VILLADSEN ALGEBRAS ARE SINGLY GENERATED

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ABSTRACT. We show that Villadsen algebras, which are not $\mathcal{Z}$-stable, are singly generated. More generally, we show that any simple unital AH algebra with diagonal maps is singly generated.

1. INTRODUCTION

The generator problem asks about the minimal number of generators for a given C*-algebra—see [13] for a survey of this problem. In particular, one wonders if a given C*-algebra is *singly* generated, and it is an interesting open question whether or not every simple separable unital C*-algebra is singly generated. In [13], it is shown that every simple separable unital *$\mathcal{Z}$-stable* C*-algebra B is singly generated (i.e., $B \otimes \mathcal{Z} \cong B$, where $\mathcal{Z}$ denotes the Jiang-Su algebra [11]). But Villadsen algebras (of the first type, Definition 2.1 below) provide examples of simple unital C*-algebras which are *not* $\mathcal{Z}$-stable, and the motivation for the current work is to show (Corollary 3.2) that these algebras are nevertheless singly generated.

Being non-$\mathcal{Z}$-stable, Villadsen algebras are not covered by the current classification theorem for C*-algebras ([3], [4], [7], [14], [9], [1], [10]). However, one regularity property they do possess is stable rank one; that is, the invertible elements in a Villadsen algebra form a norm dense subset of the algebra. Moreover, some partial classification results for Villadsen algebras are obtained in [6] using the radius of comparison (or Cuntz semigroup).

In this paper, we show that Villadsen algebras are singly generated. In fact, we show that any simple unital AH algebra with diagonal maps (Definition 2.2 below) is singly generated. To show these algebras are singly generated, we introduce the following concept: a C*-algebra B has an AF-action if it contains a simple AF algebra A and a C*-subalgebra D such that

(1) B is generated by A and D,
(2) D commutes with a certain "diagonal" subalgebra of A,

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(3) $vdv^* \in D$ for any $d \in D$ and $v \in V$,

where V denotes a special set of partial isometries in A which are intimately connected with the diagonal subalgebra in the second condition (for a precise statement, see Definition 2.3). Our main theorem (Theorem 3.1) states that any C*-algebra with an AF-action is singly generated.

In Section 2, we establish some definitions and some simple consequences of these definitions, and the remainder of the paper is dedicated to proving our main theorem at the end of Section 3.

2. Definitions

Definition 2.1. Let $(c_i)_{i\in\mathbb{N}}$, $(k_i)_{i\in\mathbb{N}}$ and $(l_i)_{i\in\mathbb{N}}$ be sequences of natural numbers, X be a compact connected metric space, and E_i be a set of cardinality k_i for each $i \in \mathbb{N}$ such that, writing $X_1 = X$ and $X_{i+1} = X_i^{c_i}$,

(1) $E_i \subseteq X_i$,
(2) the set
$$E_{i+1} \cup \left(\bigcup_{s=1}^{c_{i+1}} \pi_s(E_{i+2}) \right) \cup \left(\bigcup_{j=3}^{\infty} \bigcup_{s=1}^{c_{i+1}\cdots c_{i+j-1}} \pi_s(E_{i+j}) \right)$$
is dense in X_{i+1}, where π_s denotes the coordinate projection,
(3) $\lim\limits_{i\to\infty} \frac{l_1\cdots l_i}{(l_1+k_1)\cdots(l_i+k_i)} \neq 0$,

for each $i \in \mathbb{N}$. A *Villadsen algebra* is the limit of an inductive sequence $(B_i, \phi_i)_{i\in\mathbb{N}}$ of C*-algebras, where $B_i = \mathrm{M}_{n_{i-1}}(\mathrm{C}(X_i))$, $n_0 \in \mathbb{N}$ and $n_i = n_{i-1}(l_i + k_i)$, and the seed for ϕ_i is given by

$$\mathrm{C}(X_i) \ni f \mapsto \mathrm{diag}\Big\{ \underbrace{f\circ\pi_1, \ldots, f\circ\pi_1}_{s_{i,1}}, \ldots, \underbrace{f\circ\pi_{c_i}, \ldots, f\circ\pi_{c_i}}_{s_{i,c_i}}, f(x_{i,1}), \ldots, f(x_{i,k_i}) \Big\} \in \mathrm{M}_{l_i+k_i}\big(\mathrm{C}(X_{i+1})\big),$$

where $s_{i,t} \in \mathbb{N}$ for each $1 \leq t \leq c_i$ and $s_{i,1} + \cdots + s_{i,c_i} = l_i$.

Note that the above definition of a Villadsen algebra is more general than the original construction in [15]; in addition to the algebras in [15], Definition 2.1 also includes as a special case some algebras constructed by Goodearl in [8] (see [6, Remark 2.1]).

Given an arbitrary inductive sequence $(B_i, \phi_i)_{i\in\mathbb{N}}$ of C*-algebras, define $\phi_{i,i'} := \phi_{i'-1}\circ\cdots\circ\phi_i$ for $i' > i+1$ and $\phi_{i,i+1} := \phi_i$. Suppose $(B_i, \phi_i)_{i\in\mathbb{N}}$ is as in Definition 2.1. Then ϕ_i is unital, and a direct calculation shows that for $i' > i+1$ the seed for $\phi_{i,i'}$ is (up to permutation) given by

$$\begin{aligned}(2.1)\quad \mathrm{C}(X_i) \ni f \mapsto \operatorname{diag}\Big\{ & \underbrace{f\circ\pi_1, \ldots, f\circ\pi_{c_i\cdots c_{i'-1}}}_{l_i\cdots l_{i'-1}}, \\ & \underbrace{f\circ\pi_1(E_{i'-1}), \ldots, f\circ\pi_{c_i\cdots c_{i'-2}}(E_{i'-1})}_{l_i\cdots l_{i'-2}}, \\ & \underbrace{f\circ\pi_1(E_{i'-2}), \ldots, f\circ\pi_{c_i\cdots c_{i'-3}}(E_{i'-2})}_{l_i\cdots l_{i'-3}} 1_{l_{i'-1}+k_{i'-1}}, \\ & \underbrace{f\circ\pi_1(E_{i'-3}), \ldots, f\circ\pi_{c_i\cdots c_{i'-4}}(E_{i'-3})}_{l_i\cdots l_{i'-4}} 1_{(l_{i'-1}+k_{i'-1})(l_{i'-2}+k_{i'-2})}, \ldots, \\ & \underbrace{f\circ\pi_1(E_{i+1}), \ldots, f\circ\pi_{c_i}(E_{i+1})}_{l_i} 1_{(l_{i'-1}+k_{i'-1})\cdots(l_{i+2}+k_{i+2})}, \\ & f(E_i)1_{(l_{i'-1}+k_{i'-1})\cdots(l_{i+1}+k_{i+1})}\Big\} \in \mathrm{M}_{(l_i+k_i)\cdots(l_{i'-1}+k_{i'-1})}\big(\mathrm{C}(X_{i'})\big).\end{aligned}$$

