ON THE DYNAMICAL ASYMPTOTIC DIMENSION OF A FREE $\mathbb{Z}^d$-ACTION ON THE CANTOR SET

ZHUANG NIU AND XIAOKUN ZHOU

Abstract. Consider an arbitrary extension of a free $\mathbb{Z}^d$-action on the Cantor set. It is shown that it has dynamical asymptotic dimension at most $3^d - 1$.

1. Introduction

Dynamical Asymptotical Dimension is introduced by Guentner, Willett, and Yu in [2] to describe the complexity of a topological dynamical system:

Definition 1.1 (Definition 2.1 of [2]). Consider a group action $X \curvearrowleft \Gamma$, where X is a compact Hausdorff space and Γ is a discrete group. Its dynamical asymptotic dimension (DAD) is the smallest non-negative integer d such that for any finite set $\mathcal{F} \subseteq \Gamma$, there is an open cover $U_0 \cup U_1 \cup \cdots \cup U_d$ of X such that for each U_i, $0 \leq i \leq d$, the set

$$\left\{ \gamma \in \Gamma \;\middle|\; \begin{array}{l} \text{there exists } x \in U_i \text{ and } \gamma_1, ..., \gamma_K \in \mathcal{F} \\ \text{such that } \gamma = \gamma_1\gamma_2 \cdots \gamma_K \text{ and} \\ \text{for all } k \in \{1, ..., K\},\ x\gamma_1 \cdots \gamma_k \in U_i \end{array} \right\}$$

is finite.

If the action is free, the dynamical system $X \curvearrowleft \Gamma$ has dynamical asymptotic dimension at most d if, and only if, the following holds (see Remark 2.2(3) and Definition 1 of [2]): for any finite set $\mathcal{F} \subseteq \Gamma$ satisfying $\mathcal{F} = \mathcal{F}^{-1}$ and $e \in \mathcal{F}$, there exist an open cover $U_0 \cup U_1 \cup \cdots \cup U_d$ of X and $M > 0$ such that for each U_i, $0 \leq i \leq d$, each $x \in U_i$, the cardinality of the set

$$\mathcal{O}_x := \{y \in U_i : \exists \gamma_1, ..., \gamma_K \in \mathcal{F},\ y = x\gamma_1 \cdots \gamma_K,\ x\gamma_1 \cdots \gamma_k \in U_i,\ 1 \leq k \leq K,\ K \in \mathbb{N}\}$$

is at most M.

It is shown in [2] (Theorem 3.1) that the dynamical asymptotical dimension of any minimal $\mathbb{Z}$-action is at most 1, regardless of the space X. It is also shown in [2] that for any discrete group Γ with asymptotic dimension at most d, there is a Γ-action on the Cantor set which has dynamical asymptotical dimension at most d. In this note, we estimate the dynamical asymptotical dimension of an arbitrary $\mathbb{Z}^d$-action on the Cantor set, and we show the following theorem:

Theorem (Theorem 2.8 and Corollary 2.10). *Any extension of a free $\mathbb{Z}^d$-action on the Cantor set has dynamical asymptotic dimension at most $3^d - 1$.*

Date: October 12, 2020.

Key words and phrases. Dynamical asymptotic dimension, free $\mathbb{Z}^d$-actions, Cantor system.

The research is supported by an NSF grant (DMS-1800882).

2. Main result and its proof

2.1. **Quasi-tilings of $\mathbb{Z}^d$.** Let us start with certain quasi-tilings (see [3]) of $\mathbb{Z}^d$ by cubes:

Definition 2.1. Consider $\mathbb{Z}^d$. For any natural number l, denote by $\square_l$ the cube

$$\square_l = \{-l, -l+1, ..., l-1, l\}^d \subseteq \mathbb{Z}^d.$$

Let r, D, E be natural numbers. An (r, D, E)-tiling of $\mathbb{Z}^d$, denoted by $\mathcal{T}$, is a collection of $c_i \in \mathbb{Z}^d$ such that with

$$\mathrm{Dom}(\mathcal{T}) = \bigcup_i (c_i + \square_D),$$

then,

(1) $(c_i + \square_D) \cap (c_j + \square_D) = \varnothing$, $i \neq j$,
(2) The (Euclidean) distance between $c_i + \square_D$ and $c_j + \square_D$ is at least r if $i \neq j$, and
(3) $\square_E \cap \mathrm{Dom}(\mathcal{T}) \neq \varnothing$.

