HOMOMORPHISMS INTO SIMPLE $\mathcal{Z}$-STABLE C^*-ALGEBRAS

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Abstract. Let A and B be unital separable simple amenable C*-algebras which satisfy the Universal Coefficient Theorem. Suppose that A and B are $\mathcal{Z}$-stable and are of rationally tracial rank no more than one. We prove the following: Suppose that $\phi, \psi : A \to B$ are unital *-monomorphisms. There exists a sequence of unitaries $\{u_n\} \subset B$ such that

$$\lim_{n\to\infty} u_n^*\phi(a)u_n = \psi(a) \text{ for all } a \in A,$$

if and only if

$$[\phi] = [\psi] \text{ in } KL(A,B),\ \phi_\sharp = \psi_\sharp \text{ and } \phi^\ddagger = \psi^\ddagger,$$

where $\phi_\sharp, \psi_\sharp : \mathrm{Aff}(\mathrm{T}(A)) \to \mathrm{Aff}(\mathrm{T}(B))$ and $\phi^\ddagger, \psi^\ddagger : U(A)/CU(A) \to U(B)/CU(B)$ are the induced maps (where $\mathrm{T}(A)$ and $\mathrm{T}(B)$ are the tracial state spaces of A and B, and $CU(A)$ and $CU(B)$ are the closures of the commutator subgroups of the unitary groups of A and B, respectively). We also show that this holds if A is a rationally AH-algebra which is not necessarily simple. Moreover, for any strictly positive unit-preserving $\kappa \in KL(A,B)$, any continuous affine map $\lambda : \mathrm{Aff}(\mathrm{T}(A)) \to \mathrm{Aff}(\mathrm{T}(B))$ and any continuous group homomorphism $\gamma : U(A)/CU(A) \to U(B)/CU(B)$ which are compatible, we also show that there is a unital homomorphism $\phi : A \to B$ so that $([\phi], \phi_\sharp, \phi^\ddagger) = (\kappa, \lambda, \gamma)$, at least in the case that $K_1(A)$ is a free group.

1. Introduction

Let X and Y be two compact Hausdorff spaces, and denote by $C(X)$ (or $C(Y)$) the C*-algebra of complex-valued continuous functions on X (or Y). Any continuous map $\lambda : Y \to X$ induces a homomorphism ϕ from the commutative C*-algebra $C(X)$ into the commutative C*-algebra $C(Y)$ by $\phi(f) = f \circ \lambda$, and any homomorphism from $C(X)$ to $C(Y)$ arises this way (in this paper, by homomorphisms or isomorphisms between C*-algebras, we mean *-homomorphisms or *-isomorphisms). It should be noted that, by the Gelfand-Naimark theorem, every unital commutative C*-algebra has the form $C(X)$ as above.

For non-commutative C*-algebras, one also studies homomorphisms. Let A and B be two unital C*-algebras and let $\phi, \psi : A \to B$ be two homomorphisms. A fundamental problem in the study of C*-algebras is to determine when ϕ and ψ are (approximately) unitarily equivalent.

The last two decades saw the rapid development of classification of amenable C*-algebras, or otherwise known the Elliott program. For instance, all unital simple AH-algebras with slow dimension growth are classified by their Elliott invariant ([4]). In fact, the class of classifiable simple C*-algebras includes all unital separable amenable simple C*-algebras with the tracial rank at most one which satisfy the Universal Coefficient Theorem (the UCT) (see [11]). One of the crucial problems in the Elliott program is the so-called uniqueness theorem which usually asserts that two monomorphisms are approximately unitarily equivalent if they induce the same K-theory related maps under certain assumptions on C*-algebras involved.

Recently, W. Winter's method ([32]) greatly advances the Elliott classification program. The class of amenable separable simple C*-algebras that can be classified by the Elliott invariant has been enlarged so that it contains simple C*-algebras which no longer are assumed to have finite tracial rank. In fact, with [32], [15], [23] and [18], the classifiable C*-algebras now include any unital separable simple $\mathcal{Z}$-stable C*-algebra A satisfying the UCT such that $A\otimes U$ has the tracial rank no more than one for some UHF-algebra U (it has recently been shown, for example, $A\otimes U$ has tracial rank at most one for all UHF-algebras U of infinite type, if $A\otimes C$ has tracial rank at most one for one of infinite dimensional unital simple AF-algebra (see [26])). This class of C*-algebras is strictly larger than the class of AH-algebras without dimension growth. For example, it contains the Jiang-Su algebra $\mathcal{Z}$ itself which is projectionless and all simple unital inductive limits of so-called generalized dimension drop algebras (see [20]).

Recall that the Elliott invariant for a stably finite unital simple separable C*-algebra A is

$$\mathrm{Ell}(A) := ((K_0(A), K_0(A)_+, [1_A], \mathrm{T}(A)), K_1(A)),$$

where $(K_0(A), K_0(A)_+, [1_A], \mathrm{T}(A))$ is the quadruple consisting of the K_0-group, its positive cone, the order unit and tracial simplex together with their pairing, and $K_1(A)$ is the K_1-group.

Denote by $\mathcal{C}$ the class of all unital simple C*-algebras A for which $A\otimes U$ has tracial rank no more than one for some UHF-algebra U of infinite type. Suppose that A and B are two unital separable amenable C*-algebras in $\mathcal{C}$ which satisfy the UCT. The classification theorem in [18] states that if the Elliott invariants of A and B are isomorphic, i.e.,

$$\mathrm{Ell(A)} \cong \mathrm{Ell(B)},$$

then there is an isomorphism $\phi : A \to B$ which carries the isomorphism above.

However, the question when two isomorphisms are approximately unitarily equivalent was still left open. A more general question is: for any two such C*-algebras A and B, and, for any two homomorphisms $\phi, \psi : A \to B$, when are they approximately unitarily equivalent?

If ϕ and ψ are approximately unitarily equivalent, then one must have,

$$[\phi] = [\psi] \ \text{ in } \ KL(A, B) \ \text{ and } \ \phi_\sharp = \psi_\sharp,$$

where $\phi_\sharp, \psi_\sharp : \mathrm{Aff}(\mathrm{T}(A)) \to \mathrm{Aff}(\mathrm{T}(B))$ are the affine maps induced by ϕ and ψ, respectively. Moreover, as shown in [16], one also has

$$\phi^\ddagger = \psi^\ddagger,$$

where $\phi^\ddagger, \psi^\ddagger : U(A)/CU(A) \to U(B)/CU(B)$ are homomorphisms induced by ϕ, ψ, and $CU(A)$ and $CU(B)$ are the closures of the commutator subgroups of the unitary groups of A and B, respectively.

In this paper, we will show that the above conditions are also sufficient, that is, the maps ϕ and ψ are approximately unitarily equivalent if and only if $[\phi] = [\psi]$ in $KL(A, B)$, $\phi_\sharp = \psi_\sharp$ and $\phi^\ddagger = \psi^\ddagger$.

Not surprisingly, the proof of this uniqueness theorem is based on the methods developed in the proof of the classification result mentioned above, which can be found in [18], [17], [16], [23] and [14]. Most technical tools are developed in those papers, either directly or implicitly. In the present paper, we will collect them and then assemble them into production.

It should be noted that the above-mentioned uniqueness theorem still holds in a more general setting where the source algebra A is not necessary in the class $\mathcal{C}$. For example, it is still valid for all AH-algebras A which are not necessarily simple. In particular, A could be just $C(X)$ for any compact metric space X.

In that situation, the first version of this kind of uniqueness theorem was proved in [6], where $A = C(X)$ and B is a unital simple C*-algebra with the unique tracial state and with real rank zero, stable rank one and weakly unperforated $K_0(B)$.

Then, in [10], it was shown that, if $A = C(X)$ for some compact metric space X and B is a unital simple C*-algebra with tracial rank zero, then any unital monomorphisms ϕ and ψ from A to B are approximately unitarily equivalent if and only if $[\phi] = [\psi]$ in $KL(A, B)$ and $\phi_\sharp = \psi_\sharp$. This result was then generalized to the case that B has tracial rank no more than one with the additional condition $\phi^\ddagger = \psi^\ddagger$ in [21].

From this point of view, the main result in this paper may also be regarded as a further generalization of these uniqueness theorems. In fact, in this paper, we also allow the source algebra A to be any unital C*-algebra such that $A\otimes U$ is a unital AH-algebra for all UHF-algebra U of infinite type. One should also realize that these uniqueness theorems have a common root: The Brown-Douglass-Fillmore theorem for essentially normal operators. One version of it can be stated as follows: Two monomorphisms $\phi, \psi : C(X) \to B(H)/\mathcal{K}$—the Calkin algebra, which is a unital simple C*-algebra with real rank zero—are unitarily equivalent if and only if $[\phi] = [\psi]$ in $KK(C(X), B(H)/\mathcal{K})$.

As this research was under way, we learned that H. Matui was conducting his own investigation on the same problems. In fact, he proved the same uniqueness theorems mentioned under the assumption that $B\otimes U$ has tracial rank zero. Moreover, he actually showed the same result holds if the assumption that $B \otimes U$ has tracial rank zero is weaken to be that $B \otimes U$ has real rank zero, stable rank one and weakly unperforated $K_0(B \otimes U)$, at least for the case that quasi-traces are traces and there are only finitely many of extremal tracial states.

In [24], it is shown that, for any partially ordered simple weakly unperforated rationally Riesz group G_0 with order unit u, any countable abelian group G_1, any metrizable Choquet simple S, and any surjective affine continuous map $r : S \to S_u(G_0)$ (the state space of G_0) which preserves extremal points, there exists one (and only one up to isomorphism) unital separable simple amenable C*-algebra $A \in \mathcal{C}$ which satisfies the UCT so that $\mathrm{Ell}(A) = (G_0, (G_0)_+, u, G_1, S, r)$.

Then a natural question is: Given two unital separable simple amenable C*-algebras A, $B \in \mathcal{C}$ which satisfy the UCT, and a homomorphism Γ from $\mathrm{Ell}(A)$ to $\mathrm{Ell}(B)$, does there exist a unital homomorphism $\phi : A \to B$ which induces Γ? We will give an answer to this question. Related to the uniqueness theorem discussed earlier and also related to the question above, one may also ask the following: Given an element $\kappa \in KL(A, B)$ which preserves the unit and order, an affine map $\lambda : \mathrm{Aff}(\mathrm{T}(A)) \to \mathrm{Aff}(\mathrm{T}(B))$ and a homomorphism $\gamma : U(A)/CU(A) \to U(B)/CU(B)$ which are compatible, does there exist a unital homomorphism $\phi : A \to B$ so that $[\phi] = \kappa$, $\phi_\sharp = \lambda$ and $\phi^\ddagger = \gamma$? We will, at least, partially answer this question.

2. Preliminaries

2.1. Let A be a unital stably finite C*-algebra. Denote by $\mathrm{T}(A)$ the simplex of tracial states of A and denote by $\mathrm{Aff}(\mathrm{T}(A))$ the space of all real affine continuous functions on $\mathrm{T}(A)$. Suppose that $\tau \in \mathrm{T}(A)$ is a tracial state. We will also denote by τ the trace $\tau \otimes \mathrm{Tr}$ on $\mathrm{M}_k(A) = A \otimes \mathrm{M}_k(\mathbb{C})$ (for every integer $k \geq 1$), where Tr is the standard trace on $\mathrm{M}_k(\mathbb{C})$. A trace τ is faithful if $\tau(a) > 0$ for any $a \in A_+ \setminus \{0\}$. Denote by $\mathrm{T}_{\mathtt{f}}(A)$ the convex subset of $\mathrm{T}(A)$ consisting of all faithful tracial states.

Denote by $\mathrm{M}_\infty(A)$ the set $\cup_{k=1}^{\infty} \mathrm{M}_k(A)$, where $\mathrm{M}_k(A)$ is regarded as a C*-subalgebra of $\mathrm{M}_{k+1}(A)$ by the embedding $a \mapsto \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$. For any projection $p \in \mathrm{M}_\infty(A)$, the restriction $\tau \mapsto \tau(p)$ defines a positive affine function on $\mathrm{T}(A)$. This induces a canonical positive homomorphism $\rho_A : K_0(A) \to \mathrm{Aff}(\mathrm{T}(A))$.

Denote by $\mathrm{U}(A)$ the unitary group of A, and denote by $\mathrm{U}(A)_0$ the connected component of $\mathrm{U}(A)$ containing the identity. Let C be another unital C*-algebra and let $\phi : C \to A$ be a unital *-homomorphism. Denote by $\phi_{\mathrm{T}} : \mathrm{T}(A) \to \mathrm{T}(C)$ the continuous affine map induced by ϕ, i.e.,

$$\phi_{\mathrm{T}}(\tau)(c) = \tau \circ \phi(c)$$

for all $c \in C$ and $\tau \in \mathrm{T}(A)$. Denote by $\phi_\sharp : \mathrm{Aff}(\mathrm{T}(C)) \to \mathrm{Aff}(\mathrm{T}(A))$ the map defined by

$$\phi_\sharp(f)(\tau) = f(\phi_{\mathrm{T}}(\tau)) \quad \text{for all } \tau \in \mathrm{T}(A).$$

Definition 2.2. Let A be a unital C*-algebra. Denote by $CU(A)$ the closure of the subgroup generated by commutators of $U(A)$. If $u \in U(A)$, its image in the quotient $U(A)/CU(A)$ will be denoted by $\overline{u}$. Let B be another unital C*-algebra and let $\phi : A \to B$ be a unital homomorphism. it is clear that ϕ maps $CU(A)$ into $CU(B)$. Let $\phi^\ddagger$ denote the induced homomorphism from $U(A)/CU(A)$ into $U(B)/CU(B)$.

Let $n \geq 1$ be any integer. Denote by $U_n(A)$ the unitary group of $M_n(A)$, and denote by $CU(A)_n$ the closure of commutator subgroup of $U_n(A)$. Regard $U_n(A)$ as a subgroup of $U_{n+1}(A)$ via the embedding $u \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}$ and denote by $U_\infty(A)$ the union of all $U_n(A)$.

Consider the union $CU_\infty(A) := \bigcup_n CU_n(A)$. It is then a normal subgroup of $U_\infty(A)$, and the quotient $U(A)_\infty/CU_\infty(A)$ is in fact isomorphic to the inductive limit of $U_n(A)/CU_n(A)$ (as abelian groups). We will use $\phi^\ddagger$ for the homomorphism induced by ϕ from $U_\infty(A)/CU_\infty(A)$ into $U_\infty(B)/CU_\infty(B)$.

Definition 2.3. Let A be a unital C*-algebra, and let $u \in U(A)_0$. Let $u(t) \in C([0,1], A)$ be a piecewise-smooth path of unitaries such that $u(0) = u$ and $u(1) = 1$. Then the de la Harpe–Skandalis determinant of $u(t)$ is defined by

$$\mathrm{Det}(u(t))(\tau) = \frac{1}{2\pi i} \int_0^1 \tau(\frac{du(t)}{dt} u(t)^*) dt \text{ for all } \tau \in \mathrm{T}(A),$$

which induces a homomorphism

$$\mathrm{Det} : U(A)_0 \to \mathrm{Aff}(\mathrm{T}(A))/\overline{\rho_A(K_0(A))}.$$

The determinant Det can be extended to a map from $U_\infty(A)_0$ into $\mathrm{Aff}(\mathrm{T}(A))/\overline{\rho_A(K_0(A))}$. It is easy to see that the determinant vanishes on the closure of commutator subgroup of $U_\infty(A)$. In fact, by a result of K. Thomsen ([31]), the closure of the commutator subgroup is exactly the kernel of this map, that is, it induces an isomorphism $\overline{\mathrm{Det}} : U_\infty(A)_0/CU_\infty(A) \to \mathrm{Aff}(\mathrm{T}(A))/\overline{\rho_A(K_0(A))}$. Moreover, by ([31]), one has the following short exact sequence

$$0 \to \mathrm{Aff}(\mathrm{T}(A))/\overline{\rho_A(K_0(A))} \to U_\infty(A)/CU_\infty(A) \overset{\Pi}{\to} K_1(A) \to 0 \tag{2.1}$$

which splits (with the embedding of $\mathrm{Aff}(\mathrm{T}(A))/\overline{\rho_A(K_0(A))}$ induced by $(\overline{\mathrm{Det}})^{-1}$). We will fix a splitting map $s_1 : K_1(A) \to U_\infty(A)/CU_\infty(A)$. The notation Π and s_1 will be used late without further warning.

For each $\bar{u} \in s_1(K_1(A))$, select and fix one element $u_c \in \bigcup_{n=1}^\infty M_n(A)$ such that $\overline{u_c} = \bar{u}$. Denote this set by $U_c(A)$.

In the case that A has tracial rank at most one (see 2.8 below), by Corollary 3.4 of [31], one has

$$U_\infty(A)_0/CU_\infty(A) = U(A)_0/CU(A)$$

and thus the following splitting short exact sequence:

$$0 \to \mathrm{Aff}(\mathrm{T}(A))/\overline{\rho_A(K_0(A))} \to U(A)/CU(A) \to K_1(A) \to 0. \tag{2.2}$$

Definition 2.4. Let A be a unital C*-algebra and let C be a separable C*-algebra which satisfies the Universal Coefficient Theorem. Recall that $KL(C, A)$ is the quotient of $KK(C, A)$ modulo pure extensions. By a result of Dădărlat and Loring in [1], one has

$$KL(C, A) = \mathrm{Hom}_\Lambda(\underline{K}(C), \underline{K}(A)), \tag{2.3}$$

where

$$\underline{K}(B) = (K_0(B) \oplus K_1(B)) \oplus (\bigoplus_{n=2}^{\infty}(K_0(B, \mathbb{Z}/n\mathbb{Z}) \oplus K_1(B, \mathbb{Z}/n\mathbb{Z})))$$

for any C*-algebra B. Then, in the rest of the paper, we will identify $KL(C, A)$ with $\mathrm{Hom}_\Lambda(\underline{K}(C), \underline{K}(A))$.

Denote by $\kappa_i : K_i(C) \to K_i(A)$ the homomorphism given by κ with $i = 0, 1$, and denote by $KL(C, A)^{++}$ the set of those $\kappa \in \mathrm{Hom}_\Lambda(\underline{K}(C), \underline{K}(A))$ such that

$$\kappa_0(K_0^+(C) \setminus \{0\}) \subseteq K_0^+(A) \setminus \{0\}.$$

Denote by $KL_e(C, A)^{++}$ the set of those elements $\kappa \in KL(C, A)^{++}$ such that $\kappa_0([1_C]) = [1_A]$. Suppose that both A and C are unital, $\mathrm{T}(C) \neq \emptyset$ and $\mathrm{T}(A) \neq \emptyset$. Let $\lambda_\mathrm{T} : \mathrm{T}(A) \to \mathrm{T}(C)$ be a continuous affine map. Let $h_0 : K_0(C) \to K_0(A)$ be a positive homomorphism. We say λ_T is compatible with h_0 if for any projection $p \in \mathrm{M}_\infty(C)$, $\lambda_\mathrm{T}(\tau)(p) = \tau(h_0([p]))$ for all $\tau \in \mathrm{T}(A)$. Let $\lambda : \mathrm{Aff}(\mathrm{T_f}(C)) \to \mathrm{Aff}(\mathrm{T}(A))$ be an affine continuous map. We say λ and h_0 are compatible if h_0 is compatible to λ_T, where $\lambda_\mathrm{T} : \mathrm{T}(A) \to \mathrm{T_f}(C)$ is the map $\lambda_\mathrm{T}(\tau)(a) = \lambda(a^*)(\tau), \forall a \in C^+$ and $\tau \in \mathrm{T}(A)$, where $a^* \in \mathrm{Aff}(\mathrm{T_f}(C))$ is the affine function induced by a. We say κ and λ (or λ_T) are compatible, if κ_0 is positive and κ_0 and λ are compatible.

Denote by $KLT_e(C, A)^{++}$ the set of those pairs $(\kappa, \lambda_\mathrm{T})$ (or, (κ, λ)), where $\kappa \in KL_e(C, A)^{++}$ and $\lambda_\mathrm{T} : \mathrm{T}(A) \to \mathrm{T}_\mathfrak{f}(C)$ (or, $\lambda : \mathrm{Aff}(\mathrm{T_f}(C)) \to \mathrm{Aff}(\mathrm{T}(A))$) is a continuous affine map which is compatible with κ. If λ is compatible with κ, then λ maps $\rho_C(K_0(C))$ into $\rho_A(K_0(A))$. Therefore

λ induces a continuous homomorphism $\overline{\lambda}:\mathrm{Aff}(\mathrm{T_f}(C))/\overline{\rho_C(K_0(C))}\to\mathrm{Aff}(\mathrm{T}(A))/\overline{\rho_A(K_0(A))}$. Suppose that $\gamma: U_\infty(C)/CU_\infty(C)\to U_\infty(A)/CU_\infty(A)$ is a continuous homomorphism and $h_i: K_i(C)\to K_i(A)$ are homomorphisms for which h_0 is positive. We say that γ and h_1 are compatible if $\gamma(U_\infty(C)_0/CU_\infty(C))\subset U_\infty(A)_0/CU_\infty(A)$ and $\gamma\circ s_1=s_1\circ h_1$, we say that h_0,h_1,λ and γ are compatible, if λ and h_0 are compatible, γ and h_1 are compatible and

$$\overline{\mathrm{Det}}_A\circ\gamma|_{U_\infty(C)_0/CU_\infty(C)}=\overline{\lambda}\circ\overline{\mathrm{Det}}_C,$$

and we also say that κ,λ and γ are compatible, if $\kappa_0,\kappa_1,\lambda$ and γ are compatible.

2.5. For each prime number p, let ϵ_p be a number in $\{0,1,2,...,+\infty\}$. Then a supernatural number is the formal product $\mathfrak{p}=\prod_p p^{\epsilon_p}$. Here we insist that there are either infinitely many p in the product, or, one of ϵ_p is infinite. Two supernatural numbers $\mathfrak{p}=\prod_p p^{\epsilon_p(\mathfrak{p})}$ and $\mathfrak{q}=\prod_p p^{\epsilon_p(\mathfrak{q})}$ are relatively prime if for any prime number p, at most one of $\epsilon_p(\mathfrak{p})$ and $\epsilon_p(\mathfrak{q})$ is nonzero. A supernatural number $\mathfrak{p}$ is called of infinite type if for any prime number, either $\epsilon_p(\mathfrak{p})=0$ or $\epsilon_p(\mathfrak{p})=+\infty$. For each supernatural number $\mathfrak{p}$, there is a UHF-algebra $M_\mathfrak{p}$ associated to it, and the UHF-algebra is unique up to isomorphism (see [2]).

2.6. Denote by Q the UHF-algebra with $(K_0(Q),K_0(Q)_+,[1_A])=(\mathbb{Q},\mathbb{Q}_+,1)$ (the supernatural number associated to Q is $\prod_p p^{+\infty}$), and let $M_\mathfrak{p}$ and $M_\mathfrak{q}$ be two UHF-algebras with $M_\mathfrak{p}\otimes M_\mathfrak{q}\cong Q$ and $\mathfrak{p}=\prod_p p^{\epsilon_p(\mathfrak{p})}$ and $\mathfrak{q}=\prod_p p^{\epsilon_p(\mathfrak{q})}$ relatively prime. Then it follows that $\mathfrak{p}$ and $\mathfrak{q}$ are of infinite type. Denote by

$$\begin{aligned}\mathbb{Q}_\mathfrak{p} &= \mathbb{Z}[\frac{1}{p_1},...,\frac{1}{p_n},...]\subseteq\mathbb{Q},\quad \text{where } \epsilon_{p_n}(\mathfrak{p})=+\infty \text{ and}\\ \mathbb{Q}_\mathfrak{q} &= \mathbb{Z}[\frac{1}{p_1},...,\frac{1}{p_n},...]\subseteq\mathbb{Q},\quad \text{where } \epsilon_{p_n}(\mathfrak{q})=+\infty.\end{aligned}$$

Note that $(K_0(M_\mathfrak{p}),K_0(M_\mathfrak{p})_+,[1_{M_\mathfrak{p}}])=(\mathbb{Q}_\mathfrak{p},(\mathbb{Q}_p)_+,1)$ and $(K_0(M_\mathfrak{q}),K_0(N_\mathfrak{q})_+,[1_{M_\mathfrak{q}}])=(\mathbb{Q}_\mathfrak{q},(\mathbb{Q}_\mathfrak{q})_+,1)$. Moreover, $\mathbb{Q}_\mathfrak{p}\cap\mathbb{Q}_\mathfrak{q}=\mathbb{Z}$ and $\mathbb{Q}=\mathbb{Q}_\mathfrak{p}+\mathbb{Q}_\mathfrak{q}$.

2.7. For any pair of relatively prime supernatural numbers $\mathfrak{p}$ and $\mathfrak{q}$, define the C*-algebra $\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ by

$$\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}=\{f:[0,1]\to M_\mathfrak{p}\otimes M_\mathfrak{q};\ f(0)\in M_\mathfrak{p}\otimes 1_{M_\mathfrak{q}} \text{ and } f(1)\in 1_{M_\mathfrak{p}}\otimes M_\mathfrak{q}\}.$$

The Jiang-Su algebra $\mathcal{Z}$ is the unital inductive limit of dimension drop interval algebras with unique trace, and $(K_0(\mathcal{Z}),K_0(\mathcal{Z}),[1])=(\mathbb{Z},\mathbb{Z}^+,1)$ (see [8]). By Theorem 3.4 of [29], for any pair of relatively prime supernatural numbers $\mathfrak{p}$ and $\mathfrak{q}$ of infinite type, the Jiang-Su algebra $\mathcal{Z}$ has a stationary inductive limit decomposition:

$$\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}\longrightarrow\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}\longrightarrow\cdots\longrightarrow\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}\longrightarrow\cdots\longrightarrow\mathcal{Z}\ .$$

By Corollary 3.2 of [29], the C*-algebra $\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ absorbs the Jiang-Su algebra: $\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}\otimes\mathcal{Z}\cong\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$. A C*-algebra A is said to be *$\mathcal{Z}$-stable* if $A\otimes\mathcal{Z}\cong A$.

Definition 2.8. A unital simple C*-algebra A has tracial rank at most one, denoted by $\mathrm{TR}(A)\le 1$, if for any finite subset $\mathcal{F}\subset A$, any $\epsilon>0$, and any nonzero $a\in A^+$, there exist a nonzero projection $p\in A$ and a C*-subalgebra $I\cong\bigoplus_{i=1}^m C(X_i)\otimes M_{r(i)}$ with $1_I=p$ for some finite CW complexes X_i with dimension at most one such that

(1) $\|[x,\ p]\| \le \epsilon$ for any $x \in \mathcal{F}$,
(2) for any $x \in \mathcal{F}$, there is $x' \in I$ such that $\|pxp - x'\| \le \epsilon$, and
(3) $1-p$ is Murray-von Neumann equivalent to a projection in $\overline{aAa}$.

Moreover, if the C*-subalgebra I above can be chosen to be a finite dimensional C*-algebra, then A is said to have tracial rank zero, and in such case, we write $\mathrm{TR}(A) = 0$. It is a theorem of Guihua Gong [5] that every unital simple AH-algebra with no dimension growth has tracial rank at most one. It has been proved in [18] that every $\mathcal{Z}$-stable unital simple AH-algebra has tracial rank at most one.

Definition 2.9. Denote by $\mathcal{N}$ the class of all separable amenable C*-algebras which satisfy the Universal Coefficient Theorem (UCT). Denote by $\mathcal{C}$ the class of all simple C*-algebras A for which $\mathrm{TR}(A \otimes M_{\mathfrak{p}}) \le 1$ for some UHF-algebra $M_{\mathfrak{p}}$, where $\mathfrak{p}$ is a supernatural number of infinite type. Note, by [24], that, if $\mathrm{TR}(A \otimes M_{\mathfrak{p}}) \le 1$ for some supernatural number $\mathfrak{p}$ then $\mathrm{TR}(A \otimes M_{\mathfrak{p}}) \le 1$ for all supernatural number $\mathfrak{p}$.

Denote by $\mathcal{C}_0$ the class of all simple C*-algebras A for which $\mathrm{TR}(A \otimes M_{\mathfrak{p}}) = 0$ for some supernatural number $\mathfrak{p}$ of infinite type (and hence for all supernatural number $\mathfrak{p}$ of infinite type).

Theorem 2.10 (Theorem 5.10 [21])**.** *Let C be a unital AH-algebra and let A be a unital simple C*-algebra with $\mathrm{TR}(A) \le 1$. Suppose that $\phi, \psi : C \to A$ are two unital monomorphisms. Then ϕ and ψ are approximately unitarily equivalent if and only if*

$$[\phi] = [\psi] \text{ in } KL(C, A),$$
$$\phi_\sharp = \psi_\sharp \quad \textit{and} \quad \phi^\ddagger = \psi^\ddagger.$$

Remark 2.11. One of the main purposes of this paper is to generalize this result so that A can be allowed to be in the class $\mathcal{C}$, and C can be rationally AH; that is, $C \otimes U$ is an AH-algebra for all UHF-algebra U of infinite type.

2.12. Let A and B be two unital C*-algebras. Let $h : A \to B$ be a homomorphism and $v \in \mathrm{U}(B)$ be such that

$$[h(g),\ v] = 0 \quad \text{for any } g \in A.$$

We then have a homomorphism $\bar{h} : A \otimes \mathrm{C}(\mathbb{T}) \to B$ defined by $f \otimes g \mapsto h(f)g(v)$ for any $f \in A$ and $g \in \mathrm{C}(\mathbb{T})$. The tensor product induces two injective homomorphisms:

$$\beta^{(0)} : K_0(A) \to K_1(A \otimes \mathrm{C}(\mathbb{T})) \text{ and } \beta^{(1)} : K_1(A) \to K_0(A \otimes \mathrm{C}(\mathbb{T})).$$

The second one is the usual Bott map. Note that, in this way, one writes

$$K_i(A \otimes \mathrm{C}(\mathbb{T})) = K_i(A) \oplus \beta^{(i-1)}(K_{i-1}(A)).$$

Let us use $\widehat{\beta^{(i)}} : K_i(A \otimes \mathrm{C}(\mathbb{T})) \to \beta^{(i-1)}(K_{i-1}(A))$ to denote the quotient map.

For each integer $k \ge 2$, one also has the following injective homomorphisms:

$$\beta_k^{(i)} : K_i(A, \mathbb{Z}/k\mathbb{Z}) \to K_{i-1}(A \otimes \mathrm{C}(\mathbb{T}), \mathbb{Z}/k\mathbb{Z}), \quad i = 0, 1.$$

Thus, we write

$$K_i(A \otimes \mathrm{C}(\mathbb{T}), \mathbb{Z}/k\mathbb{Z}) = K_i(A, \mathbb{Z}/k\mathbb{Z}) \oplus \beta^{(i-1)}(K_{i-1}(A), \mathbb{Z}/k\mathbb{Z}).$$

Denote by $\widehat{\beta_k^{(i)}} : K_i(A \otimes \mathrm{C}(\mathbb{T}), \mathbb{Z}/k\mathbb{Z}) \to \beta^{(i-1)}(K_{i-1}(A), \mathbb{Z}/k\mathbb{Z})$ the map analogous to $\widehat{\beta^{(i)}}$. If $x \in \underline{K}(A)$, we use $\boldsymbol{\beta}(x)$ for $\beta^{(i)}(x)$ if $x \in K_i(A)$ and for $\beta_k^{(i)}(x)$ if $x \in K_i(A, \mathbb{Z}/k\mathbb{Z})$. Thus we have a map $\boldsymbol{\beta} : \underline{K}(A) \to \underline{K}(A \otimes \mathrm{C}(\mathbb{T}))$ as well as $\widehat{\boldsymbol{\beta}} : \underline{K}(A \otimes \mathrm{C}(\mathbb{T})) \to \boldsymbol{\beta}(\underline{K})$. Therefore, we may write $\underline{K}(A \otimes \mathrm{C}(\mathbb{T})) = \underline{K}(A) \oplus \boldsymbol{\beta}(\underline{K}(A))$. On the other hand, $\overline{h}$ induces homomorphisms

$$\overline{h}_{*i,k} : K_i(A \otimes \mathrm{C}(\mathbb{T}), \mathbb{Z}/k\mathbb{Z}) \to K_i(B, \mathbb{Z}/k\mathbb{Z}),$$

$k = 0, 2, ...,$ and $i = 0, 1$.

