ON THE SMALL BOUNDARY PROPERTY, $\mathcal{Z}$-STABILITY, AND BAUER SIMPLEXES

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Abstract. Let X be a compact metrizable space, and let Δ be a closed set of Borel probability measures on X. We study the small boundary property of the pair (X, Δ). In particular, it is shown that (X, Δ) has the small boundary property if it has a restricted version of property Gamma.

As an application, it is shown that, if A is the crossed product C*-algebra $\mathrm{C}(X) \rtimes \mathbb{Z}^d$, where $(X, \mathbb{Z}^d)$ is a free minimal topological dynamical system, or if A is an AH algebra with diagonal maps, then, A is $\mathcal{Z}$-stable if the set of extreme tracial states is compact, regardless of its dimension.

1. Introduction

The small boundary property was introduced in [7] as a dynamical system analogue of the usual definition of zero dimensional space. It was shown to be equivalent to zero mean dimension ([6]; see [4] for $\mathbb{Z}^d$-actions), and implies the $\mathcal{Z}$-stability of the crossed product C*-algebra $\mathrm{C}(X) \rtimes \mathbb{Z}$ ([3]). In this paper, let us formulate the small boundary property for a pair (X, Δ) (see Definition 2.2), where X is a compact metrizable space and Δ is a closed set of Borel probability measures on X. This clearly includes the dynamical system case where Δ consists of invariant probability measures, but also includes the examples where $\mathrm{C}(X)$ is a Cartan subalgebra of A and Δ is the trace simplex of A.

Motivated by the argument of [6], the small boundary property is characterized in the context of the uniform trace norm:

Theorem 1.1 (Theorem 2.9)**.** *(X, Δ) has the (SBP) if, and only if, for any continuous real valued function $f : X \to \mathbb{R}$ and any $\varepsilon > 0$, there is a continuous real valued function $g : X \to \mathbb{R}$ such that*

(1) $\|f - g\|_{2,\Delta} < \varepsilon$, *and*
(2) $\mu(g^{-1}(0)) < \varepsilon$ *for all* $\mu \in \Delta$.

This characterization is similar to the property of real rank zero of a C*-algebra, which asserts that any self-adjoint element can be approximated by invertible self-adjoint elements of the C*-algebra. Indeed, the (SBP) can be shown to be equivalent to zero real rank of the completion of the C*-algebra $\mathrm{C}(X)$ under the traces in Δ (Theorems 2.12 and 2.13).

As an application of the theorem above, a version of property Gamma is introduced:

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Definition 1.2 (Definition 3.3)**.** The pair (X, Δ) is said to have the weak restricted property Gamma if there is K such that for each $n \in \mathbb{N}$, there is a partition of unity

$$p_1, p_2, ..., p_n \in \ell^\infty(D)/J_{2,\Delta}$$

such that

$$\tau(p_i a p_i) \leq \frac{1}{n} K\tau(a), \quad a \in D^+,\ \tau \in \Delta_\omega,$$

where $D = \mathrm{C}(X)$.

And this property is shown to imply the (SBP):

Theorem 1.3 (Theorem 3.5)**.** *If (X, Δ) has the weak restricted property Gamma, then (X, Δ) has the (SBP).*

In the case that $\mathrm{C}(X)$ is the canonical subalgebra of $A = \mathrm{C}(X) \rtimes \mathbb{Z}^d$, where $\mathbb{Z}^d$ acts minimally and freely on X, or in the case that $\mathrm{C}(X)$ is the diagonal subalgebra of A, where A is an AH algebra with diagonal maps, it turns out that if $\Delta = \mathrm{T}(A)$ is a Bauer simplex (i.e., $\partial\Delta$ is compact), then (X, Δ) must have the (SBP), and then the C*-algebra A must be $\mathcal{Z}$-stable. Note that there is no assumption on the dimension of $\partial\Delta$:

Theorem 1.4 (Theorems 4.6 and 4.7)**.** *Let A be the C*-algebra as above. If $\mathrm{T}(A)$ is a Bauer simplex, then $A \cong A \otimes \mathcal{Z}$, where $\mathcal{Z}$ is the Jiang-Su algebra.*

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2. Small boundary property and tracial approximation

2.1. Small boundary property.

Lemma 2.1. *Let X be a compact metrizable space, and let Δ be a set of Borel probability measures. The following properties are equivalent:*

(1) *for any subsets $V \subseteq U$, where V is closed and U is open, there is a closed set V' such that $V \subseteq V' \subseteq U$, and $\mu(\partial V') = 0$, for all $\mu \in \Delta$.*
(2) *for any subsets $V \subseteq U$, where V is closed and U is open, there is an open set V' such that $V \subseteq V' \subseteq U$, and $\mu(\partial V') = 0$, for all $\mu \in \Delta$.*
(3) *for any $x \in X$ and any open set $U \ni x$, there is a closed neighborhood $V \subseteq U$ of x such that $\mu(\partial V) = 0$, for all $\mu \in \Delta$.*
(4) *for any $x \in X$ and any open set $U \ni x$, there is an open neighborhood $V \subseteq U$ of x such that $\mu(\partial V) = 0$, for all $\mu \in \Delta$.*

Proof. Since $\overline{\mathrm{int}(W)} \setminus \mathrm{int}(W) \subseteq \partial W = W \setminus \mathrm{int}(W)$ for any closed set W, (1)⇒(2) and (3)⇒(4) are straightforward.

For (2)⇒(3), choose an open set U' with $\overline{U'} \subseteq U$. Since $\{x\}$ is closed, by (2), there is an open set $V \ni x$ such that $V \subseteq U'$ and $\mu(\partial V) = 0$ for all $\mu \in \Delta$. Then the closed neighbourhood $\overline{V}$ of x satisfies (3).

For (4)⇒(1), choose an open set U' with $\overline{U'} \subseteq U$. Noting that V is compact, by (4), there is an open cover $\{V_1, V_2, ..., V_n\}$ of V such that $V_i \subseteq U'$, $i = 1, ..., n$ and $\mu(\partial V_i) = 0$, $i = 1, ..., n$, and $\mu \in \Delta$. Then $V' := V_1 \cup \cdots \cup V_n$ satisfies (1). □

Definition 2.2. Let X be a compact metrizable space, and let Δ be a closed set of probability Borel measures. Then (X, Δ) is said to have the Small Boundary Property (SBP) if one of the conditions of Lemma 2.1 is satisfied.

Lemma 2.3. *Let Δ be a closed set of Borel probability measures of X. Let $c \geq 0$, and let $V \subseteq X$ be a close set with $\mu(V) < c$ for all $\mu \in \Delta$. Then there is an open set $U \supseteq V$ such that $\mu(U) < c$ for all $\mu \in \Delta$.*

Proof. Assume the statement were not hold. There would exist a decreasing sequence of open sets (U_i) with $\overline{U_{i+1}} \subseteq U_i$ and $\bigcap_{i=1}^{\infty} U_i = V$ and a sequence of $\mu_i \in \Delta$ such that $\mu_i(U_i) \geq c$ for all $i = 1, 2, ...$ (and hence $\mu_k(U_i) \geq c$ for all $k \geq i$, $k, i = 1, 2, ...$). Since Δ is closed (hence compact), by passing to a subsequence, one may assume that (μ_i) converges to a measure $\mu_\infty \in \Delta$. Then, since

$$\mu_\infty(U_i) \geq \mu_\infty(\overline{U_{i+1}}) \geq \limsup_{k\to\infty} \mu_k(\overline{U_{i+1}}) \geq \limsup_{k\to\infty} \mu_k(U_{i+1}) \geq c,$$

one has

$$c > \mu_\infty(V) = \mu_\infty(\bigcap_{i=1}^{\infty} U_i) = \lim_{i\to\infty} \mu_\infty(U_i) \geq c,$$

which is absurd. □

Lemma 2.4 (Proposition 5.3 of [7])**.** *If (X, Δ) has the (SBP), then, for every open cover α of X and every $\varepsilon > 0$, there is a subordimate partition of unity $\phi_i : X \to [0, 1]$, $i = 1, 2, ..., |\alpha|$, such that*

(1) $\sum_{i=1}^{|\alpha|} \phi_i(x) = 1$, $x \in X$,
(2) $\mathrm{supp}(\phi_i) \subseteq U$ *for some* $U \in \alpha$, *and* $i = 1, 2, ..., |\alpha|$,
(3) $\mu(\bigcup_{i=1}^{|\alpha|} \phi_i^{-1}(0, 1)) < \varepsilon$ *for all* $\mu \in \Delta$.