Definition 2.2. Let B be the limit of an inductive sequence $(B_i, \phi_i)_{i\in\mathbb{N}}$ of unital C*-algebras, where each ϕ_i is unital. We call B a (unital) *AH algebra with diagonal maps* if $B_i = \bigoplus_{1\le j\le K_i} \mathrm{M}_{n_{i,j}}(\mathrm{C}(X_{i,j}))$, where $X_{i,j}$ is a compact connected metric space and $n_{i,j}, K_i \in \mathbb{N}$, and if for any $i' > i$ the restriction of the map $\phi_{i,i'}$ to any direct summands $\mathrm{M}_{n_{i,j}}(\mathrm{C}(X_{i,j}))$ and $\mathrm{M}_{n_{i',j'}}(\mathrm{C}(X_{i',j'}))$ has a seed of the form $f \mapsto 0$ or $f \mapsto \operatorname{diag}\{f\circ\lambda_1, \ldots, f\circ\lambda_m\}$ for some continuous maps $\lambda_1, \ldots, \lambda_m \colon X_{i',j'} \to X_{i,j}$.

If $(B_i, \phi_i)_{i\in\mathbb{N}}$ is as in Definition 2.2 and each ϕ_i is injective, it is well-known that the limit algebra B is simple if and only if, for any $i \in \mathbb{N}$ and nonzero $b \in B_i$, there is an $i_0 \ge i$ such that for every $i' \ge i_0$, $\phi_{i,i'}(b)(x) \ne 0$ for every $x \in \bigsqcup_{1\le j\le K_{i'}} X_{i',j}$ ([5, Proposition 2.3]). From this characterization of simplicity for B, one sees that the unital AF subalgebra A of B obtained as the limit of the inductive sequence $(A_i, \psi_i)_{i\in\mathbb{N}}$, where $A_i = \bigoplus_{1\le j\le K_i} \mathrm{M}_{n_{i,j}}$ and $\psi_i = \phi_i|_{A_i}$, is simple if B is.

Letting $(B_i, \phi_i)_{i\in\mathbb{N}}$ be as in Definition 2.1 once again, it is clear that the Villadsen algebra arising from this inductive sequence is an AH algebra with diagonal maps with injective connecting maps; moreover, from the above characterization of simplicity for such an algebra, one sees almost immediately that it is simple. Indeed, fix $i \in \mathbb{N}$ and let $f \in B_{i+1}$ be nonzero; then, by Condition 2 of Definition 2.1, there exists some $i_0 \ge i+1$ and $y \in E_{i_0-1}$ for which $f\circ\pi_s(y) \ne 0$; we then see from Equation (2.1) that for every $i' \ge i_0$, $\phi_{i+1,i'}(f)(x) \ne 0$ for any $x \in X_{i'}$.

Definition 2.3. Let B be a unital C*-algebra containing a simple separable unital AF subalgebra A and a separable unital C*-subalgebra D. Let $(\bigoplus_{1\le j\le K_i} \mathrm{M}_{n_{i,j}}, \phi_i)_{i\in\mathbb{N}}$ be a canonical inductive limit decomposition for A, where $n_{i,j}, K_i \in \mathbb{N}$. Denote the set of canonical matrix units for $\mathrm{M}_{n_{i,j}}$ by $V_{i,j}$, and define

$$V := \bigcup_{i\in\mathbb{N}} \bigcup_{j=1}^{K_i} V_{i,j}, \quad E_{i,j} := \{v \in V_{i,j} \mid v = v^*\}, \quad D_0 := \mathrm{C}^*\left(\bigcup_{i\in\mathbb{N}} \bigcup_{j=1}^{K_i} E_{i,j}\right).$$

We say that B has an *AF-action* if

(1) $B = \mathrm{C}^*(A, D)$,
(2) $[d, d'] = 0$ for any $d \in D$ and $d' \in D_0$,
(3) $vdv^* \in D$ for any $d \in D$ and $v \in V$.

In the sequel, to emphasize the dependence of B on A and D, we may write $B(A, D)$ for B; moreover, associated to B is the inductive limit decomposition $(\bigoplus_{1\leq j\leq K_i} \mathrm{M}_{n_{i,j}}, \phi_i)_{i\in\mathbb{N}}$ of A, which we may simply refer to as the "associated decomposition of A."

The following lemmas are straightforward consequences of this definition.

Lemma 2.1. *A simple AH algebra with diagonal maps has an AF-action. In particular, a Villadsen algebra has an AF-action.*

Proof. Let $B = \lim_{i\to\infty}(B_i, \phi_i)$ be a simple AH algebra with diagonal maps, where B_i and ϕ_i are as in Definition 2.2, and let $A = \lim_{i\to\infty}(A_i, \psi_i)$ be the AF subalgebra of B as described in the paragraph following Definition 2.2. Denote the set of canonical matrix units for $\mathrm{M}_{n_{i,j}}$ by $V_{i,j}$ and define the sets V, $E_{i,j}$, and D_0 as in Definition 2.3. Moreover, define

$$D := \mathrm{C}^*\Big(\{p \otimes f \mid p \in E_{i,j},\, f \in \mathrm{C}(X_{i,j}),\, i \in \mathbb{N},\, 1 \leq j \leq K_i\}\Big).$$