We use $\mathrm{Bott}(h, v)$ for all homomorphisms $\overline{h}_{*i,k} \circ \beta_k^{(i)}$, and we use $\mathrm{bott}_1(h, v)$ for the homomorphism $\overline{h}_{1,0} \circ \beta^{(1)} : K_1(A) \to K_0(B)$, and $\mathrm{bott}_0(h, v)$ for the homomorphism $\overline{h}_{0,0} \circ \beta^{(0)} : K_0(A) \to K_1(B)$. $\mathrm{Bott}(h, v)$ as well as $\mathrm{bott}_i(h, v)$ $(i = 0, 1)$ may be defined for a unitary v which only approximately commutes with h. In fact, given a finite subset $\mathcal{P} \subset \underline{K}(A)$, there exists a finite subset $\mathcal{F} \subset A$ and $\delta_0 > 0$ such that

$$\mathrm{Bott}(h, v)|_{\mathcal{P}}$$

is well defined if

$$\|[h(a), v]\| < \delta_0$$

for all $a \in \mathcal{F}$. See 2.11 of [14], 2.10 of [13], 2.21 of [22] for more details.

We have the following generalized Exel's formula for the traces of Bott elements.

Theorem 2.13 (Theorem 3.5 of [18])**.** *There is $\delta > 0$ satisfying the following: Let A be a unital separable simple C*-algebra with $\mathrm{TR}(A) \le 1$ and let $u, v \in U(A)$ be two unitaries such that $\|uv - vu\| < \delta$. Then $\mathrm{bott}_1(u, v)$ is well defined and*

$$\tau(\mathrm{bott}_1(u, v)) = \frac{1}{2\pi i}(\tau(\log(vuv^*u^*)))$$

for all $\tau \in \mathrm{T}(A)$.

3. Rotation maps

In this section, we collect several facts on the rotation map which are going to be used frequently in the rest of the paper. Most of them can be found in the literature.

Definition 3.1. Let A and B be two unital C*-algebras, and let ψ and ϕ be two unital monomorphisms from B to A. Then the mapping torus $M_{\phi,\psi}$ is the C*-algebra defined by

$$M_{\phi,\psi} := \{f \in \mathrm{C}([0, 1], A);\ f(0) = \phi(b) \text{ and } f(1) = \psi(b) \text{ for some } b \in B\}.$$

For any $\psi, \phi \in \mathrm{Hom}(B, A)$, denoting by π_0 the evaluation of $M_{\phi,\psi}$ at 0, we have the short exact sequence

$$0 \longrightarrow \mathrm{S}(A) \xrightarrow{\imath} M_{\phi,\psi} \xrightarrow{\pi_0} B \longrightarrow 0,$$

where $\mathrm{S}(A) = C_0((0, 1), A)$. If $\phi_{*i} = \psi_{*i}$ $(i = 0, 1)$, then the corresponding six-term exact sequence breaks down to the following two extensions:

$$\eta_i(M_{\phi,\psi}) : \quad 0 \longrightarrow K_{i+1}(A) \longrightarrow K_i(M_{\phi,\psi}) \longrightarrow K_i(B) \longrightarrow 0, \quad (i = 0, 1).$$

3.2. Suppose that, in addition,

$$\tau \circ \phi = \tau \circ \psi \text{ for all } \tau \in \mathrm{T}(A). \tag{3.1}$$

For any continuous piecewise smooth path of unitaries $u(t) \in M_{\phi,\psi}$, consider the path of unitaries $w(t) = u^*(0)u(t)$ in A. Then it is a continuous and piecewise smooth path with $w(0) = 1$ and $w(1) = u^*(0)u(1)$. Denote by $R_{\phi,\psi}(u) = \mathrm{Det}(w)$ the determinant of $w(t)$. It is clear with the assumption of (3.1) that $R_{\phi,\psi}(u)$ depends only on the homotopy class of $u(t)$. Therefore, it induces a homomorphism, denoted by $R_{\phi,\psi}$, from $K_1(M_{\phi,\psi})$ to $\mathrm{Aff}(\mathrm{T}(A))$.

Definition 3.3. Fix two unital C*-algebras A and B with $\mathrm{T}(A) \neq \emptyset$. Define $\mathcal{R}_0$ to be the subset of $\mathrm{Hom}(K_1(B), \mathrm{Aff}(\mathrm{T}(A)))$ consisting of those homomorphisms $h \in \mathrm{Hom}(K_1(B), \mathrm{Aff}(\mathrm{T}(A)))$ for which there exists a homomorphism $d : K_1(B) \to K_0(A)$ such that

$$h = \rho_A \circ d.$$

It is clear that $\mathcal{R}_0$ is a subgroup of $\mathrm{Hom}(K_1(B), \mathrm{Aff}(\mathrm{T}(A)))$.

3.4. If $[\phi] = [\psi]$ in $KK(B, A)$, then the exact sequences $\eta_i(M_{\phi,\psi})$ $(i = 0, 1)$ above split. In particular, there is a lifting $\theta : K_1(B) \to K_1(M_{\phi,\psi})$. Consider the map

$$R_{\phi,\psi} \circ \theta : K_1(B) \to \mathrm{Aff}(\mathrm{T}(A)).$$

If a different lifting θ' is chosen, then, $\theta - \theta'$ maps $K_1(B)$ into $K_0(A)$. Therefore

$$R_{\phi,\psi} \circ \theta - R_{\phi,\psi} \circ \theta' \in \mathcal{R}_0.$$

Then define

$$\overline{R}_{\phi,\psi} = [R_{\phi,\psi} \circ \theta] \in \mathrm{Hom}(K_1(B), \mathrm{Aff}(\mathrm{T}(A)))/\mathcal{R}_0.$$

If $[\phi] = [\psi]$ in $KL(B, A)$, then the exact sequences $\eta_i(M_{\phi,\psi})$ $(i = 0, 1)$ are pure, i.e., any finitely generated subgroup in the quotient groups has a lifting. In particular, for any finitely generated subgroup $G \subseteq K_1(B)$, one has a map

$$R_{\phi,\psi} \circ \theta_G : G \to \mathrm{Aff}(\mathrm{T}(A)),$$

where $\theta_G : G \to K_1(M_{\phi,\psi})$ is a lifting. Let $G \subset K_1(B)$ be a finitely generated subgroup. Denote by $\mathcal{R}_{0,G}$ the set of those elements h in $\mathrm{Hom}(G, \mathrm{Aff}(T(A)))$ such that there exists a homomorphism $d_G : G \to K_0(A)$ such that $h|_G = \rho_A \circ d_G$.

If $[\phi] = [\psi]$ in $KL(B, A)$ and $R_{\phi,\psi}(K_1(M_{\phi,\psi})) \subset \rho_A(K_0(A))$, then $\theta_G \in \mathcal{R}_{0,G}$ for any finitely generated subgroup $G \subset K_1(B)$ and any lifting θ_G. In this case, we will also write

$$\overline{R}_{\phi,\psi} = 0.$$

See 3.4 of [18] for more details.

Lemma 3.5 (Lemma 9.2 of [18])**.** *Let C and A be unital C*-algebras with $\mathrm{T}(A) \neq \emptyset$. Suppose that ϕ, $\psi : C \to A$ are two unital homomorphisms such that*

$$[\phi] = [\psi] \text{ in } KL(C, A),\ \ \phi_\sharp = \psi_\sharp \ \text{ and } \phi^\ddagger = \psi^\ddagger.$$

Then the image of $R_{\phi,\psi}$ is in the $\overline{\rho_A(K_0(A))} \subseteq \mathrm{Aff}(\mathrm{T}(A))$.

Proof. Let $z \in K_1(C)$. Suppose that $u \in U_n(C)$ for some integer $n \geq 1$ such that $[u] = z$. Note that $\psi(u)^*\phi(u) \in CU_n(A)$. Thus, by 2.3, for any continuous and piecewise smooth path of unitaries $\{w(t) : t \in [0,1]\} \subset U(A)$ with $w(0) = \psi(u)^*\phi(u)$ and $w(1) = 1$,

$$\mathrm{Det}(w)(\tau) = \int_0^1 \tau(\frac{dw(t)}{dt}w(t)^*)dt \in \overline{\rho_A(K_0(A))}. \tag{3.2}$$

Suppose that $\{(v)(t) : t \in [0,1]\}$ is a continuous and piecewise smooth path of unitaries in $U_n(A)$ with $v(0) = \phi(u)$ and $v(1) = \psi(u)$. Define $w(t) = \psi(u)^*v(t)$. Then $w(0) = \psi^*(u)\phi(u)$ and $w(1) = 1$. Thus, by (3.2),

$$R_{\phi,\psi}(z)(\tau) = \int_0^1 \tau(\frac{dv(t)}{dt}v(t)^*)dt \tag{3.3}$$

$$= \int_0^1 \tau(\psi(u)^*\frac{dv(t)}{dt}v(t)^*\psi(u))dt \tag{3.4}$$

$$= \int_0^1 \tau(\frac{dw(t)}{dt}w(t)^*)dt \in \overline{\rho_A(K_0(A))}. \tag{3.5}$$

□

3.6. Let A be a unital C*-algebra and let u and v be two unitaries with $\|u^*v - 1\| < 2$. Then $h = \frac{1}{2\pi i}\log(u^*v)$ is a well-defined self-adjoint element of A, and $w(t) := u\exp(2\pi iht)$ is a smooth path of unitaries connecting u and v. It is a straightforward calculation that for any $\tau \in \mathrm{T}(A)$,

$$\mathrm{Det}(w(t))(\tau) = \frac{1}{2\pi i}\tau(\log(u^*v)).$$

3.7. Let A be a unital C*-algebra, and let u and w be two unitaries. Suppose that $w \in U_0(A)$. Then $w = \prod_{k=0}^m \exp(2\pi ih_k)$ for some self-adjoint elements $h_0, ..., h_m$. Define the path

$$w(t) = (\prod_{k=0}^{l-1}\exp(2\pi ih_k))\exp(2\pi ih_l mt), \quad \text{if } t \in [(l-1)/m, l/m],$$

and define $u(t) = w^*(t)uw(t)$ for $t \in [0,1]$. Then, $u(t)$ is continuous and piecewise smooth, and $u(0) = u$ and $u(1) = w^*uw$. A straightforward calculation shows that $\mathrm{Det}(u(t)) = 0$.

In general, if w is not in the path-connected component containing the identity, one can consider unitaries $\mathrm{diag}(u, 1)$ and $\mathrm{diag}(w, w^*)$. Then, the same argument as above shows that there is a piecewise smooth path $u(t)$ of unitaries in $M_2(A)$ such that $u(0) = \mathrm{diag}(u, 1)$, $u(1) = \mathrm{diag}(w^*uw, 1)$, and

$$\mathrm{Det}(u(t)) = 0.$$

Lemma 3.8 (Lemma 3.5 of [14])**.** *Let B and C be two unital C^*-algebras with $\mathrm{T}(B) \neq \emptyset$. Suppose that $\phi, \psi : C \to B$ are two unital monomorphisms such that $[\phi] = [\psi]$ in $KL(C, B)$ and*

$$\tau \circ \phi = \tau \circ \psi$$

for all $\tau \in \mathrm{T}(B)$. Suppose that $u \in U_l(C)$ is a unitary and $w \in U_l(B)$ such that

$$\|(\phi \otimes \mathrm{id}_{M_l})(u)w^*(\psi \otimes \mathrm{id}_{M_l})(u^*)w - 1\| < 2.$$

Then, for any unitary $U \in U_l(M_{\phi,\psi})$ with $U(0) = (\phi \otimes \mathrm{id}_{M_l})(u)$ and $U(1) = (\psi \otimes \mathrm{id}_{M_l})(u)$, one has that

$$\frac{1}{2\pi i}\tau(\log((\phi \otimes \mathrm{id}_{M_l})(u^*)w^*(\psi \otimes \mathrm{id}_{M_l})(u)w)) - R_{\phi,\psi}([U])(\tau) \in \rho_B(K_0(B)). \tag{3.6}$$

Proof. Without loss of generality, one may assume that $u \in C$. Moreover, to prove the lemma, it is enough to show that (3.6) holds for one path of unitaries $U(t)$ in $M_2(B)$ with $U(0) = \mathrm{diag}(\phi(u), 1)$ and $U(1) = \mathrm{diag}(\psi(u), 1)$.

Let U_1 be the path of unitaries specified in 3.6 with $U_1(0) = \mathrm{diag}(\phi(u), 1)$ and $U_1(1/2) = \mathrm{diag}(w^*\psi(u)w, 1)$, and let U_2 be the path specified in 3.7 with $U_2(1/2) = \mathrm{diag}(w^*\psi(u)w, 1)$ and $U_2(1) = \mathrm{diag}(\psi(u), 1)$.

Set U the path of unitaries by connecting U_1 and U_2. Then $U(0) = \mathrm{diag}(\phi(u), 1)$ and $U(1) = \mathrm{diag}(\psi(u), 1)$. By applying 3.6 and 3.7, for any $\tau \in \mathrm{T}(B)$, one computes that

$$R_{\phi,\psi}([U]) = \mathrm{Det}(U(t))(\tau) = \mathrm{Det}(U_1(t))(\tau) + \mathrm{Det}(U_2(t))(\tau) = \frac{1}{2\pi i}\tau(\phi(u^*)w^*\psi(u)w),$$

as desired. □

4. Homotopy lemma

In this section, we collect several results from [25] on the homotopy lemma.

Definition 4.1. Let A be a unital C*-algebra. In the following, for any invertible element $x \in A$, let $\langle x \rangle$ denote the unitary $x(x^*x)^{-\frac{1}{2}}$, and let $\overline{x}$ denote the element $\overline{\langle x \rangle}$ in $U(A)/CU(A)$. Consider a subgroup $\mathbb{Z}^k \subseteq K_1(A)$, and write the unitary $\{u_1, ..., u_k\} \subseteq U_c(A)$ the unitary corresponding to the standard generators $\{e_1, e_2, ..., e_k\}$ of $\mathbb{Z}^k$. Suppose that $\{u_1, u_2, ..., u_k\} \subset M_n(A)$ for some integer $n \geq 1$. Let $\Phi : A \to B$ be a unital positive linear map and $\Phi \otimes \mathrm{id}_{M_n}$ is at least $\{u_1, ..., u_k\}$-1/4-multiplicative (hence each $\Phi \otimes \mathrm{id}_{M_n}(u_i)$ is invertible), then the map $\Phi^{\ddagger}|_{s_1(\mathbb{Z}^k)} : \mathbb{Z}^k \to U(B)/CU(B)$ is defined by

$$\Phi^{\ddagger}|_{s_1(\mathbb{Z}^k)}(e_i) = \overline{\langle \Phi \otimes \mathrm{id}_{M_n}(u_i) \rangle}, \quad 1 \leq i \leq k.$$

Thus, for any finitely generated subgroup $G \subset \overline{U_c(A)}$, there exists $\delta > 0$ and a finite subset $\mathcal{G} \subset A$ such that, for any unital δ-$\mathcal{G}$-multiplicative completely positive linear map $L : A \to B$ (for any unital C*-algebra B), the map $L^{\ddagger}$ is well defined on $s_1(G)$. (Please see 2.1 for $U_c(A)$ and s_1.)

The following theorems are taken from [25].

Theorem 4.2 (3.10 of [25])**.** *Let $C = PM_n(\mathrm{C}(X))P$, where X is a compact subset of a finite CW-complex and P a projection in $M_n(\mathrm{C}(X))$ with an integer $n \geq 1$. Let $\Delta : (0,1) \to (0,1)$ be a non-decreasing map. For any $\epsilon > 0$ and any finite subset $\mathcal{F} \subseteq C$, there exists $\delta > 0$, $\eta > 0$, $\gamma > 0$, a finite subsets $\mathcal{G} \subseteq C$, $\mathcal{P} \subseteq \underline{K}(C)$, a finite subset $\mathcal{Q} = \{x_1, x_2, ..., x_k\} \subset K_0(C)$ which generates a free subgroup and $x_i = [p_i] - [q_i]$, where $p_i, q_i \in M_m(C)$ (for some integer $m \geq 1$) are projections, satisfying the following:*

Suppose that A is a unital simple C^*-algebra with $TR(A) \leq 1$, $\phi : C \to A$ is a unital homomorphism and $u \in A$ is a unitary, and suppose that

$$\|[\phi(c), u]\| < \delta, \ \forall c \in \mathcal{G} \quad \text{and} \quad \mathrm{Bott}(\phi, u)|_{\mathcal{P}} = 0,$$

and

$$\mu_{\tau\circ\phi}(O_a) \geq \Delta(a) \ \forall \tau \in T(A \otimes D),$$

where O_a is any open ball in X with radius $\eta \leq a < 1$ and $\mu_{\tau\circ\phi}$ is the Borel probability measure defined by $\tau \circ \phi$. Moreover, for each $1 \leq i \leq k$, there is $v_i \in CU(M_m(A))$ such that

$$\|\langle(1_m - \phi(p_i) + \phi(p_i)u)(1_m - \phi(q_i) + \phi(q_i)u^*\rangle - v_i\| < \gamma.$$

Then there is a continuous path of unitaries $\{u(t) : t \in [0, 1]\}$ in A such that

$$u(0) = u, u(1) = 1, \ \textit{and} \ \|[\phi(c), u(t)]\| < \epsilon$$

for any $c \in \mathcal{F}$ and for any $t \in [0, 1]$.

Theorem 4.3 (3.14 of [25])**.** *Let $C = PM_n(\mathrm{C}(X))P$, where X is a compact subset of a finite CW-complex and P a projection in $M_n(\mathrm{C}(X))$ for some integer $n \geq 1$. Let $G \subset K_0(C)$ be a finitely generated subgroup. Write $G = \mathbb{Z}^k \oplus \mathrm{Tor}(G)$ with $\mathbb{Z}^k$ generated by*

$$\{x_1 = [p_1] - [q_1], x_2 = [p_2] - [q_2], ..., x_k = [p_k] - [q_k]\},$$

where $p_i, q_i \in M_m(C)$ (for some integer $m \geq 1$) are projections, $i = 1, ..., k$.

Let A be a simple C^*-algebra with $TR(A) \leq 1$. Suppose that $\phi : C \to A$ is a monomorphism. Then, for any finite subsets $\mathcal{F} \subseteq C$ and $\mathcal{P} \subseteq \underline{K}(C)$, any $\epsilon > 0$ and $\gamma > 0$, any homomorphism

$$\Gamma : \mathbb{Z}^k \to U_0(A)/CU(A),$$

there is a unitary $w \in A$ such that

$$\|[\phi(f), w]\| < \epsilon \quad \forall f \in \mathcal{F}$$

$$\mathrm{Bott}(\phi, w)|_{\mathcal{P}} = 0,$$

and

$$\mathrm{dist}(\overline{\langle(1_m - \phi(p_i) + \phi(p_i)w)(1_m - \phi(q_i) + \phi(q_i)w^*\rangle}, \Gamma(x_i)) < \gamma, \quad \forall 1 \leq i \leq k,$$

where $U_0(A)/CU(A)$ is identified as $U_0(M_m(A))/CU(M_m(A))$, and the distance above is understood as the distance in $U_0(M_m(A))/CU(M_m(A))$.

Theorem 4.4 (3.16 of [25])**.** *Let C be an AH-algebra, and let A be a simple C^*-algebra with $TR(A) \leq 1$. Suppose that $h : C \to A$ is a monomorphism. Then, for any $\epsilon > 0$, any finite subset $\mathcal{F} \subseteq C$ and any finite subset $\mathcal{P} \subseteq \underline{K}(C)$, there is a C^*-algebra $C' \cong PM_n(\mathrm{C}(X'))P$ for some finite CW-complex X' with $K_1(C') = \mathbb{Z}^k \oplus \mathrm{Tor}(K_1(C'))$ and a homomorphism $\iota : C' \to C$ with $\mathcal{P} \subseteq [\iota](\underline{K}(C'))$, a finite subset $\mathcal{Q} \subseteq \mathbb{Z}^k \subset K_1(C')$ and $\delta > 0$ satisfying the following: Suppose that $\kappa \in \mathrm{Hom}_\Lambda(\underline{K}(C' \otimes \mathrm{C}(\mathbb{T})), \underline{K}(A))$ with*

$$|\rho_A \circ \kappa(\boldsymbol{\beta}(x))(\tau)| < \delta, \quad \forall x \in \mathcal{Q}, \ \forall \tau \in \mathrm{T}(A).$$

Then there exists a unitary $u \in A$ such that

$$\|[h(c), u]\| < \epsilon \quad \forall c \in \mathcal{F} \quad \textit{and} \quad \mathrm{Bott}(h \circ \iota, u) = \kappa \circ \boldsymbol{\beta}.$$

Moreover, there is a sequence of C^*-algebras C_n with the form $C_n = P_nM_{r(n)}(C(X_n))P_n$, where each X_n is a finite CW-complex and $P_n \in M_{r(n)}(C(X_n))$ a projection, such that $C = \varinjlim(C_n, \phi_n)$ for a sequence of unital homomorphisms $\phi_n : C_n \to C_{n+1}$ and one may choose $C' = C_n$ and $\iota = \phi_n$ for some integer $n \geq 1$.

5. Approximately unitary equivalence

First we begin with the following lemma which is a simple combination of the uniqueness theorem 2.10 and the proof of Theorem 4.2 in [23]. In what follows, if $\mathcal{G}$ is a subset of a group, we will use $G(\mathcal{P})$ for the subgroup generated by $\mathcal{G}$.

Lemma 5.1. *Let A be a simple C^*-algebra with $\mathrm{TR}(A) \leq 1$, and let C be a unital AH-algebra. If there are monomorphisms $\phi, \psi : C \to A$ such that*

$$[\phi] = [\psi] \text{ in } KL(C, A), \ \phi_\sharp = \psi_\sharp, \text{ and } \phi^\ddagger = \psi^\ddagger,$$

then, for any $2 > \epsilon > 0$, any finite subset $\mathcal{F} \subseteq C$, any finite subset of unitaries $\mathcal{P} \subset U_n(C)$ for some $n \geq 1$, there exist a finite subset $\mathcal{G} \subset K_1(C)$ with $\overline{\mathcal{P}} \subseteq \mathcal{G}$ (where $\overline{\mathcal{P}}$ is the image of $\mathcal{P}$ in $K_1(C)$) and $\delta > 0$ such that, for any map $\eta : G(\mathcal{G}) \to \mathrm{Aff}(\mathrm{T}(A))$ with $|\eta(x)(\tau)| < \delta$ for all $\tau \in \mathrm{T}(A)$ and $\eta(x) - \overline{R}_{\phi,\psi}(x) \in \rho_A(K_0(A))$ for all $x \in \mathcal{G}$, there is a unitary $u \in A$ such that

$$\|\phi(f) - u^*\psi(f)u\| < \epsilon \quad \forall f \in \mathcal{F},$$

and $\tau(\frac{1}{2\pi i}\log((\phi \otimes \mathrm{id}_{M_n}(x^*))(u \otimes 1_{M_n})^*(\psi \otimes \mathrm{id}_{M_n}(x))(u \otimes 1_{M_n}))) = \tau(\eta([x]))$ for all $x \in \mathcal{P}$ and for all $\tau \in \mathrm{T}(A)$.

Proof. Without loss of generality, one may assume that any element in $\mathcal{F}$ has norm at most one. Let $\epsilon > 0$. Choose $\epsilon > \theta > 0$ and a finite subset $\mathcal{F} \subset \mathcal{F}_0 \subset C$ satisfying the following: For all $x \in \mathcal{P}$, $\tau(\frac{1}{2\pi i}\log(\phi(x^*)w_j^*\psi(x)w_j))$ is well defined and

$$\tau(\frac{1}{2\pi i}\log(\phi(x^*)w_j^*\psi(x)w_j)) \tag{5.1}$$

$$= \tau(\frac{1}{2\pi i}\log(\phi(x^*)v_1^*\psi(x)v_1) + \cdots + \tau(\frac{1}{2\pi i}\log(\phi(x^*)v_j^*\psi(x)v_j)) \text{ for all } \tau \in T(A), \tag{5.2}$$

whenever

$$\|\phi(f) - v_j^*\psi(f)v_j\| < \theta \text{ for all } f \in \mathcal{F}_0,$$

where v_j are unitaries in A and $w_j = v_1 \cdots v_j$, $j = 1, 2, 3$. In the above, if $x \in U_n(C)$, we denote by ϕ and ψ the extended maps $\phi \otimes \mathrm{id}_{M_n}$ and $\psi \otimes \mathrm{id}_{M_n}$, and replace w_j, and v_j by $\mathrm{diag}(w_j, ..., w_j)$ and $\mathrm{diag}(v_j, ..., v_j)$, respectively.

Let C', $\iota : C' \to C$, $\delta' > 0$ (in the place of δ) and $\mathcal{G}' \subseteq K_1(C')$ (in the place of $\mathcal{Q}$) the constant and finite subset with respect to C (in the place of C), $\mathcal{F}_0$ (in the place of $\mathcal{F}$), $\mathcal{P}$ (in the place of $\mathcal{P}$), and ψ (in the place of h) required by 4.4. Put $\delta = \delta'/2$.

Fix a decomposition $(\iota)_{*1}(C') = \mathbb{Z}^k \oplus \mathrm{Tor}((\iota)_{*1}(C'))$ (for some integer $k \geq 0$), and let $\mathcal{G}$ be a set of standard generators of $\mathbb{Z}^k$. Let $\mathcal{G}'' \subset U_m(C)$ be a finite subset containing a representative for each element of $\mathcal{G}$. Without loss of generality, one may assume that $\mathcal{P} \subseteq \mathcal{G}''$. By Theorem

5.10 of [21], the maps ϕ and ψ are approximately unitary equivalent. Hence, for any finite subset $\mathcal{Q}$ and any δ_1, there is a unitary $v \in A$ such that

$$\|\phi(f) - v^*\psi(f)v\| < \delta_1, \quad \forall f \in \mathcal{Q}.$$

By choosing $\mathcal{Q} \supseteq \mathcal{F}_0$ sufficiently large and $\delta_1 < \eta/2$ sufficiently small, the map

$$[x] \mapsto \tau(\frac{1}{2\pi i}\log(\phi^*(x)v^*\psi(x)v)),\ x \in \mathcal{G}'',$$

induces a homomorphism $\eta_1 : (\iota)_{*1}(K_1(C')) \to \mathrm{Aff}(\mathrm{T}(A))$ (note that $\eta_1(\mathrm{Tor}((\iota)_{*1}(K_1(C')))) = \{0\}$), and moreover, $\|\eta_1(x)\| < \delta$ for all $x \in \mathcal{G}$.

By Lemma 3.8, the image of $\eta_1 - \overline{R}_{\phi,\psi}$ is in $\rho(K_0(A))$. Since $\eta(x) - \overline{R}_{\phi,\psi}(x) \in \rho_A(K_0(A))$ for all $x \in \mathcal{G}$, the image $(\eta - \eta_1)((\iota)_{*1}(K_1(C')))$ is also in $\rho_A(K_0(A))$. Since $\eta - \eta_1$ factors through $\mathbb{Z}^k$, there is a map $h : (\iota)_{*1}(K_1(C')) \to K_0(A)$ such that $\eta - \eta_1 = \rho_A \circ h$. Note that $|\tau(h(x))| < 2\delta = \delta'$ for all $\tau \in \mathrm{T}(A)$ and $x \in \mathcal{G}$.

By the universal multi-coefficient theorem, there is $\kappa \in \mathrm{Hom}_\Lambda(\underline{K}(C' \otimes \mathrm{C}(\mathbb{T})), \underline{K}(A))$ with

$$\kappa \circ \boldsymbol{\beta}|_{K_1(C')} = h \circ (\iota)_{*1}.$$

Applying 4.4, there is a unitary w such that

$$\|[w, \psi(f)]\| < \theta/2, \quad \forall f \in \mathcal{F}_0,$$

and $\mathrm{Bott}(w, \psi \circ \iota) = \kappa$. In particular, $\mathrm{bott}_1(w, \psi)(x) = h(x)$ for all $x \in \mathcal{P}$.

Set $u = wv$. One then has

$$\|\phi(f) - u^*\psi(f)u\| < \theta, \quad \forall f \in \mathcal{F}_0,$$

and for any $x \in \mathcal{P}$ and any $\tau \in \mathrm{T}(A)$,

$$\begin{aligned}
\tau(\frac{1}{2\pi i}\log(\phi(x^*)u^*\psi(x)u)) &= \tau(\frac{1}{2\pi i}\log(\phi(x)v^*w^*\psi(z)wv)) \\
&= \tau(\frac{1}{2\pi i}\log(\phi(x^*)v^*\psi(x)vv^*\psi(x^*)w^*\psi(x)wv) \\
&= \tau(\frac{1}{2\pi i}\log(\phi(x^*)v^*\psi(x)v) + \tau(\frac{1}{2\pi i}\log(\psi(x^*)w^*\psi(x)w) \\
&= \eta_1([x])(\tau) + h([x])(\tau) = \eta([x])(\tau).
\end{aligned}$$

□

Remark 5.2. In the case that $\mathrm{TR}(A) = 0$, in fact one can apply Theorem 3.6 of [12] as the uniqueness theorem in which case the condition $\phi^\ddagger = \psi^\ddagger$ is not needed, and moreover, one can apply Corollary 17.9 of [13] (homotopy lemma). This special case of lemma is also observed by H. Matui in [27].

Theorem 5.3. *Let A be a simple C^*-algebra with $\mathrm{TR}(A \otimes Q) \leq 1$, and let C be a unital AH-algebra. Suppose that there are two unital monomorphisms $\phi, \psi : C \to A$ with*

$$[\phi] = [\psi] \ \text{ in } KL(C, A),\ \ \phi_\sharp = \psi_\sharp \ \text{ and } \phi^\ddagger = \psi^\ddagger.$$

Then, for any finite subset $\mathcal{F} \subseteq C$, there exists a unitray $u \in A \otimes \mathcal{Z}$ such that

$$\|\phi(x) \otimes 1 - u^*(\psi(x) \otimes 1)u\| < \epsilon, \quad \forall x \in \mathcal{F}.$$

Proof. We first note, by [24], that $TR(A \otimes M_{\mathfrak{r}}) \leq 1$ for any supernatural number.

