ON GENERATORS OF $\mathrm{C}_r^*(\mathbb{F}_2)$

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Abstract. It is shown that the reduced group C*-algebra $\mathrm{C}_r^*(\mathbb{F}_2)$ is singly generated, where $\mathbb{F}_2$ is the free group on two generators. With respect to the generator constructed in the paper, the topological free entropy dimension of $\mathrm{C}_r^*(\mathbb{F}_2)$ is shown to be 2.

1. Introduction

In [6], among other questions, Kadison asked whether every von Neumann algebra acting on a separable Hilbert space is singly generated (see [5] for a survey of Kadison's problem list). Many von Neumann algebras are known to be singly generated, and the problem has been reduced to the II_1 factor case ([19]), but it is still open in this case. For C*-algebras, the same question has been asked: whether every simple separable C*-algebra is singly generated. (Without simplicity, C*-algebras such as $\mathrm{C}(S^2)$ provide examples which are not singly generated; see [9].) Stable C*-algebras, UHF-absorbing C*-algebras, $\mathcal{Z}$-absorbing C*-algebras, and Villadsen's algebras of the first type, are known to be singly generated ([15], [11], [8], [14], and [7]), while examples are known of simple unital AH algebras which are not singly generated ([3]).

In this note, it is shown that the reduced group C*-algebra $\mathrm{C}_r^*(\mathbb{F}_2)$ is singly generated. In fact, there is a very simple generator (see (4)).

Theorem 1.1. *With g as given by (4), one has $\mathrm{C}_r^*(\mathbb{F}_2) = \mathrm{C}^*(g)$.*

With respect to the specified generator g, the topological free dimension of $\mathrm{C}_r^*(\mathbb{F}_2)$ is 2.

Theorem 1.2. *With respect to g given by (4), one has*

$$J + \frac{3}{2} + \log(2\pi) + \log\frac{3}{5} \leq \chi_{\mathrm{top}}(g) \leq J + \frac{3}{2} + \log(2\pi),$$

where

$$J = \frac{1}{2\pi}\int_0^{2\pi} \log\left(\frac{3 + \cos\theta + \sqrt{(3+\cos\theta)^2 - 1}}{2}\right) d\theta,$$

and

$$\delta_{\mathrm{top}}(g) = 2,$$

where χ_{top} and δ_{top} are topological free entropy and topological free entropy dimension, respectively.

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2. The free shift model of $\mathrm{C}_r^*(\mathbb{F}_2)$ and conditional expectations

Let us recall some basic facts concerning $\mathrm{C}_r^*(\mathbb{F}_2)$ and (reduced) amalgamated products.

2.1. **Free shift model.** Denote by

$$\lambda : \mathbb{F}_2 \to \mathbb{B}(l^2(\mathbb{F}_2))$$

the left regular representation of $\mathbb{F}_2$. Consider reduced group C*-algebra $\mathrm{C}_r^*(\mathbb{F}_2)$, the C*-algebra generated by $\lambda(\mathbb{F}_2)$.

Write $\mathbb{F}_2 = \langle u, v\rangle$. In the paper, we shall view $\mathbb{F}_2$ as the semidirect product

$$\mathbb{F}_2 = \mathbb{F}_\infty \rtimes \mathbb{Z},$$

where

$$\mathbb{F}_\infty = \left\langle w_n := u^n v u^{-n},\ n \in \mathbb{Z} \right\rangle$$

(see Example 5.4 in [13]). Then, we have the canonical decomposition

$$\mathrm{C}_r^*(\mathbb{F}_2) \cong B \rtimes_\beta \mathbb{Z},$$

where

$$B := \mathrm{C}^*\{w_n; n \in \mathbb{Z}\} \subseteq \mathrm{C}_r^*(\mathbb{F}_2) \quad \text{and} \quad \beta(w_n) = w_{n+1}. \tag{1}$$

Write

$$(B, \tau) = \mathrm{C}_r^*(\mathbb{F}_\infty) = *_{n\in\mathbb{Z}}(\mathrm{C}^*(w_n), \tau),$$

where τ is the unique normalized trace ([12]) and denote by

$$\mathbb{E}_n^B : B = \mathrm{C}_r^*(\mathbb{F}_\infty) \to \mathrm{C}^*(w_n)$$

the canonical conditional expectation. Note that

$$\mathbb{E}_n^B \circ \beta = \beta \circ \mathbb{E}_{n-1}^B, \quad n \in \mathbb{Z}. \tag{2}$$

2.2. **Powers's property.** The following property of the conditional expectation (inspired by [12]) seems to be well known to experts. Since the authors cannot find a reference, a proof is provided in the appendix.