Notice D is a separable unital C*-subalgebra of B. Furthermore, that $B = \mathrm{C}^*(A, D)$ follows from the fact that any $b \in \mathrm{M}_{n_{i,j}}(\mathrm{C}(X_{i,j}))$ may be written as a finite sum of elements of the form $v(p \otimes f)v'$ for $v, v' \in V_{i,j}$, $p \in E_{i,j}$, and $f \in \mathrm{C}(X_{i,j})$; since D is commutative and $D_0 \subseteq D$, we have $[d, d'] = 0$ for any $d \in D$ and $d' \in D_0$; finally, clearly $v(p \otimes f)v^* \in D$ for any $v \in V_{i,j}$, $p \in E_{i',j'}$, and $f \in \mathrm{C}(X_{i',j'})$ so that $vdv^* \in D$ for any $v \in V$ and $d \in D$. $\square$

Lemma 2.2. *Let $B = B(A, D)$ have an AF-action, and let C be a separable unital simple C*-algebra. Then $B \otimes C$ has an AF-action.*

Proof. Define $\mathcal{A}_i := \bigoplus_{1\leq j\leq K_i} \mathrm{M}_{n_{i,j}} \otimes \mathbb{C}1_C$ for each $i \in \mathbb{N}$, where $(\bigoplus_{1\leq j\leq K_i} \mathrm{M}_{n_{i,j}}, \phi_i)_{i\in\mathbb{N}}$ is the associated decomposition of A, so that $\mathcal{A} = \overline{\bigcup_{i\in\mathbb{N}} \mathcal{A}_i}$ is a simple AF subalgebra of $B \otimes C$. Identify the set $V_{i,j}$ of canonical matrix units for $\mathrm{M}_{n_{i,j}}$ with the set $\mathcal{V}_{i,j} = \{v \otimes 1_C \mid v \in V_{i,j}\}$ in $B \otimes C$. Then, writing $\mathcal{V} = \bigcup_{i\in\mathbb{N}} \bigcup_{1\leq j\leq K_i} \mathcal{V}_{i,j}$, $\mathcal{E}_{i,j} = \{\mathsf{v} \in \mathcal{V}_{i,j} \mid \mathsf{v} = \mathsf{v}^*\}$, $\mathcal{D}_0 = \mathrm{C}^*(\bigcup_{i\in\mathbb{N}} \bigcup_{1\leq j\leq K_i} \mathcal{E}_{i,j})$, and $\mathcal{D} = D \otimes C$, it follows that $B \otimes C = \mathrm{C}^*(\mathcal{A}, \mathcal{D})$, $[\mathsf{d}, \mathsf{d}'] = 0$ for any $\mathsf{d} \in \mathcal{D}$ and $\mathsf{d}' \in \mathcal{D}_0$, and $\mathsf{v}\mathsf{d}\mathsf{v}^* \in \mathcal{D}$ for any $\mathsf{d} \in \mathcal{D}$ and $\mathsf{v} \in \mathcal{V}$. $\square$

3. A Generator for an Algebra with an AF-action

3.1. **Preliminary Lemmas.** Let B be a unital C*-algebra, and let $a \in B$ be such that $q_i a q_j = 0$ for $i > j$, where $(q_i)_{1\le i\le n} \subseteq B$ is a finite sequence of nonzero mutually orthogonal projections summing to the identity (a is "upper triangular" with respect to (q_i)). It is well-known that $\sigma(a) \subseteq \bigcup_{1\le i\le n} \sigma(q_i a q_i)$, and the following lemma gives a similar result for infinite sequences.

Lemma 3.1.1. *Let B be a unital C*-algebra, let $a \in B$, and let $(p_i)_{i\in\mathbb{N}} \subseteq B$ be a sequence of nonzero mutually orthogonal projections such that*

(1) $(1 - \sum_{i=1}^{n} p_i) a \sum_{i=1}^{n} p_i = 0$ for each $n \in \mathbb{N}$,
(2) $\lim_{n\to\infty} \|(1 - \sum_{i=1}^{n} p_i) a (1 - \sum_{i=1}^{n} p_i)\| = 0$,
(3) $\sigma(p_i a p_i) \cap \sigma(p_{i'} a p_{i'}) = \emptyset$ for $i \neq i'$,
(4) $0 \notin \sigma(p_i a p_i)$ for any $i \in \mathbb{N}$.

Then

$$\sigma(a) \subseteq \bigcup_{i=1}^{\infty} \sigma(p_i a p_i) \cup \{0\} \quad \textit{and} \quad (p_i)_{i\in\mathbb{N}} \subseteq \mathrm{C}^*(a).$$

Proof. Define $P_n := \sum_{1\le i\le n} p_i$, and write $1 - P_n = P_n^{\perp}$ for each $n \in \mathbb{N}$; also, define $P_0 := 0$ so that $P_0^{\perp} = 1$. Notice for any $n \in \mathbb{N}$, $P_{n-1}^{\perp} = p_n + P_n^{\perp}$, $p_n P_n^{\perp} = 0$, and $P_n^{\perp}(P_{n-1}^{\perp} a P_{n-1}^{\perp}) p_n = P_n^{\perp} a P_n p_n = 0$ by Condition 1. Fixing $n \in \mathbb{N}$, it then follows from the paragraph preceding this lemma that $\sigma(P_{n-1}^{\perp} a P_{n-1}^{\perp}) \subseteq \sigma(p_n a p_n) \cup \sigma(P_n^{\perp} a P_n^{\perp})$; by induction,

$$\sigma(P_{n-1}^{\perp} a P_{n-1}^{\perp}) \subseteq \bigcup_{j=n}^{m} \sigma(p_j a p_j) \cup \sigma(P_m^{\perp} a P_m^{\perp}), \quad \forall m \geq n. \tag{3.1}$$

If $\lambda \in \sigma(P_{n-1}^{\perp} a P_{n-1}^{\perp})$ and $\lambda \notin \sigma(p_j a p_j)$ for any $j \geq n$, Equation (3.1) implies $\lambda \in \sigma(P_j^{\perp} a P_j^{\perp})$ for every $j \geq n$; then, by Condition 2, $\lambda = 0$ and

$$\sigma(P_{n-1}^{\perp} a P_{n-1}^{\perp}) \subseteq \bigcup_{j=n}^{\infty} \sigma(p_j a p_j) \cup \{0\}. \tag{3.2}$$

Taking $n = 1$, we have the desired containment for $\sigma(a)$.