Write $C = \lim_{n\to\infty}(C_n, \phi_n)$, where each C_n has the form $P_n M_{m(n)}(C(X_n))P_n$, where X_n is a finite CW-complex and $P_n \in M_{m(n)}(C(X_n))$ is a projection. Let $\mathcal{F} \subseteq C$ be a finite subset, and let $\epsilon > 0$. Without loss of generality, we may assume that $\mathcal{F} \subseteq \phi_{n,\infty}(C_n)$ for some integer $n \geq 1$. We may write $\phi_{n,\infty}(C_n) = PM_m(C(X))P$, where X is a compact subset of a finite CW-complex. Then, to simplify notation, without loss of generality, in the rest of the proof, we may assume that $C = PM_m(C(X))P$, where X is a compact subset of a finite CW complex and $P \in M_m(C(X))$ is a projection.

Fix a metric on X. For any $a \in (0, 1)$, denote by

$$\Delta(a) = \inf\{\mu_{\tau\circ\psi}(O_a);\ \tau \in T(A), O_a \text{ an open ball of radius } a \text{ in } X\}.$$

Since A is simple, one has that $0 < \Delta(a) \leq 1$ and $\Delta(a) \to 0$ as $a \to 0$.

Assume that every element in $\mathcal{F}$ has norm at most one. Let $\mathfrak{p}$ and $\mathfrak{q}$ be a pair of relatively prime supernatural numbers of infinite type with $\mathbb{Q}_{\mathfrak{p}} + \mathbb{Q}_{\mathfrak{q}} = \mathbb{Q}$. Denote by $M_{\mathfrak{p}}$ and $M_{\mathfrak{q}}$ the UHF-algebras associated to $\mathfrak{p}$ and $\mathfrak{q}$ respectively.

Let $\delta > 0$, $\gamma > 0$, $d > 0$ (in place of η), $\mathcal{G} \subseteq C$ a finite subset, $\mathcal{P} \subseteq \underline{K}(C)$ a finite subset and $\mathcal{Q} = \{x_1, ..., x_k\} \subseteq K_0(C)$ which generates a free subgroup required by Theorem 4.2 corresponding to $\mathcal{F}$, $\epsilon/2$ (in place of ϵ) and Δ. We may assume that $x_i = [p_i] - [q_i]$, where $p_i, q_i \in M_n(C)$ are projections and $i = 1, 2, ..., k$.

In the rest of of the proof, for a homomorphism $h : C' \to B'$ (for any C*-algebras C' and B'), we will use h instead of $h \otimes \mathrm{id}_{M_n} : M_n(C') \to M_n(B')$ when it is inconvenient.

Without loss of generality, we may assume that $\delta < \epsilon/2$ is small enough and $\mathcal{G}$ is large enough so that for any homomorphism $h : C \to A$, the maps $\mathrm{Bott}(h, u_j)$ and $\mathrm{Bott}(h, w_j)$ are well defined and

$$\mathrm{Bott}(h, w_j) = \mathrm{Bott}(h, u_1) + \cdots + \mathrm{Bott}(h, u_j)$$

on the subgroup generated by $\mathcal{P}$, if u_j is any unitaries with $\|[h(x), u_j]\| < \delta$ for all $x \in \mathcal{G}$, where $w_j = u_1 \cdots u_j$, $j = 1, 2, 3, 4$.

We may also assume that

$$\|h(p_i),\ u_j]\| < 1/16 \ \text{ and } \ \|h(q_i),\ u_j]\| < 1/16, \ \ 1 \leq i \leq k, j = 1, 2, 3, 4 \tag{5.3}$$

(by choosing larger $\mathcal{G}$ and smaller δ)

Let $\imath_{\mathfrak{r}} : A \to A \otimes M_{\mathfrak{r}}$ be the embedding defined by $\imath_{\mathfrak{r}}(a) = a \otimes 1$ for all $a \in A$, where $\mathfrak{r}$ is a supernatural number. Define $\phi_{\mathfrak{r}} = \imath_{\mathfrak{r}} \circ \phi$ and $\psi_{\mathfrak{r}} = \imath_{\mathfrak{r}} \circ \psi$.

For any supernatural number $\mathfrak{r} = \mathfrak{p}, \mathfrak{q}$, the C*-algebra $A \otimes M_{\mathfrak{r}}$ has tracial rank at most one. Denote by $C' = P'M_n(\mathrm{C}(X'))P'$, $\imath : C' \to C$, $\delta_{\mathfrak{r}}$ (in place of δ) and $\mathcal{Q}_{\mathfrak{r}} \subseteq K_1(C')$ (in place of $\mathcal{Q}$) which generates a free subgroup corresponding to $\delta/8$ (in place of ϵ), $\mathcal{G}$, $\mathcal{P}$ and $\psi_{\mathfrak{r}}$ required by Theorem 4.4. Let $0 < \delta_2 < \min\{\delta_{\mathfrak{p}}, \delta_{\mathfrak{q}}, \epsilon, \gamma\}$, and let $\mathcal{H} \subseteq \underline{K}(C')$ be a finite set of generators. Denoted by $\mathcal{H}_1 = \mathcal{H} \cap K_1(C')$, we may assume that $\mathcal{Q}_{\mathfrak{r}} \subset \mathcal{H}_1$. Pick a finite subset $\mathcal{U} \subset U_n(C)$ for some integer $n \geq 1$ such that any element in $\imath_{*1}(\mathcal{H}_1)$ has a representative in $\mathcal{U}$. Let $S \subset C$ be a finite subset such that, if $u = (a_{ij}) \in \mathcal{U}$, then $a_{i,j} \in S$.

Furthermore, one may assume that δ_2 is sufficiently small such that for any unitaries z_1, z_2 in a C*-algebra with tracial states, $\tau(\frac{1}{2\pi i}\log(z_i z_j^*))$ $(i, j = 1, 2, 3)$ is well defined and

$$\tau(\frac{1}{2\pi i}\log(z_1 z_2^*)) = \tau(\frac{1}{2\pi i}\log(z_1 z_3^*)) + \tau(\frac{1}{2\pi i}\log(z_3 z_2^*))$$

for any tracial state τ, whenever $\|z_1 - z_3\| < \delta_2$ and $\|z_2 - z_3\| < \delta_2$.

Let $\mathcal{Q}_1 \subset K_1(C)$ (in place of $\mathcal{G}$) and δ_3 (in place of δ) be the finite subset and constant of Lemma 5.1 with respect to $\mathcal{G} \cup S$ (in place of $\mathcal{F}$), $\mathcal{U}$ (in place of $\mathcal{P}$) and δ_2/n^2 (in place of ϵ).

By Lemma 3.5, the image of $R_{\phi,\psi}$ is in the closure of $\rho_A(K_0(A))$. Note that kernel of $R_{\phi,\psi}$ contains $\mathrm{Tor}(G(\mathcal{Q}_1))$ and $G(\mathcal{Q}_1)$ is finitely generated. There exists a homomorphism $\eta : \mathcal{Q}_1 \to \mathrm{Aff}(\mathrm{T}(A))$ such that $\eta(x) - \overline{R}_{\phi,\psi}(x) \in \rho_A(K_0(A))$ and $\|\eta(x)\| < \delta_3$ for all $x \in \mathcal{Q}_1$. Then the image of $(\imath_{\mathfrak{p}})_\sharp \circ \eta - \overline{R}_{\phi_p,\psi_{\mathfrak{p}}}$ is in $\rho_{A\otimes M_{\mathfrak{p}}}(K_0(A\otimes M_{\mathfrak{p}})))$. The same holds for $\mathfrak{q}$. By Lemma 5.1 there exist unitaries $u_{\mathfrak{p}}$ and $u_{\mathfrak{q}}$ such that

$$\left\|\phi_{\mathfrak{p}}(g) - u_{\mathfrak{p}}^*\psi_{\mathfrak{p}}(g)u_{\mathfrak{p}}\right\| < \delta_2/n^2 \ \text{ and } \ \left\|\phi_{\mathfrak{q}}(g) - u_{\mathfrak{q}}^*\psi_{\mathfrak{q}}(g)u_{\mathfrak{q}}\right\| < \delta_2/n^2, \quad \forall g \in \mathcal{G} \cup S.$$

Moreover,

$$\tau(\frac{1}{2\pi i}\log(\phi_{\mathfrak{p}}(x^*)u_{\mathfrak{p}}^*\psi_{\mathfrak{p}}(x)u_{\mathfrak{p}})) = (\imath_{\mathfrak{p}})_\sharp \circ \eta([x])(\tau) \ \text{ for all } \ \tau \in \mathrm{T}(A_{\mathfrak{p}}) \ \text{ and}$$

$$\tau(\frac{1}{2\pi i}\log(\phi_{\mathfrak{q}}(x^*)u_{\mathfrak{q}}^*\psi_{\mathfrak{q}}(x)u_{\mathfrak{q}})) = (\imath_{\mathfrak{q}})_\sharp \circ \eta([x])(\tau) \ \text{ for all } \ \tau \in \mathrm{T}(A_{\mathfrak{q}}) \tag{5.4}$$

and for all $x \in \mathcal{U}$, where we identify ϕ and ψ with $\phi \otimes \mathrm{id}_{M_n}$ and $\phi \otimes \mathrm{id}_{M_n}$, and u with $u \otimes 1_{M_n}$, respectively.

Let ∞ be the supernatural number associated with $\mathbb{Q}$. Let $e_{\mathfrak{p}} : A \otimes M_{\mathfrak{p}} \to A \otimes Q$ and $e_{\mathfrak{q}} : A \otimes M_{\mathfrak{q}} \to A \otimes Q$ be the standard embeddings. Then, one computes that, for all $x \in \mathcal{U}$, by the Exel formula (see 2.13),

$$\begin{aligned}
\tau(\mathrm{bott}_1(\psi(x)\otimes 1, u_{\mathfrak{p}}u_{\mathfrak{q}}^*)) &= \tau(\frac{1}{2\pi i}\log(u_{\mathfrak{p}}u_{\mathfrak{q}}^*(\psi(x)\otimes 1)u_{\mathfrak{q}}u_{\mathfrak{p}}^*(\psi(x^*)\otimes 1))) && (5.5)\\
&= \tau(\frac{1}{2\pi i}\log(u_{\mathfrak{q}}^*(\psi(x)\otimes 1)u_{\mathfrak{q}}u_{\mathfrak{p}}^*(\psi(x^*)\otimes 1)u_{\mathfrak{p}}) && (5.6)\\
&= \tau(\frac{1}{2\pi i}\log(u_{\mathfrak{q}}^*(\psi(x)\otimes 1)u_{\mathfrak{q}}(\phi(x^*)\otimes 1)) && (5.7)\\
&\quad +\tau(\frac{1}{2\pi i}\log((\phi(x^*)\otimes 1)u_{\mathfrak{p}}^*(\psi(x)\otimes 1)u_{\mathfrak{p}}) && (5.8)\\
&= -(e_{\mathfrak{q}})_\sharp \circ (\imath_{\mathfrak{q}})_\sharp \circ \eta([x])(\tau) + (e_{\mathfrak{p}})_\sharp \circ (\imath_{\mathfrak{p}})_\sharp \circ \eta([x])(\tau) && (5.9)\\
&= -(\imath_\infty)_\sharp \circ \eta([x])(\tau) + (\imath_\infty)_\sharp \circ \eta([x])(\tau) = 0 && (5.10)
\end{aligned}$$

for all $\tau \in T(A\otimes Q)$, where we identify ϕ and ψ with $\phi\otimes\mathrm{id}_{M_n}$ and $\psi\otimes\mathrm{id}_{M_n}$, and $u_{\mathfrak{p}}$ and $u_{\mathfrak{q}}$ with $u_{\mathfrak{p}}\otimes 1_{M_n}$ and $u_{\mathfrak{q}}$ with $u_{\mathfrak{q}}\otimes 1_{M_n}$, respectively. Therefore, the image of the map $\mathrm{bott}_1(\psi\otimes 1, u_{\mathfrak{p}}u_{\mathfrak{q}}^*)$ is in $\ker\rho_{A\otimes Q}$. Note that $K_0(A\otimes Q) \cong K_0(A)\otimes\mathbb{Q}$ is torsion free. Hence the map $\mathrm{bott}_1(\psi\otimes 1, u_{\mathfrak{p}}u_{\mathfrak{q}}^*)$ factors through the torsion-free part of $G(\imath_{*1}(\mathcal{H}_1))$. Since $\mathcal{H}_1$ is a set of generators of $K_1(C')$, one may assume that the domain of the map $\mathrm{bott}_1(\psi\otimes 1, u_{\mathfrak{p}}u_{\mathfrak{q}}^*)$ is $\imath_{*1}(K_1(C'))$. Note that there is a short exact sequence

$$0 \longrightarrow \ker \rho_A \longrightarrow K_0(A) \xrightarrow{\rho_A} \rho_A(K_0(A)) \longrightarrow 0.$$

Since $D := \mathbb{Q}$, $\mathbb{Q}_{\mathfrak{p}}$ or $\mathbb{Q}_{\mathfrak{q}}$ is flat, one has

$$0 \longrightarrow \ker \rho_A \otimes D \longrightarrow K_0(A) \otimes D \xrightarrow{\rho_A \otimes \mathrm{id}_D} \rho_A(K_0(A)) \otimes D \longrightarrow 0.$$

Since the UHF-algebra $R := Q$, $M_{\mathfrak{p}}$ or $M_{\mathfrak{q}}$ have unique trace, the map $\rho_A \otimes \mathrm{id}_D$ is the same as the map $\rho_{A\otimes R}$ if $K_0(A\otimes R)$ is identified as $K_0(A) \otimes D$ respectively.

Hence $\ker \rho_{A\otimes Q} = \ker \rho_A \otimes \mathbb{Q}$ and $\ker \rho_{A\otimes M_{\mathfrak{r}}} = (\ker \rho) \otimes \mathbb{Q}_{\mathfrak{r}}$, $\mathfrak{r} = \mathfrak{p}$, or $\mathfrak{r} = \mathfrak{q}$. Moreover, since $\mathfrak{p}$ and $\mathfrak{q}$ are relative prime, any rational number r can be written as $r = r_{\mathfrak{p}} + r_{\mathfrak{q}}$ with $r_{\mathfrak{p}} \in \mathbb{Q}_{\mathfrak{p}}$ and $r_{\mathfrak{q}} \in \mathbb{Q}_{\mathfrak{q}}$ (see 2.6). Since $\ker \rho_{A\otimes Q}$ is torsion free, $\mathrm{bott}_1((\psi \otimes 1 \circ \imath) \otimes 1, u_{\mathfrak{p}} u_{\mathfrak{q}}^*)$ maps $\mathrm{Tor}(K_1(C'))$ to zero. Write $K_1(C') = \mathbb{Z}^r \oplus \mathrm{Tor}(K_1(C'))$ and let $\{e_1, e_2, ..., e_r\}$ be a set of generators of $\mathbb{Z}^r$. Suppose that $\mathrm{bott}_1((\psi\otimes 1\circ\imath)\otimes 1, u_{\mathfrak{p}} u_{\mathfrak{q}}^*)$ maps e_i to $\sum_{j=1}^{m_i} x_{i,j} \otimes r_{i,j}$, where $x_{i,j} \in \ker \rho_A$ and $r_{i,j} \in \mathbb{Q}$, $j = 1, 2, ..., m_i$ and $i = 1, 2, ..., r$. There are $r_{i,j,\mathfrak{p}} \in \mathbb{Q}_{\mathfrak{p}}$ and $r_{i,j,\mathfrak{q}} \in \mathbb{Q}_{\mathfrak{q}}$ such that $r_{i,j} = r_{i,j,\mathfrak{p}} - r_{i,j,\mathfrak{q}}$, $j = 1, 2, ..., m_i$ and $i = 1, 2, ..., r$. Define two homomorphisms $\theta_{\mathfrak{p}} : K_1(C') \to \ker \rho_{A\otimes M_{\mathfrak{p}}}$ and $\theta_{\mathfrak{q}} : K_1(C') \to \ker \rho_{A\otimes M_{\mathfrak{q}}}$ as follows: $(\theta_{\mathfrak{r}})|_{\mathrm{Tor}(K_1(C'))} = 0$, $\mathfrak{r} = \mathfrak{p}$, $\mathfrak{q}$. Define $\theta_{\mathfrak{r}}(e_i) = \sum_{j=1}^{m_i} x_{i,j} \otimes r_{i,j,\mathfrak{r}}$ by regarding $\sum_{j=1}^{m_i} x_{i,j} \otimes r_{i,j,\mathfrak{r}}$ as an element of $K_0(A \otimes M_{\mathfrak{r}}))$, $\mathfrak{r} = \mathfrak{p}$, $\mathfrak{q}$ and $i = 1, 2, ..., r$. Then

$$\mathrm{bott}_1((\psi \otimes 1 \circ \imath) \otimes 1, u_{\mathfrak{p}} u_{\mathfrak{q}}^*) = (j_{\mathfrak{p}})_{*0} \circ \theta_{\mathfrak{p}} - (j_{\mathfrak{q}})_{*0} \circ \theta_{\mathfrak{q}},$$

where $j_{\mathfrak{r}} : A\otimes M_{\mathfrak{r}} \to A\otimes Q$ is the embedding. The same argument shows there are homomorphisms $\alpha_{\mathfrak{p}} : K_0(C') \to K_1(A \otimes M_{\mathfrak{p}})$ and $\alpha_{\mathfrak{q}} : K_0(C') \to K_1(A \otimes M_{\mathfrak{q}})$ such that

$$\mathrm{bott}_0((\psi \circ \imath) \otimes 1, u_{\mathfrak{p}} u_{\mathfrak{q}}^*) = (j_{\mathfrak{p}})_{*1} \circ \alpha_{\mathfrak{p}} - (j_{\mathfrak{q}})_{*1} \circ \alpha_{\mathfrak{q}}.$$

By the universal multi-coefficient theorem, there is $\kappa_{\mathfrak{p}} \in \mathrm{Hom}_\Lambda(\underline{K}(C' \otimes \mathrm{C}(\mathbb{T})), \underline{K}(A \otimes M_{\mathfrak{p}}))$ such that

$$\kappa_{\mathfrak{p}}|_{\boldsymbol{\beta}(K_0(C'))} = -\alpha_{\mathfrak{p}} \circ \boldsymbol{\beta}^{-1} \text{ and } \kappa_{\mathfrak{p}}|_{\boldsymbol{\beta}(K_1(C'))} = -\theta_{\mathfrak{p}} \circ \boldsymbol{\beta}^{-1}. \tag{5.11}$$

Similarly, there is $\kappa_{\mathfrak{q}} \in \mathrm{Hom}_\Lambda(\underline{K}(C' \otimes \mathrm{C}(C(\mathbb{T}))), \underline{K}(A \otimes M_{\mathfrak{q}}))$ such that

$$\kappa_{\mathfrak{q}}|_{\boldsymbol{\beta}(K_0(C'))} = -\alpha_{\mathfrak{q}} \circ \boldsymbol{\beta}^{-1} \text{ and } \kappa_{\mathfrak{q}}|_{\boldsymbol{\beta}(K_1(C'))} = -\theta_{\mathfrak{q}} \circ \boldsymbol{\beta}^{-1}. \tag{5.12}$$

To apply 4.4, we verify that

$$|\rho_{A\otimes M_{\mathfrak{p}}} \circ \kappa_{\mathfrak{p}}(\boldsymbol{\beta}(x))| = 0 < \delta_{\mathfrak{p}} \text{ for all } x \in \mathcal{Q}_{\mathfrak{p}} \text{ and} \tag{5.13}$$

$$|\rho_{A\otimes M_{\mathfrak{q}}} \circ \kappa_{\mathfrak{p}}(\boldsymbol{\beta}(x))| = 0 < \delta_{\mathfrak{q}} \text{ for all } x \in \mathcal{Q}_{\mathfrak{q}}. \tag{5.14}$$

Then, by Theorem 4.4, there are unitaries $w_{\mathfrak{p}} \in A \otimes M_{\mathfrak{p}}$ and $w_{\mathfrak{q}} \in A \otimes M_{\mathfrak{q}}$ such that

$$\|[w_{\mathfrak{p}}, \psi_{\mathfrak{p}}(g)]\| < \delta/8, \quad \|[w_{\mathfrak{q}}, \psi_{\mathfrak{q}}(g)]\| < \delta/8,$$

for any $g \in \mathcal{G}$, and

$$\mathrm{Bott}(\psi_{\mathfrak{p}} \circ \imath, w_{\mathfrak{p}}) = \kappa_{\mathfrak{p}} \circ \boldsymbol{\beta} \quad \text{and} \quad \mathrm{Bott}(\psi_{\mathfrak{q}} \circ \imath, w_{\mathfrak{q}}) = \kappa_{\mathfrak{q}} \circ \boldsymbol{\beta}.$$

Consider the unitaries $w_{\mathfrak{p}} u_{\mathfrak{p}}$ and $w_{\mathfrak{q}} u_{\mathfrak{q}}$. One then has that

$$\left\|\phi(g) \otimes 1 - u_{\mathfrak{p}}^* w_{\mathfrak{p}}^* (\psi(g) \otimes 1) w_{\mathfrak{p}} u_{\mathfrak{p}}\right\| < \delta/4 \text{ and } \left\|\phi(g) \otimes 1 - u_{\mathfrak{q}}^* w_{\mathfrak{q}}^* (\psi(g) \otimes 1) w_{\mathfrak{q}} u_{\mathfrak{q}}\right\| < \delta/4, \quad \forall g \in \mathcal{G}.$$

Hence

$$\big\|[w_{\mathfrak{p}}u_{\mathfrak{p}}u_{\mathfrak{q}}^*w_{\mathfrak{q}}^*, \psi(g)\otimes 1]\big\| < \delta/2, \quad \forall g \in \mathcal{G}.$$

In the following computation, we use $\psi\otimes 1$ for the map from C to $A\otimes Q$ induced by ψ. We have, by (5.11) and (5.12), that

$$\begin{aligned}
&(5.15) && \mathrm{bott}_0(\psi\otimes 1, w_{\mathfrak{p}}u_{\mathfrak{p}}u_{\mathfrak{q}}^*w_{\mathfrak{q}}^*)|_{K_0(C)\cap\mathcal{P}}\\
&(5.16) &=\;& \mathrm{bott}_0(\psi\otimes 1, w_{\mathfrak{p}})|_{K_0(C)\cap\mathcal{P}} + \mathrm{bott}_0(\psi\otimes 1, u_{\mathfrak{p}}u_{\mathfrak{q}}^*)|_{K_0(C)\cap\mathcal{P}} + \mathrm{bott}_0(\psi\otimes 1, w_{\mathfrak{q}}^*)|_{K_0(C)\cap\mathcal{P}}\\
&(5.17) &=\;& -(j_{\mathfrak{p}})_{*1}\circ\alpha_{\mathfrak{p}}|_{K_0(C)\cap\mathcal{P}} + ((j_{\mathfrak{p}})_{*1}\circ\alpha_{\mathfrak{p}} - (j_{\mathfrak{q}})_{*1}\circ\alpha_{\mathfrak{q}})|_{K_0(C)\cap\mathcal{P}} + (j_{\mathfrak{q}})_{*1}\circ\alpha_{\mathfrak{q}}|_{K_0(C)\cap\mathcal{P}} = 0.
\end{aligned}$$

The same computation shows that

$$\begin{aligned}
&(5.18) && \mathrm{bott}_1(\psi\otimes 1, w_{\mathfrak{p}}u_{\mathfrak{p}}u_{\mathfrak{q}}^*w_{\mathfrak{q}}^*)|_{K_1(C)\cap\mathcal{P}}\\
&(5.19) &=\;& \mathrm{bott}_1(\psi\otimes 1, w_{\mathfrak{p}})|_{K_1(C)\cap\mathcal{P}} + \mathrm{bott}_1(\psi\otimes 1, u_{\mathfrak{p}}u_{\mathfrak{q}}^*)|_{K_1(C)\cap\mathcal{P}} + \mathrm{bott}_1(\psi\otimes 1, w_{\mathfrak{q}}^*)|_{K_1(C)\cap\mathcal{P}}\\
&(5.20) &=\;& -(j_{\mathfrak{p}})_{*0}\circ\theta_{\mathfrak{p}}|_{K_1(C)\cap\mathcal{P}} + ((j_{\mathfrak{p}})_{*0}\circ\theta_{\mathfrak{p}} - (j_{\mathfrak{q}})_{*0}\theta_{\mathfrak{q}})|_{K_1(C)\cap\mathcal{P}} + (j_{\mathfrak{q}})_{*0}\circ\theta_{\mathfrak{q}}|_{K_1(C)\cap\mathcal{P}} = 0.
\end{aligned}$$

Since $K_i(A\otimes Q)$ is torsion free ($i=0,1$), the aboves imply that

$$\mathrm{Bott}(\psi\otimes 1, w_{\mathfrak{p}}u_{\mathfrak{p}}u_{\mathfrak{q}}^*w_{\mathfrak{q}}^*)|_{\mathcal{P}} = 0. \tag{5.21}$$

By the construction of Δ, it is clear that

$$\mu_{\tau\circ(\psi\otimes 1)}(O_a) \geq \Delta(a)$$

for all a, where O_a is any open ball of X with radius a; in particular, it holds for all $a \geq d$.

For each $1 \leq i \leq k$, define (see (5.3))

$$L_{i,w_{\mathfrak{p}}u_{\mathfrak{p}}} = \overline{\langle(\mathbf{1}_n - \psi(p_i)\otimes 1 + (\psi(p_i)\otimes 1)w_{\mathfrak{p}}u_{\mathfrak{p}})(\mathbf{1}_n - \psi(q_i)\otimes 1 + (\psi(q_i)\otimes 1)u_{\mathfrak{p}}^*w_{\mathfrak{p}}^*)\rangle}$$

and

$$L_{i,w_{\mathfrak{q}}u_{\mathfrak{q}}} = \overline{\langle(\mathbf{1}_n - \psi(p_i)\otimes 1 + (\psi(p_i)\otimes 1)w_{\mathfrak{q}}u_{\mathfrak{q}})(\mathbf{1}_n - \psi(q_i)\otimes 1 + (\psi(q_i)\otimes 1)u_{\mathfrak{q}}^*w_{\mathfrak{q}}^*)\rangle},$$

and define the map $\Gamma_{\mathfrak{p}} : \mathbb{Z}^k \to U(A\otimes M_{\mathfrak{p}})/CU(A\otimes M_{\mathfrak{p}})$ by $\Gamma_{\mathfrak{p}}(x_i) = L_{i,w_{\mathfrak{p}}u_{\mathfrak{p}}}$ and the map $\Gamma_{\mathfrak{q}} : \mathbb{Z}^k \to U(A\otimes M_{\mathfrak{q}})/CU(A\otimes M_q)$ by $\Gamma_{\mathfrak{q}}(x_i) = L_{i,w_{\mathfrak{q}}u_{\mathfrak{q}}}$.

By Corollary 4.3, there are unitaries $\zeta_{\mathfrak{p}} \in A\otimes M_{\mathfrak{p}}, \zeta_{\mathfrak{q}} \in A\otimes M_{\mathfrak{q}}$ such that

$$\big\|[\zeta_{\mathfrak{p}}, \psi(g)\otimes 1_{M_{\mathfrak{p}}}]\big\| < \delta/4, \quad \big\|[\zeta_{\mathfrak{q}}, \psi(g)\otimes 1_{M_{\mathfrak{q}}}]\big\| < \delta/4, \quad \forall g\in\mathcal{G}$$

$$\mathrm{Bott}(\psi\otimes 1_{M_{\mathfrak{p}}}, \zeta_{\mathfrak{p}})|_{\mathcal{P}} = 0, \quad \mathrm{Bott}(\psi\otimes 1_{M_{\mathfrak{q}}}, \zeta_{\mathfrak{q}})|_{\mathcal{P}} = 0,$$

and for any $1\leq i\leq k$,

$$\mathrm{dist}(L_{i,\zeta_{\mathfrak{p}}^*}, \Gamma_{\mathfrak{p}}(x_i)) \leq \gamma/2 \ \text{ and } \ \mathrm{dist}(L_{i,\zeta_{\mathfrak{q}}^*}, \Gamma_{\mathfrak{q}}(x_i)) \leq \gamma/2,$$

where

$$L_{i,\zeta_{\mathfrak{p}}^*} = \overline{\langle(\mathbf{1}_n - \psi(p_i)\otimes 1_{M_{\mathfrak{p}}} + (\psi(p_i)\otimes 1_{M_{\mathfrak{p}}})\zeta_{\mathfrak{p}}^*)(\mathbf{1}_n - \psi(q_i)\otimes 1_{M_{\mathfrak{p}}} + (\psi(q_i)\otimes 1_{M_{\mathfrak{p}}})\zeta_{\mathfrak{p}})\rangle},$$

and

$$L_{i,\zeta_{\mathfrak{q}}^*} = \overline{\langle(\mathbf{1}_n - \psi(p_i)\otimes 1_{M_{\mathfrak{q}}} + (\psi(p_i)\otimes 1_{M_{\mathfrak{q}}})\zeta_{\mathfrak{q}}^*)(\mathbf{1}_n - \psi(q_i)\otimes 1_{M_{\mathfrak{q}}} + (\psi(q_i)\otimes 1_{M_{\mathfrak{q}}})\zeta_{\mathfrak{q}})\rangle}.$$

In particular, if denote by $v_0 = \zeta_{\mathfrak{p}} w_{\mathfrak{p}} u_{\mathfrak{p}} u_{\mathfrak{q}}^* w_{\mathfrak{q}}^* \zeta_{\mathfrak{q}}^*$, one has that for any $1 \le i \le k$,

$$\mathrm{dist}(\overline{\langle(\mathbf{1}_n - \psi(p_i) \otimes 1_Q + (\psi(p_i) \otimes 1_Q)v_0)(\mathbf{1}_n - \psi(q_i) \otimes 1_Q + (\psi(q_i) \otimes 1_Q)v_0^*)\rangle}, \overline{1_n}) < \gamma.$$

Then, by Theorem 4.2, there is a continuous path of unitaries $v(t)$ in $A \otimes Q$ such that $v(1) = 1$ and $v(0) = v_0$, and

$$\|[v(t), \psi(x) \otimes 1_Q]\| < \epsilon/2 \quad \forall x \in \mathcal{F},\ \forall t \in [0, 1].$$

Consider the unitary $u(t) = v(t)\zeta_{\mathfrak{q}} w_{\mathfrak{q}} u_{\mathfrak{q}} \in A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$, and it has the property

$$\|\phi(f) \otimes 1 - u^*(\psi(f) \otimes 1)u\| < \epsilon, \quad \forall f \in \mathcal{F}.$$

One then embeds $\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ into $\mathcal{Z}$ to get the desired conclusion. □

Recall that $\mathcal{C}$ is the class of all simple separable C*-algebras A for which $\mathrm{TR}(\mathrm{A} \otimes \mathrm{M}_{\mathfrak{r}}) \le 1$ form some UHF-algebra $M_{\mathfrak{r}}$, where $\mathfrak{r}$ is a supernatural number of infinite type.