Theorem 2.1. *Let*

$$(A, \tau) = (M, \tau_M) * (N, \tau_N)$$

be a reduced free product of unital tracial C-algebras with faithful GNS representations. Assume that M is abelian, and let $d \in M$ be a Haar unitary (i.e., $\tau_M(d^k) = 0$, $k \in \mathbb{Z} \setminus \{0\}$). Then,*

$$\mathbb{E}_M(a) = \lim_{L\to\infty} \frac{1}{L} \sum_{k=1}^{L} d^k a d^{-k}, \quad a \in A,$$

where $\mathbb{E}_M : A \to M$ is the canonical conditional expectation.

Applying Theorem 2.1 to the free product decomposition

$$(B,\tau) = (\mathrm{C}^*(w_0),\tau) * (*_{n\in\mathbb{Z}\setminus\{0\}}(\mathrm{C}^*(w_n),\tau)),$$

one has the following corollary.

Corollary 2.2. *Let $D \subseteq B$ be a sub-C*-algebra, and suppose that there exists a Haar unitary $d \in D \cap \mathrm{C}^*(w_0)$. Then*

$$\mathbb{E}_0^B(D) \subseteq D.$$

3. $\mathrm{C}^*_r(\mathbb{F}_2)$ is singly generated

Put

$$q = \exp\big(\frac{i}{10}(w_{-1}w_0 + w_0^*w_{-1}^*)\big), \quad \text{and} \quad h = 3 + w_0 + w_0^*, \tag{3}$$

and consider the element

$$g = uqh, \tag{4}$$

or, explicitly as a function of u and v,

$$g = u\exp(\frac{i}{10}(u^{-1}vuv + v^{-1}u^{-1}v^{-1}u))(3 + v + v^{-1}).$$

Then g generates $\mathrm{C}^*_r(\mathbb{F}_2)$:

Theorem 3.1. *With g as above, one has*

$$\mathrm{C}^*_r(\mathbb{F}_2) = \mathrm{C}^*(g).$$

The proof of this theorem is based on the next four lemmas.

Note first that the element uq is unitary. Also,

$$h = |g| = (g^*g)^{1/2} \in \mathrm{C}^*(g). \tag{5}$$

Since $\|v + v^*\| \leq 2$, the element $h = 3 + v + v^*$ is invertible.

Since the unitary q is in B, composing $\mathrm{Ad}(q)$ with β (see (1)), one obtains the automorphism

$$\gamma := \beta \circ \mathrm{Ad}(q) \in \mathrm{Aut}(B). \tag{6}$$

Since $h \in B$ (see (1)), the C*-algebra

$$D := \mathrm{C}^*\{\gamma^n(h) :\ n \in \mathbb{Z}\} \tag{7}$$

is a γ-invariant sub-C*-algebra of B.

Noting that γ is implemented by the unitary

$$uq = gh^{-1} \in \mathrm{C}^*(g),$$

one obtains

$$\gamma^n(h) \in \mathrm{C}^*(g), \quad n \in \mathbb{Z}.$$

Thus,

$$D \subseteq B \cap \mathrm{C}^*(g). \tag{8}$$

Lemma 3.2. *With D as in* (7)*, one has*

$$\mathbb{E}_0^B(D) \subseteq D.$$

Proof. By Corollary (2.2), it is enough to find a Haar unitary in $D \cap \mathrm{C}^*(w_0)$. Put

$$a = w_0 + w_0^* = h - 3 \in D \cap \mathrm{C}^*(w_0)$$

and consider the unitary

$$d = \exp(2i \arccos(a/2)\big) \in D \cap \mathrm{C}^*(w_0).$$

We assert that d is a Haar unitary.

Indeed, identify $\mathrm{C}^*(w_0)$ with $\mathrm{C}(\mathbb{T})$ by $w_0 = e^{i\theta}$ where $\theta \in [0, 2\pi]$. Then

$$\arccos(\cos\theta) = \begin{cases} \theta, & 0 \le \theta \le \pi, \\ 2\pi - \theta, & \pi \le \theta \le 2\pi, \end{cases}$$

and therefore

$$\tau(d^m) = \frac{1}{2\pi}\left(\int_0^\pi e^{2im\theta}\, d\theta + \int_\pi^{2\pi} e^{-2im\theta}\, d\theta\right) = 0, \quad m \in \mathbb{Z} \setminus \{0\},$$

as asserted. □

Lemma 3.3. *With D as in* (7)*, one has*

$$w_0 \in D.$$

Proof. Writing

$$z := w_{-1} w_0,$$

one has

$$q = \exp(\frac{i}{10}(z + z^*)) = \sum_{n \in \mathbb{Z}} c_n z^n, \tag{9}$$

where

$$c_n = i^n J_n(1/5),$$

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!}(\frac{x}{2})^{n+2k}, \quad n \ge 0,$$

and

$$J_n(x) = (-1)^{-n} J_{-n}(x), \quad n < 0.$$

(see, for instance, the Jacobi-Anger expansion on pages 22–23 of [18]).