Proof. List $\alpha = \{U_1, U_2, ..., U_{|\alpha|}\}$. Using the (SBP), for each $i = 1, ..., |\alpha|$, one finds an open set $U_i' \subseteq U_i$ such that U_i', $i = 1, 2, ..., |\alpha|$, still form an open cover of X, and $\mu(\partial U_i') = 0$ for all $\mu \in \Delta$. By Lemma 2.3, there is $\delta > 0$ such that

$$\mu((\partial U_i')_\delta) < \varepsilon / |\alpha| ,$$

where $(\partial U_i')_\delta$ is the δ-neighborhood of $\partial U'$. One may assume that δ is small enough so that $(\partial U_i')_\delta \subseteq U_i$ for each $i = 1, ..., |\alpha|$.

Define

$$\psi_i = \begin{cases} 1 & x \in U_i' \\ \max(0, 1 - \delta^{-1}\mathrm{dist}(x, \partial U_i')) & \text{otherwise} \end{cases}$$

Then, define

$$\begin{cases} \phi_1(x) &= \psi_1(x), \\ \phi_2(x) &= \min(\psi_2, 1 - \phi_1(x)), \\ \phi_3(x) &= \min(\psi_3, 1 - \phi_1(x) - \phi_2(x)), \\ & \vdots \\ \phi_{|\alpha|}(x) &= \min(\psi_{|\alpha|}, 1 - \phi_1 - \cdots - \phi_{|\alpha|-1}(x)). \end{cases}$$

Then $\mathrm{supp}(\phi_i) \subseteq U_i$, $i = 1, ..., |\alpha|$, and

$$\bigcup_{i=1}^{|\alpha|} \phi_i^{-1}(0,1) \subseteq \bigcup_{i=1}^{|\alpha|} \psi_i^{-1}(0,1).$$

□

Lemma 2.5 ([6]). *(X, Δ) has the (SBP) if, and only if, for any continuous real valued function $f : X \to \mathbb{R}$ and any $\varepsilon > 0$, there is a continuous real valued function $g : X \to \mathbb{R}$ such that*

(1) $\|f - g\|_\infty < \varepsilon$,
(2) $\mu(g^{-1}(0)) < \varepsilon$ for all $\mu \in \Delta$.

Proof. Assume the properties of the lemma. In order to show that (X, Δ) has the (SBP), it is enough to show that the set

$$\{f \in \mathrm{C}_{\mathbb{R}}(X) : \mu(f^{-1}(0)) = 0,\ \forall \mu \in \Delta\}$$

is dense (G_δ) in $\mathrm{C}_{\mathbb{R}}(X)$. As then, let $x \in X$ and U be an open neighbounhood of x; pick a continuous function $f : X \to [-1, 1]$ such that $f(x) = -1$ and $f(X \setminus U) = \{1\}$. Since the set above is dense, there is g such that $\|g - f\| < 1/4$ and $\mu(g^{-1}(0)) = 0$ for all $\mu \in \Delta$. Consider the set $V = g^{-1}([-1, 0]) \subseteq U$. It is a closed neighbourhood of x with $\partial V = g^{-1}(0)$, and hence $\mu(V) = 0$ for all $\mu \in \Delta$. So (X, Δ) has the (SBP).

For each $c > 0$, consider the set

$$Z_c := \{f \in \mathrm{C}_{\mathbb{R}}(X) : \mu(f^{-1}(0)) < c,\ \forall \mu \in \Delta\}.$$

Note that the properties of the lemma implies that Z_c is dense in $\mathrm{C}_{\mathbb{R}}(X)$.

Let us show that Z_c is also open in $\mathrm{C}_{\mathbb{R}}(X)$. Indeed, let $f : X \to \mathbb{R}$ be a continuous function such that $\mu(f^{-1}(0)) < c$ for all $\mu \in \Delta$, and consider the closed set $V := f^{-1}(0)$. By Lemma 2.3, there is an open set $U \supseteq V$ such that $\mu(U) < c$ for all $\mu \in \Delta$.

Therefore the set

$$\begin{aligned} & \{f \in \mathrm{C}_{\mathbb{R}}(X) : \mu(f^{-1}(0)) = 0,\ \forall \mu \in \Delta\} \\ = & \bigcap_{n=1}^{\infty} \{f \in \mathrm{C}_{\mathbb{R}}(X) : \mu(f^{-1}(0)) \leq 1/n,\ \forall \mu \in \Delta\} \\ = & \bigcap_{n=1}^{\infty} Z_{1/n} \end{aligned}$$

is dense (G_δ) in $\mathrm{C}_{\mathbb{R}}(X)$, and (X, Δ) has the (SBP).

For the converse, assume (X, Δ) has the (SBP), and let $f : X \to \mathbb{R}$ and any $\varepsilon > 0$ be given. By Lemma 2.4, there exists a partition of unity $\phi_i : X \to [0, 1]$, $i = 1, 2, ..., N$, such that

$$\mu(\bigcup_{i=1}^{N} \phi_i^{-1}((0,1)) < \varepsilon, \quad \mu \in \Delta$$

and

$$|f(x) - f(y)| < \varepsilon/2, \quad x, y \in \phi_i^{-1}(0, 1]),\ i = 1, ..., N.$$

For each $i = 1, ..., N$, pick $x_i \in \phi_i^{-1}(0, 1]$ and the pick real numbers

$$y_i \neq 0 \quad \text{and} \quad |y_i - f(x_i)| < \varepsilon/2.$$

Define

$$g = \sum_{i=1}^{N} y_i \phi_i.$$

Then a direct calculations show that

(1) $\|f - g\|_\infty < \varepsilon$, and
(2) $\mu(g^{-1}(0)) < \varepsilon$ for all $\mu \in \Delta$,

as desired. □

Definition 2.6. Let Δ be a closed subset of probability Borel measures of X. Then define

$$\|f\|_{2,\Delta} = \sup\{(\int |f|^2 \, d\mu)^{\frac{1}{2}} : \mu \in \Delta\}, \quad f \in \mathrm{C}(X).$$

Lemma 2.7 (Markov's inequality)**.**

$$\mu(E_\varepsilon) \leq \frac{1}{\varepsilon^2} \int |f|^2 \, d\mu,$$

where $E_\varepsilon = \{x \in X : |f(x)| > \varepsilon\}$.

Proof.

$$\varepsilon^2 \mu(E_\varepsilon) = \int_{E_\varepsilon} \varepsilon^2 d\mu \leq \int_{E_\varepsilon} |f|^2 \, d\mu + \int_{E_\varepsilon^c} |f|^2 \, d\mu = \int |f|^2 \, d\mu.$$

□

Lemma 2.8. *Let f, g be the real-valued continuous functions on X satisfying*

$$\|f-g\|_{2,\Delta}^2 < \frac{\varepsilon^3}{8}$$

and

$$\mu(g^{-1}(0)) < \frac{\varepsilon}{2}, \quad \mu \in \Delta.$$

Then, there is a self-adjoint continuous function g' on X such that

(1) $\|f-g'\|_\infty < \varepsilon$*, and*
(2) $\mu((g')^{-1}(0)) < \varepsilon$ *for all* $\mu \in \Delta$.