Corollary 5.4. *Let C be a unital AH-algebra and let A be a unital separable simple $\mathcal{Z}$-stable C*-algebra in $\mathcal{C}$. Let $\phi, \psi : C \to A$ be two unital monomorphisms. Then there exists a sequence of unitaries $\{u_n\} \subset A$ such that*

$$\lim_{n\to\infty} u_n^* \psi(c) u_n = \phi(c) \ \text{ for all } c \in C,$$

if and only if

$$[\phi] = [\psi] \ \text{ in } KL(C, A),\ \phi_\sharp = \psi_\sharp \ \text{ and } \phi^\ddagger = \psi^\ddagger.$$

Proof. We only show the "if" part. Suppose that ϕ and ψ satisfy the condition. Let $\epsilon > 0$, and let $\mathcal{F} \subset C$ be a finite subset. Then, by 5.3, there exists a unitary $v \in A \otimes \mathcal{Z}$ such that

$$\|v^*(\psi(a) \otimes 1)v - \phi(a) \otimes 1\| < \epsilon/3 \ \text{ for all } a \in \mathcal{F}. \tag{5.22}$$

Let $\imath : A \to A \otimes \mathcal{Z}$ be defined by $\imath(a) = a \otimes 1$ for $a \in A$. There exists an isomorphism $j : A \otimes \mathcal{Z} \to A$ such that $j \circ \imath$ is approximately inner. So there is a unitaries $w \in A$ such that

$$\|j(\psi(a) \otimes 1) - w^*\psi(a)w\| < \epsilon/3 \ \text{ and } \ \|w^*\phi(a)w - j(\phi(a) \otimes 1)\| < \epsilon/3 \tag{5.23}$$

for all $a \in \mathcal{F}$. Let $u = wj(v)w^* \in A$; then, for $a \in \mathcal{F}$,

$$\begin{aligned}
\|u^*\psi(a)u - \phi(a)\| &= \|wj(v)^*w^*\psi(a)wj(v)w^* - \phi(a)\| && (5.24)\\
&\le \|wj(v)^*w^*\psi(a)wj(v)w^* - wj(v)^*(j(\psi(a) \otimes 1)j(v)w^*\| && (5.25)\\
&\quad + \|wj(v)^*(j(\psi(a) \otimes 1)j(v)w^* - w(j(\phi(a) \otimes 1)w^*\| && (5.26)\\
&\quad + \|w(j(\phi(a) \otimes 1)w^* - \phi(a)\| && (5.27)\\
&< \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon \ \text{ for all } a \in \mathcal{F}. && (5.28)
\end{aligned}$$

□

A version of the following is also obtained by H. Matui.

Corollary 5.5. *Let C be a unital AH-algebra and let A be a unital separable simple C*-algebra in $\mathcal{C}_0$ which is $\mathcal{Z}$-stable. Suppose that $\phi, \psi : C \to A$ are two unital monomorphisms. Then there exists a sequence of unitaries $\{u_n\} \subset A$ such that*

$$\lim_{n\to\infty} u_n^*\phi(c)u_n = \psi(c) \ \textit{for all} \ c \in C,$$

if and only if

$$[\phi] = [\psi] \ \textit{in} \ KL(C, A), \ \phi_\sharp = \psi_\sharp \ \textit{and} \ \phi^\ddagger = \psi^\ddagger.$$

Proof. The proof is exactly the same as that of 5.3 and 5.4. At where Theorem 2.10 is applied, one applies Theorem 3.6 of [12] instead. One also uses Remark 5.2. ☐

Lemma 5.6. *Let A be a unital C*-algebra such that $A\otimes M_\mathfrak{r}$ is an AH-algebra for any supernatural number $\mathfrak{r}$ of infinite type. Let $B \in \mathcal{C}$ be a unital separable C*-algebra, and let $\phi, \psi : A \to B$ be two unital monomorphisms. Suppose that*

$$[\phi] = [\psi] \ \textit{in} \ KL(A, B), \tag{5.29}$$

$$\phi_\sharp = \psi_\sharp \ \textit{and} \ \phi^\ddagger = \psi^\ddagger. \tag{5.30}$$

Let $\mathfrak{p}$ and $\mathfrak{q}$ be two relatively prime supernatural numbers of infinite type with $M_\mathfrak{p} \otimes M_\mathfrak{q} = Q$. Then, for any $\epsilon > 0$ and any finite subset $\mathcal{F} \subset A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$, there exists a unitary $v \in B \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ such that

$$\|v^*((\phi\otimes \mathrm{id})(a))v - (\psi \otimes \mathrm{id})(a)\| < \epsilon \ \text{for all} \ a \in \mathcal{F}. \tag{5.31}$$

The proof of this lemma will be lengthy and technical in nature. However, the outline is the same as that of Theorem 5.3, that is, using homotopy lemmas, one could find a certain path of unitaries in $B \otimes Q$ such that it implements the approximate equivalence above when it is regarded as a unitary in $B \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$. But since the domain C*-algebra A is only assumed to be rational tracial rank at most one, in order to apply the homotopy lemmas, one also needs to interpolate paths in $A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$, and this increases the technical difficulty of the proof.

Proof. Let $\mathfrak{r}$ be a supernatural number. Denote by $\imath_\mathfrak{r} : A \to A \otimes M_\mathfrak{r}$ the embedding defined by $\imath_\mathfrak{r}(a) = a \otimes 1$ for all $a \in A$. Denote by $\jmath_\mathfrak{r} : B \to B \otimes M_\mathfrak{r}$ the embedding defined by $\jmath_\mathfrak{r}(b) = b \otimes 1$ for all $b \in B$. Without loss of generality, one may assume that $\mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2$, where $\mathcal{F}_1 \subseteq A$ and $\mathcal{F}_2 \subseteq \mathcal{Z}_{p,q}$ are finite subsets and $1_A \in \mathcal{F}$ and $1_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}} \in \mathcal{F}_2$. Moreover, one may assume that any element in $\mathcal{F}_1$ or $\mathcal{F}_2$ has norm at most one.

Let $0 = t_0 < t_1 < \cdots < t_m = 1$ be a partition of $[0, 1]$ such that

$$\|b(t) - b(t_i)\| < \epsilon/4 \quad \forall b \in \mathcal{F}_2, \ \forall t \in [t_{i-1}, t_i], \ i = 1, ..., m. \tag{5.32}$$

Consider

$$\mathcal{E} = \{a \otimes b(t_i); \ a \in \mathcal{F}_1, b \in \mathcal{F}_2, i = 0, ..., m\} \subseteq A \otimes Q,$$

$$\mathcal{E}_\mathfrak{p} = \{a \otimes b(t_0); \ a \in \mathcal{F}_1, b \in \mathcal{F}_2\} \subseteq A \otimes M_\mathfrak{p} \subset A \otimes Q \ \text{and} \tag{5.33}$$

$$\mathcal{E}_\mathfrak{q} = \{a \otimes b(t_m); \ a \in \mathcal{F}_1, b \in \mathcal{F}_2\} \subseteq A \otimes M_\mathfrak{q} \subset A \otimes Q. \tag{5.34}$$

Since $A\otimes Q$ is an AH-algebra, without loss of generality, one may assume that the finite subset $\mathcal{E}$ is in a C*-subalgebra of $A\otimes Q$ which is isomorphic to $C := PM_n(\mathrm{C}(X))P$ (for some $n \geq 1$) for some compact metric space X. Since $PM_n(\mathrm{C}(X))P = \lim_{m\to\infty}(P_m M_n(\mathrm{C}(X_m))P_m)$, where X_m are closed subspaces of finite CW-complexes, then, without loss of generality, one may assume further that X is a closed subset of a finite CW-complex.

Fix a metric on X, and for any $a \in (0,1)$, denote by

$$\Delta(a) = \inf\{\mu_{\tau\circ(\phi\otimes\mathrm{id})}(O_a);\ \tau \in T(B), O_a \text{ an open ball of radius } a \text{ in } X\}.$$

Since B is simple, one has that $0 < \Delta(a) \le 1$.

Let $\mathcal{H} \subset C$, $\mathcal{P} \subseteq \underline{K}(C)$, $\mathcal{Q} = \{x_1, x_2, ..., x_m\} \subset K_0(C)$ which generates a free subgroup of $K_0(C)$, $\delta > 0$, $\gamma > 0$, and $d > 0$ (in the place of η) be the constants of Theorem 4.2 with respect to $\mathcal{E}$, $\epsilon/8$, and Δ. We may assume that $x_i = [p_i] - [q_i]$, where $p_i, q_i \in M_n(C)$ are projections (for some integer $n \ge 1$), $i = 1, 2, ..., m$. Moreover, we may assume that $\gamma < 1$.

Denote by ∞ the supernatural number associated with $\mathbb{Q}$. Let $\mathcal{P}_i = \mathcal{P} \cap K_i(A\otimes Q)$, $i = 0, 1$. There is a finitely generated free subgroup $G(\mathcal{P})_{i,0} \subset K_i(A)$ such that if one sets

$$G(\mathcal{P})_{i,\infty,0} = G(\{gr : g \in (\imath_\infty)_{*i}(G(\mathcal{P})_{i,0}) \text{ and } r \in D_0\}), \tag{5.35}$$

where $1 \in D_0 \subset \mathbb{Q}$ is a finite subset, then $G(\mathcal{P})_{i,\infty,0}$ contains the subgroup generated by $\mathcal{P}_i$, $i = 0, 1$. Moreover, we may assume that, if $r = k/m$, where k and m are nonzero integers, and $r \in D_0$, then $1/m \in D_0$. Let $\mathcal{P}_i' \subset K_i(A)$ be a finite subset which generates $G(\mathcal{P})_{i,0}$, $i = 0, 1$.Also denote by $\mathcal{P}' = \mathcal{P}_0' \cup \mathcal{P}_1'$.

Denote by $j : C \to A\otimes Q$ the embedding.

Write the subgroup generated by the image of $\mathcal{Q}$ in $K_0(A\otimes Q)$ as $\mathbb{Z}^k$ (for some integer $k \ge 1$). Choose $\{x_1', ..., x_k'\} \subseteq K_0(A)$ and $\{r_{ij};\ 1 \le i \le m, 1 \le j \le k\} \subseteq \mathbb{Q}$ such that

$$j_{*0}(x_i) = \sum_{j=1}^{k} r_{ij}x_j', \quad 1 \le i \le m,\ 1 \le j \le k,$$

and moreover, $\{x_1', ..., x_k'\}$ generates a free subgroup of $K_0(A)$ of rank k. Choose projections $p_j', q_j' \in M_n(A)$ such that $x_j' = [p_j'] - [q_j']$, $1 \le j \le k$. Choose an integer M such that Mr_{ij} are integers for $1 \le i \le m$ and $1 \le j \le k$. In particular Mx_i is the linear combination of x_j' with integer coefficients.

Also noting that the subgroup of $K_0(A\otimes Q)$ generated by $\{(\imath_\infty)_{*0}(x_1'), ..., (\imath_\infty)_{*0}(x_k')\}$ is isomorphic to $\mathbb{Z}^k$ and the subgroup of $K_0(A\otimes M_\mathfrak{r})$ generated by $\{(\imath_\mathfrak{r})_{*0}(x_1'), ..., (\imath_\mathfrak{r})_{*0}(x_k')\}$ has to be isomorphic to $\mathbb{Z}^k$, where $\mathfrak{r} = \mathfrak{p}$ or $\mathfrak{r} = \mathfrak{q}$.

Since $A\otimes M_\mathfrak{r}$ is an AH-algebra, one can choose a C^*-subalgebra $C_\mathfrak{r}$ of $A\otimes M_\mathfrak{r}$ which is isomorphic to $P_\mathfrak{r} M_{n_\mathfrak{r}}(\mathrm{C}(X_\mathfrak{r}))P_\mathfrak{r}$ (for some $n_\mathfrak{r} \ge 1$) such that $\mathcal{E}_\mathfrak{r} \subseteq C_\mathfrak{r}$ and projections $\{p_{1,\mathfrak{r}}', ..., p_{k,\mathfrak{r}}', q_{1,\mathfrak{r}}', ..., q_{k,\mathfrak{r}}'\} \subseteq M_n(C_\mathfrak{r})$ such that for any $1 \le j \le k$,

$$\left\|p_j' \otimes 1_{M_\mathfrak{r}} - p_{j,\mathfrak{r}}'\right\| < \gamma/(32(1 + \sum_{i,j'} |Mr_{ij'}|)) < 1 \tag{5.36}$$

and

$$\left\|q_j' \otimes 1_{M_\mathfrak{r}} - q_{j,\mathfrak{r}}'\right\| < \gamma/(32(1 + \sum_{i,j'} |Mr_{ij'}|)) < 1, \tag{5.37}$$

where $X_\mathfrak{r}$ is a closed subset of a finite CW-complex, and $\mathfrak{r} = \mathfrak{p}$ or $\mathfrak{r} = \mathfrak{q}$.

Denote by $x'_{j,\mathfrak{r}} = [p'_{j,\tau}] - [q'_{j,\mathfrak{r}}]$, $1 \le j \le k$, and denote by $G_\mathfrak{r}$ the subgroup of $K_0(C_\mathfrak{r})$ generated by $\{x'_{1,\mathfrak{r}}, ..., x'_{k,\mathfrak{r}}\}$, and write $G_\mathfrak{r} = \mathbb{Z}^r \oplus \mathrm{Tor}(G_\mathfrak{r})$. Since $G_\mathfrak{r}$ is generated by k elements, one has that $r \le k$ and $r = k$ if and only if $G_\mathfrak{r}$ is torsion free. Note that the image of $G_\mathfrak{r}$ in $K_0(A \otimes M_\mathfrak{r})$ is the group generated by $\{[p'_1 \otimes 1_{M_\mathfrak{r}}] - [q'_1 \otimes 1_{M_\mathfrak{r}}], ..., [p'_k \otimes 1_{M_\mathfrak{r}}] - [q'_k \otimes 1_{M_\mathfrak{r}}]\}$, which is isomorphic to $\mathbb{Z}^k$ (with $\{[p'_j \otimes 1_{M_\mathfrak{r}}] - [q'_j \otimes 1_{M_\mathfrak{r}}];\ 1 \le j \le k\}$ as the standard generators). Hence $G_\mathfrak{r}$ is torsion free and $r = k$.

Without loss of generality, one may assume that $\imath_\mathfrak{r}(\mathcal{P}') \subseteq \underline{K}(C_\mathfrak{r})$. Assume that $\mathcal{H}$ is sufficiently large and δ is sufficiently small such that for any homomorphism h from $A \otimes Q$ to $B \otimes Q$ and any unitary z_j $(j = 1, 2, 3, 4)$, the map $\mathrm{Bott}(h, z_j)$ and $\mathrm{Bott}(h, w_j)$ are well defined on the subgroup generated by $\mathcal{P}$ and

$$\mathrm{Bott}(h, w_j) = \mathrm{Bott}(h, z_1) + \cdots + \mathrm{Bott}(h, z_j)$$

on the subgroup generated by $\mathcal{P}$, if $\|[h(x), z_j]\| < \delta$ for any $x \in \mathcal{H}$, where $w_j = z_1 \cdots z_j$, $j = 1, 2, 3, 4$.

By choosing larger $\mathcal{H}$ and smaller δ, one may also assume that

$$\|h(p_i),\, z_j]\| < 1/16 \ \text{ and } \ \|h(q_i),\, z_j]\| < 1/16,\ \ 1 \le i \le m, j = 1, 2, 3, 4, \tag{5.38}$$

and for any $1 \le i \le m$,

$$\mathrm{dist}(\zeta_{i,z_1}^M, \prod_{j=1}^k (\zeta'_{j,z_1})^{Mr_{ij}}) < \gamma/8, \tag{5.39}$$

where

$$\zeta_{i,z_1} = \overline{\langle (\mathbf{1}_n - h(p_i) + h(p_i))z_1)(\mathbf{1}_n - h(q_i) + h(q_i))z_1^*)\rangle},$$

and

$$\zeta'_{j,z_1} = \overline{\langle (\mathbf{1}_n - h(p'_j \otimes 1_{A\otimes Q}) + h(p'_j \otimes 1_{A\otimes Q}))z_1)(\mathbf{1}_n - h(q'_j \otimes 1_{A\otimes Q}) + h(q'_j \otimes 1_{A\otimes Q}))z_1^*)\rangle}.$$

By choosing even smaller δ, without loss of generality, we may assume that

$$\mathcal{H} = \mathcal{H}^0 \otimes \mathcal{H}^\mathfrak{p} \otimes \mathcal{H}^\mathfrak{q},$$

where $\mathcal{H}^0 \subset A$, $\mathcal{H}^\mathfrak{p} \subset M_\mathfrak{p}$ and $\mathcal{H}^\mathfrak{q} \subset M_\mathfrak{q}$ are finite subsets, and $1 \in \mathcal{H}^0$, $1 \in \mathcal{H}^\mathfrak{p}$ and $1 \in \mathcal{H}^\mathfrak{q}$.

Moreover, choose $\mathcal{H}^0$, $\mathcal{H}^\mathfrak{p}$ and $\mathcal{H}^\mathfrak{q}$ even larger and δ even smaller so that for any homomorphism $h_\mathfrak{r} : A \otimes M_\mathfrak{r} \to B \otimes M_\mathfrak{r}$ and unitaries $z_1, z_2 \in B \otimes M_\mathfrak{r}$ with $\|h_\mathfrak{r}(x), z_i\| < \delta$ for any $x \in \mathcal{H}_0 \otimes \mathcal{H}_\mathfrak{r}$, one has

$$\|h_\mathfrak{r}(p'_{i,\mathfrak{r}}),\, z_j]\| < 1/16 \ \text{ and } \ \|h_\mathfrak{r}(q'_{i,\mathfrak{r}}),\, z_j]\| < 1/16,\ \ 1 \le i \le k, j = 1, 2, \tag{5.40}$$

and

$$\mathrm{dist}(\zeta_{i,z_1z_2}, \overline{(1_{B\otimes M_\mathfrak{r}})_n}) < \mathrm{dist}(\zeta_{i,z_1^*}, \zeta_{i,z_2}) + \gamma/(32(1 + \sum_{i',j} |Mr_{i'j}|)),$$

where

$$\zeta_{i,z'} = \overline{\langle (\mathbf{1}_n - h_\mathfrak{r}(p'_{i,\mathfrak{r}}) + h_\mathfrak{r}(p'_{i,\mathfrak{r}}))z')(\mathbf{1}_n - h_\mathfrak{r}(q'_{i,\mathfrak{r}}) + h_\mathfrak{r}(q'_{i,\mathfrak{r}}))(z')^*)\rangle}, \quad z' = z_1z_2, z_1^*, z_2.$$

Denote by $C' = P'M_n(\mathrm{C}(\tilde{X}))P'$, $\iota : C' \to A \otimes Q$, δ_2 (in the place of δ) the constant, $\mathcal{G} \subseteq K_1(\mathrm{C}(\tilde{X}))$ (in the place of $\mathcal{Q}$) the finite subset in Theorem 4.4 with respect to $A \otimes Q$ (in the place of C), $B \otimes Q$ (in the place of A), $\phi \otimes \mathrm{id}_Q$ (in the place of h), $\delta/4$ (in the place of ϵ), $\mathcal{H}$ (in the place of $\mathcal{F}$) and $\mathcal{P}$. Note that $\tilde{X}$ is a finite CW-complex.

Let $\mathcal{H}' \subseteq A \otimes Q$ be a finite subset and assume that δ_2 is small enough such that for any homomorphism h from $A \otimes Q$ to $B \otimes Q$ and any unitary z_j ($j = 1, 2, 3, 4$), the map $\mathrm{Bott}(h, z_j)$ and $\mathrm{Bott}(h, w_j)$ is well defined on the subgroup $[\iota](\underline{K}(C'))$ and

$$\mathrm{Bott}(h, w_j) = \mathrm{Bott}(h, z_1) + \cdots + \mathrm{Bott}(h, z_j)$$

on the subgroup $[\iota](\underline{K}(C'))$, if $\|[h(x), z_j]\| < \delta_2$ for any $x \in \mathcal{H}'$, where $w_j = z_1 \cdots z_j$, $j = 1, 2, 3, 4$. Furthermore, as above, one may assume, without loss of generality, that

$$\mathcal{H}' = \mathcal{H}^{0'} \otimes \mathcal{H}^{\mathfrak{p}'} \otimes \mathcal{H}^{\mathfrak{q}'},$$

where $\mathcal{H}^0 \subseteq \mathcal{H}^{0'} \subset A$, $\mathcal{H}^{\mathfrak{p}} \subseteq \mathcal{H}^{\mathfrak{p}'} \in M_{\mathfrak{q}}$ and $\mathcal{H}^{\mathfrak{q}} \subseteq \mathcal{H}^{\mathfrak{q}'} \subset M_{\mathfrak{q}}$ are finite subsets.

Let $\delta_2' > 0$ be a constant such that for any unitary with $\|u - 1\| < \delta_2'$, one has that $\|\log u\| < \delta_2/4$. Without loss of generality, one may assume that $\delta_2' < \delta_2/4 < \epsilon/4$ and $\delta_2' < \delta$.

Let $C_{\mathfrak{r}}' := P_{\mathfrak{r}} M_n \mathrm{C}(X_{\mathfrak{r}}')P_{\mathfrak{r}}$ (in the place of C'), $\iota_{\mathfrak{r}}' : C_{\mathfrak{r}} \to A \otimes M_{\mathfrak{r}}$ (in the place of ι), $\mathcal{R}_{\mathfrak{r}} \subset K_1(C_{\mathfrak{r}}'))$ (in the place of $\mathcal{Q}$) and $\delta_{\mathfrak{r}}$ (in the place of δ) be the finite subset and constant of Theorem 4.4 with respect to $A \otimes M_{\mathfrak{r}}$ (in the place of C), $B \otimes M_{\mathfrak{r}}$ (in the place of A), $\phi \otimes \mathrm{id}_{M_{\mathfrak{r}}}$ (in the place of h), $\mathcal{H}^{0'} \otimes \mathcal{H}^{\mathfrak{r}'}$ (in place of $\mathcal{F}$) and $(\imath_{\mathfrak{r}})_{*0}(\mathcal{P}_0') \cup (\imath_{\mathfrak{r}})_{*1}(\mathcal{P}_1')$ (in the place of $\mathcal{P}$) and $\delta_2'/8$ (in place of ϵ) ($\mathfrak{r} = \mathfrak{p}$ or $\mathfrak{r} = \mathfrak{q}$). Note that $X_{\mathfrak{r}}'$ is a finite CW-complex with $K_1(C_{\mathfrak{r}}') = \mathbb{Z}^{k_{\mathfrak{r}}} \oplus \mathrm{Tor}(K_1(C_{\mathfrak{r}}'))$. Let $\mathcal{R}_{\mathfrak{r}}^{(i)} = (\iota_{\mathfrak{r}}')_{*i}(K_i(C_{\mathfrak{r}}'))$, $i = 0, 1$. There is a finitely generated subgroup $G_{i,0,\mathfrak{r}} \subset K_i(A)$ and a finitely generated subgroup $D_{0,\mathfrak{r}} \subseteq \mathbb{Q}_{\mathfrak{r}}$ so that

$$G_{i,0,\mathfrak{r}}' := G(\{gr : g \in (\imath_{\mathfrak{r}})_{*i}(G_{i,0,\mathfrak{r}}) \text{ and } r \in D_{0,\mathfrak{r}}\})$$

contains the subgroup $\mathcal{R}_{\mathfrak{r}}^{(i)}$, $i = 0, 1$. Without loss of generality, one may assume that $D_{0,\mathfrak{p}} = \{\frac{k}{m_{\mathfrak{p}}};\ k \in \mathbb{Z}\}$ and $D_{0,\mathfrak{q}} = \{\frac{k}{m_{\mathfrak{q}}};\ k \in \mathbb{Z}\}$ for an integer $m_{\mathfrak{p}}$ divides $\mathfrak{p}$ and an integer $m_{\mathfrak{q}}$ divides $\mathfrak{q}$.

Let $\mathcal{R} \subset \underline{K}(A \otimes Q)$ be a finite subset which generates a subgroup containing

$$\frac{1}{m_{\mathfrak{p}} m_{\mathfrak{q}}}((\imath_{\mathfrak{p},\infty})_*(G_{0,0,\mathfrak{p}}' \cup G_{1,0,\mathfrak{p}}') \cup (\imath_{\mathfrak{q},\infty})_*(G_{0,0,\mathfrak{q}}' \cup G_{1,0,\mathfrak{q}}'))$$

in $\underline{K}(A \otimes Q)$, where $\imath_{\mathfrak{r},\infty}$ is the canonical embedding $A \otimes M_{\mathfrak{r}} \to A \otimes Q$, $\mathfrak{r} = \mathfrak{p}, \mathfrak{q}$. Without loss of generality, one may also assume that $\mathcal{R} \supseteq \iota_{*1}(\mathcal{G})$. Let $\mathcal{H}_{\mathfrak{r}} \subset A \otimes M_{\mathfrak{r}}$ be a finite subset and $\delta_3 > 0$ such that for any homomorphism h from $A \otimes M_{\mathfrak{r}}$ to $B \otimes M_{\mathfrak{r}}$ ($\mathfrak{r} = \mathfrak{p}$ or $\mathfrak{r} = \mathfrak{q}$) any unitary z_j ($j = 1, 2, 3, 4$), the map $\mathrm{Bott}(h, z_j)$ and $\mathrm{Bott}(h, w_j)$ are well defined on the subgroup $[\iota_{\mathfrak{r}}'](\underline{K}(C_{\mathfrak{r}}'))$ and

$$\mathrm{Bott}(h, w_j) = \mathrm{Bott}(h, z_1) + \cdots + \mathrm{Bott}(h, z_j)$$

on the subgroup generated by $[\iota_{\mathfrak{r}}'](\underline{K}(C_{\mathfrak{r}}'))$, if $\|[h(x), z_j]\| < \delta_3$ for any $x \in \mathcal{H}_{\mathfrak{r}}$, where $w_j = z_1 \cdots z_j$, $j = 1, 2, 3, 4$. Without loss of generality, we assume that $\mathcal{H}^0 \otimes \mathcal{H}^{\mathfrak{p}} \subset \mathcal{H}_{\mathfrak{p}}$ and $\mathcal{H}^0 \otimes \mathcal{H}^{\mathfrak{q}} \subset \mathcal{H}_{\mathfrak{q}}$. Furthermore, we may also assume that

$$\mathcal{H}_{\mathfrak{r}} = \mathcal{H}_{0,0} \otimes \mathcal{H}_{0,\mathfrak{r}}$$

for some finite subsets $\mathcal{H}_{0,0}$ and $\mathcal{H}_{0,\mathfrak{r}}$ with $\mathcal{H}^{0'} \subset \mathcal{H}_{0,0} \subset A$, $\mathcal{H}^{\mathfrak{p}'} \subset \mathcal{H}_{0,\mathfrak{p}} \subset M_{\mathfrak{p}}$ and $\mathcal{H}^{\mathfrak{q}'} \subset \mathcal{H}_{0,\mathfrak{q}}$. In addition, we may also assume that $\delta_3 < \delta_2/2$.

Furthermore, one may assume that δ_3 is sufficiently small such that, for any unitaries z_1, z_2, z_3 in a C*-algebra with tracial states, $\tau(\frac{1}{2\pi i}\log(z_i z_j^*))$ $(i, j = 1, 2, 3)$ is well defined and

$$\tau(\frac{1}{2\pi i}\log(z_1 z_2^*)) = \tau(\frac{1}{2\pi i}\log(z_1 z_3^*)) + \tau(\frac{1}{2\pi i}\log(z_3 z_2^*))$$

for any tracial state τ, whenever $\|z_1 - z_3\| < \delta_3$ and $\|z_2 - z_3\| < \delta_3$.

To simply notation, we also assume that, for any unitary z_j, $(j = 1, 2, 3, 4)$ the map $\mathrm{Bott}(h, z_j)$ and $\mathrm{Bott}(h, w_j)$ are well defined on the subgroup generated by $\mathcal{R}$ and

$$\mathrm{Bott}(h, w_j) = \mathrm{Bott}(h, z_1) + \cdots + \mathrm{Bott}(h, z_j)$$

on the subgroup generated by $\mathcal{R}$, if $\|[h(x), z_j]\| < \delta_3$ for any $x \in \mathcal{H}''$, where $w_j = z_1 \cdots z_j$, $j = 1, 2, ..., 4$, and assume that

$$\mathcal{H}'' = \mathcal{H}_{0,0} \otimes \mathcal{H}_{0,\mathfrak{p}} \otimes \mathcal{H}_{0,\mathfrak{q}}.$$

Let $\mathcal{R}^i = \mathcal{R} \cap K_i(A \otimes Q)$. There is a finitely generated subgroup $G_{i,0}$ of $K_i(A)$ and there is a finite subset $D_0' \subset \mathbb{Q}$ such that

$$G_{i,\infty} := G(\{gr : g \in (\imath_\infty)_{*i}(G_{i,0})) \text{ and } r \in D_0'\})$$

contains the subgroup generated by $\mathcal{R}^i$, $i = 0, 1$. Without loss of generality, we may assume that $G_{i,\infty}$ is the subgroup generated by $\mathcal{R}^i$. Note that we may also assume that $G_{i,0} \supset G(\mathcal{P})_{i,0}$ and $1 \in D_0' \supset D_0$. Moreover, we may assume that, if $r = k/m$, where m, k are relatively prime non-zero integers, and $r \in D_0'$, then $1/m \in D_0'$. We may also assume that $G_{i,0,\mathfrak{r}} \subseteq G_{i,0}$ for $\mathfrak{r} = \mathfrak{p}, \mathfrak{q}$ and $i = 0, 1$. Let $\mathcal{R}^{i'} \subset K_i(A)$ be a finite subset which generates $G_{i,0}$, $i = 0, 1$. Choose a finite subset $\mathcal{U} \subset U_n(A)$ for some n such that for any element of $\mathcal{R}^{1'}$, there is a representative in $\mathcal{U}$. Let S be a finite subset of A such that if $(z_{i,j}) \in \mathcal{U}$, then $z_{i,j} \in S$.