In other words, an (r, D, E)-tiling of $\mathbb{Z}^d$ is a quasi-tiling by cubes of size $2D+1$, such that tiles are r-separated, but they almost cover 0 up to E.

It turns out that if $D \leq E \leq 2D$, then there are $e_0 = 0, e_1, e_2, ..., e_{3^d-1} \in \mathbb{Z}^d$ such that for any (r, D, E)-tiling $\mathcal{T}$, one of $\mathcal{T}, \mathcal{T} + e_1, ..., \mathcal{T} + e_{3^d-1}$ actually covers 0:

Lemma 2.2. *For any natural number E, then there are $e_1, e_2, ..., e_s \in \mathbb{Z}^d$, where $s = 3^d - 1$, such that if $\mathcal{T}$ is an (r, D, E)-tiling of $\mathbb{Z}^d$ for some natural numbers r and D with $D \leq E \leq 2D$, then*

$$0 \in \mathrm{Dom}(\mathcal{T}) \cup \mathrm{Dom}(\mathcal{T} + e_1) \cup \cdots \cup \mathrm{Dom}(\mathcal{T} + e_s),$$

where $s = 3^d - 1$.

Proof. Set

$$\{e_0, e_1, ..., e_{3^d-1}\} = \{(n_1, n_2, ..., n_d) \in \mathbb{Z}^d : n_i \in \{0, \pm E\}\},$$

with $e_0 = (0, ..., 0)$. In order to prove the lemma, it is enough to show that if $0 \notin \mathrm{Dom}(\mathcal{T})$, then, at least one of

$$e_i, \quad i = 1, ..., 3^d - 1,$$

is in $\mathrm{Dom}(\mathcal{T})$.

Assume none of e_i was inside $\mathrm{Dom}(\mathcal{T})$. Then one asserts that

$$\square_E \cap \mathrm{Dom}(\mathcal{T}) = \varnothing.$$

This contradicts Condition (3) and hence proves the lemma.

For the assertion, assume there is $c \in \mathbb{Z}^d$ with

$$c + \square_D \subseteq \mathrm{Dom}(\mathcal{T}) \quad \text{and} \quad \square_E \cap (c + \square_D) \neq \varnothing.$$

Then there exist

$$-E \leq n_i \leq E, \quad 1 \leq i \leq d,$$

such that

$$(n_1, ..., n_d) \in c + \square_D.$$

Note that $\square_E \cap (c + \square_D) \neq \varnothing$ implies

$$-D - E \leq c_i \leq D + E, \quad 1 \leq i \leq d,\ c = (c_1, c_2, ..., c_d);$$

and also note

$$c + \square_D = \{(c_1 + s_1, c_2 + s_2, ..., c_d + s_d) : -D \leq s_i \leq D\}.$$

For each c_i, if $|c_i| \geq E$, then choose $s_i \in [-D, D]$ such that $|c_i + s_i| = E$; if $|c_i| \leq D$, then choose $s_i = -c_i$ so that $c_i + s_i = 0$; if $D \leq |c_i| \leq E$, then choose $s_i \in [-D, D]$ such that $|c_i + s_i| = E$ (note that one assumes $E \leq 2D$). With this choice of s_i, one has that $c + \square_D$ contains at least one of e_i, and so such e_i is inside $\mathrm{Dom}(\mathcal{T})$. This contradicts the assumption, and proves the assertion. $\square$

2.2. **Group actions and equivariant quasi-tilings.** Recall

Definition 2.3. Let X be a topological space and let Γ be a discrete group. By a (right) Γ-action on X, denoted by $X \curvearrowleft \Gamma$, we mean a continuous map

$$X \times \Gamma \ni (x, \gamma) \to x\gamma \in X$$

such that

$$xe = x \quad \text{and} \quad (x\gamma_1)\gamma_2 = x(\gamma_1\gamma_2), \quad x \in X,\ \gamma_1\gamma_2 \in \Gamma.$$

We say a Γ-action on X is free if $x\gamma = x$ for some $x \in X$ and $\gamma \in \Gamma$ implies $\gamma = e$.

