a topological dynamical system which has the strong URP of Definition 3.4 (i.e., Rokhlin towers form a partition of unity). Then the C*-algebra $A = \mathrm{C}(\Omega)\rtimes\Gamma$ has the following property:*

For any finite subset $\{f_1, ..., f_n\} \subseteq A$ and any $\varepsilon > 0$, there exist $f_1', ..., f_n' \in A$, a unital sub-C-algebra $C \subseteq A$ (so $1_C = 1_A$), and a positive contraction $h \in C \cap \mathrm{C}(\Omega)$ such that (with a slight abuse of notation)*

(1) $C = \bigoplus_{s=1}^{S} \mathrm{M}_{n_s}(\mathrm{C}(Z_s))$, *where $Z_s \subseteq \Omega$, $s = 1, ..., S$, are mutually disjoint clopen subsets,*
(2) $\mathrm{C}(\Omega) \subseteq C$, *and $\mathrm{C}(\Omega)$ is equal to the diagonal sub-C*-algebra of* $\bigoplus_{s=1}^{S} \mathrm{M}_{n_s}(\mathrm{C}(Z_s))$,
(3) $\|f_i - f_i'\| < \varepsilon$, $i = 1, ..., n$,
(4) $\|[h, f_i']\| < \varepsilon$, $i = 1, ..., n$,
(5) $hf_i'h \in C$, $i = 1, ..., n$, *and*
(6) $\mathrm{d}_\tau(1 - h) < \varepsilon$, $\tau \in \mathrm{T}(A)$.

Proof. The proof is similar to the proof of Theorem 3.8 of [14] (or Lemma 3.1 of [13]), but without dealing with mean dimension.

Without loss of generality, one may assume

$$f_i = \sum_{\gamma\in\mathcal{N}} f_{i,\gamma} u_\gamma$$

for a finite set $\mathcal{N} \subseteq \Gamma$ with $e \in \mathcal{N} = \mathcal{N}^{-1}$, and $f_{i,\gamma} \in \mathrm{C}(\Omega)$, $\gamma \in \mathcal{N}$. (Then the elements f_i', $i = 1, ..., n$ in the statement of the proposition will be f_i, $i = 1, ..., n$.)

Set

$$\max\{1, \|f_{i,\gamma}\| : i = 1, ..., n, \gamma \in \mathcal{N}\} = M.$$

Pick a natural number

$$L > \frac{M\,|\mathcal{N}|}{\varepsilon},$$

and choose a sufficiently large finite set $K \subseteq \Gamma$ and a sufficiently small positive number δ that if a finite set $\Gamma_0 \subseteq \Gamma$ is (K, δ)-invariant, then

$$\frac{|\Gamma_0 \setminus \mathrm{int}_{\mathcal{N}^{L+1}}(\Gamma_0)|}{|\Gamma_0|} < \frac{\varepsilon}{2}, \tag{3.1}$$

where $\mathrm{int}_F(E) := \{\gamma \in E, \gamma F \subseteq E\}$ for any finite sets $E, F \subseteq \Gamma$.

Since (Ω, Γ) has the stronger (URP) of Proposition 3.3, there exist clopen sets $Z_1, Z_2, ..., Z_S \subseteq \Omega$ and (K, δ)-invariant sets $\Gamma_1, \Gamma_2, ..., \Gamma_S \subseteq \Gamma$ such that the subsets

$$Z_s\gamma, \quad \gamma \in \Gamma_s,\ s = 1, ..., S,$$

are mutually disjoint and

$$\Omega = \bigsqcup_{s=1}^{S} \bigsqcup_{\gamma\in\Gamma_s} Z_s\gamma. \tag{3.2}$$

Consider the sub-C*-algebra

$$C := \mathrm{C}^*\{u_\gamma^* f : f \in \mathrm{C}(Z_s),\ \gamma \in \Gamma_s,\ s = 1, 2, ..., S\} \subseteq \mathrm{C}(\Omega) \rtimes \Gamma, \tag{3.3}$$

which, by Lemma 3.11 of [14], is isomorphic to

$$\bigoplus_{s=1}^{S} \mathrm{M}_{|\Gamma_s|}(\mathrm{C}(Z_s)).$$

Since the base sets Z_s, $s = 1, ..., S$, are clopen and the towers form a partition of unity ((3.2)), one has

$$\mathrm{C}(\Omega) \subseteq C. \tag{3.4}$$

Moreover, the isomorphism constructed in the proof of Lemma 3.11 of [14] sends $\mathrm{C}(\Omega)$ onto the diagonal subalgebra of $\bigoplus_{s=1}^{S} \mathrm{M}_{|\Gamma_s|}(\mathrm{C}(Z_s))$. This shows (1) and (2).

For each Γ_s, $s = 1, 2, ..., S$, define the subsets

$$\begin{cases} \Gamma_{s,L+1} &= \mathrm{int}_{\mathcal{N}^{L+1}}(\Gamma_s), \\ \Gamma_{s,L} &= \mathrm{int}_{\mathcal{N}^{L}}(\Gamma_s) \setminus \mathrm{int}_{\mathcal{N}^{L+1}}(\Gamma_s), \\ \Gamma_{s,L-1} &= \mathrm{int}_{\mathcal{N}^{L-1}}(\Gamma_s) \setminus \mathrm{int}_{\mathcal{N}^{L}}(\Gamma_s), \\ \vdots & \vdots\ \vdots \\ \Gamma_{s,0} &= \Gamma_s \setminus \mathrm{int}_{\mathcal{N}}(\Gamma_s). \end{cases}$$

Then, for any $\gamma \in \mathcal{N}$, one has

$$\Gamma_{s,l}\gamma \subseteq \Gamma_{s,l-1} \cup \Gamma_{s,l} \cup \Gamma_{s,l+1}, \quad 1 \leq l \leq L. \tag{3.5}$$

Indeed, pick an arbitrary $\gamma' \in \Gamma_{s,l}$. By construction, one has

$$\gamma'\mathcal{N}^l \subseteq \Gamma_s \quad \text{but} \quad \gamma'\mathcal{N}^{l+1} \nsubseteq \Gamma_s. \tag{3.6}$$

Therefore,

$$\gamma'\gamma\mathcal{N}^{l-1} \subseteq \gamma'\mathcal{N}^l \subseteq \Gamma_s,$$

and hence $\gamma'\gamma \in \mathrm{int}_{\mathcal{N}^{l-1}}\Gamma_s$ (since $e \in \mathcal{N}^{l-1}$).

Thus, to show (3.5), one only has to show that $\gamma'\gamma \notin \mathrm{int}_{\mathcal{N}^{l+2}}\Gamma_s$. Suppose $\gamma'\gamma\mathcal{N}^{l+2} \subseteq \Gamma_s$. Since $\mathcal{N}$ is symmetric, one has $\gamma^{-1} \in \mathcal{N}$; hence $\mathcal{N}^{l+1} \subseteq \gamma\mathcal{N}^{l+2}$ and

$$\gamma'\mathcal{N}^{l+1} \subseteq \gamma'\gamma\mathcal{N}^{l+2} \subseteq \Gamma_s,$$

which contradicts (3.6).