Denote by δ_4 and $\mathcal{Q}_{\mathfrak{r}} \subset K_1(A \otimes M_{\mathfrak{r}}) \cong K_1(A) \otimes \mathbb{Q}_{\mathfrak{r}}$ the constant and finite subset of Lemma 5.1 corresponding to $\mathcal{E}_{\mathfrak{r}} \cup \mathcal{H}_{\mathfrak{r}} \otimes 1 \cup \imath_{\mathfrak{r}}(S)$ (in the place of $\mathcal{F}$), $\imath_{\mathfrak{r}}(\mathcal{U})$ (in the place of $\mathcal{P}$) and $\frac{1}{n^2}\min\{\delta_2'/8, \delta_3/4\}$ (in the place of ϵ) ($\mathfrak{r} = \mathfrak{p}$ or $\mathfrak{r} = \mathfrak{q}$). We may assume that $\mathcal{Q}_{\mathfrak{r}} = \{x \otimes r : x \in \mathcal{Q}' \text{ and } r \in D_{\mathfrak{r}}''\}$, where $\mathcal{Q}' \subset K_1(A)$ is a finite subset and $D_{\mathfrak{r}}'' \subset \mathbb{Q}_{\mathfrak{r}}$ is also a finite subset. Let $K = \max\{|r| : r \in D_{\mathfrak{p}}'' \cup D_{\mathfrak{q}}''\}$. Since $[\phi] = [\psi]$ in $KL(A, B)$, $\phi_\sharp = \psi_\sharp$ and $\phi^\ddagger = \psi^\ddagger$, by Lemma 3.5, $\overline{R}_{\phi,\psi}(K_1(A)) \subseteq \overline{\rho_B(K_0(B))} \subset \mathrm{Aff}(\mathrm{T}(B))$. Therefore, there is a map $\eta : G(\mathcal{Q}') \to \overline{\rho_B(K_0(B))} \subset \mathrm{Aff}(\mathrm{T}(B))$ such that

$$(\eta - \overline{R}_{\phi,\psi})([z]) \in \rho_B(K_0(B)) \text{ and } \|\eta(z)\| < \frac{\delta_4}{1+K} \text{ for all } z \in \mathcal{Q}' \tag{5.41}$$

Consider the map $\phi_{\mathfrak{r}} = \phi \otimes \mathrm{id}_{M_{\mathfrak{r}}}$ and $\psi_{\mathfrak{r}} = \psi \otimes \mathrm{id}_{M_{\mathfrak{r}}}$ ($\mathfrak{r} = \mathfrak{p}$ or $\mathfrak{r} = \mathfrak{q}$). Since η vanishes on the torsion part of $G(\mathcal{Q}')$, there is a homomorphism $\eta_{\mathfrak{r}} : G((\imath_{\mathfrak{r}})_{*1}(\mathcal{Q}')) \to \overline{\rho_{B\otimes M_{\mathfrak{r}}}(K_0(B \otimes M_{\mathfrak{r}}))} \subset \mathrm{Aff}(\mathrm{T}(B \otimes M_{\mathfrak{r}}))$ such that

$$\eta_{\mathfrak{r}} \circ (\imath_{\mathfrak{r}})_{*1} = \eta. \tag{5.42}$$

Since $\overline{\rho_{B\otimes M_{\mathfrak{r}}}(K_0(B \otimes M_{\mathfrak{r}}))} = \overline{\mathbb{R}\rho_B(K_0(B))}$ is divisible, one can extend $\eta_{\mathfrak{r}}$ so it defines on $K_1(A) \otimes \mathbb{Q}_{\mathfrak{r}}$. We will continue to use $\eta_{\mathfrak{r}}$ for the extension. It follows from (5.41) that $\eta_{\mathfrak{r}}(z) - \overline{R}_{\phi_{\mathfrak{r}},\psi_{\mathfrak{r}}}(z) \in$

$\rho_{B\otimes M_{\mathfrak{r}}}(K_0(B\otimes M_{\mathfrak{r}}))$ and $\|\eta_{\mathfrak{r}}(z)\|<\delta_4$ for all $z\in\mathcal{Q}_{\mathfrak{r}}$. By Lemma 5.1, there exists a unitary $u_{\mathfrak{p}}\in B\otimes M_{\mathfrak{p}}$ such that

$$\|u_{\mathfrak{p}}^*(\phi\otimes \mathrm{id}_{M_{\mathfrak{p}}})(c)u_{\mathfrak{p}}-(\psi\otimes \mathrm{id}_{M_{\mathfrak{p}}})(c)\|<\frac{1}{n^2}\min\{\delta_2'/8,\delta_3/4\},\quad \forall c\in\mathcal{E}_{\mathfrak{p}}\cup\mathcal{H}_{\mathfrak{p}}\cup\imath_{\mathfrak{p}}(S). \tag{5.43}$$

Note that

$$\|u_{\mathfrak{p}}^*(\phi\otimes \mathrm{id}_{M_{\mathfrak{p}}})(z)u_{\mathfrak{p}}-(\psi\otimes \mathrm{id}_{M_{\mathfrak{p}}})(z)\|<\delta_3\quad\text{for any } z\in\mathcal{U}.$$

Therefore $\tau(\frac{1}{2\pi i}\log(u_{\mathfrak{p}}^*(\phi\otimes\mathrm{id}_{\mathfrak{p}})(z)u_{\mathfrak{p}}(\psi\otimes\mathrm{id}_{\mathfrak{p}})(z^*)))=\eta_{\mathfrak{p}}([z])(\tau)$ for all $z\in\imath_{\mathfrak{p}}(\mathcal{U})$, where we identify ϕ and ψ with $\phi\otimes\mathrm{id}_{M_n}$ and $\psi\otimes\mathrm{id}_{M_n}$, and $u_{\mathfrak{p}}$ with $u_{\mathfrak{p}}\otimes 1_{M_n}$, respectively.

The same argument shows that there is a unitary $u_{\mathfrak{q}}\in B\otimes M_{\mathfrak{q}}$ such that

$$\|u_{\mathfrak{q}}^*(\phi\otimes \mathrm{id}_{M_{\mathfrak{q}}})(c)u_{\mathfrak{q}}-(\psi\otimes \mathrm{id}_{M_{\mathfrak{q}}})(c)\|<\frac{1}{n^2}\min\{\delta_2'/8,\delta_3/4\},\quad \forall c\in\mathcal{E}_{\mathfrak{q}}\cup\mathcal{H}_{\mathfrak{q}}\cup\imath_{\mathfrak{p}}(S), \tag{5.44}$$

and $\tau(\frac{1}{2\pi i}\log(u_{\mathfrak{q}}^*(\phi\otimes\mathrm{id}_{\mathfrak{q}})(z)u_{\mathfrak{q}}(\psi\otimes\mathrm{id}_{\mathfrak{q}})(z^*)))=\eta_{\mathfrak{q}}([z])(\tau)$ for all $z\in\imath_{\mathfrak{q}}(\mathcal{U})$, where we identify ϕ and ψ with $\phi\otimes\mathrm{id}_{M_n}$ and $\psi\otimes\mathrm{id}_{M_n}$, and $u_{\mathfrak{q}}$ with $u_{\mathfrak{q}}\otimes 1_{M_n}$, respectively. We will also identify $u_{\mathfrak{p}}$ with $u_{\mathfrak{p}}\otimes 1_{M_{\mathfrak{q}}}$ and $u_{\mathfrak{q}}$ with $u_{\mathfrak{q}}\otimes 1_{M_{\mathfrak{p}}}$ respectively. Then $u_{\mathfrak{p}}u_{\mathfrak{q}}^*\in A\otimes Q$ and one estimates that for any $c\in\mathcal{H}_{00}\otimes\mathcal{H}_{0,\mathfrak{p}}\otimes\mathcal{H}_{\mathfrak{q}}$,

$$\|u_{\mathfrak{q}}u_{\mathfrak{p}}^*(\phi\otimes 1_Q(c))u_{\mathfrak{p}}u_{\mathfrak{q}}^*-(\phi\otimes 1_Q)(c)\|<\delta_3, \tag{5.45}$$

and hence $\mathrm{Bott}(\phi\otimes\mathrm{id}_Q,u_{\mathfrak{p}}u_{\mathfrak{q}}^*)(z)$ is well defined on the subgroup generated by $\mathcal{R}$. Moreover, for any $z\in\mathcal{U}$, by the Exel formula (see 2.13) and applying (5.42),

$$\begin{aligned}
&\tau(\mathrm{bott}_1(\phi\otimes\mathrm{id}_Q,u_{\mathfrak{p}}u_{\mathfrak{q}}^*)((\imath_\infty)_{*1}([z]))) && (5.46)\\
&= \tau(\mathrm{bott}_1(\phi\otimes\mathrm{id}_Q,u_{\mathfrak{p}}u_{\mathfrak{q}}^*)(\imath_\infty(z))) && (5.47)\\
&= \tau(\frac{1}{2\pi i}\log(u_{\mathfrak{p}}u_{\mathfrak{q}}^*(\phi\otimes\mathrm{id}_Q)(\imath_\infty(z)))u_{\mathfrak{q}}u_{\mathfrak{p}}^*(\phi\otimes\mathrm{id}_Q)(\imath_\infty(z))^*) && (5.48)\\
&= \tau(\frac{1}{2\pi i}\log(u_{\mathfrak{q}}^*(\phi\otimes\mathrm{id}_Q)(\imath_\infty(z))))u_{\mathfrak{q}}(\psi\otimes\mathrm{id}_Q)(\imath_\infty(z^*)))) && (5.49)\\
&\quad -\tau(\frac{1}{2\pi i}\log(u_{\mathfrak{p}}^*(\phi\otimes\mathrm{id}_Q)(\imath_\infty(z))u_{\mathfrak{p}}(\psi\otimes\mathrm{id}_Q)(\imath_\infty(z^*)))) && (5.50)\\
&= \eta_{\mathfrak{q}}((\imath_{\mathfrak{q}})_{*1}([z]))(\tau)-\eta_{\mathfrak{p}}((\imath_{\mathfrak{p}})_{*1}([z]))(\tau) && (5.51)\\
&= \eta([z])(\tau)-\eta([z])(\tau)=0\ \text{ for all }\ \tau\in T(B), && (5.52)
\end{aligned}$$

where we identify ϕ and ψ with $\phi\otimes\mathrm{id}_{M_n}$ and $\psi\otimes\mathrm{id}_{M_n}$, and $u_{\mathfrak{p}}$ and $u_{\mathfrak{q}}$ with $u_{\mathfrak{p}}\otimes 1_{M_n}$ and $u_{\mathfrak{q}}$ with $u_{\mathfrak{q}}\otimes 1_{M_n}$, respectively.

Now suppose that $g\in G_{1,\infty}$. Then $g=(k/m)(\imath_\infty)_{*1}([z])$ for some $z\in\mathcal{U}$, where k,m are non-zero integers. It follows that

$$\tau(\mathrm{bott}_1(\phi\otimes\mathrm{id}_Q,u_{\mathfrak{p}}u_{\mathfrak{q}}^*)(mg))=k\tau(\mathrm{bott}_1(\phi\otimes\mathrm{id}_Q,u_{\mathfrak{p}}u_{\mathfrak{q}}^*)((([z]))=0 \tag{5.53}$$

for all $\tau\in \mathrm{T}(B)$. Since $\mathrm{Aff}(\mathrm{T}(B))$ is torsion free, it follows that

$$\tau(\mathrm{bott}_1(\phi\otimes\mathrm{id}_Q,u_{\mathfrak{p}}u_{\mathfrak{q}}^*)(g)=0 \tag{5.54}$$

for all $g \in G_{1,\infty}$ and $\tau \in \mathrm{T}(B)$. Therefore, the image of $\mathcal{R}^1$ under $\mathrm{bott}_1(\phi \otimes \mathrm{id}_Q, u_{\mathfrak{p}}u_{\mathfrak{q}}^*)$ is in $\ker \rho_{B\otimes Q}$. One may write

$$G_{1,0} = \mathbb{Z}^r \oplus \mathbb{Z}/p_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/p_s\mathbb{Z},$$

where r is a non-negative integer and $p_1, ..., p_s$ are powers of primes numbers. Since $\mathfrak{p}$ and $\mathfrak{q}$ are relatively prime, one then has the decomposition

$$G_{1,0} = \mathbb{Z}^r \oplus \mathrm{Tor}_{\mathfrak{p}}(G_{1,0}) \oplus \mathrm{Tor}_{\mathfrak{q}}(G_{1,0}) \subseteq K_1(A),$$

where $\mathrm{Tor}_{\mathfrak{p}}(G_{1,0})$ consists of the torsion-elements with their orders divide $\mathfrak{p}$ and $\mathrm{Tor}_{\mathfrak{q}}(G_{1,0})$ consists of the torsion-elements with their orders divide $\mathfrak{q}$. Fix this decomposition.

Note that the restriction of $(\imath_{\mathfrak{p}})_{*1}$ to $\mathbb{Z}^r \oplus \mathrm{Tor}_{\mathfrak{q}}(G_{1,0})$ is injective and the restriction to $\mathrm{Tor}_{\mathfrak{p}}(G_{1,0})$ is zero, and the restriction of $(\imath_{\mathfrak{q}})_{*1}$ to $\mathbb{Z}^r \oplus \mathrm{Tor}_{\mathfrak{p}}(G_{1,0})$ is injective and the restriction to $\mathrm{Tor}_{\mathfrak{q}}(G_{1,0})$ is zero.

Moreover, using the assumption that $\mathfrak{p}$ and $\mathfrak{q}$ are relatively prime again, for any element $k \in (\imath_{\mathfrak{q}})_{*1}(\mathbb{Z}^r \oplus \mathrm{Tor}_{\mathfrak{p}}(G_{1,0}))$ and any nonzero integer q which divides $\mathfrak{q}$, the element k/q is well defined in $K_1(A \otimes M_{\mathfrak{q}})$; that is, there is a unique element $s \in K_1(A \otimes M_{\mathfrak{q}})$ such that $qs = k$.

Denote by $e_1, ..., e_r$ the standard generators of $\mathbb{Z}^r$. It is also clear that

$$(\imath_\infty)_{*1}(\mathrm{Tor}_{\mathfrak{p}}(G_{1,0})) = (\imath_\infty)_{*1}(\mathrm{Tor}_{\mathfrak{q}}(G_{1,0})) = 0.$$

Recall that $D_{0,\mathfrak{p}} = \{k/m_{\mathfrak{p}};\ k \in \mathbb{Z}\} \subset \mathbb{Q}_{\mathfrak{p}}$ and $D_{0,\mathfrak{q}} = \{k/m_{\mathfrak{q}};\ k \in \mathbb{Z}\} \subset \mathbb{Q}_{\mathfrak{q}}$ for an integer $m_{\mathfrak{p}}$ dividing $\mathfrak{p}$ and an integer $m_{\mathfrak{q}}$ dividing $\mathfrak{q}$. Put $m_\infty = m_{\mathfrak{p}}m_{\mathfrak{q}}$.

Consider $\frac{1}{m_\infty}\mathbb{Z}^r \in K_1(A \otimes Q)$, and for each e_i, $1 \leq i \leq r$, consider

$$\frac{1}{m_\infty}\mathrm{bott}_1(\phi \otimes \mathrm{id}_Q, u_{\mathfrak{p}}u_{\mathfrak{q}}^*)((\imath_\infty)_{*1}(e_i)) \in \ker \rho_{B\otimes Q}.$$

Since $\ker \rho_{B\otimes Q} \cong (\ker \rho_B) \otimes \mathbb{Q}$, $\ker \rho_{B\otimes M_{\mathfrak{p}}} \cong (\ker \rho_B) \otimes \mathbb{Q}_{\mathfrak{p}}$, and $\ker \rho_{B\otimes M_{\mathfrak{q}}} \cong (\ker \rho_B) \otimes \mathbb{Q}_{\mathfrak{q}}$, using the same arguments as that of Theorem 5.3, there are $g_{i,\mathfrak{p}} \in \ker \rho_{B\otimes M_{\mathfrak{p}}}$ and $g_{i,\mathfrak{q}} \in \ker \rho_{B\otimes M_{\mathfrak{q}}}$ such that

$$\mathrm{bott}_1(\phi \otimes \mathrm{id}_Q, u_{\mathfrak{p}}u_{\mathfrak{q}}^*)(\frac{1}{m_\infty}((\imath_\infty)_{*1}(e_i))) = (j_{\mathfrak{p}})_{*0}(g_{i,\mathfrak{p}}) + (j_{\mathfrak{q}})_{*0}(g_{i,\mathfrak{q}}),$$

where $g_{i,\mathfrak{p}}$ and $g_{i,\mathfrak{q}}$ are identified as their images in $K_0(A \otimes Q)$.

Note that the subgroup $(\imath_{\mathfrak{p}})_{*1}(G_{1,0})$ in $K_0(A\otimes M_{\mathfrak{p}})$ is isomorphic to $\mathbb{Z}^r \oplus \mathrm{Tor}_{\mathfrak{q}}$ and $\frac{1}{m_{\mathfrak{p}}}(\mathbb{Z}^r \oplus \mathrm{Tor}_{\mathfrak{q}})$ is well defined in $K_0(A \otimes M_{\mathfrak{p}})$, and the subgroup $(\imath_{\mathfrak{q}})_{*1}(G_{1,0})$ in $K_0(B \otimes M_{\mathfrak{q}})$ is isomorphic to $\mathbb{Z}^r \oplus \mathrm{Tor}_{\mathfrak{p}}$ and $\frac{1}{m_{\mathfrak{q}}}(\mathbb{Z}^r \oplus \mathrm{Tor}_{\mathfrak{p}})$ is well defined in $K_0(A \otimes M_{\mathfrak{q}})$. One then defines the maps $\theta_{\mathfrak{p}} : \frac{1}{m_{\mathfrak{p}}}(\imath_{\mathfrak{p}})_{*1}(G_{1,0}) \to \ker \rho_{B\otimes M_{\mathfrak{p}}}$ and $\theta_{\mathfrak{q}} : \frac{1}{m_{\mathfrak{q}}}(\imath_{\mathfrak{q}})_{*1}(G_{1,0}) \to \ker \rho_{B\otimes M_{\mathfrak{q}}}$ by

$$\theta_{\mathfrak{p}}(\frac{1}{m_{\mathfrak{p}}}(\iota_{\mathfrak{p}})_{*1}(e_i)) = m_{\mathfrak{q}}g_{i,\mathfrak{p}} \quad \text{and} \quad \theta_{\mathfrak{q}}(\frac{1}{m_{\mathfrak{q}}}(\iota_{\mathfrak{q}})_{*1}(e_i)) = m_{\mathfrak{p}}g_{i,\mathfrak{q}}$$

for $1 \leq i \leq r$ and

$$\theta_{\mathfrak{p}}|_{\mathrm{Tor}((\imath_{\mathfrak{p}})_{*1}(G_{1,0}))} = 0 \quad \text{and} \quad \theta_{\mathfrak{q}}|_{\mathrm{Tor}((\imath_{\mathfrak{q}})_{*1}(G_{1,0}))} = 0.$$

Then, for each e_i, one has

$$\begin{aligned}
& (j_{\mathfrak{p}})_{*0}\circ\theta_{\mathfrak{p}}\circ(\imath_{\mathfrak{p}})_{*1}(e_i)+(j_{\mathfrak{q}})_{*0}\circ\theta_{\mathfrak{q}}\circ(\imath_q)_{*1}(e_i)\\
= \ & m_{\mathfrak{p}}(\frac{1}{m_{\mathfrak{p}}}(j_{\mathfrak{p}})_{*0}\circ\theta_{\mathfrak{p}}\circ(\imath_{\mathfrak{p}})_{*1}(e_i))+m_{\mathfrak{q}}(\frac{1}{m_{\mathfrak{q}}}(j_{\mathfrak{q}}))_{*0}\circ\theta_{\mathfrak{q}}\circ(\imath_q)_{*1}(e_i))\\
= \ & m_{\mathfrak{p}}m_{\mathfrak{q}}((j_{\mathfrak{p}})_{*0}(g_{i,p})+(j_{\mathfrak{q}})_{*0}(g_{i,q}))\\
= \ & m_\infty\mathrm{bott}_1(\phi\otimes\mathrm{id}_Q,u_{\mathfrak{p}}u_{\mathfrak{q}}^*)\circ(\imath_\infty)_{*1}(e_i/m_\infty)\\
= \ & \mathrm{bott}_1(\phi\otimes\mathrm{id}_Q,u_{\mathfrak{p}}u_{\mathfrak{q}}^*)\circ(\imath_\infty)_{*1}(e_i).
\end{aligned}$$

Since the restriction of $\theta_{\mathfrak{p}}\circ(\imath_{\mathfrak{p}})_{*1}$, $\theta_{\mathfrak{q}}\circ(\imath_{\mathfrak{q}})_{*1}$ and $\mathrm{bott}_1(\phi\otimes\mathrm{id}_Q,u_{\mathfrak{p}}u_{\mathfrak{q}}^*)\circ(\imath_\infty)_{*1}$ to the torsion part of $G_{1,0}$ is zero, one has

$$\mathrm{bott}_1(\phi\otimes\mathrm{id}_Q,u_{\mathfrak{p}}u_{\mathfrak{q}}^*)\circ(\imath_\infty)_{*1}=(j_{\mathfrak{p}})_{*0}\circ\theta_{\mathfrak{p}}\circ(\imath_{\mathfrak{p}})_{*1}+(j_{\mathfrak{q}})_{*0}\circ\theta_{\mathfrak{q}}\circ(\imath_q)_{*1}.$$

The same argument shows that there also exist maps $\alpha_{\mathfrak{p}}:\frac{1}{m_{\mathfrak{p}}}((\imath_{\mathfrak{p}})_{*0}(G_{0,0}))\to K_1(B\otimes M_{\mathfrak{p}})$ and $\alpha_{\mathfrak{q}}:\frac{1}{m_{\mathfrak{q}}}((\imath_{\mathfrak{q}})_{*0}(G_{0,0}))\to K_1(B\otimes M_{\mathfrak{q}})$ such that

$$\mathrm{bott}_0(\phi\otimes\mathrm{id}_Q,u_{\mathfrak{p}}u_{\mathfrak{q}}^*)\circ(\imath_\infty)_{*0}=(j_{\mathfrak{p}})_{*1}\circ\alpha_{\mathfrak{p}}\circ(\imath_{\mathfrak{p}})_{*0}+(j_{\mathfrak{q}})_{*1}\circ\alpha_{\mathfrak{q}}\circ(\imath_{\mathfrak{q}})_{*0}$$

on $G_{0,0}$.

Note that $G_{i,0,\mathfrak{r}}\subseteq G_{i,0}$, $i=0,1$, $\mathfrak{r}=\mathfrak{p},\mathfrak{q}$. In particular, one has that $(\imath_{\mathfrak{r}})_{*i}(G_{i,0,\mathfrak{r}})\subseteq(\imath_{\mathfrak{r}})_{*i}(G_{i,0})$, and therefore $G'_{1,0,\mathfrak{p}}\subseteq\frac{1}{m_{\mathfrak{p}}}(\imath_{\mathfrak{p}})_{*0}(G_{1,0})$ and $G'_{1,0,\mathfrak{q}}\subseteq\frac{1}{m_{\mathfrak{q}}}(\imath_{\mathfrak{q}})_{*0}(G_{1,0})$. Then the maps $\theta_{\mathfrak{p}}$ and $\theta_{\mathfrak{q}}$ can be restricted to $G'_{1,0,\mathfrak{p}}$ and $G'_{1,0,\mathfrak{q}}$ respectively. Since the group $G'_{i,0,\mathfrak{r}}$ contains $(\iota'_{\mathfrak{r}})_{*i}(K_i(C'_{\mathfrak{r}}))$, the maps $\theta_{\mathfrak{p}}$ and $\theta_{\mathfrak{q}}$ can be restricted further to $(\iota'_{\mathfrak{p}})_{*1}(K_1(C'_{\mathfrak{p}}))$ and $(\iota'_{\mathfrak{q}})_{*1}(K_1(C'_q))$ respectively.

For the same reason, the maps $\alpha_{\mathfrak{p}}$ and $\alpha_{\mathfrak{q}}$ can be restricted to $(\iota'_{\mathfrak{p}})_{*0}(K_0(C'_{\mathfrak{p}}))$ and $(\iota'_{\mathfrak{q}})_{*0}(K_0(C'_q))$ respectively. We keep the same notation for the restrictions of these maps $\alpha_{\mathfrak{p}}$, $\alpha_{\mathfrak{q}}$, $\theta_{\mathfrak{p}}$, and $\theta_{\mathfrak{q}}$.

By the universal multi-coefficient theorem, there is $\kappa_{\mathfrak{p}}\in\mathrm{Hom}_\Lambda(\underline{K}(C'_{\mathfrak{p}}\otimes\mathrm{C}(\mathbb{T})),\underline{K}(B\otimes M_{\mathfrak{p}}))$ such that

$$\kappa_{\mathfrak{p}}|_{\boldsymbol{\beta}(K_1(C'_{\mathfrak{p}}))}=-\theta_{\mathfrak{p}}\circ(\iota'_{\mathfrak{p}})_{*1}\circ\boldsymbol{\beta}^{-1}\ \text{ and }\ \kappa_{\mathfrak{p}}|_{\boldsymbol{\beta}(K_0(C'_{\mathfrak{p}}))}=-\alpha_{\mathfrak{p}}\circ(\iota'_{\mathfrak{p}})_{*0}\circ\boldsymbol{\beta}^{-1}.$$

Similarly, there exists $\kappa_{\mathfrak{q}}\in\mathrm{Hom}_\Lambda(\underline{K}(C'_{\mathfrak{q}}\otimes\mathrm{C}(\mathbb{T}))),\underline{K}(B\otimes M_{\mathfrak{q}}))$ such that

$$\kappa_{\mathfrak{q}}|_{\boldsymbol{\beta}(K_1(C'_{\mathfrak{q}}))}=-\theta_{\mathfrak{q}}\circ(\iota'_{\mathfrak{q}})_{*1}\circ\boldsymbol{\beta}^{-1}\ \text{ and }\ \kappa_{\mathfrak{q}}|_{\boldsymbol{\beta}(K_0(C'_{\mathfrak{q}}))}=-\alpha_{\mathfrak{q}}\circ(\iota'_{\mathfrak{q}})_{*0}\circ\boldsymbol{\beta}^{-1}.$$

Note that since $g_{i,\mathfrak{r}}\in\mathrm{k}er\rho_{A\otimes M_{\mathfrak{r}}}$, $\kappa_{\mathfrak{r}}(\boldsymbol{\beta}(K_1(C'_{\mathfrak{r}})))\subseteq\ker\rho_{B\otimes M_{\mathfrak{r}}}$, $\mathfrak{r}=\mathfrak{p}$ or $\mathfrak{r}=\mathfrak{q}$. By Theorem 4.4, there exist unitaries $w_{\mathfrak{p}}\in B\otimes M_{\mathfrak{p}}$ and $w_{\mathfrak{q}}\in B\otimes M_{\mathfrak{q}}$ such that

$$\|[w_{\mathfrak{p}},(\phi\otimes\mathrm{id}_{M_{\mathfrak{p}}})(x)]\|<\delta'_2/8,\quad \|[w_{\mathfrak{q}},(\phi\otimes\mathrm{id}_{M_{\mathfrak{q}}})(y)]\|<\delta'_2/8,$$

for any $x\in\mathcal{H}^{0'}\otimes\mathcal{H}^{\mathfrak{p}'}$ and $y\in\mathcal{H}^{0'}\otimes\mathcal{H}^{\mathfrak{q}'}$, and

$$\mathrm{Bott}(\phi\otimes\mathrm{id}_{M_{\mathfrak{p}}},w_{\mathfrak{p}})\circ[\iota'_{\mathfrak{p}}]=\kappa_p\circ\boldsymbol{\beta}\quad\text{and}\quad\mathrm{Bott}(\phi\otimes\mathrm{id}_{M_{\mathfrak{q}}},w_{\mathfrak{q}})\circ[\iota'_{\mathfrak{q}}]=\kappa_q\circ\boldsymbol{\beta}.$$

For $\mathfrak{r}=\mathfrak{p}$ or $\mathfrak{r}=\mathfrak{q}$ and each $1\leq j\leq k$, define

$$\zeta_{j,w_{\mathfrak{r}}u_{\mathfrak{r}}}=\overline{\langle(\mathbf{1}_n-(\phi\otimes\mathrm{id}_{M_{\mathfrak{r}}})(p'_{j,\mathfrak{r}})+((\phi\otimes\mathrm{id}_{M_{\mathfrak{r}}})(p'_{j,\mathfrak{r}}))w_{\mathfrak{r}}u_{\mathfrak{r}})(\mathbf{1}_n-(\phi\otimes\mathrm{id}_{M_{\mathfrak{r}}})(q'_{j,\mathfrak{r}})+((\phi\otimes\mathrm{id}_{M_{\mathfrak{r}}})(q'_{j,\mathfrak{r}}))u_{\mathfrak{r}}^*w_{\mathfrak{r}}^*)\rangle}.$$

It is an element in $U(B\otimes M_{\mathfrak{r}})/CU(B\otimes M_{\mathfrak{r}})$.