Consider actions $X \curvearrowleft \Gamma$ and $Y \curvearrowleft \Gamma$. We say that $X \curvearrowleft \Gamma$ is an extension of $Y \curvearrowleft \Gamma$ (or $Y \curvearrowleft \Gamma$ is a factor of $X \curvearrowleft \Gamma$) if there is a quotient map $\pi : X \to Y$ such that

$$\pi(x\gamma) = \pi(x)\gamma, \quad x \in X,\ \gamma \in \Gamma.$$

Definition 2.4. Consider an $\mathbb{Z}^d$-action on topological space X. A set-valued map

$$X \ni x \mapsto \mathcal{T}(x) \in 2^{\mathbb{Z}^d}$$

is said to be equivariant if

$$\mathcal{T}(xn) = \mathcal{T}(x) - n,$$

where $\mathcal{T}(x) - n$ is the translation of $\mathcal{T}(x)$ by $-n$.

The map $x \mapsto \mathcal{T}(x)$ is said to be continuous if for any $R > 0$ and any $x \in X$, there is an open set $U \ni x$ such that

$$\mathcal{T}(y) \cap B_R = \mathcal{T}(x) \cap B_R, \quad y \in U,$$

where B_R is the ball in $\mathbb{Z}^d$ with center 0 and radius R.

Lemma 2.5. *Consider an $\mathbb{Z}^d$-action on a topological space X. Let $N \in \mathbb{N}$, and let $x \mapsto \mathcal{T}(x)$ be a continuous equivariant map with value (r, D, E)-tilings of $\mathbb{Z}^d$ with $r > N\sqrt{d}$. Put*

$$\Omega = \{x \in X : 0 \in \mathrm{Dom}(\mathcal{T}(x))\}.$$

Then, Ω is open. Moreover, for any $x \in X$, one has

$$\begin{aligned} &\left|\{n \in \mathbb{Z}^d : n = n_1 + \cdots + n_K,\ x(n_1 + \cdots + n_k) \in \Omega, \|n_k\|_\infty \leq N,\right. \\ &\quad 1 \leq k \leq K,\ K \in \mathbb{N}\}| \\ \leq &\quad (2D+1)^d. \end{aligned} \tag{2.1}$$

Proof. The openness of Ω follows directly from the continuity of the map $x \mapsto \mathcal{T}(x)$. Let us show the estimate (2.1).

Pick $x_0 \in \Omega$, and write $c + \Box_D$ to be the tile of $\mathcal{T}(x_0)$ containing 0. Since the function $x \mapsto \mathcal{T}(x)$ is equivariant, one has that $\mathcal{T}(xn) = \mathcal{T}(x) - n$; hence, by Condition (2), for any $n \in \mathbb{Z}^d$ with $\|n\|_\infty \leq N$, one has that either 0 is in the tile $c + \Box_D - n$ (therefore $x_0 n \in \Omega$ and $c - n \in \Box_D$) or $0 \notin \mathrm{Dom}(\mathcal{T}(x_0 n))$ (therefore $x_0 n \notin \Omega$).

Thus, if there are $n_1, n_2, ..., n_K \in \mathbb{Z}^d$ with $\|n_k\|_\infty \leq N$ and

$$n_1 x_0 \in \Omega,\ x_0(n_1 + n_2) \in \Omega, ..., x_0(n_1 + \cdots + n_K) \in \Omega,$$

one has

$$c - n_1 \in \Box_D,\ c - n_1 - n_2 \in \Box_D, ..., c - n_1 - \cdots - n_K \in \Box_D,$$

and hence

$$n = n_1 + \cdots + n_K \in c + \Box_D.$$

Since $|c + \Box_D| = |\Box_D| = (2D + 1)^d$, this proves the lemma. $\Box$

2.3. **Cantor systems and an estimate of dynamical asymptotic dimension.** Let us focus on extensions of a free $\mathbb{Z}^d$-action on the Cantor set, which is the unique compact separable Hausdorff space that is totally disconnected and perfect.