Proof. Define

$$E_\varepsilon = \{x \in X : |f(x) - g(x)| > \varepsilon/2\}.$$

Then, by the Markov's inequality,

$$\mu(E_\varepsilon) < \frac{4}{\varepsilon^2}\|f-g\|_\Delta^2 < \frac{\varepsilon}{2}, \quad \mu \in \Delta.$$

Pick an open neighborhood U of E_ε^c such that

$$|f(x) - g(x)| < \varepsilon, \quad x \in U,$$

and pick a continuous function $h : X \to [0,1]$ such that

$$h|_{E_\varepsilon^c} = 1 \quad \text{and} \quad h|_{U^c} = 0.$$

Define

$$g'(x) = h(x)g(x) + (1-h(x))f(x).$$

Then

$$|f(x) - g'(x)| < \varepsilon, \quad x \in X.$$

Moreover, since

$$(g')^{-1}(0) \subseteq g^{-1}(0) \cup E_\varepsilon,$$

one has

$$\mu((g')^{-1}(0)) \leq \mu(g^{-1}(0)) + \mu(E_\varepsilon) < \varepsilon,$$

as desired. □

Together with Lemma 2.5, we immediately have the following criterion for the (SBP).

Theorem 2.9. (X, Δ) *has the (SBP) if, and only if, for any continuous real valued function $f : X \to \mathbb{R}$ and any $\varepsilon > 0$, there is a continuous real valued function $g : X \to \mathbb{R}$ such that*

(1) $\|f-g\|_{2,\Delta} < \varepsilon$*, and*
(2) $\mu(g^{-1}(0)) < \varepsilon$ *for all $\mu \in \Delta$; this is equivalent to that there is $\delta > 0$ such that $\tau_\mu(\chi_\delta(g)) < \varepsilon$, $\mu \in \Delta$, where*

$$\chi_\delta(t) = \begin{cases} 1, & |t| < \delta/2, \\ 0, & |t| > \delta, \\ \textit{linear}, & \textit{otherwise}. \end{cases}$$

We like to point out the following corollary:

Corollary 2.10. *Consider (X, Δ_1) and (Y, Δ_2), and assume there is an embedding $\phi : \mathrm{C}(X) \to \mathrm{C}(Y)$ such that*

(1) $\Delta_1 \subseteq \phi^*(\Delta_2)$,
(2) $\phi(\mathrm{C}(X))$ *is dense in* $\mathrm{C}(Y)$ *with respect to* $\|\cdot\|_{2,\Delta_2}$.

If (Y, Δ_2) has the (SBP), then (X, Δ_1) also has the (SBP).

Proof. Identifying $\mathrm{C}(X)$ as a sub-C*-algebra of $\mathrm{C}(Y)$, and identifying Δ_1 as a subset of Δ_2.

Let $f : X \to \mathbb{R}$ be continuous, and let $\varepsilon > 0$ be arbitrary. Since (Y, Δ) has the (SBP), by Theorem 2.9, there is $g' : Y \to \mathbb{R}$ such that

(1) $\|f - g'\|_{2,\Delta} < \varepsilon$,
(2) $\mu((g')^{-1}(0)) < \varepsilon$ for all $\mu \in \Delta_2$.

Since Δ_2 and $(g')^{-1}(0)$ are closed, by Lemma 2.3, there is an open neighbourhood of $(g')^{-1}(0)$ such that its measure is uniformly smaller than ε with respect to Δ. It then follows that there is $\delta > 0$ such that

$$\tau(\chi_\delta(g')) < \varepsilon, \quad \tau \in \Delta_2,$$

where

$$\chi_\delta(t) = \begin{cases} 1, & |t| < \delta/2, \\ 0, & |t| > \delta, \\ \text{linear}, & \text{otherwise}. \end{cases}$$

Since $\mathrm{C}(X)$ is dense inside $\mathrm{C}(Y)$ with respect to $\|\cdot\|_{2,\Delta_2}$, there is a self-adjoint element $g \in \mathrm{C}(X)$ which is sufficiently close to g' (with respect to $\|\cdot\|_{2,\Delta_2}$) such that

$$\|f - g\|_{2,\Delta_2} < \varepsilon \quad \text{and} \quad \tau(\chi_\delta(g)) < \varepsilon, \quad \mu \in \Delta_2.$$

In particular $\mu((g)^{-1}(0)) < \varepsilon$ for all $\mu \in \Delta_1$. By Theorem 2.9 again, (X, Δ_1) has the (SBP). □

2.2. **Real rank zero.** Let us characterize the (SBP) using real rank zero.

Lemma 2.11. *Assume that (X, Δ) has the (SBP). Let $f \in \mathrm{C}_{\mathbb{R}}(X)$, and let $\varepsilon > 0$. Then there are sequences $(g_n), (h_n) \subseteq \mathrm{C}_{\mathbb{R}}(X)$ such that for each $n = 1, 2, ...$,*

(1) $\|f - g_n\| < \varepsilon$,
(2) $\tau_\mu(|g_n h_n - 1|^2) < 1/2^n$ *for all* $\mu \in \Delta$,
(3) $\|h_n\| < 4/\varepsilon$,
(4) $\|g_n - g_{n+1}\|_{2,\Delta} < 1/2^n$, *and* $\|h_n - h_{n+1}\|_{2,\Delta} < 1/2^n$.

Proof. Without loss of generality, one may assume that $\varepsilon < 1$ and $\|f\| = 1$

Choose an open cover $\{U_1, ..., U_N\}$ of X such that

$$|f(x) - f(y)| < \varepsilon/2, \quad x, y \in U_i, \ i = 1, ..., N.$$

By Lemma 2.4, there is a partition of unity $\phi_i^{(1)} : X \to [0, 1]$, subordinate to $U_1, U_2, ..., U_N$ such that

$$\mu(\bigcup_{i=1}^{N}(\phi_i^{(1)})^{-1}((0,1))) < \varepsilon/2N(4/\varepsilon + 5)^2, \quad \mu \in \Delta. \tag{2.1}$$

Consider the collection of open sets

$$U_i^{(1)} = (\phi_i^{(1)})^{-1}((0,1]), \quad i = 1, 2, ..., N,$$

which is an open cover of X satisfying

$$U_i^{(1)} \subseteq U_i, \quad i = 1, ..., N. \tag{2.2}$$

Using Lemma 2.4 again, one obtains a partition of unity $\phi_i^{(2)} : X \to [0,1]$, subordinate to $U_1^{(1)}, U_2^{(1)}, ..., U_N^{(1)}$ such that

$$\mu(\bigcup_{i=1}^{N}(\phi_i^{(2)})^{-1}((0,1))) < \varepsilon/2^2N(4/\varepsilon+5)^2, \quad \mu \in \Delta.$$

Note that, for each $i = 1, ..., N$, by (2.2), $\phi_i^{(2)}(x) = 1$ for all x satisfying $\phi_i^{(1)}(x) = 1$, one then has

$$(\phi_i^{(2)})^{-1}((0,1)) \subseteq (\phi_i^{(1)})^{-1}(\{1\}) \subseteq \bigcup_{i=1}^{N}(\phi_i^{(1)})^{-1}((0,1)),$$

and therefore, by (2.1),

$$\left\|\phi_i^{(1)} - \phi_i^{(2)}\right\|_{2,\Delta} < \varepsilon/2N(4/\varepsilon+5)^2 < \varepsilon/2N.$$