The series (9) converges absolutely in norm. It follows that

$$\begin{aligned} q(w_0 + w_0^*)q^* &= \sum_{n,m \in \mathbb{Z}} c_n \overline{c_m}(z^n w_0 z^{-m} + z^n w_0^* z^{-m}) \\ &= \sum_{n,m \in \mathbb{Z}} c_n \overline{c_m}((w_{-1}w_0)^n w_0 (w_{-1}w_0)^{-m} + (w_{-1}w_0)^n w_0^* (w_{-1}w_0)^{-m}). \end{aligned}$$

Since w_{-1} and w_0 are freely independent, we have

$$\mathbb{E}^B_{-1}((w_{-1}w_0)^n w_0(w_{-1}w_0)^{-m}) = \begin{cases} w_{-1}^*, & n=0,\ m=1 \\ 0, & \text{otherwise} \end{cases},$$

and

$$\mathbb{E}^B_{-1}((w_{-1}w_0)^n w_0^*(w_{-1}w_0)^{-m}) = \begin{cases} w_{-1}, & n=1,\ m=0 \\ 0, & \text{otherwise} \end{cases}.$$

It then follows that

$$\begin{aligned} \mathbb{E}^B_{-1}(q(w_0+w_0^*)q^*) &= c_1\overline{c_0}w_{-1} + c_0\overline{c_1}w_{-1}^* \\ &= iJ_1(1/5)\overline{J_0(1/5)}w_{-1} + J_0(1/5)\overline{iJ_1(1/5)}w_{-1}^* \\ &= i\kappa(w_{-1}-w_{-1}^*), \end{aligned}$$

where

$$\kappa = J_0(1/5)J_1(1/5) > 0.$$

Together with (2), one has

$$\begin{aligned} \mathbb{E}^B_0(\gamma(h)) &= \mathbb{E}^B_0(\beta(qhq^*)) \\ &= \beta(\mathbb{E}^B_{-1}(q(3+w_0+w_0^*)q^*)) \\ &= 3+i\kappa\beta(w_{-1}-w_{-1}^*) \\ &= 3+i\kappa(w_0-w_0^*). \end{aligned}$$

Note that $\gamma(h) \in D$ (see (7)). It follows from Lemma 3.2 that

$$w_0 - w_0^* = (\mathbb{E}^B_0(\gamma(h)) - 3)/i\kappa \in D.$$

Since $w_0 + w_0^* = h - 3 \in D$, one has

$$w_0 = \frac{1}{2}(w_0+w_0^*) + \frac{1}{2}(w_0-w_0^*) \in D,$$

as asserted. □

Lemma 3.4. *For any self-adjoint elements a, b in a C*-algebra, one has*

$$\|e^{ia} - e^{ib}\| \le \|a-b\|.$$

Proof. By Duhamel's formula (see, for instance, [4], pages 161, 162),

$$e^{ia} - e^{ib} = i\int_0^1 e^{i(1-s)a}(a-b)e^{isb}ds.$$

The lemma follows. □

Lemma 3.5. *With D as in (7), one has*

$$q \in D.$$

Proof. By Lemma 3.3, $w_0 \in D$. Since D is γ-invariant, one has (see (6))

$$q^* w_{-1} q = q^* \beta^{-1}(w_0) q = \gamma^{-1}(w_0) \in D. \tag{10}$$

Define the maps

$$Q : B \ni b \mapsto \exp(\frac{i}{10}(bw_0 + w_0^* b^*)) \in B$$

and

$$\Phi : B \ni b \mapsto Q(b) q^* w_{-1} q Q(b)^* \in B.$$

(Recall that by definition w_n and q are in B.)

Then, for $b_1, b_2 \in B$, by Lemma 3.4, one has

$$\begin{aligned}
\|\Phi(b_1) - \Phi(b_2)\| &= \|Q(b_1) q^* w_{-1} q Q(b_1)^* - Q(b_2) q^* w_{-1} q Q(b_2)^*\| \\
&\leq \|Q(b_1) q^* w_{-1} q Q(b_1)^* - Q(b_2) q^* w_{-1} q Q(b_1)^*\| \\
&\quad + \|Q(b_2) q^* w_{-1} q Q(b_1)^* - Q(b_2) q^* w_{-1} q Q(b_2)^*\| \\
&\leq 2\|Q(b_1) - Q(b_2)\| \\
&\leq \frac{1}{5}\|(b_1 w_0 + w_0^* b_1^*) - (b_2 w_0 + w_0^* b_2^*)\| \\
&\leq \frac{2}{5}\|b_1 - b_2\|.
\end{aligned}$$

Note that, by (3) (the definition of q),

$$Q(w_{-1}) = \exp(\frac{i}{10}(w_{-1} w_0 + w_0^* w_{-1}^*)) = q,$$

and so

$$\Phi(w_{-1}) = q q^* w_{-1} q q^* = w_{-1}.$$

Then, for each $n \in \mathbb{N}$, iterating the inequality above, one has

$$\|\Phi^n(0) - w_{-1}\| = \|\Phi^n(0) - \Phi^n(w_{-1})\| \leq (\frac{2}{5})^n \|0 - w_{-1}\|.$$