Also note that

$$\Gamma_{s,L+1}\gamma \subseteq \Gamma_{s,L+1} \cup \Gamma_{s,L}. \tag{3.7}$$

For each $\gamma \in \Gamma_s$, define

$$\ell(\gamma) = l, \quad \text{if } \gamma \in \Gamma_{s,l}.$$

By (3.5) and (3.7), the function ℓ satisfies

$$|\ell(\gamma'\gamma) - \ell(\gamma)| \leq 1, \quad \gamma' \in \mathcal{N},\ \gamma \in \Gamma_{s,1} \cup \cdots \cup \Gamma_{s,L+1}. \tag{3.8}$$

Consider the function

$$h := \sum_{s=1}^{S} \sum_{l=1}^{L+1} \sum_{\gamma \in \Gamma_{s,l}} \frac{l-1}{L}(\chi_{Z_s} \circ \gamma^{-1}) = \sum_{s=1}^{S} \sum_{l=1}^{L+1} \sum_{\gamma \in \Gamma_{s,l}} \frac{l-1}{L} u_\gamma^* \chi_{Z_s} u_\gamma \in \mathrm{C}(\Omega) \cap C.$$

By (3.1), for all $\mu \in \mathcal{M}_1(\Omega, \Gamma)$,

$$\mu(X \setminus h^{-1}(1)) \leq \max\{\frac{|\Gamma_s \setminus \mathrm{int}_{\mathcal{N}^{L+1}}(\Gamma_s)|}{|\Gamma_s|} : s = 1, ..., S\} < \varepsilon,$$

and therefore

$$\mathrm{d}_\tau(1-h) < \varepsilon, \quad \tau \in \mathrm{T}(A).$$

This shows (6).

Note that, by the construction of C (see (3.3)),

$$\chi_{Z_s} u_\gamma \in C, \quad \gamma \in \Gamma_s.$$

Hence, for each $\gamma' \in \mathcal{N}$, since $\gamma\gamma' \in \Gamma_s$, $\gamma \in \Gamma_{s,l}$, $l = 1, 2, ..., L+1$, one has

$$hu_{\gamma'} = \sum_{s=1}^{S} \sum_{l=1}^{L+1} \sum_{\gamma \in \Gamma_{s,l}} \frac{l-1}{L} u_\gamma^* \chi_{Z_s} u_{\gamma\gamma'} = \sum_{s=1}^{S} \sum_{l=1}^{L+1} \sum_{\gamma \in \Gamma_{s,l}} \frac{l-1}{L} (u_\gamma^* \chi_{Z_s})(\chi_{Z_s} u_{\gamma\gamma'}) \in C,$$

and therefore,

$$hu_\gamma h \in C, \quad \gamma \in \mathcal{N}.$$

Hence, by (3.4),

$$hf_i h \in C, \quad 1 \leq i \leq n.$$

This shows (5).

Note that, for each $\gamma' \in \mathcal{N}$, by (3.8),

$$\begin{aligned}
&\left\| u_{\gamma'}^* h u_{\gamma'} - h \right\| \\
= \ & \left\| \sum_{s=1}^{S} \sum_{l=1}^{L+1} \sum_{\gamma \in \Gamma_{s,l}} \frac{l-1}{L} \chi_{Z_s} \circ (\gamma'\gamma)^{-1} - \sum_{s=1}^{S} \sum_{l=1}^{L+1} \sum_{\gamma \in \Gamma_{s,l}} \frac{l-1}{L} \chi_{Z_s} \circ \gamma^{-1} \right\| \\
= \ & \max\{\left| \frac{\ell(\gamma'\gamma) - 1}{L} - \frac{\ell(\gamma) - 1}{L} \right| : \gamma \in \Gamma_s \setminus \Gamma_{s,0},\ s = 1, 2, ..., S\} \\
< \ & \frac{1}{L} < \frac{\varepsilon}{M\,|\mathcal{N}|},
\end{aligned}$$

and hence

$$\|hf_i - f_ih\| < \varepsilon, \quad i = 1, 2, ..., n. \tag{3.9}$$

This shows (4). □

3.3. **Property (COS).** Recall ([14]) that a topological dynamical system (X, Γ) is said to have (λ, m)-Cuntz comparison on open sets, where $\lambda \in (0, +\infty)$ and $m \in \mathbb{N}$, if for all $f, g \in \mathrm{C}(X)^+$ such that

$$\mathrm{d}_\tau(f) < \lambda \mathrm{d}_\tau(g), \quad \tau \in \mathcal{M}_1(X, \Gamma),$$

one has $f \precsim 1_m \otimes g$ in $\mathrm{C}(X) \rtimes \Gamma$, where $\mathcal{M}_1(X, \Gamma)$ is the set of all invariant probability Borel measures on X. (The property (COS) in [14] was formulated using open sets.)

The property (COS) can be considered for general pairs of C*-algebras: Let A be a unital C*-algebra and let $D \subseteq A$ be a unital sub-C*-algebra. Then (D, A) is said to have the property (λ, m)-(COS) if for all positive contractions $a, b \in D$ such that

$$\mathrm{d}_\tau(a) < \lambda \mathrm{d}_\tau(b), \quad \tau \in \mathrm{T}(A),$$

one has $a \precsim 1_m \otimes b$ in A.

One should compare the property (COS) to the dynamical comparison property considered in [10] and [6]. It is straightforward to check that the dynamical comparison property implies the property (COS), but it is not clear if the converse holds.

With the Rokhlin partition property of Proposition 3.3, the property (COS) follows directly from Corollary 8.8 of [14] or Theorem 6.3 of [6] for the case that Ω is totally disconnected. For the reader's convenience, we provide a proof below.

Proposition 3.9 (Corollary 8.8 of [14], Theorem 6.3 of [6])**.** *Let (Ω, Γ) be a dynamical system which has the strong URP of Definition 3.4 (i.e., Rokhlin towers form a partition of unity). Then the pair of C*-algebras $(\mathrm{C}(\Omega), \mathrm{C}(\Omega) \rtimes \Gamma)$ has the property $(\frac{1}{4}, 1)$-(COS).*

Moreover, if Ω is totally disconnected (as in the case that $\Omega = \widehat{l^\infty(\Gamma)} = \beta\Gamma$ or Ω is the universal minimal set of Γ), $(\mathrm{C}(\Omega), \mathrm{C}(\Omega) \rtimes \Gamma)$ has the property $(1, 1)$-(COS) (indeed, it has the dynamical comparison property).