Repeating this process, one obtains partitions of unity $\phi_i^{(n)} : X \to [0,1]$, $i = 1, 2, ..., N$, subordinate to $U_1, U_2, ..., U_N$ such that

$$\mu(\bigcup_{i=1}^{N}(\phi_i^{(n)})^{-1}((0,1))) < \varepsilon/N2^n(4/\varepsilon+5)^2, \quad \mu \in \Delta. \tag{2.3}$$

and

$$\left\|\phi_i^{(n)} - \phi_i^{(n+1)}\right\|_{2,\Delta} < \varepsilon/2^nN. \tag{2.4}$$

For each $i = 1, ..., N$, pick $x_i \in U_i$ and then pick a real number y_i such that

$$|y_i| > \varepsilon/4 \quad \text{and} \quad |y_i - f(x_i)| < \varepsilon/2.$$

Define

$$g_n = \sum_{i=1}^{N} y_i\phi_i^{(n)} \quad \text{and} \quad h_n = \sum_{i=1}^{N} \frac{1}{y_i}\phi_i^{(n)}.$$

Then

$$\|f - g_n\| < \varepsilon$$

and

$$|h(x)| \leq \sum_{i=1}^{N} \frac{\phi_i(x)}{|y_i|} \leq \frac{4}{\varepsilon}, \quad x \in X.$$

Note that $(g_nh_n)(x) = 1$ whenever $x \in \bigsqcup_{i=1}^{N}(\phi_i^{(n)})^{-1}(1)$, and then, together with (2.3), for all $\mu \in \Delta$,

$$\tau_\mu(|g_nh_n - 1|^2) < (\frac{4\|g_n\|}{\varepsilon} + 1)^2 \cdot \frac{\varepsilon}{2^nN(\frac{4}{\varepsilon}+5)^2} < (\frac{4(1+\varepsilon)}{\varepsilon} + 1)^2 \frac{1}{2^n(\frac{4}{\varepsilon}+5)^2} = \frac{1}{2^n}.$$

By (2.4),

$$\|g_n - g_{n+1}\|_{2,\Delta} = \left\|\sum_{i=1}^{N} y_i(\phi_i^{(n)} - \phi_i^{(n+1)})\right\|_{2,\Delta} \leq \sum_{i=1}^{N}|y_i|\left\|\phi_i^{(n)} - \phi_i^{(n+1)}\right\|_{2,\Delta} \leq 1/2^n$$

and

$$\|h_n - h_{n+1}\|_{2,\Delta} = \left\|\sum_{i=1}^{N} \frac{1}{y_i}(\phi_i^{(n)} - \phi_i^{(n+1)})\right\|_{2,\Delta} \leq \sum_{i=1}^{N}\left|\frac{1}{y_i}\right|\left\|\phi_i^{(n)} - \phi_i^{(n+1)}\right\|_{2,\Delta} \leq 1/2^n,$$

as desired. □

Consider $l^\infty(\mathrm{C}(X))$ and consider the ideal

$$J_{2,\Delta} := \{(f_1, f_2, ...) \in l^\infty(\mathrm{C}(X)) : \lim_{n\to\omega}\sup\{\tau(|f_n|^2) : \tau \in \Delta\} = 0\}.$$

Theorem 2.12. *Then the C*-algebra* $\ell^\infty(\mathrm{C}(X))/J_{2,\Delta}$ *has real rank zero if, and only if,* (X, Δ) *has the (SBP).*

Proof. Assume $\ell^\infty(\mathrm{C}(X))/J_{2,\Delta}$ has real rank zero. Let $f \in \mathrm{C}_{\mathbb{R}}(X)$, and let $\varepsilon > 0$ be arbitrary. Consider the image of the constant sequence $\overline{(f, f, ...)} \in l^\infty(\mathrm{C}(X))/J_{\omega,\Delta}$. By the real rank zero assumption, there is an invertible self-adjoint element $\overline{(g_1, g_2, ...)} \in l^\infty(\mathrm{C}(X))/J_{\omega,\Delta}$ such that

$$\left\|\overline{(f, f, ...)} - \overline{(g_1, g_2, ...)}\right\| < \varepsilon.$$

Then there is $\delta > 0$ such that

$$\chi_\delta(\overline{(g_1, g_2, ...)}) = 0.$$

Hence

$$\mathrm{dist}((f - g_1, f - g_2, ...), J_{2,\Delta}) < \varepsilon \quad \text{and} \quad (\chi_\delta(g_1), \chi_\delta(g_2), ...) \in J_{2,\Delta},$$

and then, with sufficiently large n,

$$\|f - g_n\|_{2,\Delta} < \varepsilon \quad \text{and} \quad \tau(\chi_\delta(g_n)) < \varepsilon, \quad \tau \in \Delta.$$

By Theorem 2.9, (X, Δ) has the (SBP).

Now, assume (X, Δ) has the (SBP), and let us show that $l^\infty(\mathrm{C}(X))/J_{\omega,\Delta}$ has real rank zero. Let $f \in l^\infty(\mathrm{C}(X))/J_{\omega,\Delta}$ be a self-adjoint element, and let $\varepsilon > 0$ be arbitrary. Pick a self-adjoint representative $(f_1, f_2, ...)$ of f. By Lemma 2.11, there are self-adjoint elements $g_1, g_1, ...$ and $h_1, h_2, ...$ in $\mathrm{C}(X)$ such that for each $n = 1, 2, ...,$

(1) $\|f_n - g_n\| < \varepsilon$,
(2) $\tau_\mu(|g_nh_n - 1|^2) < 1/n$ for all $\mu \in \Delta$, and
(3) $\|h_n\| < 4/\varepsilon$.

Then, with $g := \overline{(g_1, g_2, ...)}$ and $h := \overline{(h_1, h_2, ...)}$ in $l^\infty(\mathrm{C}(X))/J_{\omega,\Delta}$, one has

$$\|f - g\| < \varepsilon \quad \text{and} \quad gh = 1.$$

That is, g is invertible. Since ε is arbitrary, this shows that $l^\infty(\mathrm{C}(X))/J_{\omega,\Delta}$ has real rank zero. □

Inside $l^\infty(\mathrm{C}(X))/J_{2,\Delta}$, consider the sub-C*-algebra consisting of equivalence classes of bounded sequences of $\mathrm{C}(X)$ which is $\|\cdot\|_{2,\Delta}$-Cauchy. This C*-algebra is independent of ω, and denote it by $\overline{\mathrm{C}(X)}^\Delta$.

Theorem 2.13. *The pair (X, Δ) has the (SBP) if, and only if, $\overline{\mathrm{C}(X)}^\Delta$ has real rank zero.*

Proof. Assume $\overline{\mathrm{C}(X)}^\Delta$ has real rank zero. Let $f \in \mathrm{C}_\mathbb{R}(X)$, and let $\varepsilon > 0$ be arbitrary. Consider the image of the constant sequence $\overline{(f, f, ...)} \in l^\infty(\mathrm{C}(X))/J_{\omega,\Delta}$. By the real rank zero assumption, there is an invertible self-adjoint element $\overline{(g_1, g_2, ...)} \in l^\infty(\mathrm{C}(X))/J_{\omega,\Delta}$ such that

$$\left\|\overline{(f, f, ...)} - \overline{(g_1, g_2, ...)}\right\| < \varepsilon.$$

Then there is $\delta > 0$ such that

$$\chi_\delta(\overline{(g_1, g_2, ...)}) = 0.$$

Hence

$$\mathrm{dist}((f - g_1, f - g_2, ...), J_{2,\Delta}) < \varepsilon \quad \text{and} \quad (\chi_\delta(g_1), \chi_\delta(g_2), ...) \in J_{2,\Delta},$$

and then, with sufficiently large n, one has

$$\|f - g_n\|_{2,\Delta} < \varepsilon \quad \text{and} \quad \tau(\chi_\delta(g_n)) < \varepsilon, \quad \tau \in \Delta.$$

By Theorem 2.9, (X, Δ) has the (SBP).