That is,

$$w_{-1} = \lim_{n \to \infty} \Phi^n(0). \tag{11}$$

A straightforward calculation shows that

$$\Phi(w_0),\ \Phi(q^* w_{-1} q) \in \mathrm{C}^*(w_0, q^* w_{-1} q)$$

and

$$\Phi(0) \in \mathrm{C}^*(w_0, q^* w_{-1} q).$$

Therefore,

$$\Phi^n(0) \in \mathrm{C}^*(w_0, q^* w_{-1} q), \quad n \in \mathbb{N}.$$

Hence, by (11),

$$w_{-1} \in \mathrm{C}^*(w_0, q^* w_{-1} q).$$

Since $w_0 \in D$ (by Lemma 3.3) and $q^* w_{-1} q \in D$ (see (10)), one has

$$\mathrm{C}^*(w_0, q^* w_{-1} q) \subseteq D,$$

and therefore

$$q = \exp(\frac{i}{10}(w_{-1}w_0 + w_0^* w_{-1}^*)) \in D,$$

as desired. □

Let us prove the main theorem.

Proof of Theorem 3.1. By (5), Lemma 3.5, and (8), respectively, one has

$$h, q \in \mathrm{C}^*(g),$$

and therefore

$$u = uqhh^{-1}q^{-1} = gh^{-1}q^{-1} \in \mathrm{C}^*(g).$$

By Lemma 3.3 and (8),

$$v = w_0 \in D \subseteq \mathrm{C}^*(g).$$

Thus,

$$\mathrm{C}_r^*(\mathbb{F}_2) = \mathrm{C}^*(u, v) \subseteq \mathrm{C}^*(g) \subseteq \mathrm{C}_r^*(\mathbb{F}_2),$$

as stated. □

Remark 3.6. The formula (4) was provided by ChatGPT. Is there some fundamental mechanism behind this formula?

4. Free entropy and free entropy dimension of the generator

As a further revelation of the usefulness of ChatGPT, let us determine, or rather estimate the topological free entropy and topological free entropy dimension of $\mathrm{C}_r^*(g)$, with respect to the generator g given by (4).

Let τ denote the canonical trace on $\mathrm{C}_r^*(\mathbb{F}_2)$, and retain the notation

$$w_{-1} = u^{-1}vu, \qquad q = \exp\left(\frac{i}{10}(w_{-1}v + v^* w_{-1}^*)\right), \qquad h = 3 + v + v^*, \qquad g = uqh.$$

Thus, $g^*g = h^2$ and $1 \leq h \leq 5$. The microstates free entropy below is computed in the tracial W^*-probability space $(L(\mathbb{F}_2), \tau)$. Since microstates free entropy and topological free entropy are defined for self-adjoint tuples, throughout this section we use the conventions

$$\chi(g) = \chi^{\mathrm{sa}}(\Re(g), \Im(g)), \qquad \chi_{\mathrm{top}}(g) = \chi_{\mathrm{top}}^{\mathrm{sa}}(\Re(g), \Im(g)),$$

and similarly for the corresponding free entropy dimensions. Equivalently, one may use the non-self-adjoint microstate convention of [10]: its Euclidean measure on $\mathrm{M}_k(\mathbb{C})$ and normalization term $\log k$ agree with the product Euclidean measure on $(M_k^{\mathrm{sa}})^2$ and its normalization term $\log k$ under the map $T \mapsto (\Re T, \Im T)$.

Put

$$c_{\mathrm{U}} := \frac{3}{4} + \frac{1}{2}\log(2\pi) \tag{12}$$

and

$$\begin{aligned}\mathcal{J} &:= \frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\log(3+\cos\theta+\cos\varphi)\,d\theta\,d\varphi \\ &= \frac{1}{2\pi}\int_0^{2\pi}\log\left(\frac{3+\cos\theta+\sqrt{(3+\cos\theta)^2-1}}{2}\right)d\theta.\end{aligned} \tag{13}$$

Theorem 4.1. *With Voiculescu's normalization of microstates free entropy and topological free entropy, one has*

$$\mathcal{J}+2c_{\mathrm{U}}+\log\frac{3}{5} \;\le\; \chi_{\mathrm{top}}(g) \;\le\; \chi(g) \;\le\; \mathcal{J}+2c_{\mathrm{U}}. \tag{14}$$

Consequently,

$$\delta_{\mathrm{top}}(\Re g,\Im g)=\delta_0(\Re g,\Im g;\tau)=\delta(\Re g,\Im g;\tau)=2. \tag{15}$$

The argument below gives bounds (those of (14)) rather than the exact value of either entropy. The gap in (14) is $\log(5/3)$; it comes from a uniform lower bound for the Jacobian of the twisting map $U\mapsto Uq(U,V)$.