First, for any free $\mathbb{Z}^d$-action on the Cantor set, equivariant continuous (r, D, E)-tiling-valued functions always exist:

Proposition 2.6. *Consider a free $\mathbb{Z}^d$-action on X where X is the Cantor set, and let $N \in \mathbb{N}$ be arbitrary. Then, there are natural numbers r, D, E with $r > N\sqrt{d}$ and $D \leq E \leq 2D$, and a continuous equivariant map $x \mapsto \mathcal{T}(x)$ on X such that each $\mathcal{T}(x)$ a (r, D, E)-tiling of $\mathbb{Z}^d$.*

Proof. The construction is similar to that of Lemma 3.4 of [1].

Pick a natural number $r > N\sqrt{d}$, and then pick a natural number $L > 2r$. Since the action is free and X is the Cantor set, by a compactness argument, one obtains mutually disjoint clopen sets $U_1, U_2, ..., U_s$, such that

$$X = U_1 \cup U_2 \cup \cdots \cup U_s,$$

and for each U_i, $1 \leq i \leq s$, the open sets

$$U_i n, \quad n \in \Box_{2L},$$

are mutually disjoint.

Start with U_1. For each $x \in X$, put

$$\begin{cases} \mathcal{C}_1(x) & = & \{n \in \mathbb{Z}^d : xn \in U_1\}, \\ \cdots & \cdots & \cdots \\ \mathcal{C}_i(x) & = & \mathcal{C}_{i-1}(x) \cup \{n \in \mathbb{Z}^d : xn \in U_i,\ (n + \Box_L) \cap (\mathcal{C}_{i-1}(x) + \Box_L) = \varnothing\}, \\ \cdots & \cdots & \cdots \\ \mathcal{C}_s(x) & = & \mathcal{C}_{s-1}(x) \cup \{n \in \mathbb{Z}^d : xn \in U_s,\ (n + \Box_L) \cap (\mathcal{C}_{s-1}(x) + \Box_L) = \varnothing\}. \end{cases}$$

Since U_1 is clopen, the map $x \mapsto \mathcal{C}_1(x)$ is continuous in the sense that for any x and any $R > 0$, there is a neighbourhood W of x such that

$$\mathcal{C}_1(y) \cap B_R = \mathcal{C}_1(x) \cap B_R, \quad y \in W.$$

Consider the map $x \mapsto \mathcal{C}_2(x)$. Fix $x \in X$, $R > 0$. Since U_2 is clopen, there is a neighbourhood W of x such that

$$\{n \in \mathbb{Z}^d : xn \in U_2\} \cap B_R = \{n \in \mathbb{Z}^d : yn \in U_2\} \cap B_R, \quad y \in W.$$

Note that $x \mapsto \mathcal{C}_1(x)$ is continuous, then the neighbourhood W can be chosen so that

$$(\mathcal{C}_1(x) + \Box_L) \cap B_R = (\mathcal{C}_1(x) + \Box_L) \cap B_R, \quad y \in W,$$

and therefore for any $y \in W$,

$$\begin{aligned} & \{xn \in U_2,\ (n + \Box_L) \cap (\mathcal{C}_1(x) + \Box_L) = \varnothing\} \cap B_R \\ = \ & \{yn \in U_2,\ (n + \Box_L) \cap (\mathcal{C}_1(y) + \Box_L) = \varnothing\} \cap B_R. \end{aligned}$$

Together with the continuity of $x \mapsto \mathcal{C}_1(x)$, this shows that $x \mapsto \mathcal{C}_2(x)$ is continuous.