Proof. Let $f, g \in \mathrm{C}(\Omega)^+$ be positive elements such that

$$\mathrm{d}_\mu(f) < \frac{1}{4}\mathrm{d}_\mu(g), \quad \mu \in \mathcal{M}_1(\Omega, \Gamma).$$

Denote by E and F the open supports of f and g respectively. Then

$$\mu(E) < \frac{1}{4}\mu(F), \quad \mu \in \mathcal{M}_1(\Omega, \Gamma). \tag{3.10}$$

To prove the assertion, it is enough to show that for any $\varepsilon > 0$, one has

$$(f - \varepsilon)_+ \precsim g.$$

First note that $\mu(F) > 0$ for all μ. Since the set of invariant measures is compact, there is $\delta > 0$ such that

$$\mu(F) > \delta, \quad \mu \in \mathcal{M}_1(\Omega, \Gamma). \tag{3.11}$$

For the given ε, pick a compact set $E' \subseteq E$ such that

$$(f-\varepsilon)_+(x) = 0, \quad x \notin E'.$$

Then, we assert that there is (K, δ) such that if a finite set $\Gamma' \subseteq \Gamma$ is (K, δ)-invariant, then, for all $x \in \Omega$,

$$\frac{1}{|\Gamma'|}\left|\{\gamma \in \Gamma' : x\gamma \in E'\}\right| < \frac{1}{4}\frac{1}{|\Gamma'|}\left|\{\gamma \in \Gamma' : x\gamma \in F\}\right|, \tag{3.12}$$

and

$$\delta < \frac{1}{|\Gamma'|}\left|\{\gamma \in \Gamma' : x\gamma \in F\}\right|. \tag{3.13}$$

Let us only show (3.12) (for suitable (K, δ)), using (3.10); (3.13) can be proved in a similar way by using (3.11). If (3.12) did not hold (for any (K, δ)), there would exist a Følner sequence Γ_n, $n = 1, 2, ...$, such that for each Γ_n, there is $x_n \in \Omega$ such that

$$\frac{1}{|\Gamma_n|}\left|\{\gamma \in \Gamma_n : x_n\gamma \in E'\}\right| \geq \frac{1}{4}\frac{1}{|\Gamma_n|}\left|\{\gamma \in \Gamma_n : x_n\gamma \in F\}\right|. \tag{3.14}$$

Consider the discrete measures

$$\mu_n := \frac{1}{|\Gamma_n|}\sum_{\gamma\in\Gamma_n}\delta_{x_n\gamma}, \quad n = 1, 2, ...,$$

and pick an accumulation point, say μ_∞. Since (Γ_n) is a Følner sequence, μ_∞ is invariant under the action of Γ. By (3.14),

$$\mu_n(F) \leq 4\mu_n(E'), \quad n = 1, 2, ...,$$

and hence (note that F is open and E' is closed),

$$\begin{aligned} \mu_\infty(F) &\leq \liminf_{n\to\infty}\mu_n(F) \\ &\leq 4\liminf_{n\to\infty}\mu_n(E') \\ &\leq 4\limsup_{n\to\infty}\mu_n(E') \\ &\leq 4\mu_\infty(E'), \end{aligned}$$

which contradicts (3.10), and so (3.12) holds.

Now, with (K, δ) as assured above, by (1) and (2) of Proposition 3.8, there is a unital sub-C*-algebra $\mathrm{C}(\Omega) \subseteq C \subseteq \mathrm{C}(\Omega) \rtimes \Gamma$ such that

$$C = \bigoplus_{s=1}^{S}\mathrm{M}_{|\Gamma_s|}(\mathrm{C}(Z_s)),$$

with $\mathrm{C}(\Omega)$ the diagonal subalgebra, for (K, δ)-invariant subsets Γ_s, $s = 1, ..., S$, with $|\Gamma_s| > 4/\delta$. By (3.12), one has, for all $x \in \bigsqcup_{s=1}^{S} Z_s$,

$$\begin{aligned} \mathrm{rank}(f-\varepsilon)_+(x) &= |\{\gamma \in \Gamma_s : (f-\varepsilon)(x\gamma) > 0| \\ &\leq |\{\gamma \in \Gamma_s : x\gamma \in E'\}| \\ &< \frac{1}{4}|\{\gamma \in \Gamma_s : x\gamma \in F\}| \\ &= \frac{1}{4}\mathrm{rank}(g(x)). \end{aligned}$$

By (3.13),

$$4 < \delta\, |\Gamma_s| < |\{\gamma \in \Gamma_s : x\gamma \in F\}| = \mathrm{rank}(g(x)).$$

Then, by Lemma 2.2,

$$(f-\varepsilon)_+ \precsim g,$$

as desired.

If Ω is totally disconnected, the base sets $Z_1, ..., Z_S$ are also totally disconnected. Then, on using Lemma 2.3 instead of Lemma 2.2, the argument above shows that the dynamical system $(\mathrm{C}(\Omega), \mathrm{C}(\Omega) \rtimes \Gamma)$ actually has the property $(1,1)$-(COS).

If, moreover, f and g are projections (hence E and F are clopen sets), the same argument above and the second part of Lemma 2.3 provide a unitary $u \in C$ such that u is locally a permutation unitary, u^*fu is a diagonal projection, and

$$u^*fu \leq g.$$

Since permutations are induced by group actions and diagonal projections correspond to clopen subsets of Ω, the subequivalence above implies that there are a partition $E = E_1 \sqcup \cdots \sqcup E_n$ and group elements $\gamma_1, ..., \gamma_n$ such that

$$E_i\gamma_i \subseteq F \quad \text{and} \quad E_i\gamma_i \cap E_j\gamma_j = \varnothing, \quad i, j = 1, ..., n,\ i \neq j.$$

So, (Ω, Γ) has dynamical comparison in this case. □

4. Strict comparison

Consider a dynamical system (Ω, Γ) with the URP and (COS). Assume that Ω is totally disconnected (in particular, (Ω, Γ) has zero mean dimension). If (Ω, Γ) is free and minimal, and Ω is metrizable, then, by Theorem 4.8 of [14], the C*-algebra $\mathrm{C}(\Omega) \rtimes \Gamma$ has strict comparison.

In this section, we shall show that strict comparison holds in a more general setting. In particular, the C*-algebras $l^\infty(\Gamma) \rtimes \Gamma$ and $\mathrm{C}(M) \rtimes \Gamma$ have strict comparison.

Moreover, the simple C*-algebra $\mathrm{C}(M) \rtimes \Gamma$ has stable rank one and real rank zero, and is approximately divisible.