Let $n \in \mathbb{N}$ be arbitrary again. By Equation (3.2) and Conditions 3 and 4, $\sigma(p_n a p_n) \cap \sigma(P_n^{\perp} a P_n^{\perp}) = \emptyset$. It follows that

$$p_n \in \mathrm{C}^*(P_{n-1}^{\perp} a P_{n-1}^{\perp}); \tag{3.3}$$

we refer the reader to [12, Theorem 1] and [2, p. 22] for the details. If $p_1, \ldots, p_{n-1} \in \mathrm{C}^*(a)$, expanding $P_{n-1}^{\perp} a P_{n-1}^{\perp}$ in terms of $p_1, \ldots, p_{n-1}$ reveals that $P_{n-1}^{\perp} a P_{n-1}^{\perp} \in \mathrm{C}^*(a)$. Now, taking $n = 1$ in Equation (3.3), we see $p_1 \in \mathrm{C}^*(a)$; thus $(p_i)_{i\in\mathbb{N}} \subseteq \mathrm{C}^*(a)$ by induction. $\square$

Lemma 3.1.2. *Let B be a unital C^*-algebra, and suppose B contains a unital C^*-subalgebra D and a finite set $\{v_k\}_{1\leq k\leq n}$ of nonzero partial isometries such that*

(a) v_1 is the range projection of each v_k, i.e., $v_1 = v_k v_k^*$,
(b) the source projections form a partition of unity for B, i.e., $(v_k^* v_k)(v_{k'}^* v_{k'}) = 0$ for $k \neq k'$ and $\sum_{1\leq k\leq n}(v_k^* v_k) = 1$,
(c) the source projections commute with D, i.e., $[v_k^* v_k, d] = 0$ for any $d \in D$,
(d) $v_k d v_k^* \in D$ for any $d \in D$.

Then, the map

$$\Phi\colon\ \mathrm{M}_n\left((v_1v_1^*)D(v_1v_1^*)\right) \to \mathrm{C}^*(D, v_1, \ldots, v_n), \quad [b_{i,j}]_{i,j=1}^n \mapsto \sum_{i=1}^n\sum_{j=1}^n v_i^* b_{i,j} v_j$$

is a $*$-isomorphism.

Proof. It is clear that Φ is linear and preserves adjoints. For multiplicativity, notice

$$\begin{aligned}
\Phi\big([b_{i,j}]\big)\Phi\big([c_{i,j}]\big) &= \Big(\sum_{i=1}^n\sum_{j=1}^n v_i^* b_{i,j} v_j\Big)\Big(\sum_{i=1}^n\sum_{j=1}^n v_i^* c_{i,j} v_j\Big) \\
&= \Big(\sum_{i=1}^n v_i^* b_{i,1} v_1 + \cdots + \sum_{i=1}^n v_i^* b_{i,n} v_n\Big)\Big(\sum_{j=1}^n v_1^* c_{1,j} v_j + \cdots + \sum_{j=1}^n v_n^* c_{n,j} v_j\Big) \\
&= \Big(\sum_{i=1}^n v_i^* b_{i,1} v_1\Big)\Big(\sum_{j=1}^n v_1^* c_{1,j} v_j\Big) + \cdots + \Big(\sum_{i=1}^n v_i^* b_{i,n} v_n\Big)\Big(\sum_{j=1}^n v_n^* c_{n,j} v_j\Big) \\
&= \sum_{i=1}^n\sum_{j=1}^n v_i^*\Big(\sum_{k=1}^n b_{i,k}c_{k,j}\Big)v_j \\
&= \Phi\big([a_{i,j}]\big),
\end{aligned}$$

where $a_{i,j} = \sum_{1\leq k\leq n} b_{i,k}c_{k,j}$ and where the third equality is a result of Condition (b). Hence, $\Phi([b_{i,j}])\Phi([c_{i,j}]) = \Phi([b_{i,j}][c_{i,j}])$.

Now, notice $v_1 = v_1v_1^* = v_1 1 v_1^* \in D$ by Condition (a) and Condition (d). Fixing some $1 \leq k \leq n$ and defining $[b_{i,j}]$ such that $b_{1,k} = (v_1v_1^*)v_1(v_1v_1^*) = v_1$ and $b_{i,j} = 0$ otherwise, it follows from the previous sentence that $[b_{i,j}] \in \mathrm{M}_n((v_1v_1^*)D(v_1v_1^*))$. We then see v_k is in the image of Φ since $\Phi([b_{i,j}]) = v_1^*v_1v_k = v_1v_k = v_kv_k^*v_k = v_k$. Moreover, for any $d \in D$,

$$d = \Big(\sum_{i=1}^n v_i^*v_i\Big)d\Big(\sum_{i=1}^n v_i^*v_i\Big) = \sum_{i=1}^n (v_i^*v_i)d(v_i^*v_i) \tag{3.4}$$

by Condition (b) and Condition (c); by Condition (a),

$$\sum_{i=1}^n (v_i^*v_i)d(v_i^*v_i) = \sum_{i=1}^n v_i^*(v_1v_1^*)d_i(v_1v_1^*)v_i, \tag{3.5}$$

where $d_i = v_i d v_i^* \in D$ for each $1 \le i \le n$. Putting Equation (3.4) and Equation (3.5) together, we see d is in the image of Φ since $\Phi([c_{i,j}]) = d$ when $c_{k,k} = (v_1 v_1^*) d_k (v_1 v_1^*)$, for each $1 \le k \le n$, and $c_{i,j} = 0$ otherwise. Thus Φ is onto.

Finally, notice if $\Phi([b_{i,j}]) = 0$, then

$$0 = v_k \Phi\big([b_{i,j}]\big) v_l^* = \sum_{i=1}^{n} \sum_{j=1}^{n} v_k v_i^* b_{i,j} v_j v_l^* = \delta_{k,i} b_{i,j} \delta_{j,l}$$

for every $1 \le k, l \le n$, where δ denotes the Kronecker delta function; in particular, Φ is injective. □

Lemma 3.1.3 below references a result from the paper of Olsen and Zame [12]; we reproduce a version of it here for the reader's convenience.