Once $\chi_{\mathrm{top}}(g)$ is shown to be finite ((14)), it follows from [17] that the topological free dimension must be 2.

4.1. **The upper bound of** (14)**.** We first compare topological entropy and tracial microstates entropy. Voiculescu's free-capacity inequality [17, Section 3], together with uniqueness of the tracial state on $\mathrm{C}_r^*(\mathbb{F}_2)$ and the identification $\mathrm{C}^*(\Re g,\Im g)=\mathrm{C}_r^*(\mathbb{F}_2)$, gives

$$\chi_{\mathrm{top}}(g)\le\chi(g). \tag{16}$$

Indeed, if sufficiently fine norm-microstate conditions did not force a prescribed finite collection of τ-moment conditions, a sequence of counterexamples and a free ultrafilter would produce a tracial state on $C_r^*(\mathbb{F}_2)$ different from τ.

Let r be a Haar unitary which is $*$-free from h. The maximality theorem for the microstates free entropy of R-diagonal elements [10, Theorem 2.1] gives

$$\chi(g)\le\chi(rh)=\chi^{\mathrm{sa}}\left(\frac{h^2}{2}\right)+c_{\mathrm{U}}. \tag{17}$$

We calculate the last one-variable entropy. With respect to τ, the spectral distribution of v is normalized Haar measure on the circle. Set

$$H(\theta)=3+2\cos\theta, \qquad P(\theta)=\frac{1}{2}H(\theta)^2.$$

The one-variable formula for free entropy yields

$$\chi^{\mathrm{sa}}\left(\frac{h^2}{2}\right)=\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\log|P(\theta)-P(\varphi)|\,d\theta\,d\varphi+c_{\mathrm{U}}. \tag{18}$$

Now

$$P(\theta)-P(\varphi)=\big(H(\theta)-H(\varphi)\big)\frac{H(\theta)+H(\varphi)}{2}.$$

Writing $\lambda = e^{i\theta}$ and $\mu = e^{i\varphi}$, we also have

$$H(\theta) - H(\varphi) = (\lambda - \mu)\left(1 - \frac{1}{\lambda\mu}\right).$$

The logarithmic potential of normalized Haar measure on the circle vanishes on the circle. Consequently,

$$\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} \log|H(\theta) - H(\varphi)|\, d\theta\, d\varphi = 0.$$

The remaining term in (18) is precisely $\mathcal{J}$. Hence

$$\chi^{\mathrm{sa}}\left(\frac{h^2}{2}\right) = \mathcal{J} + c_{\mathrm{U}}. \tag{19}$$

Equations (16), (17), and (19) prove the two upper inequalities in (14).

4.2. **A Jacobian estimate.** Equip $M_k(\mathbb{C})$ with the real Hilbert–Schmidt inner product

$$\langle A, B\rangle = \Re\,\mathrm{Tr}(AB^*)$$

and use the induced Riemannian metric on $\mathrm{U}(k)$. Let μ_k be normalized Haar measure on $\mathrm{U}(k)$ and let C_k be its Riemannian volume. Thus Riemannian volume is $C_k\mu_k$, and

$$\lim_{k\to\infty}\left(\frac{1}{k^2}\log C_k + \frac{1}{2}\log k\right) = c_{\mathrm{U}}; \tag{20}$$

see, for example, [10, Lemma 2.8].

For $U, V \in \mathrm{U}(k)$, define

$$\begin{aligned} Z(U,V) &= U^*VUV, \\ Q(U,V) &= \exp\left(\frac{i}{10}(Z(U,V) + Z(U,V)^*)\right), \\ H(V) &= 3 + V + V^*, \\ \Phi_k(U,V) &= UQ(U,V)H(V). \end{aligned} \tag{21}$$

Lemma 4.2. *For fixed $V \in \mathrm{U}(k)$, the map*

$$\psi_V \colon \mathrm{U}(k) \longrightarrow \mathrm{U}(k), \qquad \psi_V(U) = UQ(U,V),$$

is injective, and every singular value of its differential is at least $3/5$. Consequently, the real Jacobian of

$$\Theta_k(U,V) = (UQ(U,V), V)$$

is at least $(3/5)^{k^2}$.