Repeat this argument, one shows that the map $x \mapsto \mathcal{C}_s(x)$ is continuous.

Let us show that the map $x \mapsto \mathcal{C}_s(x)$ is equivariant. Start with $x \mapsto \mathcal{C}_1(x)$. Let $n \in \mathbb{Z}^d$ and consider xn. Since $xm \in U_1$ if and only if $x(n + m - n) \in U_1$, one has

$$\mathcal{C}_1(xn) = \mathcal{C}_1(x) - n.$$

A similar argument shows that $\mathcal{C}_2(x), ..., \mathcal{C}_s(x)$ are equivariant.

One asserts that

$$(c_1 + \Box_L) \cap (c_2 + \Box_L) = \varnothing, \quad c_1 \neq c_2,\ c_1, c_2 \in \mathcal{C}_s(x).$$

Indeed, since $U_1 n$, $n \in \Box_{2L}$, are mutually disjoint, one has that

$$(c + \Box_{2L}) \cap \mathcal{C}_1(x) = c, \quad c \in \mathcal{C}_1(x),$$

and thus

$$(c_1 + \Box_L) \cap (c_2 + \Box_L) = \varnothing, \quad c_1 \neq c_2,\ c_1, c_2 \in \mathcal{C}_1(x).$$

Now, pick

$$c_1, c_2 \in \mathcal{C}_2(x) = \mathcal{C}_1(x) \cup \{n \in \mathbb{Z}^d : xn \in U_2,\ (n + \Box_L) \cap (\mathcal{C}_1(x) + \Box_L) = \varnothing\}.$$

If $c_1, c_2 \in \mathcal{C}_1(x)$, then as shown above,

$$(c_1 + \Box_L) \cap (c_2 + \Box_L) = \varnothing.$$

Assume that

$$c_1, c_2 \in \{n \in \mathbb{Z}^d : xn \in U_2,\ (n + \Box_L) \cap (\mathcal{C}_1(x) + \Box_L) = \varnothing\} \subseteq \{n \in \mathbb{Z}^d : xn \in U_2\}.$$

Then, since $U_2 n$, $n \in \Box_{2L}$, are mutually disjoint, the same argument as that of $\mathcal{C}_1(x)$ shows that

$$(c_1 + \Box_L) \cap (c_2 + \Box_L) = \varnothing.$$

Assume that $c_1 \in \mathcal{C}_1$ and $c_2 \in \{n \in \mathbb{Z}^d : xn \in U_2,\ (n + \Box_L) \cap (\mathcal{C}_1(x) + \Box_L) = \varnothing\}$. Then the equation

$$(c_1 + \Box_L) \cap (c_2 + \Box_L) = \varnothing$$

just follows from the definition.

Repeat this argument for $\mathcal{C}_3(x), ..., \mathcal{C}_s(x)$, and this proves the assertion.

Note that for the given x, there exists a U_i containing x. Therefore, either

$$\Box_L \cap (\mathcal{C}_{i-1}(x) + \Box_L) \neq \varnothing \quad \text{or} \quad 0 \in \mathcal{C}_i(x).$$

In particular, one always has that $\Box_L \cap (\mathcal{C}_i(x) + \Box_L) \neq \varnothing$, and hence

$$\Box_L \cap (\mathcal{C}_s(x) + \Box_L) \neq \varnothing.$$

To summarize, setting $\mathcal{C}(x) = \mathcal{C}_s(x)$, one obtains a continuous equivariant map $x \mapsto \mathcal{C}(x)$ satisfying

(1) $(c_i + \Box_L) \cap (c_j + \Box_L) = \varnothing$, $c_i \neq c_j$, $c_i, c_j \in \mathcal{C}_s(x)$ and
(2) $\Box_L \cap (\mathcal{C}_s(x) + \Box_L) \neq \varnothing$;

hence it satisfies

(3) $(c_i + \Box_{L-r}) \cap (c_j + \Box_{L-r}) = \varnothing$, $c_i \neq c_j$, $c_i, c_j \in \mathcal{C}_s(x)$,
(4) $\Box_{L+r} \cap (\mathcal{C}_s(x) + \Box_{L-r}) \neq \varnothing$;

and, moreover

(5) the (Euclidean) distance between $c_i + \Box_{L-r}$ and $c_j + \Box_{L-r}$ is at least r if $c_i \neq c_j$.