Lemma (Olsen and Zame)**.** *Let A be a unital C^*-algebra generated by the $k(k+1)/2$ invertible self-adjoint elements $a_1, \dots, a_{k(k+1)/2}$ with pairwise disjoint spectra. Then, $\mathrm{M}_k(A)$ is generated by the upper triangular matrix*

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_k \\ 0 & a_{k+1} & \cdots & a_{2k-1} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{k(k+1)/2} \end{bmatrix}.$$

Lemma 3.1.3. *Let B, D, and $\{v_k\}_{1 \le k \le n}$ be as in Lemma 3.1.2. Let m be a positive integer such that $n > 2m-1$, and let $\{d_1, \dots, d_m\} \subseteq D \setminus \{0\}$ be a subset of self-adjoint elements. Then, there exists an invertible element $\mathfrak{g} \in B$ such that $d_1, \dots, d_m \in \mathrm{C}^*(\mathfrak{g})$.*

Proof. As in Equation (3.4) and Equation (3.5), we can write $d_i \in \{d_1, \dots, d_m\}$ as

$$d_i = \sum_{j=1}^{n} v_j^* (v_1 v_1^*) d_{i,j} (v_1 v_1^*) v_j, \quad d_{i,j} = v_j d_i v_j^* \in D.$$

Consider the self-adjoint elements $(v_1 v_1^*) d_{i,j} (v_1 v_1^*) \in (v_1 v_1^*) D (v_1 v_1^*)$ for $1 \le i \le m$ and $1 \le j \le n$. Suppose the distinct nonzero such elements constitute a set S', and let $S = S' \cup \{v_1 v_1^*\}$. Denoting the cardinality of S by N, $\mathrm{C}^*(S)$ is a unital C*-algebra generated by $N \le nm + 1 \le n(n+1)/2$ self-adjoint elements; it is then an simple consequence of the continuous function calculus that $\mathrm{C}^*(S)$ is generated by $n(n+1)/2$ invertible self-adjoint elements with disjoint spectra, say $a_1, \dots, a_{n(n+1)/2}$. It follows from Olsen and Zame that

$\mathrm{M}_n(\mathrm{C}^*(S))$ is generated by an element g of the form

$$g = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \\ 0 & a_{n+1} & \cdots & a_{2n-1} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{n(n+1)/2} \end{bmatrix};$$

notice g is invertible since its diagonal entries are (see the paragraph preceding Lemma 3.1.1).

Now, consider the map $\Phi\colon \mathrm{M}_n((v_1v_1^*)D(v_1v_1^*)) \to \mathrm{C}^*(D, v_1, \ldots, v_n)$ from Lemma 3.1.2. Clearly $\Phi(\mathrm{C}^*(g)) = \mathrm{C}^*(\Phi(g))$, and $\Phi(g)$ is invertible. Fix $1 \le i \le m$; since $\mathrm{C}^*(g)$ contains the element $[a_{k,l}]$, where $a_{j,j} = (v_1v_1^*)d_{i,j}(v_1v_1^*)$ for $1 \le j \le n$ and $a_{k,l} = 0$ otherwise, $\Phi(\mathrm{C}^*(g))$ contains the element $\Phi([a_{k,l}]) = d_i$. Writing $\mathfrak{g} = \Phi(g)$, the result follows. □

3.2. **Lemmas Pertaining Specifically to Algebras with AF-actions.** Let $B = B(A, D)$ have an AF-action, and let $(\bigoplus_{1\le j\le K_i} \mathrm{M}_{n_{i,j}}, \phi_i)_{i\in\mathbb{N}}$ be the associated decomposition of A. Fix $i_0 \in \mathbb{N}$, and denote the multiplicity of the embedding of $\mathrm{M}_{n_{i_0,j'}}$ into $\mathrm{M}_{n_{i,j}}$ via $\phi_{i_0,i}$ by $m_{i_0,i;j',j}$, where $i > i_0$, $1 \le j' \le K_{i_0}$, and $1 \le j \le K_i$. For a positive integer N, note that one can always find an $i > i_0$ such that $m_{i_0,i;j',j} > N$ for each $1 \le j' \le K_{i_0}$ and $1 \le j \le K_i$; this is a simple consequence of the fact that A is simple.

Lemma 3.2.1. *Let $B = B(A, D)$ have an AF-action, and let $(\bigoplus_{1\le j\le K_i} \mathrm{M}_{n_{i,j}}, \phi_i)_{i\in\mathbb{N}}$ be the associated decomposition of A; denote the set of canonical matrix units for $\mathrm{M}_{n_{i,j}}$ by $V_{i,j}$ and the subset of $V_{i,j}$ consisting of all self-adjoint elements by $E_{i,j}$. Let $p \in E_{i',j'}$ for some $i' \in \mathbb{N}$ and some $1 \le j' \le K_{i'}$, and let $\{d_1, \ldots, d_m\} \subseteq pDp$ be a subset of self-adjoint elements. Then, there exists an invertible element $\mathfrak{g} \in pBp$ such that $d_1, \ldots, d_m \in \mathrm{C}^*(\mathfrak{g})$.*

Proof. Let i be such that $m_{i',i;j',j} > 2m - 1$ for each $1 \le j \le K_i$, and write $m_{i',i;j',j} = M_j$ for convenience. For each $1 \le j \le K_i$, there is a subset $\{v_{j,k}\}_{1\le k\le M_j} \subseteq V_{i,j}$ of cardinality M_j such that $v_{j,1}$ is the range projection of each $v_{j,k}$ and $p = \sum_{1\le j\le K_i} \sum_{1\le k\le M_j} (v_{j,k}^* v_{j,k})$; writing $p_j = \sum_{1\le k\le M_j}(v_{j,k}^* v_{j,k})$, it follows that the source projections of the members of the set $\{v_{j,k}\}_{1\le k\le M_j}$ form a partition of unity for p_jBp_j. Moreover, notice $v_{j,k}$ and $v_{j,k}^*$ commute with p_j so that $v_{j,k}p_jdp_jv_{j,k}^* = p_jv_{j,k}dv_{j,k}^*p_j \in p_jDp_j$ for any $1 \le k \le M_j$ and $d \in D$; also, the source projections of the members of the set $\{v_{j,k}\}_{1\le k\le M_j}$ are contained in D_0 so that $(v_{j,k}^*v_{j,k})p_jdp_j = p_jdp_j(v_{j,k}^*v_{j,k})$ for any $d \in D$. We conclude that for each $1 \le j \le K_i$, p_jBp_j is a unital C^*-algebra containing a unital C^*-subalgebra p_jDp_j and a finite set $\{v_{j,k}\}_{1\le k\le M_j}$ of nonzero partial isometries such that

(a) $v_{j,1} = v_{j,k}v_{j,k}^*$,
(b) $(v_{j,k}^*v_{j,k})(v_{j,k'}^*v_{j,k'}) = 0$ for $k \ne k'$ and $\sum_{1\le k\le M_j}(v_{j,k}^*v_{j,k}) = p_j$,
(c) $[v_{j,k}^*v_{j,k}, d'] = 0$ for any $d' \in p_jDp_j$,
(d) $v_{j,k}d'v_{j,k}^* \in p_jDp_j$ for any $d' \in p_jDp_j$.