Proof. For $U_1, U_2 \in \mathrm{U}(k)$,

$$\|Z(U_1,V) - Z(U_2,V)\|_2 \le 2\|U_1 - U_2\|_2.$$

The Duhamel formula, valid for every unitarily invariant norm (see Lemma 3.4 and proof), gives $\|e^{iA} - e^{iB}\|_2 \le \|A - B\|_2$ for self-adjoint A, B. It follows that

$$\|Q(U_1,V) - Q(U_2,V)\|_2 \le \frac{2}{5}\|U_1 - U_2\|_2.$$

Therefore

$$\|\psi_V(U_1) - \psi_V(U_2)\|_2 \geq \frac{3}{5}\|U_1 - U_2\|_2.$$

This proves injectivity. Differentiating the last inequality along smooth curves shows that every singular value of $D\psi_V$ is at least $3/5$. Since $\dim_{\mathbb{R}} \mathrm{U}(k) = k^2$ and $D\Theta_k$ is block triangular, the asserted Jacobian bound follows. □

Set

$$\mathcal{P}_k(V) = \frac{1}{2}H(V)^2. \tag{22}$$

Lemma 4.3. *Suppose that the eigenvalues of V are $e^{i\theta_1}, \dots, e^{i\theta_k}$, and put $h_j = 3 + 2\cos\theta_j$. At every regular point of $\mathcal{P}_k$, its real Jacobian is*

$$\operatorname{Jac}\mathcal{P}_k(V) = \prod_{i,j=1}^{k} \frac{h_i + h_j}{2}\left|e^{i\theta_i} - e^{-i\theta_j}\right|. \tag{23}$$

Proof. By unitary equivariance, it is enough to calculate at a diagonal V. On the diagonal tangent direction which changes θ_i, the absolute derivative of $h_i^2/2$ is

$$2h_i|\sin\theta_i| = h_i|e^{i\theta_i} - e^{-i\theta_i}|.$$

For $i < j$, write the tangent vector at V as VK, where $K^* = -K$, and put $K_{ij} = z$. The corresponding (i,j) entry of $DH(V)$ is $(e^{i\theta_i} - e^{-i\theta_j})z$, and hence that of $D\mathcal{P}_k(V)$ is

$$\frac{h_i + h_j}{2}(e^{i\theta_i} - e^{-i\theta_j})z.$$

The determinant on this real two-dimensional off-diagonal tangent plane is the square of the modulus of the displayed scalar. Multiplication over all diagonal directions and all pairs $i < j$ gives (23). □

The polar-coordinate map

$$\mathrm{U}(k) \times M_k^{\mathrm{sa},+} \longrightarrow M_k(\mathbb{C}), \qquad (W,P) \longmapsto W(2P)^{1/2}, \tag{24}$$

has Jacobian one when the first factor has its Hilbert–Schmidt Riemannian volume and the second and target have Euclidean Lebesgue measure. Equivalently, when normalized Haar measure is used on the first factor, the Jacobian factor is C_k; see the polar-decomposition calculation in [10, Section 2]. The factorization

$$(U,V) \xrightarrow{\Theta_k} (UQ(U,V),V) \longmapsto (UQ(U,V),\mathcal{P}_k(V)) \xrightarrow{(24)} \Phi_k(U,V)$$

therefore combines Lemmas 4.2 and 4.3 into a lower bound for $\operatorname{Jac}\Phi_k$.

4.3. **Phase averaging and the lower bound of** (14)**.** For V as in Lemma 4.3, put

$$A_k(V) := \frac{1}{k^2} \sum_{i,j=1}^{k} \log\left(\frac{h_i + h_j}{2}\right), \tag{25}$$

$$B_k(V) := \frac{1}{k^2} \sum_{i,j=1}^{k} \log |e^{i\theta_i} - e^{-i\theta_j}|. \tag{26}$$

Thus

$$\frac{1}{k^2} \log \mathrm{Jac}\, \mathcal{P}_k(V) = A_k(V) + B_k(V). \tag{27}$$

If the empirical eigenvalue measures of V_k converge weakly to normalized Haar measure on the circle, then

$$\sup_{t \in \mathbb{R}} |A_k(e^{it} V_k) - \mathcal{J}| \longrightarrow 0. \tag{28}$$

This follows because the function $(\theta, \varphi) \mapsto \log(3 + \cos\theta + \cos\varphi)$ is continuous, and the family of its simultaneous translates is compact in $C(\mathbb{T}^2)$.

The singular term B_k is controlled by averaging the scalar phase. For every V outside a null set,

$$\frac{1}{2\pi} \int_0^{2\pi} B_k(e^{it} V)\, dt = 0, \qquad B_k(e^{it} V) \leq \log 2. \tag{29}$$

Indeed, each summand has the form

$$\log |e^{i(\theta_i + \theta_j + 2t)} - 1|,$$

the integral of which is zero. It follows from (29) that, for every $\eta > 0$,

$$\frac{1}{2\pi} \left| \{ t \in [0, 2\pi] : B_k(e^{it} V) \geq -\eta \} \right| \geq \frac{\eta}{\log 2 + \eta}. \tag{30}$$