4.1. **Strict comparison.**

Lemma 4.1. *Let (Ω, Γ) be a dynamical system which has the strong URP of Definition 3.4 (i.e., Rokhlin towers form a partition of unity). Assume that Ω is totally disconnected, and $|\Gamma| = \infty$.*

Let $a \in A = \mathrm{C}(\Omega) \rtimes \Gamma$ be a positive contraction such that

$$\tau(a) > 0, \quad \tau \in \mathrm{T}(A).$$

Then there is a positive element $h \in \mathrm{C}(\Omega)$ such that

$$h \precsim a \quad \textit{and} \quad \tau(h) > 0, \quad \tau \in \mathrm{T}(A).$$

Moreover, h can be chosen such that 0 is an isolated point of $\mathrm{sp}(h)$, and for any given $\varepsilon > 0$, h can be chosen such that also

$$\mathrm{d}_\tau(h) < \varepsilon, \quad \tau \in \mathrm{T}(A).$$

Proof. Let $\varepsilon > 0$ be given. Since $\mathrm{T}(A)$ is compact, there is $\delta > 0$ such that

$$\tau(\mathbb{E}(a)) = \tau(a) > \delta, \quad \tau \in \mathrm{T}(A).$$

Choose $\delta' < \frac{\delta}{4}$, and choose a positive contraction $a' = \sum_{\gamma \in \mathcal{G}} f_\gamma u_\gamma$, where $\mathcal{G} \subseteq \Gamma$ is a finite set such that $\mathcal{G} = \mathcal{G}^{-1}$, such that

$$\|a - a'\| < \delta'. \tag{4.1}$$

Then

$$\tau((\mathbb{E}(a') - \delta')_+) \approx_{\delta'} \tau(\mathbb{E}(a')) \approx_{\delta'} \tau(\mathbb{E}(a)) > \delta,$$

and hence

$$\tau((\mathbb{E}(a') - \delta')_+) > \delta - 2\delta' > \frac{\delta}{2}, \quad \tau \in \mathrm{T}(A).$$

Since (Ω, Γ) has the strong version of the Rokhlin property, a compactness argument (applied to $\mathrm{T}(A)$) shows that there is a Rokhlin partition (Z_s, Γ_s), $s = 1, ..., S$, such that each Γ_s is sufficiently invariant that

$$\frac{1}{|\Gamma_s|} \sum_{\gamma \in \Gamma_s} (\mathbb{E}(a') - \delta')_+(x\gamma) > \frac{\delta}{2}, \quad x \in Z_s, \tag{4.2}$$

$$\frac{|\partial_{\mathcal{G}} \Gamma_s|}{|\Gamma_s|} < \frac{\delta}{4}, \quad s = 1, ..., S, \tag{4.3}$$

and

$$|\Gamma_s| > \frac{1}{\varepsilon}, \quad s = 1, ..., S. \tag{4.4}$$

Since each Z_s is totally disconnected and the function $(\mathbb{E}(a') - \delta')_+$ is continuous, by (4.2) and (4.3), for each $x \in Z_s$, there are $\gamma_x \in \mathrm{int}_{\mathcal{G}}(\Gamma_s)$ and a clopen neighbourhood $Z_{s,x} \subseteq Z_s$ such that

$$(\mathbb{E}(a') - \delta')_+(y\gamma_x) > \frac{\delta}{2} - \frac{\delta}{4} = \frac{\delta}{4}, \quad y \in Z_{s,x}.$$

Then another compactness argument (applied to Z_s) shows that there exist a partition $Z_s = Z_{s,1} \sqcup \cdots \sqcup Z_{s,N_s}$ and $\gamma_{s,1}, ..., \gamma_{s,N_s} \in \mathrm{int}_{\mathcal{G}}(\Gamma_s)$ such that

$$(\mathbb{E}(a') - \delta')_+(x\gamma_{s,n}) > \frac{\delta}{4}, \quad x \in Z_{s,n},\ n = 1, ..., N_s.$$

Therefore, the tower (Z_s, Γ_s) can be split into thinner towers,

$$(Z_{s,1}, \Gamma_s), ..., (Z_{s,N_s}, \Gamma_s),$$

such that for each $(Z_{s,n}, \Gamma_s)$, there is $\gamma_{s,n} \in \mathrm{int}_{\mathcal{G}}(\Gamma_s)$ such that

$$(\mathbb{E}(a') - \delta')_+(x) > \frac{\delta}{4}, \quad x \in Z_{s,n}\gamma_{s,n}. \tag{4.5}$$

Consider the projection

$$p := \sum_{s=1}^{S} \sum_{n=1}^{N} \chi_{Z_{s,n}\gamma_{s,n}}$$

corresponding to the level sets

$$Z_{s,n}\gamma_{s,n}, \quad n = 1, ..., N_s,\ s = 1, ..., S.$$

It follows from (4.4) that

$$\tau(p) < \max\{\frac{1}{|\Gamma_s|} : s = 1, ..., S\} < \varepsilon, \tag{4.6}$$

and it follows from (4.5) that

$$\tau(p(\mathbb{E}(a') - \delta')_+ p) > \frac{\delta}{4\,|\Gamma_s|} > 0, \quad \tau \in \mathrm{T}(A). \tag{4.7}$$

Note that, since $\gamma_{s,n} \in \mathrm{int}_{\mathcal{G}}(\Gamma_s)$ one has

$$pu_\gamma p = 0, \quad \gamma \in \mathcal{G} \setminus \{e\},$$

and hence

$$pa'p = p\mathbb{E}(a')p.$$

Then, since p is a projection and p commutes with $\mathbb{E}(a')$,

$$(pa'p - \delta')_+ = (p\mathbb{E}(a')p - \delta')_+ = p(\mathbb{E}(a') - \delta')_+ p \in l^\infty(\Gamma). \tag{4.8}$$

By (4.1),

$$pa'p \approx_{\delta'} pap,$$

and hence

$$(pa'p - \delta')_+ \precsim pap \precsim a.$$

By (4.8) and (4.7),

$$\tau((pa'p - \delta')_+) = \tau(p(\mathbb{E}(a') - \delta')_+ p) > 0, \quad \tau \in \mathrm{T}(A).$$

By (4.5), 0 is an isolated point of the spectrum of $(pa'p - \delta')_+$. By (4.6),

$$\tau((pa'p - \delta')_+) \leq \tau(p) < \varepsilon, \quad \tau \in \mathrm{T}(A).$$

Thus, $h := (pa'p - \delta')_+$ is the desired element. □

Lemma 4.2. *Let (Ω, Γ) be a dynamical system which has the strong URP of Definition 3.4 (i.e., Rokhlin towers form a partition of unity). Assume that Ω is totally disconnected, and $|\Gamma| = \infty$.*

Let $A = \mathrm{C}(\Omega) \rtimes \Gamma$, and let $a \in A$ be a positive element such that

$$\tau(a) > 0, \quad \tau \in \mathrm{T}(A).$$

Then a is full.

Proof. First, observe that the projection of the base sets $p := p_1 + \cdots + p_S$ in Proposition 3.3 is a full element of $\mathrm{C}(\Omega) \rtimes \Gamma$. Since $|\Gamma| = \infty$, the trace of p can be chosen to be arbitrarily small.

Let $I \subseteq A$ be a two-sided closed ideal which contains a. By Lemma 4.1, there is a projection $h \in \mathrm{C}(\Omega)$ such that

$$h \precsim a \quad \text{and} \quad \tau(h) > 0, \quad \tau \in \mathrm{T}(A).$$

In particular, $h \in I$.