For each $1 \leq j \leq K_i$, consider the self-adjoint elements $p_j d_1 p_j, \dots, p_j d_m p_j \in p_j D p_j$; take the distinct nonzero such elements and form a set $S_j \subseteq p_j D p_j \setminus \{0\}$. Then, $|S_j|$ (the cardinality of S_j) is a positive integer such that $M_j > 2m - 1 \geq 2|S_j| - 1$. By Lemma 3.1.3, there exists an invertible element $g_j \in p_j B p_j$ such that $p_j d_1 p_j, \dots, p_j d_m p_j \in \mathrm{C}^*(g_j)$ for each $1 \leq j \leq K_i$.

Assuming $\sigma(g_j) \cap \sigma(g_{j'}) = \emptyset$ for $j \neq j'$ (which we may by the functional calculus), we claim that the C*-algebra generated by $\mathfrak{g} = \sum_{1 \leq j \leq K_i} g_j \in pBp$ contains $d_1, \dots, d_m$. Indeed, $\mathfrak{g}$ is "diagonal" with respect to the sequence $(p_j)_{1 \leq j \leq K_i}$, and it is a simple corollary of Lemma 3.1.1 that $(p_j)_{1 \leq j \leq K_i} \subseteq \mathrm{C}^*(\mathfrak{g})$; hence $g_j \in \mathrm{C}^*(\mathfrak{g})$, hence $p_j d_1 p_j, \dots, p_j d_m p_j \in \mathrm{C}^*(\mathfrak{g})$, for each $1 \leq j \leq K_i$. But, notice $d_l = \sum_{1 \leq j \leq K_i} p_j d_l p_j$ for each $1 \leq l \leq m$ (since D and D_0 commute). The result follows. □

Lemma 3.2.2. *Let $B(A, D)$, $(\bigoplus_{1 \leq j \leq K_i} \mathrm{M}_{n_{i,j}}, \phi_i)_{i \in \mathbb{N}}$, $V_{i,j}$, and $E_{i,j}$ be as in Lemma 3.2.1. Fix $i \in \mathbb{N}$. For each $1 \leq j \leq K_i$, let $p_j \in E_{i,j}$ and $\{v_{j,k}\}_{1 \leq k \leq n_{i,j}} \subseteq V_{i,j}$ be a subset of cardinality $n_{i,j}$ such that $v_{j,1} = p_j$ is the range projection of each $v_{j,k}$. Then, given a self-adjoint element $d \in D$, there exists a subset $G = \{g_j\}_{1 \leq j \leq K_i} \subseteq B$ such that*

(1) $g_j \in p_j B p_j$,
(2) $0 \notin \sigma(g_j)$,
(3) $d \in \mathrm{C}^*(\{v_{1,k}\}_{1 \leq k \leq n_{i,1}}, \dots, \{v_{K_i,k}\}_{1 \leq k \leq n_{i,K_i}}, G)$.

Proof. Notice the source projections of the elements of the set $\{v_{j,k}\}_{1 \leq k \leq n_{i,j}}$ exhaust the elements of $E_{i,j}$ for each $1 \leq j \leq K_i$ so that the projections in the set $\bigcup_{1 \leq j \leq K_i} \{v_{j,k}^* v_{j,k}\}_{1 \leq k \leq n_{i,j}}$ form a partition of unity for B. We can then write d as

$$
\begin{aligned}
d &= \Big(\sum_{j=1}^{K_i} \sum_{k=1}^{n_{i,j}} (v_{j,k}^* v_{j,k}) \Big) d \Big(\sum_{j=1}^{K_i} \sum_{k=1}^{n_{i,j}} (v_{j,k}^* v_{j,k}) \Big) \\
&= \sum_{j=1}^{K_i} \sum_{k=1}^{n_{i,j}} (v_{j,k}^* v_{j,k}) d (v_{j,k}^* v_{j,k}) \\
&= \sum_{j=1}^{K_i} \sum_{k=1}^{n_{i,j}} v_{j,k}^* (p_j v_{j,k}) d (v_{j,k}^* p_j) v_{j,k} \\
&= \sum_{j=1}^{K_i} \sum_{k=1}^{n_{i,j}} v_{j,k}^* (p_j d_{j,k} p_j) v_{j,k}
\end{aligned} \tag{3.6}
$$

where $d_{j,k} = v_{j,k} d v_{j,k}^*$. Consider the subset $\{p_j d_{j,1} p_j, \dots, p_j d_{j,n_{i,j}} p_j\} \subseteq p_j D p_j$ of self-adjoint elements for each $1 \leq j \leq K_i$; Lemma 3.2.1 implies there exists an invertible element $g_j \in p_j B p_j$ such that $p_j d_{j,1} p_j, \dots, p_j d_{j,n_{i,j}} p_j \in \mathrm{C}^*(g_j)$. Defining $G := \{g_1, \dots, g_{K_i}\}$, the final assertion follows from the last equality in Equation (3.6). □

We remark that the subsets $\{v_{j,k}\}_{1\leq k\leq n_{i,j}} \subseteq V_{i,j}$ from the previous lemma generate the finite-dimensional C*-algebra $\bigoplus_{1\leq j\leq K_i} \mathrm{M}_{n_{i,j}}$; that is,

$$\bigoplus_{j=1}^{K_i} \mathrm{M}_{n_{i,j}} = \mathrm{C}^*\left(\{v_{1,k}\}_{k=1}^{n_{i,1}}, \dots, \{v_{K_i,k}\}_{k=1}^{n_{i,K_i}}\right)$$