Now, assume (X, Δ) has the (SBP). To show that $\overline{\mathrm{C}(X)}^\Delta$ has real rank zero, it is enough to show that each constance sequence (f) with $f \in \mathrm{C}_\mathbb{R}(X)$ can be approximated by an invertible element within ε. This follows from Lemma 2.11 directly as the sequences (g_n) and (f_n) are $\|\cdot\|_{2,\Delta}$-Cauchy. □

Corollary 2.14. *Consider (X, Δ_1) and (Y, Δ_2) such that $\overline{X}^{\Delta_1} \cong \overline{Y}^{\Delta_2}$. Then (X, Δ_1) has the (SBP) if, and only if, (Y, Δ_2) has the (SBP).*

3. A restricted version of property Gamma and the small boundary property

In this section, let us use the characterization of the (SBP) in Theorem 2.9 to connect it to a version of property Gamma (Theorem 3.5).

Recall ([2]) that a C*-algebra A is said to have the uniform McDuff property if for each n, there is a unital embedding

$$\mathrm{M}_n(\mathbb{C}) \to (\ell^2(A)/J_{2,\Delta}) \cap A',$$

and the C*-algebra A is said to have the uniform property Gamma if for each n, there is a partition of unity

$$p_1, p_2, ..., p_n \in \ell^2(A)/J_{2,\Delta} \cap A'$$

such that

$$\tau(p_iap_i) = \frac{1}{n}\tau(a), \quad a \in A,\ i = 1, 2, ..., n, \tau \in \mathrm{T}(A)_\omega,$$

where $\mathrm{T}(A)_\omega$ is the set of traces of $\ell^\infty(A)$ in the form

$$\tau((a_i)) = \lim_{i\to\omega} \tau_i(a_i), \quad \tau_i \in \mathrm{T}(A),$$

and a is regarded as the constant sequence $(a) \in \ell^\infty(A)$.

Let us introduce the following restricted versions of these properties:

Definition 3.1. Let A be a C*-algebra and let $D \subseteq A$ be a sub-C*-algebra. Then the pair (D, A) is said to have the restricted McDuff property if for each $n \in \mathbb{N}$, there is a unital embedding

$$\phi : \mathrm{M}_n(\mathbb{C}) \to (\ell^\infty(A)/J_{2,\Delta}) \cap A'$$

such that

$$\phi(e_{ii}) \in \ell^2(D)/J_{2,\Delta}, \quad i = 1, ..., n.$$

The pair (D, A) is said to have the restricted property Gamma if for each $n \in \mathbb{N}$, there is a partition of unity

$$p_1, p_2, ..., p_n \in (\ell^\infty(D)/J_{2,\Delta}) \cap A'$$

such that

$$\tau(p_iap_i) = \frac{1}{n}\tau(a), \quad a \in A,\ \tau \in \mathrm{T}(A)_\omega.$$

The pair (D, A) is said to have the weak restricted property Gamma if there is $K > 0$ such that for each $n \in \mathbb{N}$, there is a partition of unity

$$p_1, p_2, ..., p_n \in \ell^\infty(D)/J_{2,\Delta}$$

such that

$$\tau(p_iap_i) \leq \frac{1}{n}K\tau(a), \quad a \in D^+,\ \tau \in \mathrm{T}(A)_\omega.$$

Remark 3.2. It is clear that the restricted McDuff property implies restricted property Gamma (hence the weak restricted property Gamma).

Next, let us formulate the weak restricted property Gamma without the ambient C*-algebra A, just for a pair (D, Δ), where $D = \mathrm{C}(X)$ and Δ is a closed set of Borel probability measures.

Definition 3.3. The pair (D, Δ) (or (X, Δ)) is said to have the weak restricted property Gamma if there is K such that for each $n \in \mathbb{N}$, there is a partition of unity

$$p_1, p_2, ..., p_n \in \ell^\infty(D)/J_{2,\Delta}$$

such that

$$\tau(p_iap_i) \leq \frac{1}{n}K\tau(a), \quad a \in D^+,\ \tau \in \Delta_\omega.$$

The restricted property Gamma for (D, Δ) is defined similarly.

The following lemma is straightforward:

Lemma 3.4. *The pair (D, Δ) has the weak restricted property Gamma if, and only if, there is $K > 0$ such that for any finite set $\mathcal{F} \subseteq D^+$, any $\varepsilon > 0$, and any $n \in \mathbb{N}$, there are positive contractions $a_1, ..., a_n$ such that*

(1) $\|a_i a_j\|_{2,\Delta} < \varepsilon$ *if* $i \neq j$,
(2) $\|a_i - a_i^2\|_{2,\Delta} < \varepsilon$, $i = 1, ..., n$,
(3) $\|1 - (a_1^2 + \cdots + a_n^2)\|_{2,\Delta} < \varepsilon$,
(4) $\tau(a_i f a_i) < \frac{1}{n} K\tau(f) + \varepsilon$, $i = 1, ..., n$, $f \in \mathcal{F}$, $\tau \in \Delta$.

It turns out that the weak restricted property Gamma implies the (SBP):

Theorem 3.5. *Let D be a commutative C*-algebra and let Δ be a closed set of probability Borel measures. If (D, Δ) has the weak restricted property Gamma, then (D, Δ) has the (SBP).*

The converse holds in the case that $D \subseteq D \rtimes \Gamma$, where Γ is a discrete amenable group acting freely on D.

Let us start with some preparations:

Lemma 3.6. *Let $f \in \mathrm{C}(X)$ be a self-adjoint element, and let $\varepsilon > 0$. Then there exists $n \in \mathbb{N}$, and self-adjoint elements $f_1, f_2, ..., f_n \in \mathrm{C}(X)$ such that*

(1) $\|f - f_i\| < \varepsilon$, $i = 1, 2, ..., n$, *and*
(2)
$$\frac{1}{n}|\{1 \leq i \leq n : f_i(x) = 0\}| < \varepsilon, \quad x \in X.$$

Proof. Choose an open cover $\mathcal{U}$ of X such that
$$\|f(x) - f(y)\| < \varepsilon, \quad x, y \in U,\ U \in \mathcal{U}.$$
Choose a partition of unity $\{\phi_U : U \in \mathcal{U}\}$, subordinated to $\mathcal{U}$, and choose $x_U \in U$ for each $U \in \mathcal{U}$. Consider the function
$$g := \sum_{U \in \mathcal{U}} f(x_U)\phi_U,$$
and it is straightforward to verify that
$$\|f - g\| < \varepsilon.$$
Also note that g has a factorization
$$X \xrightarrow{\Phi} N(\mathcal{U}) \xrightarrow{\Psi} \mathbb{R}$$
where $N(\mathcal{U})$ is the nerve simplex of $\mathcal{U}$, the map $\Phi : X \to N(\mathcal{U})$ is given by
$$x \mapsto \sum_{U \in \mathcal{U}} \phi_U(x)[U],$$
and the map $\Psi : N(\mathcal{U}) \to \mathbb{R}$ is the linear map
$$\sum_{U \in \mathcal{U}} \alpha_U[U] \mapsto \sum_{U\mathcal{U}} f(x_U)\alpha_U.$$

Pick $n > \dim(N(\mathcal{U}))/\varepsilon$, and consider the map

$$(\Psi, ..., \Psi) : N(\mathcal{U}) \to \mathbb{R}^n.$$

By Lemma 5.7 of [4], there is

$$(\Psi_1, ...\Psi_n) : N(\mathcal{U}) \to \mathbb{R}^n$$

such that

$$\|\Psi - \Psi_i\| < \varepsilon, \quad 1 \leq i \leq n$$

and

$$|\{1 \leq i \leq n : \Psi_i(y) = 0\}| < \dim(N(\mathcal{U})), \quad y \in N(\mathcal{U}).$$

Then

$$(f_1, ..., f_n) := (\Psi_1 \circ \Phi, ..., \Psi_n \circ \Phi)$$

satisfies the Lemma. □

Remark 3.7. The open cover $\mathcal{U}$ can be chosen to have order 1, and hence the simplex $N(\mathcal{U})$ to have dimension 1. Therefore, the number n in the proof can be chosen to be $\lfloor 1/\varepsilon \rfloor + 1$, which is independent of X.