Since $\mathrm{T}(A)$ is compact, there is $\delta > 0$ such that $\tau(h) > \delta$, $\tau \in \mathrm{T}(A)$. By the observation made at the beginning of the proof, there is a full projection $p \in \mathrm{C}(\Omega)$ such that

$$\tau(p) < \delta, \quad \tau \in \mathrm{T}(A).$$

By the property (COS) (Proposition 3.9), one has

$$p \precsim h,$$

which implies $p \in I$. Since p is full, one has $I = A$. □

The following lemma certainly is well known:

Lemma 4.3. *Let A be a C*-algebra. Let $a \in A$ be a positive element, and let $p \in \overline{aAa}$ be a projection. Then, there are positive elements $a', q \in \overline{aAa}$ such that q is a projection, $p \sim q$, $a' \perp q$, and a is Cuntz equivalent to $a' + q$.*

Proof. Passing to a separable sub-C*-algebra of A which contains a and p, one may assume that A is separable. Choose an approximate unit $(e_n)_{n=1}^{\infty}$ for $\overline{aAa}$ such that $e_ne_{n+1} = e_{n+1}e_n = e_n$, $n = 1, 2, \ldots$. Since p is a projection (so it is a compact element), there is a projection $q \in \overline{e_{n-1}Ae_{n-1}}$ for a sufficiently large n such that $p \sim q$. Note that $e_nq = q$. Then

$$a' := (e_{n+1} - q) + \sum_{i=1}^{\infty} \frac{1}{2^i}(e_{n+1+i} - e_{n+i})$$

and q has the desired property. □

Proposition 4.4. *Let (Ω, Γ) be a dynamical system which has the strong URP of Definition 3.4 (i.e., Rokhlin towers form a partition of unity), and assume that Ω is totally disconnected.*

If $a, b \in A := \mathrm{C}(\Omega) \rtimes \Gamma$ are positive elements such that

$$\mathrm{d}_\tau(a) < \mathrm{d}_\tau(b), \quad \tau \in \mathrm{T}(A), \tag{4.9}$$

then $a \precsim b$.

Moreover, if (Ω, Γ) is minimal, then the statement above holds for all positive elements $a, b \in A \otimes \mathcal{K}$.

Proof. The proof is similar to the proof of Theorem 4.8 of [14].

By [2], we may assume that $a, b \in \mathrm{M}_\infty(A)$. Let $a, b \in \mathrm{M}_n(A)$, where n is an arbitrary natural number if (Ω, Γ) is minimal, and $n = 1$ for the general case.

By Lemma 2.1 and the compactness of $\mathrm{T}(A)$, upon replacing a by $(a-\varepsilon)_+$ for an arbitrary $\varepsilon > 0$, we may assume that there is $\delta > 0$ such that

$$\mathrm{d}_\tau(a) + \delta < \mathrm{d}_\tau(b), \quad \tau \in \mathrm{T}(A).$$

Note that $\mathrm{d}_\tau(b) > 0$ for all $\tau \in \mathrm{T}(A)$, and hence $\tau(b) > 0$ for all $\tau \in \mathrm{T}(A)$.

In the case that (Ω, Γ) is minimal (so n is arbitrary), since $b \in \mathrm{M}_n(A)$ is not zero, pick a non-zero diagonal element b_0. Since A is simple in this case, one has $\tau(b_0) > 0$ for all $\tau \in \mathrm{T}(A)$. Note that $b_0 \precsim b$ since $b_0 = ebe$ for a rank one projection e. In the general case (when $n = 1$), just set $b_0 = b$.

By Lemma 4.1, there is a projection $p' \in \mathrm{C}(\Omega)$ such that $p' \precsim b_0 \precsim b$ and

$$0 < \tau(p') < \frac{\delta}{2}, \quad \tau \in \mathrm{T}(A). \tag{4.10}$$

Then there is a projection $p \in \overline{b\mathrm{M}_n(A)b}$ such that p is Murray-von Neumann equivalent to p'.

Pick mutually orthogonal and mutually equivalent non-zero projections $p'_1, ..., p'_n \leq p'$ (in the case that $n = 1$, just choose $p'_1 = p'$). Note that

$$\tau(p'_1) > \delta_1, \quad \tau \in \mathrm{T}(A), \tag{4.11}$$

for some $\delta_1 > 0$.

By Lemma 4.3, there are positive elements $\tilde{b}, q \in \overline{b\mathrm{M}_n(A)b}$ such that q is a projection, q is Murray-von Neumann equivalent to p, $\tilde{b} \perp q$, and $\tilde{b} + q \sim b$. Therefore, by (4.10),

$$\mathrm{d}_\tau(a) + \frac{\delta}{2} < \mathrm{d}_\tau(\tilde{b}), \quad \tau \in \mathrm{T}(A). \tag{4.12}$$

In particular, $\tau(\tilde{b}) > 0$, $\tau \in \mathrm{T}(A)$. By Lemma 4.2, the positive element $\tilde{b}$ is full in A (note that $n = 1$ for the possibly non-simple case). Thus, the Cuntz class of $\tilde{b}$ is a strong order unit of the Cuntz semigroup of $\mathrm{C}(\Omega) \rtimes \Gamma$ (i.e., for each positive element $s \in \mathrm{M}_\infty(\mathrm{C}(\Omega) \rtimes \Gamma)$, there is $m \in \mathbb{N}$ such that $s \precsim \tilde{b} \otimes 1_m$). By (4.12) and the proof of Proposition 3.2 of [20], there is $N \in \mathbb{N}$ such that

$$a \otimes 1_{N+1} \precsim \tilde{b} \otimes 1_N. \tag{4.13}$$

Let $\varepsilon > 0$ be arbitrary. Then, there is $(x_{i,j})_{i,j=1,...,N+1}$ such that

$$\left\| a \otimes 1_{N+1} - (x_{i,j})^*(\tilde{b} \otimes 1_N)(x_{i,j}) \right\| < \varepsilon. \tag{4.14}$$

Denote by $\delta' := \delta(\varepsilon, \|(x_{i,j})\|, N+1)$ the constant of Lemma 4.6 of [14] with respect to $\|(x_{i,j})\|$ and ε.

Since $(\mathrm{C}(\Omega), \Gamma)$ has the URP, by Proposition 3.8, for any $\varepsilon' > 0$ (to be determined), there exist a unital sub-C*-algebra C, $h \in C$, a', b', and $x'_{i,j}$, $i, j = 1, ..., N+1$, in $\mathrm{M}_n(A)$ such that

$$C \cong \bigoplus_{s=1}^{S} \mathrm{M}_{n_s}(\mathrm{C}(Z_s)),$$

where $Z_s \subseteq \Omega$, $s = 1, ..., S$, are clopen subsets,

$$\left\|x'_{i,j} - x_{i,j}\right\|, \ \|a' - a\|, \ \left\|b' - \tilde{b}\right\| < \varepsilon' < \frac{\varepsilon^3}{16 \max\{1, \|(x_{i,j})\|^6\}}, \quad i, j = 1, ..., N+1,$$

$$\left\|[x'_{i,j}, h \otimes 1_n]\right\|, \ \|[a', h \otimes 1_n]\|, \ \|[b', h \otimes 1_n]\| < \varepsilon' < \delta', \quad i, j = 1, ..., N+1, \tag{4.15}$$

$$hx'_{i,j}h, ha'h, hb'h \in \mathrm{M}_n(C), \quad i, j = 1, ..., N+1, \tag{4.16}$$

and

$$\mathrm{d}_\tau(1 - h) < \delta_1, \quad \tau \in \mathrm{T}(A).$$

Then ε' can be chosen to be sufficiently small that

$$\left\|a' \otimes 1_{N+1} - (x'_{i,j})^*(b' \otimes 1_N)(x'_{i,j})\right\| < \varepsilon, \tag{4.17}$$

and

$$\left\|b' - \tilde{b}\right\| < \frac{\varepsilon^3}{8 \max\{1, \left\|(x'_{i,j})\right\|^6\}}. \tag{4.18}$$