For this, it is sufficient to show $V_{i,j} \subseteq \mathrm{C}^*(\{v_{j,k}\}_{1\leq k\leq n_{i,j}})$ for each $1 \leq j \leq K_i$. Indeed, fix j, and let $V_{i,j} = \{e_{s,t} \mid 1 \leq s, t \leq n_{i,j}\}$. Then, $e_{t_k,t_k} = v_{j,k}^* v_{j,k}$ for each $1 \leq k \leq n_{i,j}$, where $1 \leq t_k \leq n_{i,j}$ and $t_k \neq t_{k'}$ for $k \neq k'$; thus $v_{j,k} = e_{s_k,t_k}$ for some $1 \leq s_k \leq n_{i,j}$. But, $v_{j,1}$ is the range projection of each $v_{j,k}$ so that $v_{j,1} = e_{s_k,s_k} = e_{s_1,t_1}$; hence $s_1 = t_1 = s_k$ for each $1 \leq k \leq n_{i,j}$ so that $v_{j,k} = e_{s_1,t_k}$. Then, to obtain $e_{s,t}$ for some $1 \leq s, t \leq n_{i,j}$, take the product $v_{j,k}^* v_{j,k'}$ for k and k' such that $t_k = s$ and $t_{k'} = t$.

3.3. **Main Results.** Let $B = B(A, D)$ have an AF-action, and let $(\bigoplus_{1\leq j\leq K_i} \mathrm{M}_{n_{i,j}}, \phi_i)_{i\in\mathbb{N}}$ be the associated decomposition of A; denote the set of canonical matrix units for $\mathrm{M}_{n_{i,j}}$ by $V_{i,j}$ and the subset of $V_{i,j}$ consisting of all self-adjoint elements by $E_{i,j}$. To prove our main theorem, that every C*-algebra with an AF-action is singly generated, we will construct a generator for the C*-algebra B. We now define the sets of elements from which we will build a generator for B.

Let $(s_i)_{i\in\mathbb{N}}$ be a sequence of natural numbers such that $n_{s_1,j} > 1$ for each $1 \leq j \leq K_{s_1}$ and $m_{s_i,s_{i+1};j',j} > 1$ for each $1 \leq j' \leq K_{s_i}$ and $1 \leq j \leq K_{s_{i+1}}$; for convenience, write

$$\mathsf{K}_i = K_{s_i}, \quad \mathsf{N}_{i,j'} = n_{s_i,j'}, \quad \mathsf{M}_{i+1;j',j} = m_{s_i,s_{i+1};j',j}, \qquad 1 \leq j' \leq \mathsf{K}_i,\ 1 \leq j \leq \mathsf{K}_{i+1};$$

moreover, define $\mathsf{K}_0 := 1$ and $\mathsf{M}_{1;j',j} := \mathsf{N}_{1,j}$ for each $1 \leq j' \leq \mathsf{K}_0$ and $1 \leq j \leq \mathsf{K}_1$.

Inductively on i, construct sets of projections $Q_{i;j',j} = \{q_{i;j',j,k}\}_{1\leq k\leq \mathsf{M}_{i;j',j}}$ for each $1 \leq j' \leq \mathsf{K}_{i-1}$ and $1 \leq j \leq \mathsf{K}_i$ of cardinality $\mathsf{M}_{i;j',j}$ such that

- (Q1) $Q_{i;j',j} \subseteq E_{s_i,j}$ for each $1 \leq j' \leq \mathsf{K}_{i-1}$,
- (Q2) $p_{i-1,j'} = \sum_{1\leq j\leq \mathsf{K}_i} \sum_{q\in Q_{i;j',j}} q$ for each $1 \leq j' \leq \mathsf{K}_{i-1}$, where $p_{0,1}$ is the identity and $p_{i-1,j'} = q_{i-1;\mathsf{K}_{i-2},j',\mathsf{M}_{i-1;\mathsf{K}_{i-2},j'}}$ for $i > 1$.

We take $Q_{1;1,j} = E_{s_1,j}$ for $1 \leq j \leq \mathsf{K}_1$ to start the induction. To each set of projections $Q_{i;j',j}$, associate a set of partial isometries $W_{i;j',j} = \{w_{i;j',j,k}\}_{1\leq k\leq \mathsf{M}_{i;j',j}}$ of cardinality $\mathsf{M}_{i;j',j}$ such that

- (W1) $W_{i;j',j} \subseteq V_{s_i,j}$ for each $1 \leq j' \leq \mathsf{K}_{i-1}$,
- (W2) the range projection of each member of $W_{i;j',j}$ is $w_{i;1,j,1}$ for $1 \leq j' \leq \mathsf{K}_{i-1}$,
- (W3) the source projection of $w_{i;j',j,k}$ is $q_{i;j',j,k}$.

For each $i \in \mathbb{N}$ and $1 \leq j \leq \mathsf{K}_i$, choose additional sets of partial isometries $U_{i,j} = \{v_{i;j,k}\}_{1\leq k\leq \mathsf{N}_{i,j}}$ of cardinality $\mathsf{N}_{i,j}$ such that

- (U1) $W_{i;j',j} \subseteq U_{i,j} \subseteq V_{s_i,j}$ for each $1 \leq j' \leq \mathsf{K}_{i-1}$, (U2) $v_{i;j,1} = w_{i;1,j,1}$ is the range projection of each member of $U_{i,j}$.

We then have the following lemma.