Define the map $\Gamma_{\mathfrak{r}}:\mathbb{Z}^k\to U(B\otimes M_{\mathfrak{p}})/CU(B\otimes M_{\mathfrak{p}})$ by

$$\Gamma_{\mathfrak{r}}(x'_{j,\mathfrak{r}})=\zeta_{j,w_{\mathfrak{r}}u_{\mathfrak{r}}},\quad 1\le j\le k.$$

Applying Corollary 4.3 to $C_{\mathfrak{r}}$ (in the place of C), $G(x'_{1,\mathfrak{r}},...,x'_{k,\mathfrak{r}})$ (in the place of G), $B\otimes M_{\mathfrak{r}}$ (in the place of A), and $(\phi\otimes\mathrm{id}_{M_{\mathfrak{r}}})|_{C_{\mathfrak{r}}}$ (in the place of ϕ), there is a unitary $c_{\mathfrak{r}}\in B\otimes M_{\mathfrak{r}}$ such that

$$\|[c_{\mathfrak{r}},(\phi\otimes\mathrm{id}_{M_{\mathfrak{r}}})(x)]\|<\delta'_2/16$$

for any $x\in\mathcal{H}^{0'}\otimes\mathcal{H}^{\mathfrak{r}'}$,

$$\mathrm{Bott}(\phi\otimes\mathrm{id}_{M_{\mathfrak{r}}},c_{\mathfrak{r}})|_{\imath_{\mathfrak{r}}(\mathcal{P}')}=0,$$

and

$$\mathrm{dist}(\zeta_{j,c_{\mathfrak{r}}^*},\Gamma_{\mathfrak{r}}(x_{j,\mathfrak{r}}))\le\gamma/(32(1+\sum_{i,j}|Mr_{ij}|)),\quad 1\le j\le k, \tag{5.55}$$

where

$$\zeta_{j,c_{\mathfrak{r}}^*}=\overline{\langle(\mathbf{1}_n-(\phi\otimes\mathrm{id}_{M_{\mathfrak{r}}})(p'_{j,\mathfrak{r}})+((\phi\otimes\mathrm{id}_{M_{\mathfrak{r}}})(p'_{j,\mathfrak{r}}))c_{\mathfrak{r}}^*)(\mathbf{1}_n-(\phi\otimes\mathrm{id}_{M_{\mathfrak{r}}})(q'_{j,\mathfrak{r}})+((\phi\otimes\mathrm{id}_{M_{\mathfrak{r}}})(q'_{j,\mathfrak{r}}))c_{\mathfrak{r}})\rangle}.$$

Put $v_{\mathfrak{r}}=c_{\mathfrak{r}}w_{\mathfrak{r}}u_{\mathfrak{r}}$. Then, by (5.40) and (5.55), for $1\le j\le k$,

$$\mathrm{dist}(\zeta_{j,v_{\mathfrak{r}}},\overline{(1_{B\otimes M_{\mathfrak{r}}})_n})<\mathrm{dist}(\zeta_{j,c_{\mathfrak{r}}^*},\zeta_{j,w_{\mathfrak{r}}u_{\mathfrak{r}}})+\gamma/(32(1+\sum_{i,j}|Mr_{ij}|))<\gamma/(16(1+\sum_{i,j}|Mr_{ij}|)), \tag{5.56}$$

where

$$\zeta_{j,v_{\mathfrak{r}}}=\overline{\langle(\mathbf{1}_n-(\phi\otimes\mathrm{id}_{M_{\mathfrak{r}}})(p'_{j,\mathfrak{r}})+((\phi\otimes\mathrm{id}_{M_{\mathfrak{r}}})(p'_{j,\mathfrak{r}}))v_{\mathfrak{r}})(\mathbf{1}_n-(\phi\otimes\mathrm{id}_{M_{\mathfrak{r}}})(q'_{j,\mathfrak{r}})+((\phi\otimes\mathrm{id}_{M_{\mathfrak{r}}})(q'_{j,\mathfrak{r}}))v_{\mathfrak{r}}^*)\rangle}.$$

Recall that $[x'_j]=[p'_j]-[q'_j]$. Define

$$\zeta_{x'_j,v_{\mathfrak{r}}}=\overline{\langle(\mathbf{1}_n-\phi(p'_j)\otimes 1_{M_{\mathfrak{r}}}+(\phi(p'_j)\otimes 1_{M_{\mathfrak{r}}})v_{\mathfrak{r}})(\mathbf{1}_n-\phi(q'_j)\otimes 1_{M_{\mathfrak{r}}}+(\phi(q'_j)\otimes 1_{M_{\mathfrak{r}}})v_{\mathfrak{r}}^*)\rangle}.$$

By (5.36) and (5.37), one has

$$\mathrm{dist}(\zeta_{x'_j,v_{\mathfrak{r}}},\zeta_{j,v_r})<\gamma/(16(1+\sum_{i,j'}|Mr_{ij'}|)),$$

and hence by (5.56),

$$\mathrm{dist}(\zeta_{x'_j,v_{\mathfrak{r}}},\overline{(1_{B\otimes M_{\mathfrak{r}}})_n})<\gamma/(8(1+\sum_{i,j'}|Mr_{ij'}|)).$$

Regard $\zeta_{x'_j,v_{\mathfrak{r}}}$ as its image in $B\otimes Q$, one has

$$\mathrm{dist}(\zeta_{x'_j,v_{\mathfrak{r}}},\overline{(1_{B\otimes Q})_n})<\gamma/(8(1+\sum_{i,j'}|Mr_{ij'}|)),$$

and hence for any $1\le i\le m$,

$$\mathrm{dist}(\prod_{j=1}^k(\zeta_{x'_j,v_{\mathfrak{r}}})^{Mr_{ij}},\overline{(1_{B\otimes Q})_n})<\gamma/8.$$

By (5.39), one has

$$\mathrm{dist}(\overline{\langle(1-(\phi\otimes \mathrm{id}_Q)(p_i)+(\phi\otimes \mathrm{id}_Q)(p_i)v_{\mathfrak{r}})(1-(\phi\otimes \mathrm{id}_Q)(q_i)+(\phi\otimes \mathrm{id}_Q)(q_i)v_{\mathfrak{r}}^*)\rangle^M}, \overline{(1_{B\otimes Q})_n}) < \gamma/4,$$

and then, by Theorem 6.10 (and Theorem 6.11) of [11],

$$\begin{aligned} & \mathrm{dist}(\overline{\langle(1-(\phi\otimes \mathrm{id}_Q)(p_i)+(\phi\otimes \mathrm{id}_Q)(p_i)v_{\mathfrak{r}})(1-(\phi\otimes \mathrm{id}_Q)(q_i)+(\phi\otimes \mathrm{id}_Q)(q_i)v_{\mathfrak{r}}^*)\rangle}, \overline{(1_{B\otimes Q})_n}) \\ < \;& \gamma/(4M) < \gamma/4. \end{aligned}$$

In particular,

$$\begin{aligned} & \mathrm{dist}(\overline{\langle(1-(\phi\otimes \mathrm{id}_Q)(p_i)+(\phi\otimes \mathrm{id}_Q)(p_i)v_{\mathfrak{q}}v_p^*)(1-(\phi\otimes \mathrm{id}_Q)(q_i)+(\phi\otimes \mathrm{id}_Q)(q_i)v_pv_q^*)\rangle}, \overline{(1_{B\otimes Q})_n}) \\ \le \;& \mathrm{dist}(\overline{\langle(1-(\phi\otimes \mathrm{id}_Q)(p_i)+(\phi\otimes \mathrm{id}_Q)(p_i)v_{\mathfrak{q}})(1-(\phi\otimes \mathrm{id}_Q)(q_i)+(\phi\otimes \mathrm{id}_Q)(q_i)v_{\mathfrak{q}}^*)\rangle}, \overline{(1_{B\otimes Q})_n}) \\ & +\mathrm{dist}(\overline{\langle(1-(\phi\otimes \mathrm{id}_Q)(p_i)+(\phi\otimes \mathrm{id}_Q)(p_i)v_{\mathfrak{p}})(1-(\phi\otimes \mathrm{id}_Q)(q_i)+(\phi\otimes \mathrm{id}_Q)(q_i)v_{\mathfrak{p}}^*)\rangle}, \overline{(1_{B\otimes Q})_n}) \\ < \;& \gamma/2. \end{aligned}$$

That is

$$\mathrm{dist}(\zeta_{i,v_{\mathfrak{q}}v_{\mathfrak{p}}^*}, \overline{1_n}) < \gamma/2, \tag{5.57}$$

where $\zeta_{i,v_{\mathfrak{q}}v_{\mathfrak{p}}^*} = \overline{\langle(1-(\phi\otimes \mathrm{id}_Q)(p_i)+(\phi\otimes \mathrm{id}_Q)(p_i)v_{\mathfrak{q}}v_p^*)(1-(\phi\otimes \mathrm{id}_Q)(q_i)+(\phi\otimes \mathrm{id}_Q)(q_i)v_pv_q^*)\rangle}$.

Moreover, one also has

$$\left\|\psi\otimes \mathrm{id}_Q(x)-v_{\mathfrak{p}}^*(\phi\otimes \mathrm{id}_Q(x))v_{\mathfrak{p}}\right\| < \delta_2'/4, \quad \forall x\in \mathcal{H}^{0'}\otimes \mathcal{H}^{p'}\otimes \mathcal{H}^{\mathfrak{q}'} \text{ and}$$

$$\left\|\psi\otimes \mathrm{id}_Q(x)-v_{\mathfrak{q}}^*(\phi\otimes \mathrm{id}_Q)(x)v_{\mathfrak{q}}\right\| < \delta_2'/4, \quad \forall x\in \mathcal{H}^{0'}\otimes \mathcal{H}^{p'}\otimes \mathcal{H}^{q'}.$$

Hence

$$\left\|[v_{\mathfrak{p}}v_{\mathfrak{q}}^*, \phi(x)\otimes 1_Q]\right\| < \delta_2'/2 < \delta_2, \quad \forall x\in\mathcal{H}'.$$

Thus $\mathrm{Bott}(\phi\otimes \mathrm{id}_Q, v_{\mathfrak{p}}v_{\mathfrak{q}}^*)$ is well defined on the subgroup generated by $\mathcal{P}$. Moreover, a direct calculation shows that

$$\begin{aligned} & \mathrm{bott}_1(\phi\otimes \mathrm{id}_Q, v_{\mathfrak{p}}v_{\mathfrak{q}}^*)\circ(\imath_\infty)_{*1}(z) \\ = \;& \mathrm{bott}_1(\phi\otimes \mathrm{id}_Q, c_{\mathfrak{p}})\circ(\imath_\infty)_{*1}(z)+\mathrm{bott}_1(\phi\otimes \mathrm{id}_Q, w_{\mathfrak{p}})\circ(\imath_\infty)_{*1}(z) \\ & +\mathrm{bott}(\phi\otimes \mathrm{id}_Q, u_{\mathfrak{p}}u_{\mathfrak{q}}^*))\circ(\imath_\infty)_{*1}(z)+\mathrm{bott}_1(\phi\otimes \mathrm{id}_Q, w_{\mathfrak{q}}^*)\circ(\imath_\infty)_{*1}(z) \\ & +\mathrm{bott}_1(\phi\otimes \mathrm{id}_Q, c_{\mathfrak{q}}^*)\circ(\imath_\infty)_{*1}(z) \\ = \;& (j_{\mathfrak{p}})_{*0}\circ\mathrm{bott}_1(\phi\otimes \mathrm{id}_{M_{\mathfrak{p}}}, c_{\mathfrak{p}})\circ(\imath_{\mathfrak{p}})_{*1}(z)+(j_{\mathfrak{p}})_{*0}\circ\mathrm{bott}_1(\phi\otimes \mathrm{id}_{M_{\mathfrak{p}}}, w_{\mathfrak{p}})\circ(\imath_{\mathfrak{p}})_{*1}(z) \\ & +\mathrm{bott}(\phi\otimes \mathrm{id}_Q, u_{\mathfrak{p}}u_{\mathfrak{q}}^*)\circ(\imath_\infty)_{*1}(z)+(j_{\mathfrak{q}})_{*0}\circ\mathrm{bott}_1(\phi\otimes \mathrm{id}_{M_{\mathfrak{q}}}, w_{\mathfrak{q}}^*)\circ(\imath_{\mathfrak{q}})_{*1}(z) \\ & +(j_q)_{*0}\circ\mathrm{bott}_1(\phi\otimes \mathrm{id}_{M_{\mathfrak{q}}}, c_{\mathfrak{q}}^*)\circ(\imath_{\mathfrak{q}})_{*1}(z) \\ = \;& (j_{\mathfrak{p}})_{*0}\circ\mathrm{bott}_1(\phi\otimes \mathrm{id}_{M_{\mathfrak{p}}}, w_{\mathfrak{p}})\circ(\imath_{\mathfrak{p}})_{*1}(z)+\mathrm{bott}(\phi\otimes \mathrm{id}_Q, u_{\mathfrak{p}}u_{\mathfrak{q}}^*)\circ(\imath_\infty)_{*1}(z) \\ & +(j_{\mathfrak{q}})_{*0}\circ\mathrm{bott}_1(\phi\otimes \mathrm{id}_{M_{\mathfrak{q}}}, w_{\mathfrak{q}}^*)\circ(\imath_{\mathfrak{q}})_{*1}(z) \\ = \;& -(j_{\mathfrak{p}})_{*0}\circ\theta_{\mathfrak{p}}\circ(\imath_{\mathfrak{p}})_{*1}(z)+((j_{\mathfrak{p}})_{*0}\circ\theta_{\mathfrak{p}}\circ(\imath_p)_{*1}+(j_{\mathfrak{q}})_{*0}\circ\theta_{\mathfrak{q}}\circ(\imath_q)_{*1})-(j_{\mathfrak{q}})_{*0}\circ\theta_{\mathfrak{q}}\circ(\imath_q)_{*1}(z) \\ = \;& 0 \quad \text{ for all } z\in G(\mathcal{P})_{1,0}. \end{aligned}$$

The same argument shows that $\mathrm{bott}_0(\phi\otimes \mathrm{id}_Q, v_{\mathfrak{p}}v_{\mathfrak{q}}^*)=0$ on $G(\mathcal{P})_{0,0}$. Now, for any $g\in G(\mathcal{P})_{1,\infty,0}$, there is $z\in G(\mathcal{P})_{1,0}$ and integers k,m such that $(k/m)z=g$. From the above,

$$\mathrm{bott}_1(\phi\otimes \mathrm{id}_Q, v_{\mathfrak{p}}v_{\mathfrak{q}}^*)(mg) = k\mathrm{bott}_1(\phi\otimes \mathrm{id}_Q, v_{\mathfrak{p}}v_{\mathfrak{q}}^*)(z)=0. \tag{5.58}$$

Since $K_0(B\otimes Q)$ is torsion free, it follows that

$$\mathrm{bott}_1(\phi\otimes \mathrm{id}_Q, v_{\mathfrak{p}}v_{\mathfrak{q}}^*)(g)=0$$

for all $g\in G(\mathcal{P})_{1,\infty,0}$. So it vanishes on $\mathcal{P}\cap K_1(A\otimes Q)$. Similarly,

$$\mathrm{bott}_0(\phi\otimes \mathrm{id}_Q, v_{\mathfrak{p}}v_{\mathfrak{q}}^*)|_{\mathcal{P}\cap K_0(A\otimes Q)}=0$$

on $\mathcal{P}\cap K_0(A\otimes Q)$.

Since $K_i(B\otimes Q, \mathbb{Z}/m\mathbb{Z})=\{0\}$ for all $m\ge 2$, we conclude that

$$\mathrm{Bott}(\phi\otimes \mathrm{id}_Q, v_{\mathfrak{p}}v_{\mathfrak{q}}^*)|_{\mathcal{P}}=0$$

on the subgroup generated by $\mathcal{P}$.

Since $[\phi]=[\psi]$ in $KL(A,B)$, $\phi_\sharp=\psi_\sharp$ and $\phi^\ddagger=\psi^\ddagger$, one has that

$$[\phi\otimes \mathrm{id}_Q]=[\psi\otimes \mathrm{id}_Q] \text{ in } KL(A\otimes Q, B\otimes Q), \tag{5.59}$$

$$(\phi\otimes \mathrm{id}_Q)_\sharp=(\psi\otimes \mathrm{id}_Q)_\sharp \text{ and } (\phi\otimes \mathrm{id}_Q)^\ddagger=(\psi\otimes \mathrm{id}_Q)^\ddagger. \tag{5.60}$$

Therefore, by 5.10 of [21], $\phi\otimes \mathrm{id}_Q$ and $\psi\otimes \mathrm{id}_Q$ are approximately unitarily equivalent. Thus there exists a unitary $u\in B\otimes Q$ such that

$$\|u^*(\phi\otimes \mathrm{id}_Q)(c)u-(\psi\otimes \mathrm{id}_Q)(c)\|<\delta_2'/8 \text{ for all } c\in \mathcal{E}\cup\mathcal{H}'. \tag{5.61}$$

It follows that

$$\left\|uv_{\mathfrak{p}}^*(\phi(c)\otimes 1_Q)v_{\mathfrak{p}}u^*-\psi(c)\otimes 1_Q\right\|<\delta_2'/2+\delta_2'/8 \quad \forall c\in\mathcal{G}'.$$

By the choice of δ_2' and $\mathcal{H}'$, $\mathrm{Bott}(\phi\otimes \mathrm{id}_Q, v_{\mathfrak{p}}u^*)$ is well defined on $[\iota](\underline{K}(C'))$, and

$$|\tau(\mathrm{bott}_1(\phi\otimes \mathrm{id}_Q, v_{\mathfrak{p}}u^*)(z))|<\delta_2/2, \quad \forall\tau\in \mathrm{T}(B), \forall z\in\mathcal{G}.$$

By Theorem 4.4, there exists a unitary $y_{\mathfrak{p}}\in B\otimes Q$ such that

$$\|[y_{\mathfrak{p}}, (\phi\otimes \mathrm{id}_Q)(h)]\|<\delta/2, \quad \forall h\in\mathcal{H},$$

and $\mathrm{Bott}(\phi\otimes \mathrm{id}_Q, y_{\mathfrak{p}})=\mathrm{Bott}(\phi\otimes \mathrm{id}_Q, v_{\mathfrak{p}}u^*)$ on the subgroup generated by $\mathcal{P}$.

For each $1\le i\le m$, define

$$\begin{aligned}&\zeta_{i,y_{\mathfrak{p}}uv_{\mathfrak{p}}^*}\\ &=\overline{\langle(\mathbf{1}_n-(\phi\otimes \mathrm{id}_Q)(p_i)+((\phi\otimes \mathrm{id}_Q)(p_i))y_{\mathfrak{p}}uv_{\mathfrak{p}}^*)(\mathbf{1}_n-(\phi\otimes \mathrm{id}_Q)(q_i)+((\phi\otimes \mathrm{id}_Q)(q_i))v_{\mathfrak{p}}u^*y_{\mathfrak{p}}^*)\rangle},\end{aligned}$$

and define the map $\Gamma:\mathbb{Z}^m\to U(B\otimes Q)/CU(B\otimes Q)$ by $\Gamma(x_i)=\zeta_{i,y_{\mathfrak{p}}uv_{\mathfrak{q}}^*}$.

Applying Corollary 4.3 to C and $G(\mathcal{Q})$, there is a unitary $c\in B\otimes Q$ such that

$$\|[c, (\phi\otimes \mathrm{id}_Q)(h)]\|<\delta/4, \quad \forall h\in\mathcal{H}$$

$$\mathrm{Bott}(\phi\otimes \mathrm{id}_Q, c)|_{\mathcal{P}}=0$$

and for any $1\le i\le k$,

$$\mathrm{dist}(\zeta_{i,c^*}', \Gamma(x_i))\le\gamma/2,$$

where

$$\zeta'_{i,c^*} = \overline{\langle(\mathbf{1}_n - (\phi\otimes \mathrm{id}_Q)(p_i) + (\phi\otimes \mathrm{id}_Q)(p_i)c^*)(\mathbf{1}_n - (\phi\otimes \mathrm{id}_Q)(q_i) + (\phi\otimes \mathrm{id}_Q)(q_i)c)\rangle}.$$

Consider the unitary $v = cy_{\mathfrak{p}}u$, one has that

$$\|[v, (\phi\otimes \mathrm{id}_Q)(h)]\| < \delta, \text{ for all } h \in \mathcal{H} \text{ and } \mathrm{Bott}(\phi\otimes \mathrm{id}_Q, vv_{\mathfrak{p}}^*) = 0$$

on the subgroup generated by $\mathcal{P}$, and for any $1 \le i \le m$,

$$\mathrm{dist}(\zeta_{i,vv_{\mathfrak{p}}^*}, \overline{1_n}) < \gamma/2, \tag{5.62}$$

where

$$\zeta_{i,vv_{\mathfrak{p}}^*} = \overline{\langle(\mathbf{1}_n - (\phi\otimes \mathrm{id}_Q)(p_i) + ((\phi\otimes \mathrm{id}_Q)(p_i))vv_{\mathfrak{p}}^*)(\mathbf{1}_n - (\phi\otimes \mathrm{id}_Q)(q_i) + ((\phi\otimes \mathrm{id}_Q)(q_i))v_{\mathfrak{p}}v^*)\rangle}.$$

By the construction of Δ, it is clear that

$$\mu_{\tau\circ(\psi\otimes 1)}(O_a) \ge \Delta(a)$$

for all a, where O_a is any open ball of X with radius a; in particular, it holds for all $a \ge d$. Applying Theorem 4.2 to C and $(\phi\otimes \mathrm{id}_Q)|_C$, one obtains a continuous path of unitaries $v(t)$ in $B\otimes Q$ such that $v(0) = 1$ and $v(t_1) = vv_{\mathfrak{p}}^*$, and

$$\|[z_p(t), (\phi\otimes \mathrm{id}_Q)(c)]\| < \epsilon/2 \quad \forall x \in \mathcal{E},\ \forall t \in [0, t_1]. \tag{5.63}$$

Note that

$$\begin{aligned}
\mathrm{Bott}(\phi\otimes \mathrm{id}_Q, v_{\mathfrak{q}}v^*) &= \mathrm{Bott}(\phi\otimes \mathrm{id}_Q, v_{\mathfrak{q}}v_{\mathfrak{p}}^*v_{\mathfrak{p}}v^*) && (5.64)\\
&= \mathrm{Bott}(\phi\otimes \mathrm{id}_Q, v_{\mathfrak{q}}v_{\mathfrak{p}}^*) + \mathrm{Bott}(\phi\otimes \mathrm{id}_Q, v_{\mathfrak{p}}v^*) && (5.65)\\
&= 0 + 0 = 0 && (5.66)
\end{aligned}$$

on the subgroup generated by $\mathcal{P}$, and for any $1 \le i \le m$,

$$\begin{aligned}
&\mathrm{dist}(\zeta_{i,v_{\mathfrak{q}}v^*}, \overline{1}) && (5.67)\\
\le\ &\mathrm{dist}(\zeta_{i,v_{\mathfrak{q}}v_{\mathfrak{p}}^*}, \overline{1}) + \mathrm{dist}(\zeta_{i,v_{\mathfrak{p}}v^*}, \overline{1}) && (5.68)\\
=\ &\gamma, \quad \text{(by (5.57) and (5.62))} && (5.69)
\end{aligned}$$

where

$$\zeta_{i,v_qv^*} = \overline{\langle(1 - (\phi\otimes \mathrm{id}_Q)(p_i) + (\phi\otimes \mathrm{id}_Q)(p_i)v_{\mathfrak{q}}v^*)(1 - (\phi\otimes \mathrm{id}_Q)(q_i) + (\phi\otimes \mathrm{id}_Q)(q_i)vv_{\mathfrak{q}}^*)\rangle}$$

Since

$$\|[vv_{\mathfrak{q}}^*, (\phi\otimes \mathrm{id}_Q)(c)]\| < \delta, \quad \forall c \in \mathcal{H},$$

Theorem 4.2 implies that there is a path of unitaries $z_{\mathfrak{q}}(t) : [t_{m-1}, 1] \to \mathrm{U}(A\otimes Q)$ such that $z_{\mathfrak{q}}(t_{m-1}) = vv_{\mathfrak{q}}^*$, $z_{\mathfrak{q}}(1) = 1$ and

$$\|[z_{\mathfrak{q}}(t), \phi\otimes \mathrm{id}_Q(c)]\| < \epsilon/8, \quad \forall t \in [t_{m-1}, 1],\ \forall c \in \mathcal{E}. \tag{5.70}$$

Consider the unitary

$$v(t)=\begin{cases} z_{\mathfrak{p}}(t)v_{\mathfrak{p}}, & \text{if } 0\le t\le t_1,\\ v, & \text{if } t_1\le t\le t_{m-1},\\ z_{\mathfrak{q}}(t)v_{\mathfrak{q}}, & \text{if } t_{m-1}\le t\le t_m.\end{cases}$$

Then, for any t_i, $0\le i\le m$, one has that

$$\|v^*(t_i)(\phi\otimes \mathrm{id}_Q)(c)v(t_i)-(\psi\otimes\mathrm{id}_Q)(c)\|<\epsilon/2,\quad \forall c\in\mathcal{E}. \tag{5.71}$$

Then for any $t\in[t_j,t_{j+1}]$ with $1\le j\le m-2$, one has

$$\|v^*(t)(\phi\otimes\mathrm{id}(a\otimes b(t)))v(t)-\psi\otimes\mathrm{id}(a\otimes b(t))\| \tag{5.72}$$
$$= \|v^*(\phi(a)\otimes b(t))v-\psi(a)\otimes b(t)\| \tag{5.73}$$
$$< \|v^*(\phi(a)\otimes b(t_j))v-\psi(a)\otimes b(t_j)\|+\epsilon/4 \tag{5.74}$$
$$< \epsilon/4+\epsilon/4<\epsilon/2. \tag{5.75}$$

For any $t\in[0,t_1]$, one has that for any $a\in\mathcal{F}_1$ and $b\in\mathcal{F}_2$,

$$\|v^*(t)(\phi\otimes\mathrm{id}(a\otimes b(t)))v(t)-\psi\otimes\mathrm{id}(a\otimes b(t))\| \tag{5.76}$$
$$= \|v_{\mathfrak{p}}^*z_{\mathfrak{p}}^*(t)(\phi(a)\otimes b(t))z_{\mathfrak{p}}(t)v_{\mathfrak{p}}-\psi(a)\otimes b(t)\| \tag{5.77}$$
$$< \|v_{\mathfrak{p}}^*z_{\mathfrak{p}}^*(t)(\phi(a)\otimes b(t_0))z_{\mathfrak{p}}(t)v_{\mathfrak{p}}-\psi(a)\otimes b(t_0)\|+\epsilon/2 \tag{5.78}$$
$$< \|v_{\mathfrak{p}}^*(\phi(a)\otimes b(t_0))v_{\mathfrak{p}}-\psi(a)\otimes b(t_0)\|+3\epsilon/4 \tag{5.79}$$
$$< 3\epsilon/4+\epsilon/4=\epsilon. \tag{5.80}$$

The same argument shows that for any $t\in[t_{m-1},1]$, one has that for any $a\in\mathcal{F}_1$ and $b\in\mathcal{F}_2$,

$$\|v^*(t)(\phi\otimes\mathrm{id}(a\otimes b(t)))v(t)-\psi\otimes\mathrm{id}(a\otimes b(t))\|<\epsilon. \tag{5.81}$$

Therefore, one has

$$\|v(\phi\otimes\mathrm{id}(f))v-\psi\otimes\mathrm{id}(f)\|<\epsilon \ \text{ for all } \ f\in\mathcal{F}.$$

□

Remark 5.7. In fact, using the same argument as the lemma above, one has the following: Let A and B be two unital stably finite C*-algebras. Assume that, for any UHF-algebra U of infinite type,

(1) the approximately unitarily equivalence classes of the monomorphisms from $A\otimes U$ to $B\otimes U$ is classified by the induced elements in $KL(A\otimes U, B\otimes U)$, the induced maps on traces, together with the induced maps from $U_\infty(A\otimes U)/CU_\infty(A\otimes U)$ to $U_\infty(B\otimes U)/CU_\infty(B\otimes U)$,
(2) $B\otimes U$ satisfies Theorem 4.4 with respect to any embedding of $A\otimes U$,
(3) $B\otimes U$ satisfies a homotopy lemma, such as Theorem 4.2 or Lemma 8.4 of [17], for any embedding of $A\otimes U$ to $B\otimes U$,

then, for any monomorphisms $\phi,\psi: A\to B$, the maps $\phi\otimes\mathrm{id}$ and $\psi\otimes\mathrm{id}$ from $A\otimes\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ to $B\otimes\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ are approximately unitarily equivalent if and only if

$$[\phi]=[\psi]\ \text{ in }\ KL(A,B),\ \phi_\sharp=\psi_\sharp\ \text{ and }\ \phi^\ddagger=\psi^\ddagger. \tag{5.82}$$

Theorem 5.8. *Let A be a $\mathcal{Z}$-stable C^*-algebra such that $A\otimes M_{\mathfrak{r}}$ is an AH-algebra for any supernatural number $\mathfrak{r}$ of infinite type, and let $B\in\mathcal{C}$ be a unital separable $\mathcal{Z}$-stable C^*-algebras. If ϕ and ψ are two monomorphisms from A to B with*

$$[\phi]=[\psi]\ \text{ in }\ KL(A,B),\ \ \phi_\sharp=\psi_\sharp\ \text{ and }\ \phi^\ddagger=\psi^\ddagger, \tag{5.83}$$

then, for any $\epsilon>0$ and any finite subset $\mathcal{F}\subseteq A$, there exists a unitary $u\in B$ such that

$$\|u^*\phi(a)u-\psi(a)\|<\epsilon\ \text{ for all }\ a\in\mathcal{F}. \tag{5.84}$$

Proof. Let $\alpha: A\to A\otimes\mathcal{Z}$ and $\beta:\mathcal{Z}\to\mathcal{Z}\otimes\mathcal{Z}$ be isomorphisms. Consider the map

$$\Gamma_A:\ A\xrightarrow{\alpha}A\otimes\mathcal{Z}\xrightarrow{\mathrm{id}\otimes\beta}A\otimes\mathcal{Z}\otimes\mathcal{Z}\xrightarrow{\alpha^{-1}\otimes\mathrm{id}}A\otimes\mathcal{Z}\ .$$

Then Γ is an isomorphism. However, since β is approximately unitarily equivalent to the map

$$\mathcal{Z}\ni a\mapsto a\otimes 1\in\mathcal{Z}\otimes\mathcal{Z},$$

the map Γ_A is approximately unitarily equivalent to the map

$$A\ni a\mapsto a\otimes 1\in A\otimes\mathcal{Z}.$$

Hence the map $\Gamma_B\circ\phi\circ\Gamma_A$ is approximately unitarily equivalent to $\phi\otimes\mathrm{id}_{\mathcal{Z}}$. The same argument shows that $\Gamma_B\circ\psi\circ\Gamma_A$ is approximately unitarily equivalent to $\psi\otimes\mathrm{id}_{\mathcal{Z}}$. Thus, in order to prove the theorem, it is enough to show that $\phi\otimes\mathrm{id}_{\mathcal{Z}}$ is approximately unitarily equivalent to $\psi\otimes\mathrm{id}_{\mathcal{Z}}$.

Since $\mathcal{Z}$ is an inductive limit of C*-algebras $\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$, it is enough to show that $\phi\otimes\mathrm{id}_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}}$ is approximately unitarily equivalent to $\psi\otimes\mathrm{id}_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}}$, and this follows from Lemma 5.6. ☐

6. The range of approximate equivalence classes of homomorphisms

Now let A and B be two unital C*-algebras in $\mathcal{N}\cap\mathcal{C}$. Theorem 5.8 states that two unital monomorphisms are approximately unitarily equivalent if they induce the same element in $KLT_e(A,B)^{++}$ and the same map on $U(A)/CU(A)$. In this section, we will discuss the following problem: Suppose that one has $\kappa\in KLT_e(A,B)^{++}$ and a continuous homomorphism $\gamma: U(A)/CU(A)\to U(B)/CU(B)$ which is compatible with κ. Is there always a unital monomorphism $\phi: A\to B$ such that ϕ induces κ and $\phi^\ddagger=\gamma$? At least in the case that $K_1(A)$ is free, Theorem 6.10 states that such ϕ always exists.

Lemma 6.1. *Let A and B be two unital infinite dimensional separable stably finite C*-algebras whose tracial simplexes are non-empty. Let $\gamma: U_\infty(A)/CU_\infty(A)\to U_\infty(B)/CU_\infty(B)$ be a continuous homomorphism, $h_i: K_i(A)\to K_i(B)$ ($i=0,1$) be homomorphisms for which h_0 is positive, and let $\lambda:\mathrm{Aff}(\mathrm{T}(A))\to\mathrm{Aff}(\mathrm{T}(B))$ be an affine map so that (h_0,h_1,λ,γ) are compatible. Let $\mathfrak{p}$ be a supernatural number. Then γ induces a unique homomorphism $\gamma_{\mathfrak{p}}: U_\infty(A_{\mathfrak{p}})/CU_\infty(A_{\mathfrak{p}})\to U_\infty(B_{\mathfrak{p}})/CU_\infty(B_{\mathfrak{p}})$ which is compatible with $(h_{\mathfrak{p}})_i$ ($i=0,1$) and $\gamma_{\mathfrak{p}}$, where $A_{\mathfrak{p}}=A\otimes M_{\mathfrak{p}}$ and $B_{\mathfrak{p}}=B\otimes M_{\mathfrak{p}}$, and $(h_{\mathfrak{p}})_i: K_i(A)\otimes\mathbb{Q}_{\mathfrak{p}}\to K_i(B)\otimes\mathbb{Q}_{\mathfrak{p}}$ is induced by h_i*

($i = 0, 1$). Moreover, the diagram

$$\begin{array}{ccc} U_\infty(A)/CU_\infty(A) & \xrightarrow{\gamma} & U_\infty(B)/CU_\infty(B) \\ \downarrow_{\imath_{\mathfrak{p}}^{\ddagger}} & & \downarrow_{(\imath'_{\mathfrak{p}})^{\ddagger}} \\ U_\infty(A_{\mathfrak{p}})/CU(A_{\mathfrak{p}}) & \xrightarrow{\gamma_{\mathfrak{p}}} & U_\infty(B_{\mathfrak{p}})/CU_\infty(B_{\mathfrak{p}}) \end{array}$$

commutes, where $\imath_{\mathfrak{p}} : A \to A_{\mathfrak{p}}$ and $\imath'_{\mathfrak{p}} : B \to B_{\mathfrak{p}}$ are the maps induced by $a \mapsto a \otimes 1$ and $b \mapsto b \otimes 1$, respectively.