By (4.11),

$$\mathrm{d}_\tau(1 - h) < \delta_1 < \mathrm{d}_\tau(p'_1), \quad \tau \in \mathrm{T}(A),$$

and by the property (COS) (Proposition 3.9),

$$(1 - h) \otimes 1_n \precsim p' \sim p \sim q. \tag{4.19}$$

By (4.17), (4.15) and (4.16), it follows from Lemma 4.6 of [14] that there is $c \in \mathrm{M}_n(C)$ such that

$$((h \otimes 1_n)a'(h \otimes 1_n) - 102\varepsilon)_+ \otimes 1_{N+1} \precsim_C c \otimes 1_N$$

and

$$c \precsim (h \otimes 1_n)(b' - \frac{\varepsilon^3}{8 \max\{1, \|x'\|^6\}})_+(h \otimes 1_n). \tag{4.20}$$

In particular

$$\mathrm{rank}(((h \otimes 1_n)a'(h \otimes 1_n) - 102\varepsilon)_+(x)) \leq \mathrm{rank}(c(x)), \quad x \in \widehat{\mathrm{M}_n(C)} = \bigsqcup_{s=1}^{S} Z_s.$$

Since Ω is zero dimensional, the clopen sets $Z_1, ..., Z_S$ are also zero dimensional. By Lemma 2.3, one has

$$((h \otimes 1_n)a'(h \otimes 1_n) - 102\varepsilon)_+ \precsim_C c. \tag{4.21}$$

Then

$$\begin{aligned} a &\approx_{\varepsilon} a' \\ &\approx_{\varepsilon} ((1-h)^{\frac{1}{2}} \otimes 1_n)a'((1-h)^{\frac{1}{2}} \otimes 1_n) + (h^{\frac{1}{2}} \otimes 1_n)a'(h^{\frac{1}{2}} \otimes 1_n) \\ &\approx_{102\varepsilon} ((1-h)^{\frac{1}{2}} \otimes 1_n)a'((1-h)^{\frac{1}{2}} \otimes 1_n) + ((h^{\frac{1}{2}} \otimes 1_n)a'(h^{\frac{1}{2}} \otimes 1_n) - 102\varepsilon)_+, \end{aligned}$$

and by (4.21), (4.19), (4.20), and (4.18),

$$\begin{aligned} & ((1-h)^{\frac{1}{2}} \otimes 1_n)a'((1-h)^{\frac{1}{2}} \otimes 1_n) + ((h^{\frac{1}{2}} \otimes 1_n)a'(h^{\frac{1}{2}} \otimes 1_n) - 102\varepsilon)_+ \\ \precsim\ & (1-h) \otimes 1_n + c \\ \precsim\ & q \oplus (b' - \frac{\varepsilon^3}{8\max\{1, \|x'\|^6\}})_+ \\ \precsim\ & q + \tilde{b} \precsim b, \end{aligned}$$

which implies $(a - 104\varepsilon)_+ \precsim b$. Since ε is arbitrary, one has $a \precsim b$, as desired. □

Since $l^\infty(\Gamma)$ has real rank zero, its spectrum $\widehat{l^\infty(\Gamma)} = \beta\Gamma$ is totally disconnected. Therefore, we have the following corollary:

Theorem 4.5. *Let Γ be a countable discrete amenable group. Let $A = l^\infty(\Gamma) \rtimes \Gamma$ or $A = \mathrm{C}(M) \rtimes \Gamma$, where M is the universal minimal set of Γ.*

If $a, b \in A \otimes \mathcal{K}$ are positive elements such that

$$\mathrm{d}_\tau(a) < \mathrm{d}_\tau(b), \quad \tau \in \mathrm{T}(A),$$

then $a \precsim b$.

Proof. By [2], one may assume that $a, b \in \mathrm{M}_\infty(A)$.

It follows from Proposition 3.3 and Corollary 3.6 that both dynamical systems have the strong URP of Proposition 3.3. The statement of theorem for the universal minimal set (M, Γ) follows directly from Proposition 4.4.

Let us consider $l^\infty(\Gamma) \rtimes \Gamma$. Note that

$$\begin{aligned} l^\infty(\Gamma \times \mathbb{Z}/n\mathbb{Z}) \rtimes (\Gamma \times \mathbb{Z}/n\mathbb{Z}) &\cong (l^\infty(\Gamma) \otimes l^\infty(\mathbb{Z}/n\mathbb{Z})) \rtimes (\Gamma \times \mathbb{Z}/n\mathbb{Z}) \\ &\cong (l^\infty(\Gamma) \rtimes \Gamma) \otimes (l^\infty(\mathbb{Z}/n\mathbb{Z}) \rtimes \mathbb{Z}/n\mathbb{Z}) \\ &\cong \mathrm{M}_n(l^\infty(\Gamma) \rtimes \Gamma). \end{aligned}$$

Then, the comparison of positive elements $a, b \in \mathrm{M}_n(l^\infty(\Gamma) \rtimes \Gamma)$ follows from Proposition 4.4 applied to the dynamical system $(l^\infty(\Gamma \times \mathbb{Z}/n\mathbb{Z}), \Gamma \times \mathbb{Z}/n\mathbb{Z})$. □

Remark 4.6. In general, there exist positive elements $p \in A = l^\infty(\Gamma) \rtimes \Gamma$ and tracial states $\tau_1, \tau_2 \in \mathrm{T}(A)$ such that $\tau_1(p) = 0$ but $\tau_2(p) \neq 0$. For example, one can take $\Gamma = \mathbb{Z}$ and $p = \chi_{(-\infty, 0]}$. Then it is straightforward to construct two Følner sequences such that the density of the set $(-\infty, 0]$ along one Følner sequence is 0 (so there is a trace τ_0 such that $\tau_0(p) = 0$), while its density along the other Følner sequence is 1 (so there is another trace τ_1 such that $\tau_1(p) = 1$). In particular, this implies that A/J in general is not simple, where $J = \{a \in A : \tau(a^*a) = 0,\ \tau \in \mathrm{T}(A)\}$.