Corollary 3.8. *Let $f \in \mathrm{C}(X)$ be a self-adjoint element, and let $\varepsilon > 0$. Then there exist $n \in \mathbb{N}$, $\delta > 0$, and self-adjoint elements $f_1, f_2, ..., f_n \in \mathrm{C}(X)$ such that*

(1) $\|f - f_i\| < \varepsilon$, $i = 1, 2, ..., n$, *and*
(2)
$$\frac{1}{n}|\{1 \leq i \leq n : |f_i(x)| < \delta\}| < \varepsilon, \quad x \in X.$$

In particular

$$\frac{1}{n}(\tau((f_1)_\delta) + \cdots + \tau((f_n)_\delta)) < \varepsilon, \quad \tau \in \mathrm{T}(\mathrm{C}(X)),$$

where $(f)_\delta = \chi_\delta(f)$ with $\chi_\delta : \mathbb{R} \to \mathbb{R}$ defined by

$$\chi_\delta(t) = \begin{cases} 1, & |t| < \delta/2, \\ 0, & |t| > \delta, \\ \textit{linear}, & \textit{otherwise}. \end{cases}$$

Proof. By the lemma above, there exist $n \in \mathbb{N}$, and self-adjoint elements $f_1, f_2, ..., f_n \in \mathrm{C}(X)$ such that

(1) $\|f - f_i\| < \varepsilon$, $i = 1, 2, ..., n$, and
(2)
$$\frac{1}{n}|\{1 \leq i \leq n : |f_i(x)| = 0\}| < \varepsilon, \quad x \in X.$$

Since $f_1, ...f_n$ are continuous, for each $x \in X$, there exist a neighbourhood $U_x \ni x$ and $\delta_x > 0$ such that

$$\frac{1}{n}|\{1 \leq i \leq n : |f_i(y)| < \delta_x\}| < \varepsilon, \quad y \in U_x.$$

Then the corollary follows from the compactness of X. □

It worth pointing out that the corollary above holds for a general unital C*-algebra.

Corollary 3.9. *Let A be a unital C*-algebra. Let $f \in A$ be a self-adjoint element, and let $\varepsilon > 0$. Then there exist $n \in \mathbb{N}$, $\delta > 0$, and self-adjoint elements $f_1, f_2, ..., f_n \in A$ such that*

(1) $\|f - f_i\| < \varepsilon$, $i = 1, 2, ..., n$, *and*
(2)
$$\frac{1}{n}(\tau((f_1)_\delta) + \cdots + \tau((f_n)_\delta)) < \varepsilon, \quad \tau \in \mathrm{T}(A)$$

Proof. It follows directly from Corollary 3.8 applying to the sub-C*-algebra $D := \mathrm{C}^*\{1, f\}$. □

Proof of Theorem 3.5. To show that (D, Δ) have the (SBP), by Corollary 2.9, it is enough to show that for any self-adjoint element $f \in D$ and any $\varepsilon > 0$, there is a self-adjoint element $g \in D$ such that

(1) $\|f - g\|_{2,\Delta} < \varepsilon$, and
(2) $\mu(g^{-1}\{0\}) < \varepsilon$, $\mu \in \Delta$.

By Corollary 3.8, for the given ε, there exist $n \in \mathbb{N}$ and selfadjoint elements
$$f_1, f_2, ..., f_n \in D$$
such that
$$\|f - f_i\| < \varepsilon, \quad i = 1, 2, ..., n, \tag{3.1}$$
and there is $\delta > 0$ such that
$$\frac{1}{n}(\tau((f_1)_\delta) + \cdots + \tau((f_n)_\delta)) < \varepsilon/K, \quad \tau \in \mathrm{T}(\mathrm{C}(X)),$$
where K is the constant of the weak restricted property Gamma. In particular, regarding $(f_1)_\delta, ..., (f_n)_\delta$ as constant sequences of D, we have
$$\frac{1}{n}(\tau((f_1)_\delta) + \cdots + \tau((f_n)_\delta)) < \varepsilon/K, \quad \tau \in \Delta_\omega. \tag{3.2}$$

By the weak restricted property Gamma, there are
$$p_1, p_2, ..., p_n \in \ell^\infty(D)/J_{2,\Delta}$$
such that
$$\tau(p_i a p_i) \leq \frac{1}{n} K \tau(a), \quad a \in D^+, \ \tau \in \Delta_\omega. \tag{3.3}$$

Consider
$$g := p_1 \overline{(f_1)} p_1 + \cdots + p_n \overline{(f_n)} p_n \in \ell^\infty(D)/J_{2,\Delta}.$$
By (3.1),
$$\|f - g\|_{2,\Delta} = \left\| p_1 \overline{(f - f_1)} p_1 + \cdots + p_n \overline{(f - f_n)} p_n \right\|_{2,\Delta} < \varepsilon.$$
Note that, for each $\tau \in \Delta_\omega$, by (3.3),
$$\tau(p_i \overline{((f_i)_\delta)} p_i) \leq \frac{1}{n} \tau((f_i)_\delta), \quad i = 1, ..., n, \ \tau \in \Delta_\omega,$$

and hence, together with (3.2),

$$\begin{aligned} \tau((g)_\delta) &= \tau(p_1\overline{((f_1)_\delta)}p_1) + \cdots + \tau(p_n\overline{((f_n)_\delta)}p_n) \\ &\leq \frac{1}{n}K(\tau((f_1)_\delta) + \cdots + \tau((f_n)_\delta)) \\ &< \varepsilon. \end{aligned}$$

Pick a representative sequence $g = \overline{(g_k)}$ with g_k, $k = 1, 2, ...$, selfadjoint elements of D. With k sufficiently large, the function g_k satisfies

(1) $\|f - g_k\|_{2,\Delta} < \varepsilon$, and
(2) $\tau((g_k)_\delta) < \varepsilon$, $\tau \in \Delta$,

as desired.

Now, assume (D, A) has the small boundary property, where $A = D \rtimes \Gamma$ and (D, Γ) is free. It follows from [5] that (D, A) has the restricted property Gamma, and hence the weak restricted property Gamma. ☐

Corollary 3.10. *Consider a C*-pair (D, A). If A has the restricted uniform McDuff property, then (D, A) has the (SBP).*

Corollary 3.11. *Consider C*-pairs (D_1, A_1) and (D_2, A_2), and assume that one of them, say (D_1, A_1), has the restricted property Gamma, then $(D_1 \otimes D_2, A_1 \otimes A_2)$ has the (SBP).*

Proof. Note that $\partial\mathrm{T}(A_1 \otimes A_2) = \partial\mathrm{T}(A_1) \times \partial\mathrm{T}(A_2)$. Then, using Lemma 3.4, it is easy to see that $D_1 \otimes D_2$ has the weak restricted property Gamma with respect to $\mathrm{T}(A)$ (say, if (D_1, A_1) has the weak restricted property Gamma, then consider the corresponding elements $a_1 \otimes 1_{D_2}$, ..., $a_n \otimes 1_{D_2}$ in $D_1 \otimes D_2$). ☐

Remark 3.12. Does $(D_1 \otimes D_2, A_1 \otimes A_2)$ always have the weak restricted property Gamma?