We now pass to topological microstates. Let U_k, V_k be independent Haar-distributed unitary matrices. Strong asymptotic freeness [2] implies that (U_k, V_k) converges almost surely strongly to the canonical free Haar pair (u, v). The circle of automorphisms

$$\alpha_t(u) = u, \qquad \alpha_t(v) = e^{it} v$$

preserves both the norm and the canonical trace, and

$$\Phi_k(U, e^{it} V)$$

is the matrix expression corresponding to $\alpha_t(g)$. Strong convergence, a finite-net argument in t, and the uniform continuity of the expressions in (21) therefore have the following consequence. For every fixed topological microstate neighborhood of g, there are phase-invariant Borel sets

$$\Omega_k \subseteq \mathrm{U}(k)^2, \qquad (\mu_k \times \mu_k)(\Omega_k) \longrightarrow 1,$$

such that

(1) $\Phi_k(U, V)$ belongs to that microstate neighborhood whenever $(U, V) \in \Omega_k$;
(2) $A_k(e^{it} V) \geq \mathcal{J} - \eta$ for every t whenever $(U, V) \in \Omega_k$, after increasing k if necessary.

Here phase-invariance means that $(U,V)\in\Omega_k$ if and only if $(U,e^{it}V)\in\Omega_k$ for every t.

Define

$$E_k=\{(U,V)\in\Omega_k : B_k(V)\geq -\eta\}.$$

By (30) and phase-invariance,

$$(\mu_k\times\mu_k)(E_k)\geq\frac{\eta}{\log 2+\eta}(\mu_k\times\mu_k)(\Omega_k). \tag{31}$$

Moreover, (27) gives

$$\operatorname{Jac}\mathcal{P}_k(V)\geq\exp\bigl(k^2(\mathcal{J}-2\eta)\bigr),\qquad (U,V)\in E_k. \tag{32}$$

It remains to control multiplicity. The polar decomposition of $\Phi_k(U,V)$ determines $W=UQ(U,V)$ and $P=\mathcal{P}_k(V)$. For almost every P, its spectrum is simple. For such a P, the equation

$$3+V+V^*=(2P)^{1/2}$$

has at most 2^k solutions. Indeed, V commutes with $V+V^*$; hence it preserves each one-dimensional spectral subspace of $(2P)^{1/2}-3$, and on each such subspace there are at most two choices for its eigenvalue. For each fixed V, Lemma 4.2 shows that U is uniquely determined by W. Thus the multiplicity of Φ_k is at most 2^k almost everywhere.

Let λ_k denote Euclidean Lebesgue measure on $M_k(\mathbb{C})$. The area formula, the two Jacobian lemmas, (32), and (31) give

$$\lambda_k(\Phi_k(E_k))\geq 2^{-k}\left(\frac{3}{5}\right)^{k^2}e^{k^2(\mathcal{J}-2\eta)}C_k^2(\mu_k\times\mu_k)(E_k). \tag{33}$$

Since $\Phi_k(E_k)$ is contained in the prescribed topological microstate neighborhood of g, we take $k^{-2}\log$, add the normalization term $\log k$, use (20), and then let k tend to ∞. The factors 2^{-k} and $(\mu_k\times\mu_k)(E_k)$ do not contribute on the k^2 scale. We obtain

$$\chi_{\mathrm{top}}(g)\geq\mathcal{J}-2\eta+2c_{\mathrm{U}}+\log\frac{3}{5}.$$

Letting $\eta\downarrow 0$ proves the lower inequality in (14).

Finally, finite topological free entropy for a two-variable self-adjoint tuple forces its topological free entropy dimension to be 2 ([17]). Likewise, finite microstates free entropy forces both δ and δ_0 to equal the number of self-adjoint variables ([16]). This proves (15).

Appendix A. Proof of Theorem 2.1

We follow the argument of [1], proving that $\mathrm{C}_r^*(\mathbb{Z}_2*\mathbb{Z}_3)$ is simple, has unique trace, and is not quasidiagonal (compare also the early paper of Powers, [12]). First, recall the following lemma:

Lemma A.1 (Lemma 2.2 of [1]). *Let $\mathcal{H}=\mathcal{M}\oplus\mathcal{M}^\perp$ be a direct sum decomposition of a Hilbert space $\mathcal{H}$. Suppose that $x\in\mathbb{B}(\mathcal{H})$ has the form $\begin{pmatrix}0 & *\\ * & *\end{pmatrix}$ and $u_i\in\mathbb{B}(\mathcal{H})$, $i=1,\ldots,n$, are unitaries*

such that $u_i u_j^*$ *have the form* $\begin{pmatrix} * & * \\ * & 0 \end{pmatrix}$ *when* $i \neq j$. *Then*

$$\|\sum_{i=1}^{n} u_i^* x u_i\| \leq 2\sqrt{n}\|x\|.$$

Let us also recall the construction of the reduced free product of tracial C*-algebras. Let (M, τ_M) and (N, τ_N) be unital tracial C*-algebras with faithful GNS representations. A reduced word is a product $a_1 \cdots a_n$, where each a_j is contained in $\ker \tau_M$ or $\ker \tau_N$, and adjacent letters are contained in different factors. The canonical conditional expectation $\mathbb{E}_M : A \to M$ is defined so that $\mathbb{E}_M(a) = a$ for each $a \in M$ and $\mathbb{E}_M(b) = 0$ for each reduced word containing at least a letter in $\ker \tau_N$.