Thus, each $\mathcal{C}(x)$ is an $(r, L-r, L+r)$ tiling. Since $L > 2r$, one has $L + r < 2(L - r)$, and this proves the statement of the proposition. $\Box$

Corollary 2.7. *Consider a free $\mathbb{Z}^d$-action on X where X is the Cantor set, and let $N \in \mathbb{N}$ be arbitrary. Then, there exist continuous equivariant maps*

$$x \mapsto \mathcal{T}_i(x), \quad i = 0, 1, ..., 3^d - 1,$$

with each $\mathcal{T}_i(x)$ a (r, D, E)-tilings of $\mathbb{Z}^d$ for some $r, D, E \in \mathbb{N}$ with $r > N\sqrt{d}$, such that, if put

$$\Omega_i = \{x \in X : 0 \in \mathrm{Dom}(\mathcal{T}_i(x))\}, \quad i = 0, 1, ..., 3^d - 1,$$

then

$$\Omega_0 \cup \Omega_1 \cup \cdots \cup \Omega_{3^d-1} = X.$$

Proof. It follows from Proposition 2.6 that there are natural numbers r, D, E with

$$r > N\sqrt{d} \quad \text{and} \quad D \leq E \leq 2D,$$

and a continuous equivariant map $x \mapsto \mathcal{T}_0(x)$ on X such that each $\mathcal{T}_0(x)$ a (r, D, E)-tiling of $\mathbb{Z}^d$.

Consider the translations of the function $\mathcal{T}_0$:

$$\mathcal{T}_1 = \mathcal{T}_0 + e_1,\ \mathcal{T}_2 = \mathcal{T}_0 + e_2,\ ..., \mathcal{T}_{3^d-1} = \mathcal{T}_0 + e_{3^d-1},$$

where $e_1, ..., e_{3^d-1}$ are the vectors (with repect to E) obtained from Lemma 2.2. Since $D \leq E \leq 2D$, it follows from Lemma 2.2 that for any $x \in X$, one has

$$0 \in \mathrm{Dom}(\mathcal{T}_0(x)) \cup \mathrm{Dom}(\mathcal{T}_1(x)) \cup \cdots \cup \mathrm{Dom}(\mathcal{T}_{3^d-1}(x)),$$

and thus

$$\Omega_0 \cup \Omega_1 \cup \cdots \cup \Omega_{3^d-1} = X,$$

as desired. $\Box$

Theorem 2.8. *The dynamical asymptotic dimension of any free $\mathbb{Z}^d$-action on the Cantor set is at most $3^d - 1$.*

Proof. Let $N \in \mathbb{N}$ be arbitrary. It follows from Corollary 2.7 that there exist continuous equivariant maps

$$x \mapsto \mathcal{T}_i(x), \quad i = 0, 1, ..., 3^d - 1,$$

with each $\mathcal{T}_i(x)$ a (r, D, E)-tilings of $\mathbb{Z}^d$ for some r, D, E with $r > N\sqrt{d}$ with

$$\Omega_0 \cup \Omega_1 \cup \cdots \cup \Omega_{3^d-1} = X,$$

where

$$\Omega_i = \{x \in X : 0 \in \mathrm{Dom}(\mathcal{T}_i(x))\}, \quad i = 0, 1, ..., 3^d - 1,$$

which is open.