4. The case that $\mathrm{T}(A)$ is a Bauer simplex

As an application of Theorem 3.5, in this section, let us show that if $A = \mathrm{C}(X) \rtimes \mathbb{Z}^d$ or is an AH algebra with diagonal maps, and if $\partial\mathrm{T}(A)$ is compact, then A has the (SBP) (Theorem 4.6), and hence is $\mathcal{Z}$-stable.

Lemma 4.1. *Consider a C*-pair (D, A) where D is a unital commutative sub-C*-algebra of A, and assume that $\mathrm{T}(A)$ a Bauer simplex. If $p_1, p_2, ..., p_n \in (\ell^\infty(D)/J_{2,D}) \cap A'$, where $n \in \mathbb{N}$, is a partition of unity such that*

$$\tau(p_i) = \frac{1}{n}, \quad \tau \in \mathrm{T}(A)_\omega,\ i = 1, ..., n.$$

then,

$$\tau(p_i a p_i) = \frac{1}{n}\tau(a), \quad a \in A, \quad \tau \in \mathrm{T}(A)_\omega,\ i = 1, ..., n. \tag{4.1}$$

Proof. The proof is the same as that of Proposition 3.2 of [2]. Let $a \in A$ and $1 \leq i \leq n$ be arbitrary. Then there is a partition of unity $p_1, p_2, ..., p_n \in (\ell^\infty(D)/J_{2,D})/A'$ such that

$$\tau(p_i) = \frac{1}{n}, \quad \tau \in \mathrm{T}_\omega(A),\ i = 1, ..., n. \tag{4.2}$$

Pick a representative $(b_k) \in \ell^\infty(D)$ for p_i. Since $p_i \in (\ell^\infty(A)/J_{2,\Delta}) \cap A'$, one has

$$\lim_{k\to\omega} \|ab_k - b_k a\|_{2,\Delta} = 0, \quad a \in A.$$

Since $\mathrm{T}(A)$ is a Bauer simplex, by Proposition 3.1 of [2],

$$\lim_{k\to\omega} \sup_{\tau\in\partial\Delta} |\tau(ab_k) - \tau(a)\tau(b_k)| = 0, \quad a \in A.$$

By (4.2),

$$\lim_{k\to\omega} \sup_{\tau\in\Delta} \left|\tau(b_k) - \frac{1}{n}\right| = 0,$$

and hence

$$\lim_{k\to\omega} \sup_{\tau\in\partial\Delta} \left|\tau(ab_k) - \frac{1}{n}\tau(a)\right| = 0, \quad a \in A,$$

which implies (4.1). □

Before diving to the main result of this section (Theorems 4.6 and 4.7), let us take a detour to formulate a version of Lemma 4.1 without using the ambient algebra A (Lemma 4.5 below), which might be interesting by its own.

Definition 4.2. A subset $\Delta \subseteq \mathrm{T}(D)$ is said to be ample for a nonzero positive contractions $c = \overline{(c_1, c_2, ...)} \in \ell^\infty(D)/J_{2,\Delta}$ if for any $\tau_1, \tau_2, ... \in \Delta$, the tracial states

$$D \ni x \mapsto \frac{1}{(\tau_n)_\omega(c)}(\tau_n)_\omega(xc) = \frac{\lim_{n\to\omega}\tau_n(xc_n)}{\lim_{n\to\omega}\tau_n(c_n)} \in \mathbb{C}$$

and

$$D \ni x \mapsto \frac{1}{1-(\tau_n)_\omega(c)}(\tau_n)_\omega(x(1-c)) = \frac{\lim_{n\to\omega}\tau_n(x(1-c_n))}{1-\lim_{n\to\omega}\tau_n(c_n)} \in \mathbb{C}$$

are still in Δ (when $(\tau_n)_\omega(c) = 0$ or 1, only the well-defined trace is considered above), where $(\tau_n)_\omega$ is the limit trace of (τ_n) on $\ell^\infty(D)$.

Remark 4.3. Note that for a C*-pair (D, A), the set $\mathrm{T}(A)|_D$ is ample for any positive contraction $c \in (\ell^\infty(D)/J_{2,D}) \cap A'$.

Lemma 4.4. *If Δ is ample for $c = \overline{(c_n)} \in \ell^\infty(D)/J_{2,\Delta}$ and $\partial\Delta$ is compact, then*

$$\lim_{k\to\omega} \sup_{\tau\in\partial\Delta} |\tau(ac_k) - \tau(a)\tau(c_k)| = 0, \quad a \in D.$$

Proof. Assume the statement were not true, there exist $a \in D$, $\varepsilon > 0$ and a sequence $(\tau_n) \subseteq \partial\Delta$ such that

$$|\tau_n(ac_n) - \tau_n(a)\tau_n(c_n))| \geq \varepsilon, \quad n = 1, 2, ...$$

Consider $(\tau_n)_\omega \in \Delta_\omega$. Then

$$(\tau_n)_\omega(ac) \neq (\tau_n)_\omega(a)(\tau_n)_\omega(c). \tag{4.3}$$

Note that, for any $x \in D$,

$$(\tau_n)_\omega(x) = (\tau_n)_\omega(c) \cdot \frac{(\tau_n)_\omega(xc)}{(\tau_n)_\omega(c)} + (1 - (\tau_n)_\omega(c)) \cdot \frac{(\tau_n)_\omega(x(1-c))}{1-(\tau_n)_\omega(c)}. \tag{4.4}$$

Since Δ is ample for c, we have that

$$\frac{(\tau_n)_\omega(\cdot c)}{(\tau_n)_\omega(c)}, \quad \frac{(\tau_n)_\omega(\cdot(1-c))}{1-(\tau_n)_\omega(c)} \in \Delta.$$

Since $(\tau_n) \subseteq \partial\Delta$ and $\partial\Delta$ is compact, $(\tau_n)_\omega|_D \in \partial\Delta$; in particular, it is an extreme point of Δ. Therefore (note that $(\tau_n)_\omega(c) \neq 0$), by (4.4),

$$(\tau_n)_\omega(x) = \frac{(\tau_n)_\omega(xc)}{(\tau_n)_\omega(c)}, \quad x \in D,$$

which is

$$(\tau_n)_\omega(xc) = (\tau_n)_\omega(x)(\tau_n)_\omega(c), \quad x \in D.$$

This contradicts (4.3). □

Then, we have the following version of Lemma 4.1 for (D, Δ):

Lemma 4.5. *Consider a pair (D, Δ) with Δ a Bauer simplex. Assume that, for each $n \in \mathbb{N}$, there is a partition of unity $p_1, p_2, ..., p_n \in \ell^\infty(D)/J_{2,\Delta}$ such that*

$$\tau(p_i) = \frac{1}{n}, \quad \tau \in \Delta_\omega,\ i = 1, ..., n. \tag{4.5}$$

and, and Δ is ample for each p_i, $i = 1, ..., n$. Then, the partition of unity $p_1, p_2, ..., p_n$ above satisfies

$$\tau(p_i a p_i) = \frac{1}{n}\tau(a), \quad a \in D, \quad \tau \in \Delta_\omega,\ i = 1, ..., n. \tag{4.6}$$

So, (D, Δ) has the restricted property Gamma.