In fact, there is no faithful trace on $A/\mathcal{K}$ when $|\Gamma| = \infty$, as $l^\infty(\Gamma)/c_0(\Gamma) \subseteq A/\mathcal{K}$ contains an uncountable family of mutually orthogonal projections. So, none of the invariant means (actually none of the states of $A/\mathcal{K}$) is faithful on $\beta\Gamma \setminus \Gamma$. Hence, for each invariant mean τ, the invariant ideal

$$l^\infty(\Gamma) \cap \{a \in A : \tau(a^*a) = 0\}$$

is proper, and therefore induces an invariant closed subset of $\beta\Gamma \setminus \Gamma$. (On the other hand, for any non-empty closed invariant subset of $\Omega' \subseteq \beta\Gamma \setminus \Gamma$, there is always an invariant mean with support inside Ω'.)

Does A/J contain an uncountable family of mutually orthogonal contractions (or even projections)?

4.2. **The C*-algebra of the universal minimal set.** Let us consider the simple C*-algebra $\mathrm{C}(M) \rtimes \Gamma$ (or the simple C*-algebra $\mathrm{C}(\Omega) \rtimes \Gamma$, where (Ω, Γ) is a minimal dynamical system which satisfies the strong URP and Ω is a zero-dimensional compact space, but not necessarily metrizable). Let us show that this C*-algebra has stable rank one and real rank zero, and is approximately divisible.

First, let us show that the canonical image of $\mathrm{K}_0(\mathrm{C}(\Omega) \rtimes \Gamma)$ is dense in $\mathrm{Aff}(\mathrm{T}(\mathrm{C}(\Omega) \rtimes \Gamma))$ (with respect to the uniform convergence topology) if (Ω, Γ) has the strong URP and Ω is zero dimensional (this also holds for the non-simple C*-algebra $l^\infty(\Gamma) \rtimes \Gamma$).

Lemma 4.7. *Let (Ω, Γ) be a topological dynamical system which has the strong URP of Definition 3.4 (i.e., Rokhlin towers form a partition of unity), and denote by A the crossed product $\mathrm{C}(\Omega) \rtimes \Gamma$. Assume that Ω is totally disconnected. Then the canonical image of $K_0(A)$ is dense in $(\mathrm{Aff}(\mathrm{T}(A)), \|\cdot\|_\infty)$. Indeed, for any $\rho \in \mathrm{Aff}^+(\mathrm{T}(A))$ and $\varepsilon > 0$, there is a diagonal projection in $\mathrm{M}_\infty(\mathrm{C}(\Omega))$ such that $|\rho(\tau) - \tau(p)| < \varepsilon$, $\tau \in \mathrm{T}(A)$.*

Proof. Let $\rho : \mathrm{T}(A) \to \mathbb{R}$ be a positive continuous affine function, and let $\varepsilon > 0$. By Kadison's representation theorem, there is a self-adjoint element $a \in A$ such that

$$\rho(\tau) = \tau(a), \quad \tau \in \mathrm{T}(A).$$

By Proposition 3.8 and the assumption that Ω is totally disconnected, there is a finite dimensional C*-algebra $C \subseteq A$ and a self-adjoint element $a' \in C$ such that

$$|\tau(a) - \tau(a')| < \varepsilon/2, \quad \tau \in \mathrm{T}(A).$$

Since $|\Gamma| = \infty$, the C*-algebra C can be chosen such that the order of each minimal direct summand is at least $1/\varepsilon$. Since ρ is positive, one has $\tau(a') > -\varepsilon/2$ for all $\tau \in \mathrm{T}(A)$. Then, a compactness argument shows that C can be chosen such that

$$\tau(a') > -\varepsilon, \quad \tau \in \mathrm{T}(C). \tag{4.22}$$

As, otherwise, there exist a Følner sequence (Γ_n), a sequence $(x_n) \subseteq \Omega$, finite dimensional C*-algebras (C_n), and self-adjoint elements $a'_n \in C_n$ such that

$$\tau_{x_n,\Gamma_n}(a'_n) \leq -\varepsilon,$$

where τ_{x_n,Γ_n} is the probability measure

$$\frac{1}{|\Gamma_n|}\sum_{\gamma\in\Gamma_n}\delta_{x_n\gamma}.$$

Note that the probability measures τ_{x_n,Γ_n}, $n=1,2,...$, have an accumulation point, say τ_∞, which is necessarily an invariant probability measure (so it induces a trace of A, which will still be denoted by τ_∞). Then, with n sufficiently large,

$$\tau_\infty(a)\approx_{\varepsilon/2}\tau_\infty(a'_n)\approx_{\varepsilon/4}\tau_{x_n,\Gamma_n}(a)\leq-\varepsilon,$$

which implies that $\rho(\tau_\infty)=\tau_\infty(a)<0$, a contradiction to the positivity of ρ.

Then, since C is finite dimensional and such that the order of each minimal direct summand is at least $1/\varepsilon$, by (4.22), there is a projection $p\in C$ such that

$$|\tau(p)-\tau(a')|<\varepsilon,\quad \tau\in\mathrm{T}(C),$$

and thus,

$$|\tau(p)-\rho(\tau)|<2\varepsilon,\quad \tau\in\mathrm{T}(A),$$

as desired. □

Lemma 4.8. *Let (Ω,Γ) be a topological dynamical system which has the strong URP of Definition 3.4 (i.e., there are Rokhlin towers forming a partition of unity), and assume that Ω is totally disconnected. Then, for any countable set $\mathcal{F}\subseteq\mathrm{C}(\Omega)$, there is a separable unital sub-C*-algebra $C=\mathrm{C}(\Omega')\subseteq\mathrm{C}(\Omega)$ which is invariant under the action of Γ such that*

1. $\mathcal{F}\subseteq\mathrm{C}(\Omega')$,
2. Ω' *is totally disconnected,*
3. (Ω',Γ) *has the strong URP.*

Proof. The statement follows from the Blackadar argument.

Let $K_1\subseteq K_2\subseteq\cdots\subseteq\Gamma$ be a sequence of finite set such that $\bigcup_{i=1}^\infty K_i=\Gamma$, and let $\varepsilon_1>\varepsilon_2>\cdots$ be a sequence of positive numbers such that $\lim_{i\to\infty}\varepsilon_i=0$. Since (Ω,Γ) has the URP with Rokhlin towers forming a partition of unity, for each $i=1,2,...$, there are projection $p_{i,s}\in\mathrm{C}(\Omega)$, and finite sets $\Gamma_{i,s}\subseteq\Gamma$, $s=1,...,S_i$, such that

1. $\Gamma_{i,s}$, $s=1,...,S_i$, are (K_i,ε_i)-invariant, and
2. $\sum_{s=1}^{S_i}\sum_{\gamma\in\Gamma_{i,s}}\gamma(p_{i,s})=1_{\mathrm{C}(\Omega)}$.

Set

$$C_0:=\mathrm{C}^*\{\gamma(p_{i,s}),\gamma(f):s=1,...,S_i,\ i=1,2,...,f\in\mathcal{F},\ \gamma\in\Gamma\}\subseteq\mathrm{C}(\Omega).$$

It is a separable sub-C*-algebra which is invariant under Γ and contains $\{p_{i,s}:s=1,...,S_i,\ i=1,2,...\}$ and $\mathcal{F}$.