For $k \in \{M, N\}$, set

$$H_k = L^2(k, \tau_k), \quad H_k' = H_k \ominus \mathbb{C}\widehat{1},$$

where $\widehat{a}$ denotes the GNS vector of $a \in k$, and the inner product in H_k is of course

$$\langle \widehat{a}, \widehat{b} \rangle := \tau_k(b^* a).$$

The reduced free-product Hilbert space is

$$\mathcal{F} = \mathbb{C}\Omega \oplus \bigoplus_{\substack{n \in \mathbb{N},\ k_1, \ldots, k_n \in \{M,N\} \\ k_j \neq k_{j+1} (1 \leq j < n)}} H_{k_1}' \otimes \cdots \otimes H_{k_n}', \tag{34}$$

where Ω is a unit vector which is called the vacuum.

The (reduced) free-product C*-algebra

$$(A, \tau) = (M, \tau_M) * (N, \tau_N)$$

is the C*-algebra generated by the left actions of M and N on $\mathcal{F}$. In fact, if $a \in \ker \tau_k$, then

$$a\Omega = \widehat{a}, \quad a(\xi_1 \otimes \cdots \otimes \xi_n) = \widehat{a} \otimes \xi_1 \otimes \cdots \otimes \xi_n, \quad \text{when } \ \xi_1 \in H_j',\ j \neq k.$$

Lemma A.2. *Let*

$$(A, \tau) = (M, \tau_M) * (N, \tau_N)$$

be a reduced free product of unital tracial C-algebras with faithful GNS representations. Let* $d \in M$ *be a Haar unitary. If* $x = a_1 \cdots a_n$ *is a reduced word with* $a_1, a_n \in \ker \tau_N$, *then for each* $L \in \mathbb{N}$, *we have*

$$\|\sum_{k=1}^{L} d^k x d^{-k}\| \leq 2\sqrt{L}\|x\|.$$

Proof. Consider

$$\mathcal{M} := \overline{\operatorname{span}\{\xi_1 \otimes \cdots \otimes \xi_m : \xi_1 \in H_M',\ m \in \mathbb{N}\}}.$$

Then

$$\mathcal{M}^\perp = \overline{\operatorname{span}\{\Omega,\ \xi_1 \otimes \cdots \otimes \xi_m : \xi_1 \in H_N',\ m \in \mathbb{N}\}}.$$

Let P be the orthogonal projection onto $\mathcal{M}$. If an elementary tensor begins in H_M', then the rightmost letter a_n, which belongs to $\ker \tau_N$, acts by creation. Every preceding letter also acts

by creation, since the word is reduced. The resulting tensor begins in H'_N, because $a_1 \in \ker \tau_N$. Thus $x\mathcal{M} \subseteq \mathcal{M}^\perp$ and $PxP = 0$.

For $i \neq j$, the Haar property gives $d^{j-i} \in \ker \tau_M$. This element sends the vacuum and every elementary tensor beginning in H'_N into $\mathcal{M}$. Consequently, $(1-P)d^{j-i}(1-P) = 0$. Apply Lemma A.1 with $u_i = d^{-i}$, $1 \leq i \leq L$. Then $u_i u_j^* = d^{j-i}$, and its conclusion is exactly

$$\left\| \sum_{i=1}^{L} d^i x d^{-i} \right\| = \left\| \sum_{i=1}^{L} u_i^* x u_i \right\| \leq 2\sqrt{L}\|x\|.$$

□

Proof of Theorem 2.1. For $L \in \mathbb{N}$, define

$$A_L(a) = \frac{1}{L} \sum_{k=1}^{L} d^k a d^{-k}, \quad a \in A.$$

Every element b of the algebraic $*$-algebra generated by M and N can be written in the form

$$b = m + \sum_{j=1}^{r} m_{0,j} x_j m_{1,j},$$

where $m, m_{0,j}, m_{1,j} \in M$, $1 \leq j \leq r$, and each x_j is a reduced word beginning and ending in $\ker \tau_N$. Thus every x_j satisfies both endpoint conditions of Lemma A.2. Hence, by Lemma A.2 and noting that M is abelian, one has

$$\begin{aligned} \|A_L(b) - \mathbb{E}_M(b)\| &= \left\| \sum_{j=1}^{r} m_{0,j} A_L(x_j) m_{1,j} \right\| \\ &\leq \frac{2}{\sqrt{L}} \sum_{j=1}^{r} \|m_{0,j}\| \|x_j\| \|m_{1,j}\| \to 0, \end{aligned}$$

as $L \to \infty$. Since A_L and $\mathbb{E}_M$ are contractions, we have

$$\|A_L(a) - \mathbb{E}_M(a)\| \to 0, \quad a \in A,$$

as desired. □

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