Proof. Denote by $A_0 = A$, $A_{\mathfrak{p}} = A \otimes M_{\mathfrak{p}}$, $B_0 = B$ and $B_{\mathfrak{p}} = B \otimes M_{\mathfrak{p}}$. By a result of K. Thomsen ([31]), using the de la Harpe and Skandalis determinant, one has the following short exact sequences:

$$0 \to \mathrm{Aff}(\mathrm{T}(A_i))/\overline{\rho_A(K_0(A_i))} \to U_\infty(A_i)/CU_\infty(A_i) \to K_1(A_i) \to 0, \;\; i = 0, \mathfrak{p},$$

and

$$0 \to \mathrm{Aff}(\mathrm{T}(B_i))/\overline{\rho_A(K_0(B_i))} \to U_\infty(B_i)/CU_\infty(B_i) \to K_1(B_i) \to 0, \;\; i = 0, \mathfrak{p}.$$

Note that, in all these cases, $\mathrm{Aff}(\mathrm{T}(A_i))/\overline{\rho_A(K_0(A_i))}$ and $\mathrm{Aff}(\mathrm{T}(B_i))/\overline{\rho_A(K_0(B_i))}$ are divisible groups, $i = 0, \mathfrak{p}$. Therefore the exact sequences above splits. Fix splitting maps $s_i : K_1(A_i) \to U_\infty(A)/CU_\infty(A_i)$ and $s'_i : K_1(B_i) \to U_\infty(B_i)/CU_\infty(B_i)$, $i = 0, \mathfrak{p}$, for the above two splitting short exact sequences. Let $\imath_{\mathfrak{p}} : A \to A_{\mathfrak{p}}$ be the homomorphism defined by $\imath_{\mathfrak{p}}(a) = a \otimes 1$ for all $a \in A$ and $\imath'_{\mathfrak{p}} : B \to B_{\mathfrak{p}}$ be the homomorphism defined by $\imath'_{\mathfrak{p}}(b) = b \otimes 1$ for all $b \in B$. Let $\imath_{\mathfrak{p}}^{\ddagger} : U_\infty(A)/CU_\infty(A) \to U_\infty(A_{\mathfrak{p}})/CU_\infty(A)$ and $(\imath'_{\mathfrak{p}})^{\ddagger} : U_\infty(B)/CU_\infty(B) \to U_\infty(B_{\mathfrak{p}})/CU_\infty(B_{\mathfrak{p}})$ be the induced maps. The map $\imath_{\mathfrak{p}}$ induces the following commutative diagram:

$$\begin{array}{ccccccc} 0 \to & \mathrm{Aff}(\mathrm{T}(A))/\overline{\rho_A(K_0(A))} & \to U_\infty(A)/CU_\infty(A) & \to K_1(A) & \to 0 \\ & \downarrow_{\overline{(\imath_{\mathfrak{p}})_\sharp}} & \downarrow_{\imath_{\mathfrak{p}}^{\ddagger}} & \downarrow_{(\imath_{\mathfrak{p}})_{*1}} & \\ 0 \to & \mathrm{Aff}(\mathrm{T}(A_{\mathfrak{p}}))/\overline{\rho_A(K_0(A_{\mathfrak{p}}))} & \to U_\infty(A_i)/CU_\infty(A_{\mathfrak{p}}) & \to K_1(A_{\mathfrak{p}}) & \to 0. \end{array}$$

Since there is only one tracial state on $M_{\mathfrak{p}}$, one may identify $\mathrm{T}(A)$ with $\mathrm{T}(A_{\mathfrak{p}})$ and $\mathrm{T}(B)$ with $\mathrm{T}(B_{\mathfrak{p}})$. One may also identify $\overline{\rho_{A_{\mathfrak{p}}}(K_0(A_{\mathfrak{p}}))}$ with $\overline{\mathbb{R}\rho_A(K_0(A))}$ which is the closure of those elements $r\widehat{[p]}$ with $r \in \mathbb{R}$. Note that $(h_{\mathfrak{p}})_i : K_i(A \otimes M_{\mathfrak{p}}) \to K_i(B \otimes M_{\mathfrak{p}})$ $(i = 0, 1)$ is given by the Künneth formula. Since γ is compatible with λ, γ maps $\overline{\mathbb{R}\rho_A(K_0(A))}/\overline{\rho_A(K_0(A))}$ into $\overline{\mathbb{R}\rho_B(K_0(B))}/\overline{\rho_B(K_0(B))}$. Note that

$$\ker(\imath_{\mathfrak{p}})_{*1} = \{x \in K_1(A) : px = 0 \;\text{ for some factor } p \text{ of } \mathfrak{p}\} \;\text{ and} \tag{6.1}$$

$$\ker(\imath'_{\mathfrak{p}})_{*1} = \{x \in K_1(B) : px = 0 \;\text{ for some factor } p \text{ of } \mathfrak{p}\}. \tag{6.2}$$

Therefore

$$\ker(\imath_{\mathfrak{p}}^{\ddagger}) = \{x + s_0(y) : x \in \overline{\mathbb{R}\rho_A(K_0(A))}/\overline{\rho_A(K_0(A))}, y \in \ker((\imath_{\mathfrak{p}})_{*1})\} \;\text{ and} \tag{6.3}$$

$$\ker(\imath'_{\mathfrak{p}})^{\ddagger} = \{x + s'_0(y) : x \in \overline{\mathbb{R}\rho_A(K_0(B))}/\overline{\rho_B(K_0(B))},\, y \in \ker((\imath'_{\mathfrak{p}})_{*1})\}. \tag{6.4}$$

If $y \in \ker((\imath_{\mathfrak{p}})_{*1})$, then, for some factor p of $\mathfrak{p}$, $py = 0$. It follows that $p\gamma(s_0(y)) = 0$. Therefore $\gamma(s_0(y))$ must be in $\ker((\imath'_{\mathfrak{p}})^{\ddagger})$. It follows that

$$\gamma(\ker(\imath_{\mathfrak{p}}^{\ddagger})) \subset \ker((\imath'_{\mathfrak{p}})^{\ddagger}). \tag{6.5}$$

This implies that γ induces a unique homomorphism $\gamma_{\mathfrak{p}}$ such that the following diagram commutes:

$$\begin{array}{ccc} U_\infty(A)/CU_\infty(A) & \xrightarrow{\gamma} & U_\infty(B)/CU_\infty(B) \\ \downarrow_{\imath_{\mathfrak{p}}^{\ddagger}} & & \downarrow_{(\imath'_{\mathfrak{p}})^{\ddagger}} \\ U_\infty(A_{\mathfrak{p}})/CU_\infty(A_{\mathfrak{p}}) & \xrightarrow{\gamma_{\mathfrak{p}}} & U_\infty(B_{\mathfrak{p}})/CU_\infty(B_{\mathfrak{p}}). \end{array}$$

The lemma follows. □

Lemma 6.2. *Let A and B be two unital infinite dimensional separable stably finite C*-algebras whose tracial simplexes are non-empty. Let $\gamma : U_\infty(A)/CU_\infty(A) \to U_\infty(B)/CU_\infty(B)$ be a continuous homomorphism, $h_i : K_i(A) \to K_i(B)$ $(i = 0, 1)$ be homomorphisms and $\lambda : \mathrm{Aff}(\mathrm{T}(A)) \to \mathrm{Aff}(\mathrm{T}(B))$ be an affine homomorphism which are compatible. Let $\mathfrak{p}$ and $\mathfrak{q}$ be two relatively prime supernatural numbers such that $M_{\mathfrak{p}} \otimes M_{\mathfrak{q}} = Q$. Denote by ∞ the supernatural number associated with the product $\mathfrak{p}$ and $\mathfrak{q}$. Let $E_B : B \to B \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ be the embedding defined by $E_B(b) = b \otimes 1$, $\forall b \in B$. Then*

$$(\pi_t \circ E_B)^{\ddagger} \circ \gamma = \gamma_\infty \circ \imath_\infty^{\ddagger} \ \text{ for all } t \in (0,1), \tag{6.6}$$

$$(\pi_0 \circ E_B)^{\ddagger} \circ \gamma = \gamma_{\mathfrak{p}} \circ \imath_{\mathfrak{p}}^{\ddagger}, \ \text{ and} \tag{6.7}$$

$$(\pi_1 \circ E_B)^{\ddagger} \circ \gamma = \gamma_{\mathfrak{q}} \circ \imath_{\mathfrak{q}}^{\ddagger}, \tag{6.8}$$

with the notation of 6.1, where $\pi_t : \mathcal{Z}_{\mathfrak{p},\mathfrak{q}} \to Q$ is the point-evaluation at t.

Proof. Fix $z \in U_\infty(B)/CU_\infty(B)$. Let $u \in U_n(B)$ for some integer $n \geq 1$ such that $\overline{u} = z$ in $U_\infty(B)/CU_\infty(B)$. Then

$$E_B^{\ddagger}(z) = \overline{u \otimes 1}. \tag{6.9}$$

In other words, $E_B^{\ddagger}(z)$ is represented by $w(t) \in M_n(B \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}})$ for which

$$w(t) = u \otimes 1 \ \text{ for all } t \in [0,1]. \tag{6.10}$$

Therefore, for any $t \in (0,1)$, $\pi_t \circ E_B^{\ddagger}(z)$ may be written as

$$\pi_t \circ E_B^{\ddagger}(z) = \overline{u \otimes 1} \ \text{ in } U_\infty(B \otimes Q)/CU_\infty(B \otimes Q). \tag{6.11}$$

This implies that

$$\pi_t \circ E_B^{\ddagger}(z) = (\imath_\infty)^{\ddagger}(z) \ \text{ for all } z \in U_\infty(B)/CU_\infty(B), \tag{6.12}$$

where $\imath_\infty : B \to B \otimes Q$ is defined by $\imath_\infty(b) = b \otimes 1$ for all $b \in B$. It follows from 6.1 that

$$(\pi_t \circ E_B)^{\ddagger} \circ \gamma = \gamma_\infty \circ \imath_\infty^{\ddagger} \ \text{ for all } t \in (0,1). \tag{6.13}$$

The identities (6.7) and (6.8) for end points exactly follow from the same arguments. □

The following is standard (see the proof of 9.6 of [18]).

Lemma 6.3. *Let C and A be two unital separable stably finite C*-algebras, and let $\phi_1, \phi_2, \phi_3 : C \to A$ be three unital homomorphisms. Suppose that*

$$[\phi_1] = [\phi_2] = [\phi_3] \ \text{ in } KL(C,A), \tag{6.14}$$

$$(\phi_1)_\sharp = (\phi_2)_\sharp = (\phi_3)_\sharp. \tag{6.15}$$

Then

$$\overline{R}_{\phi_1,\phi_3} = \overline{R}_{\phi_1,\phi_2} + \overline{R}_{\phi_2,\phi_3}. \tag{6.16}$$

Lemma 6.4. *(cf. Theorem 4.2 of* [23]*) Let A be a unital infintie dimensional separable simple C^*-algebra with $\mathrm{T}(A) \le 1$, let $C \subset A$ be a unital C^*-subalgebra which is a unital AH-algebra an let $\imath : C \to A$ be the embedding. For any $\lambda \in \mathrm{Hom}(K_0(C), \overline{\rho_A(K_0(A))})$, there exists $\phi \in \overline{\mathrm{Inn}}(C, A)$ such that there are homomorphisms $\theta_i : K_i(C) \to K_i(M_{\imath,\phi})$ with $(\pi_0)_{*i}\theta_i = \mathrm{id}_{K_i(C)}$, $i = 0, 1$, and the rotation map $R_{\imath,\phi} : K_1(C) \to \mathrm{Aff}(T(A))$ is given by*

$$R_{\imath,\phi}(x) = \rho_A(x - \theta_1((\pi_0)_{*1}(x)) + \lambda \circ (\pi_0)_{*1}(x)) \ \ \textit{for all} \ \ x \in K_1(M_{\imath,\phi}). \tag{6.17}$$

In other words,

$$[\phi] = [\imath] \ \ \textit{in} \ \ KK(C, A) \tag{6.18}$$

and the rotation map $R_{\phi,\psi} : K_1(M_{\imath,\psi}) \to \mathrm{A}ff(T(A))$ is given by

$$R_{\imath,\phi}(a, b) = \rho_A(a) + \lambda(b) \tag{6.19}$$

for some identification of $K_1(M_{\imath,\psi})$ with $K_0(A) \oplus K_1(C)$.

Proof. This follows from the proof of Theorem 4.2 of [23]. In Theorem 4.2 of [23], it is assumed that $\rho_A(K_0(A))$ is dense in $\mathrm{Aff}(T(A))$. However, in fact, it is the condition $\lambda(K_1(C)) \subset \overline{\rho_A(K_0(A))}$ that is used. Note that, by Theorem 3.10 of [25], A has property (B1) and (B2) associated C and a constant Δ_C (3.6 and 3.8 of [23]) . Thus this lemma follows exactly the same proof. □

Lemma 6.5. *Let A be a unital AH-algebra and let B be a unital separable simple amenable C^*-algebra with $TR(B) \le 1$. Suppose that $\phi_1, \phi_2 : A \to B$ are two monomorphisms such that*

$$[\phi_1] = [\phi_2] \ \ \textit{in} \ \ KK(A, B), \ \ (\phi_1)_\sharp = (\phi_2)_\sharp \ \ \textit{and} \ \ \phi_1^\ddagger = \phi_2^\ddagger. \tag{6.20}$$

Then there exists a monomorphism $\beta : \phi_2(A) \to B$ such that $[\beta \circ \phi_2] = [\phi_2]$ in $KK(A, B)$, $(\beta \circ \phi_2)_\sharp = \phi_{2\sharp}$, $(\beta \circ \phi_2)^\ddagger = \phi_2^\ddagger$ and $\beta \circ \phi_2$ is asymptotically unitarily equivalent to ϕ_1. Moreover, if $H_1(K_0(A), K_1(B)) = K_1(B)$, they are strongly asymptotically unitarily equivalent, where $H_1(K_0(A), K_1(B)) = \{x \in K_1(B) : \psi([1_A]) = x \ \ \mathrm{f}or \ \ some \ \ \psi \in \mathrm{Hom}(K_0(A), K_1(B))\}$.

Proof. By Lemma 6.4, there is a monomorphism $\beta \in \overline{\mathrm{Inn}}(\phi_2(A), B)$ such that $[\beta] = [\imath]$ in $KK(\phi_2(A), B)$ and

$$\overline{R}_{\imath,\beta} = -\overline{R}_{\phi_1,\phi_2}$$

where $\imath$ is the embedding of $\phi_2(A)$ to B and $\overline{R}_{\imath,\beta}$ is viewed as a homomorphism from $K_1(A) = K_1(\phi_2(A))$ to $\mathrm{Aff}(\mathrm{T}(B))$. In other words

$$\overline{R}_{\phi_2,\beta\circ\phi_2} = -\overline{R}_{\phi_1,\phi_2}. \tag{6.21}$$

One also has that

$$[\phi_2] = [\beta \circ \phi_2] \ \ \text{in} \ \ KK(A, B), \tag{6.22}$$

$$(\beta \circ \phi_2)_\sharp = (\phi_2)_\sharp \ \ \text{and} \ \ (\beta \circ \phi_2)^\ddagger = \phi_2^\ddagger. \tag{6.23}$$

Thus

$$[\phi_1] = [\beta \circ \phi_2] \text{ in } KK(A, B), \tag{6.24}$$

$$(\phi_1)_\sharp = (\beta \circ \phi_2)_\sharp \text{ and } \phi_1^\ddagger = (\beta \circ \phi_2)^\ddagger. \tag{6.25}$$

It follows from 6.3 and (6.21) that

$$\overline{R}_{\phi_1, \beta\circ\phi_2} = \overline{R}_{\phi_1,\phi_2} + \overline{R}_{\phi_2, \beta\circ\phi_2} = 0.$$

Therefore, it follows from Theorem 4.2 of [25] that the map ϕ_1 and $\beta \circ \phi_2$ are asymptotically unitarily equivalent.

In the case that $H_1(K_0(A), K_1(B)) = K_1(B)$, it follows from Theorem 4.4 of [25] that $\beta \circ \phi_2$ and ϕ_1 are strongly asymptotically unitarily equivalent. □

Lemma 6.6. *Let C and A be two unital separable stably finite C*-algebras. Suppose that $\phi, \psi : C \to A$ are two unital monomorphisms such that*

$$[\phi] = [\psi] \text{ in } KL(C, A), \ \phi_\sharp = \psi_\sharp \text{ and } \overline{R}_{\phi,\psi} = 0. \tag{6.26}$$

Suppose that $\{U(t) : t \in [0, 1)\}$ is a piecewise smooth and continuous path of unitaries in A with $U(0) = 1$ such that

$$\lim_{t\to 1} U^*(t)\phi(u)U(t) = \psi(u) \tag{6.27}$$

for some $u \in U(C)$ and suppose that there exists $w \in U(A)$ such that $\psi(u)w^* \in U_0(A)$. Let

$$Z = Z(t) = U^*(t)\phi(u)U(t)w^* \text{ if } t \in [0, 1)$$

and $Z(1) = \psi(u)w^*$. Suppose also that there is a piecewise smooth continuous path of unitaries $\{z(s) : s \in [0, 1]\}$ in A such that $z(0) = \phi(u)w^*$ and $z(1) = 1$. Then, for any piecewise smooth continuous path $\{Z(t, s) : s \in [0, 1]|\} \subset C([0, 1], A)$ of unitaries such that $Z(t, 0) = Z(t)$ and $Z(t, 1) = 1$, there is $f \in \rho_A(K_0(A))$ such that

$$\frac{1}{2\pi\sqrt{-1}} \int_0^1 \tau(\frac{dZ(t,s)}{ds} Z(t,s)^*)ds = \frac{1}{2\pi\sqrt{-1}} \int_0^1 \tau(\frac{dz(s)}{ds} z(s)^*)ds + f(\tau) \tag{6.28}$$

for all $t \in [0, 1]$ and $\tau \in T(A)$.

Proof. Define

$$Z_1(t, s) = \begin{cases} U^*(t - 2s)\phi(u)U(t - 2s)w^* & \text{for } s \in [0, t/2) \\ \phi(u)w^* & \text{for } s \in [t/2, 1/2) \\ z(2s - 1) & \text{for } s \in [1/2, 1] \end{cases} \tag{6.29}$$

for $t \in [0, 1)$ and define

$$Z_1(1, s) = \begin{cases} \psi(u)w^* & \text{for } s = 0 \\ U^*(1 - 2s)\phi(u)U(1 - 2s)w^* & \text{for } s \in (0, 1/2) \\ z(2s - 1) & \text{for } s \in [1/2, 1]. \end{cases} \tag{6.30}$$

Thus $\{Z_1(t,s) : s\in[0,1]\}\subset C([0,1],A)$ is a piecewise smooth continuous path of unitaries such that $Z_1(t,0)=Z(t)$ and $Z_1(t,1)=1$. Thus, there is an element $f_1\in\rho_A(K_0(A))$, such that

$$f_1(\tau)=\frac{1}{2\pi\sqrt{-1}}\int_0^1\tau(\frac{dZ(t,s)}{ds}Z(t,s)^*)ds-\frac{1}{2\pi\sqrt{-1}}\int_0^1\tau(\frac{dZ_1(t,s)}{ds}Z_1(t,s)^*ds \tag{6.31}$$

for all $\tau\in T(A)$ an for all $t\in[0,1]$.

On the other hand, let $V(t)=U(t)^*\phi(u)U(t)$ for $t\in[0,1)$ and $V(1)=\psi(u)$. For any $s\in[0,1)$, since $U(0)=1$, $U(t)\in U(C([0,s],A))_0$ (for $t\in[0,s]$). There there are $a_1,a_2,...,a_k\in U([0,s],A)_{s.a.}$ such that

$$U(t)=\prod_{j=1}^k\exp(ia_j(t))\ \text{ for all }\ t\in[0,s]$$

Then a straightforward calculation shows that

$$\int_0^s\frac{dV(t)}{dt}V^*(t)dt=0. \tag{6.32}$$

We also have

$$\frac{1}{2\pi\sqrt{-1}}\int_0^1\tau(\frac{dV(t)}{dt}V^*(t))dt=R_{\phi,\psi}([V])(\tau)=:f(\tau)\in\rho_A(K_0(A)) \tag{6.33}$$

for all $\tau\in T(A)$.

Then

$$\frac{1}{2\pi\sqrt{-1}}\int_0^{1/2}\tau(\frac{dZ_1(1,s)}{ds}Z_1(1,s)^*)ds = \frac{1}{2\pi\sqrt{-1}}\int_0^{1/2}\tau(\frac{dV(2s-1)}{ds}V(2s-1)^*)ds \tag{6.34}$$

$$= R_{\phi,\psi}([V])(\tau)=f(\tau)\ \text{ for all }\ \tau\in T(A). \tag{6.35}$$

One computes that, for any $\tau\in T(A)$ and for any $t\in[0,1)$, by applying (6.34),

$$\frac{1}{2\pi\sqrt{-1}}\int_0^1\tau(\frac{dZ_1(t,s)}{ds}Z_1(t,s)^*)ds \tag{6.36}$$

$$=\frac{1}{2\pi\sqrt{-1}}[\int_0^{t/2}\tau(\frac{(d(U^*(t-2s)\phi(u)U(t-2s)w^*)}{ds}(U^*(t-2s)\phi(u)U(t-2s)w^*)^*)ds+ \tag{6.37}$$

$$\int_{t/2}^{1/2}\tau(\frac{dZ_1(t,s)}{ds}Z_1(t,s)^*)ds+\int_{1/2}^1\tau(\frac{dz(s-1)}{ds}z(2s-1)^*)ds] \tag{6.38}$$

$$=\frac{1}{2\pi\sqrt{-1}}[\int_0^{t/2}\frac{dV(t-2s)}{ds}V(t-2s)^*ds+\int_{1/2}^1\tau(\frac{dz(2s-1)}{ds}z(2s-1)^*)ds] \tag{6.39}$$

$$=0+\frac{1}{2\pi\sqrt{-1}}\int_{1/2}^1\tau(\frac{dz(2s-1)}{ds}z(2s-1)^*)ds \tag{6.40}$$

$$=\frac{1}{2\pi\sqrt{-1}}\int_0^1\tau(\frac{dz(s)}{ds}z(s)^*)ds. \tag{6.41}$$

It then follows from (6.34) that

$$\frac{1}{2\pi\sqrt{-1}}\int_0^1 \tau(\frac{dZ_1(1,s)}{ds}Z_1(1,s)^*ds \tag{6.42}$$

$$= \frac{1}{2\pi\sqrt{-1}}[\int_0^{1/2} \tau(\frac{dZ_1(1,s)}{ds}Z_1(1,s)^*ds + \int_{1/2}^1 \tau(\frac{dz(2s-1)}{ds}z(2s-1)^*)ds] \tag{6.43}$$

$$= f(\tau) + \frac{1}{2\pi\sqrt{-1}}\int_0^1 \tau(\frac{dz(s)}{ds}z(s)^*)ds \tag{6.44}$$

The lemma follows. □

Remark 6.7. Note that the lemma 6.6 applies to $M_n(C)$ and $M_n(A)$ for all integer $n \geq 1$. So it works for all $u \in U_n(C)$.

Lemma 6.8. *Let A be a unital C^*-algebra satisfying that $A \otimes M_{\mathfrak{r}}$ is an AH-algebra for all supernatural number $\mathfrak{r}$ with infinite type (in particular, all AH-algebra satisfies this property), and let B be a unital simple C^*-algebra in $\mathcal{N} \cap \mathcal{C}$. Let $\kappa \in KL_e(A,B)^{++}$ and $\lambda : \mathrm{Aff}(\mathrm{T}(A)) \to \mathrm{Aff}(\mathrm{T}(B))$ be an affine homomorphism which are compatible (see Definition 2.4). Then there exists a unital homomorphism $\phi : A \to B$ such that*

$$[\phi] = \kappa \text{ and } (\phi)_\sharp = \lambda.$$

Moreover, if $\gamma \in U_\infty(A)/CU_\infty(A) \to U_\infty(B)/CU_\infty(B)$ is a continuous homomorphism which is compatible with κ and λ, then one may also require that

$$\phi^\ddagger|_{U_\infty(A)_0/CU_\infty(A)} = \gamma|_{U_\infty(A)_0/CU_\infty(A)} \text{ and } (\phi)^\ddagger \circ s_1 = \gamma \circ s_1 - \bar{h}, \tag{6.45}$$

where $s_1 : K_1(A) \to U_\infty(A)/CU_\infty(A)$ is a splitting map (see 2.3), and

$$\bar{h} : K_1(A) \to \overline{\mathbb{R}\rho_B(K_0(B))}/\overline{\rho_B(K_0(B))}$$

is a homomorphism.

Moreover,

$$(\phi \otimes \mathrm{id}_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}})^\ddagger \circ s_1 = E_B \circ \gamma \circ s_1 - \bar{h}, \tag{6.46}$$

where E_B is as defined in 6.2.

Proof. Let $\mathfrak{p}$ and $\mathfrak{q}$ be two relative prime supernatural numbers of infinite type such that $Q = M_{\mathfrak{p}} \otimes M_{\mathfrak{q}}$. Let $A_{\mathfrak{p}} = A \otimes M_{\mathfrak{p}}$, $A_{\mathfrak{q}} = A \otimes M_{\mathfrak{q}}$, $B_{\mathfrak{p}} = B \otimes M_{\mathfrak{p}}$ and $B_{\mathfrak{q}} = B \otimes M_{\mathfrak{q}}$. Then $A_{\mathfrak{p}}$ and $A_{\mathfrak{q}}$ are AH-algebras, and $TR(B_{\mathfrak{p}}) \leq 1$ and $TR(B_{\mathfrak{q}}) \leq 1$. Let $\kappa_{\mathfrak{p}} \in KL(A_{\mathfrak{p}}, B_{\mathfrak{p}})$, $\kappa_{\mathfrak{q}} \in KL(A_{\mathfrak{p}}, B_{\mathfrak{p}})$, $\lambda_{\mathfrak{p}} : \mathrm{Aff}(\mathrm{T}(A_{\mathfrak{p}})) \to \mathrm{Aff}(\mathrm{T}(B_{\mathfrak{p}}))$, $\lambda_{\mathfrak{q}} : \mathrm{Aff}(\mathrm{T}(A_{\mathfrak{q}})) \to \mathrm{Aff}(\mathrm{T}(B_{\mathfrak{q}}))$, $\gamma_{\mathfrak{p}} : U(A_{\mathfrak{p}})/CU(A_{\mathfrak{p}}) \to U(B_{\mathfrak{p}})/CU(B_{\mathfrak{p}})$ and $\gamma_{\mathfrak{q}} : U(A_{\mathfrak{q}})/CU(A_{\mathfrak{q}}) \to U(B_{\mathfrak{q}})/CU(B_{\mathfrak{q}})$ be induced by κ, λ and γ, respectively. Note that $A_{\mathfrak{p}}$, A_q, $B_{\mathfrak{p}}$ and B_q are all unital AH-algebras. Moreover, since $M_{\mathfrak{r}} \cong M_{\mathfrak{r}} \oplus M_{\mathfrak{r}}$, for any supernatural number $\mathfrak{r}$ of infinite type, $B_{\mathfrak{p}}$ and $B_{\mathfrak{q}}$ are unital simple AH-algebras of slow dimension growth. It follows from Corollary 6.11 of [21] that there is a unital homomorphism $\phi_{\mathfrak{p}} : A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ such that

$$[\phi_{\mathfrak{p}}] = \kappa_{\mathfrak{p}} \text{ in } KL(A_{\mathfrak{p}}, B_{\mathfrak{p}}), (\phi_{\mathfrak{p}})^\ddagger = \gamma_{\mathfrak{p}} \text{ and } (\phi_{\mathfrak{p}})_\sharp = \lambda_{\mathfrak{p}}. \tag{6.47}$$

For the same reason, there is also a unital homomorphism $\psi_{\mathfrak{q}}: A_{\mathfrak{q}} \to B_{\mathfrak{q}}$ such that

$$[\psi_{\mathfrak{q}}] = \kappa_{\mathfrak{q}} \text{ in } KL(A_{\mathfrak{q}}, B_{\mathfrak{q}}), (\psi_{\mathfrak{q}})^{\ddagger} = \gamma_{\mathfrak{q}} \text{ and } (\psi_{\mathfrak{q}})_{\sharp} = \lambda_{\mathfrak{q}}. \tag{6.48}$$

Define $\phi = \phi_{\mathfrak{p}} \otimes \mathrm{id}_{M_{\mathfrak{q}}}$ and $\psi = \psi_{\mathfrak{q}} \otimes \mathrm{id}_{M_{\mathfrak{p}}}$. From above, one has that

$$[\phi] = [\psi] \text{ in } KL(A \otimes Q, B \otimes Q), \phi_{\sharp} = \psi_{\sharp} \text{ and } \phi^{\ddagger} = \psi^{\ddagger}.$$

Since both $K_i(B \otimes Q)$ are divisible ($i = 0, 1$), one actually has

$$[\phi] = [\psi] \text{ in } KK(A \otimes Q, B \otimes Q).$$

It follows from 6.5 that there is $\beta_0 \in \overline{\mathrm{Inn}}(\psi(A \otimes Q), B \otimes Q)$ such that if $\imath_{\psi(A\otimes Q)}$ denotes the embedding of $\psi(A \otimes Q)$ into $B \otimes Q$,

$$[\beta_0] = [\imath_{\psi(A\otimes Q)}] \text{ in } KK(\psi(A \otimes Q), B \otimes Q), \tag{6.49}$$

$$(\beta_0)_{\sharp} = (\imath_{\psi(A\otimes Q)})_{\sharp} \text{ and } (\beta_0)^{\ddagger} = (\imath_{\psi(A\otimes Q)})^{\ddagger} \tag{6.50}$$

such that ϕ and $\beta_0 \circ \psi$ are strongly asymptotically unitarily equivalent (since in this case $H_1(K_0(A\otimes Q), K_1(B\otimes Q)) = K_1(B\otimes Q)$). Note that one may identify $T(B_{\mathfrak{q}}), T(B_{\mathfrak{p}})$ and $T(B\otimes Q)$. Moreover,

$$\overline{\rho_{B\otimes Q}(K_0(B \otimes Q))} = \overline{\mathbb{R}\rho_B(K_0(B))} = \overline{\rho_{B_{\mathfrak{q}}}(K_0(B_{\mathfrak{q}}))}.$$

Denote by $\imath_{\mathfrak{p}}: B_{\mathfrak{q}} \to B \otimes Q$ the embedding $a \mapsto a \otimes 1_{M_{\mathfrak{p}}}$, and note that the image of $\imath_{\mathfrak{p}} \circ \psi_{\mathfrak{q}}$ is in the image of ψ. Thus, by 3.5, $R_{\beta_0\circ\imath_{\mathfrak{p}}\circ\psi_{\mathfrak{q}},\imath_{\mathfrak{p}}\circ\psi_{\mathfrak{q}}}$ is in $\mathrm{Hom}((K_1(M_{\beta_0\circ\imath_{\mathfrak{p}}\circ\psi_{\mathfrak{q}},\imath_{\mathfrak{p}}\circ\psi_{\mathfrak{q}}}), \overline{\rho_{B_{\mathfrak{q}}}(K_0(B_{\mathfrak{q}}))})$. Note that

$$[\beta_0 \circ \imath_{\mathfrak{p}} \circ \psi_{\mathfrak{q}}] = [\imath_{\mathfrak{p}} \circ \psi_{\mathfrak{q}}] \quad \text{in } KK(A_{\mathfrak{q}}, B_{\mathfrak{q}}).$$

By 6.4, there exists $\alpha \in \overline{\mathrm{Inn}}(\psi_{\mathfrak{q}}(A_{\mathfrak{q}}), B_{\mathfrak{q}})$ such that

$$[\alpha] = [\imath_{\psi_{\mathfrak{q}}(A_{\mathfrak{q}})}] \quad \text{in } KK(B_{\mathfrak{q}}, B_{\mathfrak{q}}),$$

where $\imath_{\psi_{\mathfrak{q}}(A_{\mathfrak{q}})}$ is the embedding of $\psi_{\mathfrak{q}}(A_{\mathfrak{q}})$ into $B_{\mathfrak{q}}$, and

$$\overline{R}_{\alpha,\imath_{\psi_{\mathfrak{q}}(A_{\mathfrak{q}})}} = -\overline{R}_{\beta_0\circ\imath_{\mathfrak{p}}\circ\psi_{\mathfrak{q}},\imath_{\mathfrak{p}}\circ\psi_{\mathfrak{q}}}.$$

As computed in the proof of 6.5, one has that

$$[\imath_{\mathfrak{p}} \circ \alpha \circ \psi_q] = [\beta_0 \circ \imath_{\mathfrak{p}} \circ \psi_{\mathfrak{q}}] \text{ in } KK(A_q, B \otimes Q), \tag{6.51}$$

$$(\imath_{\mathfrak{p}} \circ \alpha \circ \psi_q)_{\sharp} = (\beta_0 \circ \imath_{\mathfrak{p}} \circ \psi_{\mathfrak{q}})_{\sharp} \text{ and } (\imath_{\mathfrak{p}} \circ \alpha \circ \psi_q)^{\ddagger} = (\beta_0 \circ \imath_{\mathfrak{p}} \circ \psi_{\mathfrak{q}})^{\ddagger}, \tag{6.52}$$

and

$$\overline{R}_{\imath_{\mathfrak{p}}\circ\alpha\circ\psi_q,\beta_0\circ\imath_{\mathfrak{p}}\circ\psi_{\mathfrak{q}}} = 0.$$

It follows from 7.2 and Theorem 4.2 of [25] that $\imath_{\mathfrak{p}} \circ \alpha \circ \psi_q$ and $\beta_0 \circ \imath_{\mathfrak{p}} \circ \psi_q$ are strongly asymptotically unitarily equivalent.