Proof. Let $a \in D$ and $1 \leq i \leq n$ be arbitrary. Pick a representative $(c_k) \in \ell^\infty(D)$ for p_i. By Lemma 4.4,

$$\lim_{k\to\omega} \sup_{\tau\in\partial\Delta} |\tau(ac_k) - \tau(a)\tau(c_k)| = 0, \quad a \in D.$$

Together with (4.5), one has

$$\lim_{k\to\omega} \sup_{\tau\in\partial\Delta} \left|\tau(ac_k) - \tau(a)\frac{1}{n}\right| = 0, \quad a \in D,$$

which implies (4.6). □

Theorem 4.6. *Let (X, Γ) be a free and minimal topological dynamical system with the (URP), where Γ is a discrete amenable group. Assume that $\mathcal{M}_1(X, \Gamma)$ is a Bauer simplex (but with no restriction on $\dim(\partial\mathcal{M}_1(X, \Gamma))$). Then (X, Γ) has the (SBP). In particular, if (X, Γ) also has the (COS), the C*-algebra $\mathrm{C}(X) \rtimes \Gamma$ is $\mathcal{Z}$-stable.*

Proof. If $|\Gamma| < \infty$, then X is a finite set, and hence (X, Γ) has the (SBP). Therefore, we can assume that $|\Gamma| = \infty$.

In this case, since (X, Γ) has the (URP), for any $n \in \mathbb{N}$, there is a partition of unity $p_1, p_2, ..., p_n \in (\ell^\infty(D)/J_{2,D}) \cap A'$ such that

$$\tau(p_i) = \frac{1}{n}, \quad \tau \in (\mathrm{T}(A))_\omega,\ i = 1, ..., n.$$

By Lemma 4.1, and $(D, \mathrm{T}(A))$ has the restricted property Gamma. By Theorem 3.5, $(D, \mathrm{T}(A))$ has the (SBP). ☐

Now, let us consider an AH algebra with diagonal maps A, which is an inductive limit

$$A_1 \longrightarrow A_2 \longrightarrow \cdots \longrightarrow A = \varinjlim A_n,$$

where each $A_i = \bigoplus_j \mathrm{M}_{n_{i,j}}(\mathrm{C}(X_{i,j}))$, and the connecting maps have the form

$$f \mapsto \mathrm{diag}\{f \circ \lambda_1, ..., f \circ \lambda_n\}.$$

Since the connecting maps preserve the diagonal subalgebras, their inductive limit D is an commutative subalgebra of A. Let us assume A is simple. Then $n_{i,j}$ are arbitrarily large if $i \to \infty$. Hence a standard argument shows that A has the property that, for any $n \in \mathbb{N}$, any finite set $\mathcal{F}$, and any $\varepsilon > 0$, there are mutually orthogonal projections $p_1, ..., p_n \in D$ such that $p_1 + \cdots + p_n = 1$ and

$$\|[p_i, a]\|_{2,\Delta} < \varepsilon, \quad i = 1, ..., n,\ a \in \mathcal{F}.$$

Theorem 4.7. *Let A be a simple unital AH algebra with diagonal maps. If $T(A)$ is a Bauer simplex, then (D, A) has the (SBP). (By Proposition 4.8 below, the C*-algebra A is $\mathcal{Z}$-stable.)*

Proof. Since $\mathrm{T}(A)$ is a Bauer simplex, by Lemma 4.1, $(D, \mathrm{T}(A))$ has the restricted property Gamma, and by Theorem 3.5, $(D, \mathrm{T}(A))$ has the (SBP). ☐

Similar to the transformation group C*-algebra $\mathrm{C}(X) \rtimes \Gamma$ (with the (URP) and (COS)), an AH algebra with the (SBP) is also $\mathcal{Z}$-absorbing:

Proposition 4.8. *Let A be a simple unital AH algebra with diagonal maps, and denote by $D \subseteq A$ the standard diagonal sub-C*-algebra. If $(D, \mathrm{T}(A))$ has the (SBP), then A locally has slow dimension growth and hence is $\mathcal{Z}$-stable.*

Proof. Let $\varepsilon > 0$ be arbitrary, and let $\mathcal{F} \in A$ be a finite subset. Without loss of generality, one may assume that $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1$ with $\mathcal{F}_0 \subseteq D \cong \mathrm{C}(X)$ and $\mathcal{F}_1$ consists of finitely many matrix units of a building block. Passing to a finite open cover of X and use the (SBP), the same argument of Lemma 2.4 shows that one may assume further that

$$\mathcal{F}_0 = \{\phi_1, ..., \phi_L\}$$

where $\phi_1, ..., \phi_L$ is a partition of unity of X with the property

$$\mu_\tau(\phi_i^{-1}(0, 1)) < \frac{\varepsilon}{L}, \quad i = 1, ..., L,\ \tau \in \mathrm{T}(A).$$

Then pick a positive contraction $g \in \mathrm{C}(X)$ such that

$$\tau(g) < \varepsilon \quad \text{and} \quad g\chi_{\varepsilon/4,1-\varepsilon/4}(\phi_i) = \chi_{\varepsilon/4,1-\varepsilon/4}(\phi_i), \quad i = 1, ..., L,\ \tau \in \mathrm{T}(A),$$

where $\chi_{\varepsilon/4,1-\varepsilon/4}$ is a positive function vanishing outside $(\varepsilon/4, 1-\varepsilon/4)$, but is 1 on $[\varepsilon/2, 1-\varepsilon/2]$.

With a sufficiently large k, choose positive contractions $g' \in D_k$ and $\phi_i \in D_k$, $i = 1, ..., L$ such that

$$\|g - g'\| < \varepsilon/4 \quad \text{and} \quad \|\phi_i' - \phi_i\| < \varepsilon/4, \quad i = 1, ..., L.$$

Without loss of generality, we may assume that

$$\tau(g') < \varepsilon, \quad \tau \in \mathrm{T}(A_k),$$

and

$$\mathcal{F}_1 \subseteq A_k.$$

Then

$$\chi_{\varepsilon,1-\varepsilon}((\phi_i' - \varepsilon)_+)h_{1-\varepsilon/4}(g') = \chi_{\varepsilon,1-\varepsilon}((\phi_i' - \varepsilon)_+)$$

where $h_{1-\varepsilon/4}$ is a continuous function which is 1 on $1-\varepsilon/4$, 0 at 0, and linear in between, and also note that

$$\left\|h_{1-\varepsilon/4}(g') - g'\right\| < \varepsilon/4.$$

Then consider

$$\theta_{\varepsilon,1-\varepsilon}((\phi_i' - \varepsilon)_+), \quad i = 1, ..., L,$$

where θ is a positive function which is 0 on $(0, \varepsilon)$, 1 on $(1-\varepsilon, \infty)$, and linear in between. Then

$$\mu_\tau(\bigcup_{i=1}^{\infty}(\theta_{\varepsilon,1-\varepsilon}((\phi_i' - \varepsilon)_+))^{-1}(0,1)) < \tau(h_{1-\varepsilon/4}(g')) < \varepsilon/2, \quad \tau \in \mathrm{T}(A_k). \tag{4.7}$$

Then, consider the C*-algebra

$$C := \mathrm{C}^*\{\theta_{\varepsilon,1-\varepsilon}((\phi_i' - \varepsilon)_+)), v_j : i = 1, ..., L, v_j,\ j = 1, ..., d \text{ are standard matrix unit of } A_k\} \subseteq A_k.$$

Note that $\mathcal{F}_0 \subseteq_\varepsilon C$.

Write $A_k \cong \bigoplus \mathrm{M}_{n_j}(\mathrm{C}(Z_j))$. Then, by (4.7), a similar argument as Theorem 4.5 of [3] shows that

$$\mathrm{C} \cong \bigoplus_j \mathrm{M}_{n_j}(\mathrm{C}(Z_j'))$$

with $\dim(Z_j') \leq (\varepsilon/2)d_j$. □

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