Since $r > N\sqrt{d}$, by Lemma 2.5, for any $i = 0, 1, ..., 3^d$, one has

$$\begin{aligned} & \big| n \in \mathbb{Z}^d : n = n_1 + \cdots + n_K,\ x(n_1 + \cdots + n_k) \in \Omega_i,\ \|n_k\|_\infty \leq N, \\ & 1 \leq k \leq K,\ K \in \mathbb{N}\}| \\ \leq \ & (2D+1)^d < +\infty. \end{aligned}$$

That is, the dynamical asymptotic dimension of $X \curvearrowleft \mathbb{Z}^d$ is at most $3^d - 1$. □

Lemma 2.9. *Let $X \curvearrowleft \Gamma$ be an extension of a free action $Y \curvearrowleft \Gamma$. Then the dynamical asymptotic dimension of $X \curvearrowleft \Gamma$ is at most the dynamical asymptotic dimension of $Y \curvearrowleft \Gamma$.*

Proof. Let $d \in \mathbb{Z}$ such that the dynamical asymptotical dimension of $Y \curvearrowleft \Gamma$ is at most d. Let $\Gamma_0 \subseteq \Gamma$ be finite. Then, together with the freeness of $Y \curvearrowleft \Gamma$, there exist an open cover $U_0 \cup U_1 \cup \cdots \cup U_d$ of Y and $M > 0$ such that for each U_i, $0 \leq i \leq d$, $y_0 \in U_i$, one has that

$$|\{\gamma_1 \cdots \gamma_K : \exists \gamma_1, ..., \gamma_K \in \Gamma_0,\ y_0\gamma_1 \cdots \gamma_k \in U_i,\ 1 \leq k \leq K,\ K \in \mathbb{N}\}| \leq M. \tag{2.2}$$

Consider the open sets

$$\pi^{-1}(U_0),\ \pi^{-1}(U_1),\ ...,\ \pi^{-1}(U_d),$$

where $\pi : X \to Y$ is the quotient map, and note that they form an open cover of X. For each $0 \leq i \leq d$, pick an arbitrary $x_0 \in \pi^{-1}(U_i)$ and assume there are $\gamma_1, ..., \gamma_K \in \Gamma_0$ for some $K \in \mathbb{N}$ such that

$$x_0 \in \pi^{-1}(U_i),\ x_0\gamma_1 \in \pi^{-1}(U_i),\ ...,\ x_0\gamma_1\gamma_2 \cdots \gamma_K \in \pi^{-1}(U_i).$$

Applying the quotient map π, one has

$$\pi(x_0) \in U_i,\ \pi(x_0)\gamma_1 \in U_i,\ ...,\ \pi(x_0)\gamma_1\gamma_2 \cdots \gamma_K \in U_i,$$

and, by (2.2), this implies

$$\big|\{\gamma_1 \cdots \gamma_K : \exists \gamma_1, ..., \gamma_K \in \Gamma_0,\ x_0\gamma_1 \cdots \gamma_k \in \pi^{-1}(U_i),\ 1 \leq k \leq K,\ K \in \mathbb{N}\}\big| \leq M.$$

Thus, the dynamical asymptotic dimension of $X \curvearrowleft \Gamma$ is at most d. □

Then, the following is a straightforward corollary of Theorem 2.8:

Corollary 2.10. *The dynamical asymptotic dimension of any extension of a free $\mathbb{Z}^d$-action on the Cantor set is at most $3^d - 1$.*

References

[1] T. Downarowicz and D. Huczek. Dynamical quasitilings of amenable groups. *Bull. Pol. Acad. Sci. Math.*, 66(1):45–55, 2018. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=3782587, doi:10.4064/ba8128-1-2018.

[2] E. Guentner, R. Willett, and G. Yu. Dynamic asymptotic dimension: relation to dynamics, topology, coarse geometry, and C^*-algebras. *Math. Ann.*, 367(1-2):785–829, 2017. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=3606454, doi:10.1007/s00208-016-1395-0.

[3] D. S. Ornstein and B. Weiss. Entropy and isomorphism theorems for actions of amenable groups. *J. Analyse Math.*, 48:1–141, 1987. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=910005, doi:10.1007/BF02790325.

Department of Mathematics, University of Wyoming, Laramie, Wyoming, USA, 82071
Email address: zniu@uwyo.edu

Department of Mathematics, University of Wyoming, Laramie, Wyoming, USA, 82071
Email address: xzhou3@uwyo.edu