Pick a countable set $\mathcal{G}_0\subseteq C_0$ which is dense in the set of self-adjoint elements of C_0. Since $\mathrm{C}(\Omega)$ has real rank zero, there is a countable set $\mathcal{H}_0\subseteq\mathrm{C}(\Omega)$ consisting of invertible self-adjoint elements such that $\mathcal{G}_0\subseteq\overline{\mathcal{H}_0}$. Consider the separable C*-algebra

$$C_1:=\mathrm{C}^*\{C_0,\gamma(\mathcal{H}_0):\gamma\in\Gamma\}\subseteq\mathrm{C}(\Omega).$$

Then the C*-algebra C_1 is invariant under the action of Γ, $C_0 \subseteq C_1$, and any self-adjoint element of C_0 can be approximated by invertible self-adjoint elements of C_1.

Repeating this process, one obtains a sequence of Γ-invariant separable unital sub-C*-algebras

$$C_0 \subseteq C_1 \subseteq \cdots \subseteq \mathrm{C}(\Omega)$$

such that each self-adjoint element of C_n, $n = 0, 1, ...,$ can be approximated by invertible self-adjoint elements of C_{n+1}. Then the C*-algebra

$$C := \overline{\bigcup_{n=0}^{\infty} C_n}$$

is Γ-invariant, separable, and has real rank zero.

It is clear that $\mathcal{F} \subseteq C$. Since for each $i = 1, 2, ...,$

$$\{p_{i,s} : s = 1, ..., S_i\} \subseteq C,$$

the dynamical system (C, Γ) has partitions by Rokhlin towers which are (K_i, ε_i)-invariant, which is the strong URP. □

Note that, the quotient system (Ω', Γ) is minimal if (Ω, Γ) is minimal.

Theorem 4.9. *Let* (Ω, Γ) *be a minimal topological dynamical system which has the strong URP of Definition 3.4 (i.e., there are Rokhlin towers forming a partition of unity), and assume that* Ω *is totally disconnected.*

Then, for any countable set $\mathcal{F} \subseteq \mathrm{C}(\Omega) \rtimes \Gamma$*, there is a unital simple separable nuclear* $\mathcal{Z}$*-absorbing C*-algebra* $B \subseteq \mathrm{C}(\Omega) \rtimes \Gamma$ *which satisfies the UCT and has real rank zero such that*

$$\mathcal{F} \subseteq B.$$

In particular, the C-algebra* $\mathrm{C}(\Omega) \rtimes \Gamma$ *has stable rank one and real rank zero, and is approximately divisible.*

Proof. Approximating each element of $\mathcal{F}$ with elements in the algebraic crossed product, one may assume that each element of $\mathcal{F}$ is a $\mathrm{C}(\Omega)$-valued function on Γ with finite support. Then, denote by $\mathcal{F}' \subseteq \mathrm{C}(\Omega)$ the countable set of all the coefficients of elements of $\mathcal{F}$ (i.e., their values as functions $\Gamma \to \mathrm{C}(\Omega)$). By Lemma 4.8, there is a Γ-invariant separable sub-C*-algebra $\mathrm{C}(\Omega') \subseteq \mathrm{C}(\Omega)$ such that $\mathcal{F}' \subseteq \mathrm{C}(\Omega')$.

Regarding $B := \mathrm{C}(\Omega') \rtimes \Gamma$ as a sub-C*-algebra of $\mathrm{C}(\Omega) \rtimes \Gamma$, one then has $\mathcal{F} \subseteq B$. The C*-algebra B is simple since (Ω', Γ) is minimal and has the strong URP (hence $(\mathrm{C}(\Omega'), \Gamma)$ satisfies Kishimoto's condition). It is also nuclear and satisfies the UCT since Γ is amenable.

Since Ω' is zero dimensional, the system (Ω', Γ) has zero mean dimension. It then follows from Theorem 4.8 of [13] that $B \cong B \otimes \mathcal{Z}$. (The freeness assumption of Theorem 4.8 of [13] actually is not necessary. This assumption is not used in the proof of Proposition 3.1 and Lemma 4.1 of [13]. When the freeness is used in Corollary 3.2 and Theorem 4.8, it is strict comparison that is needed. But, by Proposition 4.4, strict comparison actually just follows from the URP.)

Since B is finite, it has stable rank one ([21]). By Lemma 4.7, the image of $\mathrm{K}_0(B)$ is uniformly dense in $\mathrm{Aff}(\mathrm{T}(B))$, and hence, by Theorem 7.2 of [21], the C*-algebra B has real rank zero.

Therefore, B is a unital simple separable nuclear $\mathcal{Z}$-absorbing C*-algebra which satisfies the UCT. It is classified by the conventional Elliott invariant. Since B has real rank zero, it is isomorphic to an AH algebra, and is approximately divisible. □

Remark 4.10. If (Ω, Γ) is not minimal, the crossed product C*-algebra might fail to have stable rank one; see, for example, [16] and [1] for Cantor systems of $\mathbb{Z}$-actions. The uniform Roe algebra $A = l^\infty(\Gamma) \rtimes \Gamma$ also provides such examples: Assume $\mathbb{Z} \subseteq \Gamma$, and consider the projection

$$p = \chi_{(-\infty,0]} \in l^\infty(\Gamma).$$

To see the stable rank of A is not 1, it is enough to show that the stable rank of the hereditary subalgebra pAp is not 1. Denote by u the canonical unitary corresponding to $1 \in \mathbb{Z}$. Then the element

$$v := pup$$

satisfies

$$vv^* = p(upu^*)p = p, \quad \text{and} \quad v^*v = p(u^*pu)p = u^*pu = \chi_{(-\infty,-1]}.$$

So, v is an isometry. Its image in the quotient $pAp/(pAp \cap \mathcal{K})$, where $\mathcal{K} = c_0(\Gamma) \rtimes \Gamma$, is a unitary with non-zero index

$$[p - vv^*]_0 - [p - v^*v]_0 = -[\chi_{\{-1\}}]_0 \in \mathrm{K}_0(\mathcal{K}) \cong \mathbb{Z}.$$

Thus, the stable rank of pAp is not 1. (Indeed, if $\Gamma = \mathbb{Z}^d$, it is shown in [12] that the stable rank of A is 2.) It is also shown in [12] that the real rank of $l^\infty(\Gamma) \rtimes \Gamma$ in general fails to be zero due to non-zero exponential maps.

Remark 4.11. What are the K-groups and the trace simplex of $\mathrm{C}(M) \rtimes \Gamma$? Note that, since $\mathrm{C}(M) \rtimes \Gamma$ has real rank zero, its trace simplex is canonically isomorphic to the state space of its order-unit K_0-group.

Remark 4.12. Let Γ_1 and Γ_2 be discrete amenable groups, and consider their universal minimal sets (M_1, Γ_1) and (M_2, Γ_2). When are these systems topologically orbit equivalent?

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