Consider maps

$$(\beta_0 \circ \imath_{\mathfrak{p}} \circ \psi_{\mathfrak{q}}) \otimes \mathrm{id}_{M_{\mathfrak{p}}},\ \imath \circ \beta_0 \circ \psi : A \otimes M_{\mathfrak{q}} \otimes M_{\mathfrak{p}} \to (B \otimes M_{\mathfrak{q}} \otimes M_{\mathfrak{p}}) \otimes M_{\mathfrak{p}},$$

where $\imath : B \otimes Q \to (B \otimes Q) \otimes M_{\mathfrak{p}}$ is the embedding $b \to b \otimes 1_{M_{\mathfrak{p}}}$ for all $b \in B \otimes Q$.

Identify $\beta_0\circ\psi(B\otimes M_{\mathfrak{q}}\otimes M_{\mathfrak{p}})\otimes M_{\mathfrak{p}}$ with $\beta_0\circ\psi(B)\otimes\beta_0\circ\psi(M_{\mathfrak{q}})\otimes\beta_0\circ\psi(M_{\mathfrak{p}})\otimes M_{\mathfrak{p}}$, and consider the automorphism θ on $\beta_0\circ\psi(B)\otimes\beta_0\circ\psi(M_{\mathfrak{q}})\otimes\beta_0\circ\psi(M_{\mathfrak{p}})\otimes M_{\mathfrak{p}}$ defined by

$$\theta:\ a\otimes b\otimes c\otimes d\mapsto a\otimes b\otimes d\otimes c.$$

Then

$$[\theta|_{\beta_0(M_q)\otimes\beta_0(M_{\mathfrak{p}})\otimes M_{\mathfrak{p}}}]=[\mathrm{id}_{\beta_0(M_q)\otimes\beta_0(M_{\mathfrak{p}})\otimes M_{\mathfrak{p}}}]\ \text{ in }\ KK(\beta_0(M_q)\otimes\beta_0(M_{\mathfrak{p}})\otimes M_{\mathfrak{p}},\beta_0(M_q)\otimes\beta_0(M_{\mathfrak{p}})\otimes M_{\mathfrak{p}}).$$

Since $K_1(\beta_0(M_q)\otimes\beta_0(M_{\mathfrak{p}})\otimes M_{\mathfrak{p}})=\{0\}$, it follows from Theorem 4.2 of [25] that $\theta|_{\beta_0(M_q)\otimes\beta_0(M_{\mathfrak{p}})\otimes M_{\mathfrak{p}}}$ is strongly asymptotically unitarily equivalent to the identity map. Therefore θ is strongly asymptotically unitarily equivalent to the identity map. Note that for any $a\in A$, $b\in M_{\mathfrak{q}}$, and $c\in M_{\mathfrak{p}}$, one has

$$\begin{aligned}\theta(((\beta_0\circ\imath_{\mathfrak{p}}\circ\psi_{\mathfrak{q}})\otimes\mathrm{id}_{M_{\mathfrak{p}}})(a\otimes b\otimes c)) &= \theta(\beta_0(\psi_{\mathfrak{q}}(a\otimes b)\otimes 1_{M_{\mathfrak{p}}})\otimes c) &(6.53)\\ &= \beta_0(\psi_{\mathfrak{q}}(a\otimes b)\otimes c)\otimes 1_{M_{\mathfrak{p}}} &(6.54)\\ &= \imath\circ\beta_0\circ\psi(a\otimes b\otimes c). &(6.55)\end{aligned}$$

Thus, the map $(\beta_0\circ\imath_{\mathfrak{p}}\circ\psi_{\mathfrak{q}})\otimes\mathrm{id}_{M_{\mathfrak{p}}}$ is strongly asymptotically unitarily equivalent to $\imath\circ\beta_0\circ\psi$.

Define a map $\Psi_q: A\otimes M_{\mathfrak{q}}\otimes M_{\mathfrak{p}}\to B\otimes M_{\mathfrak{q}}\otimes M_{\mathfrak{p}}\otimes M_{\mathfrak{p}}$ by

$$\Psi_{\mathfrak{q}}: a\otimes b\otimes c\mapsto\alpha(\psi_{\mathfrak{q}}(a\otimes b))\otimes c\otimes 1_{M_{\mathfrak{p}}}. \qquad (6.56)$$

Note that for all $a\otimes b\otimes c\in A\otimes M_{\mathfrak{q}}\otimes M_{\mathfrak{p}}$,

$$((\imath_{\mathfrak{p}}\circ\alpha\circ\psi_q)\otimes\mathrm{id}_{M_{\mathfrak{p}}})(a\otimes b\otimes c)=\alpha(\psi_{\mathfrak{q}}(a\otimes b))\otimes 1_{M_{\mathfrak{p}}}\otimes c \qquad (6.57)$$

Then the same argument as above shows that Ψ_q is strongly asymptotically unitarily equivalent to $(\imath_{\mathfrak{p}}\circ\alpha\circ\psi_q)\otimes\mathrm{id}_{M_{\mathfrak{p}}}$.

Since ϕ and $\beta_0\circ\psi$ are strongly asymptotically unitarily equivalent, one has that the map $\imath\circ\phi$ is strongly asymptotically unitarily equivalent to $\imath\circ\beta_0\circ\psi$, and hence strongly asymptotically unitarily equivalent to $(\beta_0\circ\imath_{\mathfrak{p}}\circ\psi_{\mathfrak{q}})\otimes\mathrm{id}_{M_{\mathfrak{p}}}$, and therefore strongly asymptotically unitarily equivalent to $(\imath_{\mathfrak{p}}\circ\alpha\circ\psi_q)\otimes\mathrm{id}_{M_{\mathfrak{p}}}$. It follows that the map $\imath\circ\phi$ is strongly asymptotically unitarily equivalent to $\Psi_{\mathfrak{q}}$. Thus there is a continuous path of unitaries $\{w(t):\ t\in[0,1)\}$ in $B\otimes M_{\mathfrak{q}}\otimes M_{\mathfrak{p}}\otimes M_{\mathfrak{p}}$ with $w(0)=1$ such that

$$\lim_{t\to 1}w^*(t)(\imath\circ\phi(a))w(t)=\Psi_{\mathfrak{q}}(a),\quad\forall a\in A\otimes Q.$$

Pick an isomorphism $\chi': M_{\mathfrak{p}}\otimes M_{\mathfrak{p}}\to M_{\mathfrak{p}}$, and consider the induced isomorphism $\chi: B\otimes M_{\mathfrak{q}}\otimes M_{\mathfrak{p}}\otimes M_{\mathfrak{p}}\to B\otimes M_{\mathfrak{q}}\otimes M_{\mathfrak{p}}$. Note that $(\chi')^{-1}$ is strongly asymptotically unitarily equivalent to the map $\imath': M_{\mathfrak{p}}\to M_{\mathfrak{p}}\otimes M_{\mathfrak{p}}$ defined by $a\mapsto 1\otimes a$. Then, it is straightforward to verify that $\chi\circ\imath\circ\phi$ is strongly asymptotically unitarily equivalent to ϕ, and $\chi\circ\Psi_{\mathfrak{q}}$ is strongly asymptotically unitarily equivalent to $(\alpha\circ\psi_q)\otimes\mathrm{id}_{M_{\mathfrak{p}}}$. Thus, there is a continuous path of unitaries $u(t)$ in $B\otimes M_{\mathfrak{p}}\otimes M_{\mathfrak{q}}$ (one can be made it into piecewise smooth—see Lemma 4.1 of [18]) such that $u(0)=1$ and

$$\lim_{t\to 1}\mathrm{ad}\,u(t)\circ\phi(a)=(\alpha\circ\psi_{\mathfrak{q}})\otimes\mathrm{id}_{M_{\mathfrak{p}}}(a)\ \text{ for all }\ a\in A\otimes Q. \qquad (6.58)$$

This provides a unital homomorphism $\Phi: A\otimes\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}\to B\otimes\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ such that, for each $t\in(0,1)$,

$$\pi_t\circ\Phi(a)=\mathrm{ad}\,u(t)\circ\phi(a(t))\ \text{ for all }\ a\in A\otimes\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}. \qquad (6.59)$$

Denote by ϑ a unital embedding $\mathcal{Z} \to \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$, and let $j : \mathcal{Z}_{\mathfrak{p},\mathfrak{q}} \to \mathcal{Z}$ be a unital homomorphism induced by the stationary inductive limit

$$\mathcal{Z}_{\mathfrak{p},\mathfrak{q}} \stackrel{\vartheta}{\to} \mathcal{Z}_{\mathfrak{p},\mathfrak{q}} \stackrel{\vartheta}{\to} \mathcal{Z}_{\mathfrak{p},\mathfrak{q}} \stackrel{\vartheta}{\to} \cdots \to \quad \mathcal{Z}$$

given by 3.4 of [29], where the map ϑ is regarded as its restriction to $\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$.

As in the proof of 7.1 of [32] (note that it follows from the same proof that Proposition 4.6 of [32] also works for homomorphisms which are not necessary being injective),

$$((\mathrm{id}_B \otimes j) \circ \Phi \circ (\mathrm{id}_A \otimes \vartheta))_{*i} = \kappa_i, \quad i = 0, 1, \tag{6.60}$$

$$((\mathrm{id}_B \otimes j) \circ \Phi \circ (\mathrm{id}_A \otimes \vartheta))_\sharp = \lambda. \tag{6.61}$$

In fact, one has that

$$\Phi_\sharp(a \otimes b)(\tau \otimes \mu) = \gamma(a(\tau))\mu(b) \text{ for all } a \in A_{s.a.} \text{ and } b \in (\mathcal{Z}_{\mathfrak{p},\mathfrak{q}})_{s.a.}. \tag{6.62}$$

By considering $((\mathrm{id}_B \otimes j) \circ \Phi \circ (\mathrm{id}_A \otimes i)) \otimes \mathrm{id}_{C(X_k)} : A \otimes C(X_k) \to B \otimes C(X_k)$ for some suitable compact metric spaces X_k, the same argument shows that, in fact,

$$[(\mathrm{id}_B \otimes j) \circ \Phi \circ (\mathrm{id}_A \otimes \vartheta)] = \kappa. \tag{6.63}$$

Define the map $H = (\mathrm{id}_B \otimes j) \circ \Phi \circ (\mathrm{id}_A \otimes \vartheta)$. Then $[H] = \kappa$ in $KL(A, B)$ and $H_\sharp = \lambda$.

Note that it follows from (6.62) that

$$\Phi^\ddagger|_{U(A)_0/CU(A)} = E_B^\ddagger \circ \gamma|_{U(A)_0/CU(A)}. \tag{6.64}$$

Let $z \in U(A)/CU(A)$. Then, one has

$$H^\ddagger = \gamma_\infty = \imath_\infty^\ddagger \circ \gamma. \tag{6.65}$$

On the other hand, for each $z \in U(A)/CU(A)$, there is a unitary $w \in B \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ such that

$$\pi_t(w) = \pi_{t'}(w) \text{ for all } t, t' \in [0, 1] \text{ and } E_B^\ddagger \circ \gamma(z) = \overline{w}. \tag{6.66}$$

Since $\pi_t(w) \in B$ is constant, one may use w for its evaluation at t. Let $v_0 \in U(A)$ be such that $\overline{v_0} = z$. For any $t \in (0, 1)$, define

$$Z(t) = \pi_t \circ \Phi(v_0)w^* = u(t)^*\phi(v_0)u(t)w^*. \tag{6.67}$$

Let $Z(t, s)$ be a piecewise smooth continuous path of unitaries in $B \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ such that $Z(t, 0) = Z(t)$ and $Z(t, 1) = 1$. Denote by τ_0 the unique tracial state in $\mathrm{T}(M_{\mathfrak{r}})$, where $\mathfrak{r}$ is a supernatural number. For each $s_\mu \in \mathrm{T}(\mathcal{Z}_{\mathfrak{p},\mathfrak{q}})$, one may write

$$s_\mu(a) = \int_0^1 \tau_0(a(t))d\mu(t),$$

where μ is a probability Borel measure on $[0, 1]$.

Then, for $\tau \in T(B)$ and $s_\mu \in T(\mathcal{Z}_{\mathfrak{p},\mathfrak{q}})$, by applying 6.6,

$$\begin{aligned}
\text{(6.68)}\quad \mathrm{Det}(Z)(\tau\otimes s_\mu) &= \frac{1}{2\pi\sqrt{-1}}\int_0^1 (\tau\otimes s_\mu)(\frac{dZ(t,s)}{ds}Z(t,s)^*)ds \\
\text{(6.69)}\quad &= \frac{1}{2\pi\sqrt{-1}}\int_0^1\int_0^1 (\tau\otimes \tau_0)(\frac{dZ(t,s)}{ds}Z(t,s)^*)d\mu(t)ds \\
\text{(6.70)}\quad &= \int_0^1 (\frac{1}{2\pi\sqrt{-1}}\int_0^1 (\tau\otimes \tau_0)(\frac{dZ(t,s)}{ds}Z(t,s)^*))ds)d\mu(t) \\
\text{(6.71)}\quad &= \int_0^1 \mathrm{Det}(\phi(v_0)w^*)(\tau)d\mu(t)+f(\tau) \ \text{ for some } f\in \rho_B(K_0(B)).
\end{aligned}$$

By 6.2 and (6.59),

$$\text{(6.72)}\quad \mathrm{Det}(Z)(\tau\otimes s_\mu)=\mathrm{Det}(\phi(v_0)w^*)(\tau)+f(\tau)\in \overline{\mathbb{R}\rho_B(K_0(B))}\subseteq \mathrm{Aff}(\mathrm{T}(B\otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}).$$

Thus, $\Phi^\ddagger(z)(E_B\circ\lambda(z)^*)$ defines a homomorphism from $U(A)/CU(A)$ into $\overline{\mathbb{R}\rho_B(K_0(B))}/\overline{\rho_B(K_0(B))}$ which will be denoted by h. By (6.63),

$$\text{(6.73)}\quad h|_{U(A)_0/CU(A)}=0.$$

Thus h induces a homomorphism $\bar{h}: K_1(A)\to \overline{\mathbb{R}\rho_B(K_0(B))}/\overline{\rho_B(K_0(B))}$. □

In [18], it was shown that, given two unital separable simple C*-algebras A and B in $\mathcal{N}\cap\mathcal{C}$, if there is an isomorphism on the Elliott invariant, i.e.,

$$(K_0(A),K_0(A)_+,[1_A],K_1(A),\mathrm{T}(A),r_A)\cong (K_0(B),K_0(B)_+,[1_B],\mathrm{T}(B),r_B),$$

then $A\cong B$. The following corollary is a more general statement.

Corollary 6.9. *Let A and B be two unital separable C^*-algebras in $\mathcal{N}\cap\mathcal{C}$. Suppose that there is a homomorphism $\kappa_i: K_i(A)\to K_i(B)$ such that κ_0 is order preserving and $\kappa_0([1_A])\le[1_B]$ and there is a continuous affine map $\lambda:\mathrm{Aff}(\mathrm{T}(A))\to\mathrm{Aff}(\mathrm{T}(B))$ which is compatible with κ_0. Then there is a homomorphism $\phi: A\to B$ such that*

$$(\phi)_{*i}=\kappa_i,\ \ i=0,1 \ \textit{ and } \ \phi_\sharp=\lambda.$$

Proof. Consider the splitting short exact sequence:

$$0\to \mathrm{Ext}_{\mathbb{Z}}(K_*(A),K_{*+1}(B))\to KK(A,B)\to \mathrm{Hom}(K_*(A),K_*(B))\to 0.$$

There exists an element $\kappa\in KK(A,B)$ such that the image of κ in $Hom(K_*(A),K_*(B))$ is exactly the same as that κ_*. Let $\bar\kappa$ in $KL(A,B)$ be the image of κ. There is a projection $p\in B$ such that $[p]=\kappa_0([1_A])$. Let $B_1=pBp$. Then $\bar\kappa\in KL_e(A,B_1)^{++}$ and λ and $\bar\kappa$ are compatible. It follows from 6.8 that there is a unital homomorphism $\phi: A\to B_1\subset B$ such that

$$[\phi]=\bar\kappa \ \text{ and } \ \phi_\sharp=\lambda.$$

□

Theorem 6.10. *Let C be a unital C^*-algebra such that $C\otimes M_{\mathfrak{r}}$ is an AH-algebra for all supernatural number $\mathfrak{r}$ with infinite type, and let A be a unital simple C^*-algebra in $\mathcal{N}\cap\mathcal{C}$ which is $\mathcal{Z}$-stable. Then, for any $\kappa\in KLT_e(C,A)^{++}$ and a continuous homomorphism $\gamma: U_\infty(C)/CU_\infty(C)\to U_\infty(A)/CU_\infty(A)$ which are compatible, there is a unital monomorphism $\phi: C\to A$ such that*

$$([\phi],\phi_\sharp)=\kappa \ \text{ and } \ \phi^\ddagger=\gamma,$$

provided that

(1) *$K_1(C)$ is a free group, or*
(2) $\overline{\mathbb{R}\rho_A(K_0(A))}/\overline{\rho_A(K_0(A))}=\{0\}$, *or*
(3) $\overline{\mathbb{R}\rho_A(K_0(A))}/\overline{\rho_A(K_0(A))}$ *is torsion free and $K_1(C)$ is finitely generated.*

Proof. It follows from 6.8 that there is a unital monomorphism $\psi: C\to A$ such that

$$(\psi,\psi_\sharp)=\kappa,\ \ \psi^\ddagger|_{U(C)_0/CU(C)}=\lambda|_{U(C)_0/CU(C)} \ \text{ and } \ (\psi\otimes \mathrm{id}_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}})^\ddagger\circ s_1=E_B^\ddagger\circ\gamma\circ s_1-\bar{h}, \tag{6.74}$$

where $\bar{h}: K_1(C)\to \overline{\mathbb{R}\rho_A(K_0(A))}/\overline{\rho_A(K_0(A))}$ is a homomorphism. If $K_1(C)$ is free, there exists a homomorphism $h_1: K_1(C)\to\overline{\mathbb{R}\rho_A(K_0(A))}$ which induces h_1. In the case that $\overline{\mathbb{R}\rho_A(K_0(A))}/\overline{\rho_A(K_0(A))}$ is torsion free and $K_1(C)$ is finitely generated, then one also obtains a such h_1. Since $\overline{\mathbb{R}\rho_A(K_0(A))}$ is torsion free, h_1 induces a homomorphism $\bar{h}_1: K_1(C)/(\mathrm{Tor}(K_1(C)))\to\overline{\mathbb{R}\rho_A(K_0(A))}$. Since the map from $K_1(C)/(\mathrm{Tor}(K_1(C)))\to (K_1(A)/(\mathrm{Tor}(K_1(A)))\otimes\mathbb{Q}_{\mathfrak{p}}$ is injective, one obtains a homomorphism $h_{1,\mathfrak{p}}: K_1(C\otimes M_{\mathfrak{p}})\to\overline{\mathbb{R}\rho_A(K_0(A))}$ such that

$$h_1=h_{1,\mathfrak{p}}\circ(\imath_{\mathfrak{p}})_{*1}, \tag{6.75}$$

where $\imath_{\mathfrak{r}}: A\to A\otimes M_{\mathfrak{r}}$ is the embedding so that $\imath_r(a)=a\otimes 1$ for all $a\in A$ ($\mathfrak{r}$ is a supernatural number). Similarly, there is a homomorphism $h_{1,\mathfrak{q}}: K_1(C\otimes M_{\mathfrak{q}})\to\overline{\mathbb{R}\rho_A(K_0(A))}$ such that

$$h_1=h_{1,\mathfrak{q}}\circ(\imath_{\mathfrak{q}})_{*1}. \tag{6.76}$$

Put $C_{\mathfrak{r}}'=(\psi\otimes\mathrm{id}_{M_{\mathfrak{r}}})(C\otimes M_{\mathfrak{r}}))$, where $\mathfrak{r}$ is a supernatural number. It follows from 6.4 that there is a monomorphism $\beta_0\in\overline{\mathrm{Inn}}(C_{\mathfrak{p}}',A_{\mathfrak{p}})$ such that

$$[\beta_0]=[\imath_{C_{\mathfrak{p}}'}] \ \text{ in } \ KK(C_{\mathfrak{p}}',A_{\mathfrak{p}}),\ \ (\beta_0)_\sharp=\imath_{C_{\mathfrak{p}}'\sharp},\ \ \beta_0^\ddagger=\imath_{C_{\mathfrak{p}}'}{}^\ddagger \ \text{ and } \ \overline{R}_{\psi\otimes\mathrm{id}_{M_{\mathfrak{p}}},\beta_0\circ(\psi\otimes\mathrm{id}_{M_{\mathfrak{p}}})}=h_{1,\mathfrak{p}}, \tag{6.77}$$

where $\imath_{C_{\mathfrak{p}}'}$ is the embedding of $C_{\mathfrak{p}}'$.

Similarly, there is a monomorphism $\beta_1\in\overline{\mathrm{Inn}}(C_{\mathfrak{q}}',A_{\mathfrak{q}})$ such that

$$[\beta_1]=[\imath_{C_{\mathfrak{q}}'}] \ \text{ in } \ KK(C_{\mathfrak{q}}',A_{\mathfrak{q}}),\ \ (\beta_1)_\sharp=\imath_{C_{\mathfrak{q}}'\sharp},\ \ \beta_1^\ddagger=\imath_{C_{\mathfrak{q}}'}{}^\ddagger \ \text{ and } \ \overline{R}_{\psi\otimes\mathrm{id}_{M_{\mathfrak{q}}},\beta_1\circ(\psi\otimes\mathrm{id}_{M_{\mathfrak{q}}})}=h_{1,\mathfrak{q}}, \tag{6.78}$$

where $\imath_{C_{\mathfrak{q}}'}$ is the embedding of $C_{\mathfrak{q}}'$.

As in the proof of 6.8, by applying 6.5 and its proof, one has a monomorphism $\beta_2\in\overline{\mathrm{Inn}}(\beta_1\circ(\psi\otimes\mathrm{id}_{M_{\mathfrak{q}}})(C_{\mathfrak{q}}),A_{\mathfrak{q}})$ and a piecewise smooth continuous path of unitaries $\{U(t): t\in[0,1)\}$ of $A\otimes Q$ such that $U(0)=1$ and

$$[\beta_2\circ\beta_1\circ(\psi\otimes\mathrm{id}_{M_{\mathfrak{q}}}))]=[\beta_0\circ(\psi\otimes\mathrm{id}_{M_{\mathfrak{p}}})] \quad \text{in } KK(C_{\mathfrak{q}},A_{\mathfrak{q}}), \tag{6.79}$$

$$(\beta_2\circ\beta_1\circ(\psi\otimes\mathrm{id}_{M_{\mathfrak{q}}})))_\sharp=(\beta_0\circ(\psi\otimes\mathrm{id}_{M_{\mathfrak{p}}}))_\sharp \tag{6.80}$$

and

$$(\beta_2 \circ \beta_1 \circ (\psi \otimes \mathrm{id}_{M_{\mathfrak{q}}})))^{\ddagger} = (\beta_0 \circ (\psi \otimes \mathrm{id}_{M_{\mathfrak{p}}}))^{\ddagger}. \tag{6.81}$$

Moreover, if denote by $\psi_0 = \beta_0 \circ (\psi \otimes \mathrm{id}_{M_{\mathfrak{p}}})$ and $\psi_1 = \beta_2 \circ \beta_1 \circ (\psi \otimes \mathrm{id}_{M_{\mathfrak{q}}}))$, one has that

$$\lim_{t \to 1} U(t)^* (\psi_0 \otimes \mathrm{id}_{M_{\mathfrak{q}}})(a) U(t) = (\psi_1 \otimes \mathrm{id}_{M_{\mathfrak{p}}})(a) \tag{6.82}$$

for all $a \in A \otimes Q$. In particular,

$$\overline{R}_{\psi_0 \otimes \mathrm{id}_{M_{\mathfrak{q}}}, \psi_1 \otimes \mathrm{id}_{M_{\mathfrak{p}}}} = 0. \tag{6.83}$$

Let $\Phi : A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}} \to A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ be defined by

$$\Phi(a \otimes b)(t) = U^*(t)((\psi_0 \otimes \mathrm{id}_{M_{\mathfrak{q}}}(a \otimes b(t))U(t) \text{ for all } t \in [0,1) \text{ and} \tag{6.84}$$

$$\Phi(a \otimes b)(1) = \psi_1 \otimes \mathrm{id}_{M_{\mathfrak{p}}}(a \otimes b(1)), \tag{6.85}$$

for all $a \otimes b \in A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$.

We claim that

$$\Phi^{\ddagger} \circ (E_A \circ \psi)^{\ddagger} \circ s_1 = (E_A)^{\ddagger} \circ \gamma \circ s_1. \tag{6.86}$$

To compute $\Phi^{\ddagger}$, let $x \in s_1(K_1(C))$ and $v_0 \in U(C)$ such that $\overline{v_0} = x$. There is $w \in U(A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}})/CU(A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}})$ such that $w(t) = w(t')$ for all $t, t' \in [0,1]$ and

$$E_A^{\ddagger} \circ \gamma \circ s_1(x) = \overline{w}. \tag{6.87}$$

Let $Z = (\Phi \circ (\psi \otimes \mathrm{id}_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}})(v_0))w^* \in A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$. Note that $Z \in U(A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}})_0$. Suppose that there is a piecewise smooth continuous path $\{Z(t,s) : s \in [0,1]\} \subset A \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ such that $Z(t,0) = Z(t)$ and $Z(t,1) = 1$. Then

$$\mathrm{Det}(Z(t,s)) \tag{6.88}$$

$$= \mathrm{Det}(\Phi \circ ((\psi \otimes \mathrm{id}_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}})(v_0))(\psi \otimes \mathrm{id}_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}}(v_0)^*) + \mathrm{Det}((\psi \otimes \mathrm{id}_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}})(v_0) w^*) \tag{6.89}$$

$$= \mathrm{Det}(\Phi \circ ((\psi \otimes \mathrm{id}_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}})(v_0))(\psi \otimes \mathrm{id}_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}}(v_0)^*) + \bar{h} \circ s_1(x). \tag{6.90}$$

It follows from 6.6 that

$$\mathrm{Det}(\Phi \circ ((\psi \otimes \mathrm{id}_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}})(v_0))(\psi \otimes \mathrm{id}_{\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}}(v_0)^*) = \mathrm{Det}(\beta_0 \circ \psi(v_0)\psi(v_0)^*) + \rho_A(K_0(A)) \tag{6.91}$$

$$= R_{\beta_0 \circ \psi, \psi}([v_0]) + \rho_A(K_0(A)) \tag{6.92}$$

$$= -h_{1,\mathfrak{p}} \circ s_1(x) + \rho_A(K_0(A)). \tag{6.93}$$

Therefore, by (6.75) and by (6.88),

$$\mathrm{Det}(Z(t,s))(\tau \otimes s_\mu) \in \rho_A(K_0(A)).$$

This proves the claim.

Regard ψ as a map to $A \otimes \mathcal{Z}$. Denote by $j : \mathcal{Z}_{\mathfrak{p},\mathfrak{q}} \to \mathcal{Z}$ the unital homomorphism induced by the stationary inductive limit decomposition of $\mathcal{Z}$, and denote by $\vartheta : \mathcal{Z} \to \mathcal{Z}_{\mathfrak{p}.\mathfrak{q}}$ the unital embedding induced by tensoring $\mathcal{Z}$ ($\mathcal{Z}_{\mathfrak{p},\mathfrak{q}}$ is $\mathcal{Z}$-stable). Consider

$$\phi = (\mathrm{id}_A \otimes j) \circ \Phi \circ (\mathrm{id}_A \otimes \vartheta) \circ \psi.$$

One then checks that

$$[\psi] = [\phi] \text{ in } KL(C, A), \;\; \phi_\sharp = \psi_\sharp \text{ and } \phi^\ddagger = \gamma.$$

□

Remark 6.11. It follows from Proposition 3.6 of [19] that, if $TR(A) \leq 1$, then

$$\overline{\mathbb{R}\rho_A(K_0(A))}/\overline{\rho_A(K_0(A))} = \{0\}.$$

So Theorem 6.10 recovers a version of Theorem 8.6 of [18].

Now suppose that in 6.10,

$$U_\infty(C)/CU_\infty(C) = U_\infty(C)_0/CU_\infty(C) \oplus G_1 \oplus \mathrm{Tor}(K_1(C)),$$

where G_1 is identified with a free subgroup of $K_1(C)$. From the proof of Theorem 6.10, we see that, if $\kappa \in KLT_e(C, A)^{++}$ and $\gamma : U_\infty(C)/CU_\infty(C) \to U(A)/CU(A)$ which is compatible to κ are given, there is a unital monomorphism $\phi : C \to A$ such that $([\phi], \phi_\sharp) = \kappa$ and

$$\phi|_{U_\infty(C)_0/CU_\infty(C)\oplus G_1} = \gamma|_{U_\infty(C)_0/CU_\infty(C)\oplus G_1}$$

and

$$\phi^\ddagger(z) - \gamma(z) \in \overline{\mathbb{R}\rho_A(K_0(A))}/\overline{\rho_A(K_0(A))}$$

for all $z \in \mathrm{Tor}(K_1(C))$.

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