Lemma 3.3.1. *For each $i \in \mathbb{N}$ and $1 \le j \le \mathsf{K}_{i+1}$, one has*

$$U_{i+1,j} \subseteq \mathrm{C}^*\big(\bigoplus_{j=1}^{\mathsf{K}_i} \mathrm{M}_{\mathsf{N}_{i,j}}, W_{i+1;1,j}, \dots, W_{i+1;\mathsf{K}_i,j}\big).$$

Proof. Let $V_{s_{i,j'}} = \{e_{j';l,k} \mid 1 \le l,k \le \mathsf{N}_{i,j'}\}$ for $1 \le j' \le \mathsf{K}_i$ and $V_{s_{i+1,j}} = \{f_{j;l,k} \mid 1 \le l,k \le \mathsf{N}_{i+1,j}\}$ for $1 \le j \le \mathsf{K}_{i+1}$. Then for each j there exist positive integers $L_{j',j,k,t}$ for $1 \le j' \le \mathsf{K}_i$, $1 \le k \le \mathsf{N}_{i,j'}$, and $1 \le t \le \mathsf{M}_{i+1;j',j}$ such that

(a) $1 \le L_{j',j,k,t} \le \mathsf{N}_{i+1,j}$,
(b) $L_{j_1',j,k_1,t_1} \ne L_{j_2',j,k_2,t_2}$ for $j_1' \ne j_2'$ or $k_1 \ne k_2$ or $t_1 \ne t_2$,
(c) $e_{j';l,k} = \sum_{j=1}^{\mathsf{K}_{i+1}} \sum_{t=1}^{\mathsf{M}_{i+1;j',j}} f_{j;L_{j',j,l,t},L_{j',j,k,t}}$.

Thus, since for some $1 \le J_{j'} \le \mathsf{N}_{i,j'}$ we have $q_{i;\mathsf{K}_{i-1},j',\mathsf{M}_{i;\mathsf{K}_{i-1},j'}} = e_{j';J_{j'},J_{j'}}$, it follows from Condition (c) that $Q_{i+1;j',j} = \{f_{j;L_{j',j,J_{j'},t},L_{j',j,J_{j'},t}} \mid 1 \le t \le \mathsf{M}_{i+1;j',j}\}$ for $1 \le j' \le \mathsf{K}_i$; hence, for some $I_j \in \{L_{j',j,J_{j'},t} \mid 1 \le j' \le \mathsf{K}_i,\ 1 \le t \le \mathsf{M}_{i+1;j',j}\}$, we have $W_{i+1;j',j} = \{f_{j;I_j,L_{j',j,J_{j'},t}} \mid 1 \le t \le \mathsf{M}_{i+1;j',j}\}$ and $U_{i+1,j} = \{f_{j;I_j,k} \mid 1 \le k \le \mathsf{N}_{i+1,j}\}$. But Conditions (a) and (b) imply that the set of numbers $\{L_{j',j,k,t} \mid 1 \le j' \le \mathsf{K}_i,\ 1 \le k \le \mathsf{N}_{i,j'},\ 1 \le t \le \mathsf{M}_{i+1;j',j}\}$ exhausts the integers in the interval $[1, \mathsf{N}_{i+1,j}]$. Hence, we can rewrite $U_{i+1,j}$ as

$$U_{i+1,j} = \{f_{j;I_j,L_{j',j,k,t}} \mid 1 \le j' \le \mathsf{K}_i,\ 1 \le k \le \mathsf{N}_{i,j'},\ 1 \le t \le \mathsf{M}_{i+1;j',j}\}.$$

Finally, since $f_{j;I_j,L_{j',j,J_{j'},t}} e_{j';J_{j'},k} = f_{j;I_j,L_{j',j,k,t}}$ for $1 \le j' \le \mathsf{K}_i$, $1 \le k \le \mathsf{N}_{i,j'}$, and $1 \le t \le \mathsf{M}_{i+1;j',j}$, the result follows. $\square$

Let $\{d_1, d_2, \dots\}$ be a subset of self-adjoint generators for D. Then, for each $i \in \mathbb{N}$, by Lemma 3.2.2, there exists a subset $G_i = \{g_{i,j}\}_{1 \le j \le \mathsf{K}_i} \subseteq B$ such that

(G1) $g_{i,j} \in w_{i;1,j,1} B w_{i;1,j,1}$,
(G2) $0 \notin \sigma(g_{i,j})$,
(G3) $d_i \in \mathrm{C}^*(U_{i,1}, \dots, U_{i,\mathsf{K}_i}, G_i)$;

moreover, we may assume (using the functional calculus)

(G4) $\sigma(g_{i,j}) \cap \sigma(g_{i',j'}) = \emptyset$ for $i \ne i'$ or $j \ne j'$,
(G5) $\|g_{i,j}\| \le 2^{-i-j-2}$.

Also, let $\Lambda = \{\lambda_{i;j',j,k} \mid i \in \mathbb{N},\ 1 \le j' \le \mathsf{K}_{i-1},\ 1 \le j \le \mathsf{K}_i,\ 1 \le k \le \mathsf{M}_{i;j',j}\}$ be a set of mutually different positive real numbers such that $\Lambda \cap (\bigcup_{i \in \mathbb{N}} \bigcup_{1 \le j \le \mathsf{K}_i} \sigma(g_{i,j})) = \emptyset$ and

$$\sum_{j'=1}^{\mathsf{K}_{i-1}} \sum_{j=1}^{\mathsf{K}_i} \sum_{k=1}^{\mathsf{M}_{i;j',j}} \lambda_{i;j',j,k} \le 2^{-i-5}, \quad \forall i \in \mathbb{N}.$$

With the necessary ingredients now defined, we claim that a generator for $B(A,D)$ is given by $\mathfrak{G} = \sum_{i\in\mathbb{N}} \mathfrak{G}_i$, where

$$\mathfrak{G}_i = \begin{cases} \sum_{j=1}^{\mathsf{K}_i} \left(g_{i,j} + \sum_{k=2}^{\mathsf{M}_{i;1,j}-1} \lambda_{i;1,j,k} q_{i;1,j,k} + \sum_{k=2}^{\mathsf{M}_{i;1,j}} \lambda_{i;1,j,k} w_{i;1,j,k} \right), & \mathsf{K}_{i-1} = 1 \\ \sum_{j=1}^{\mathsf{K}_i} \left(g_{i,j} + \sum_{k=2}^{\mathsf{M}_{i;1,j}} \lambda_{i;1,j,k} q_{i;1,j,k} + \sum_{k=2}^{\mathsf{M}_{i;1,j}} \lambda_{i;1,j,k} w_{i;1,j,k} \right. & \\ \quad + \sum_{j'=2}^{\mathsf{K}_{i-1}-1} \sum_{k=1}^{\mathsf{M}_{i;j',j}} \left(\lambda_{i;j',j,k} q_{i;j',j,k} + \lambda_{i;j',j,k} w_{i;j',j,k} \right) & \\ \quad \left. + \sum_{k=1}^{\mathsf{M}_{i;\mathsf{K}_{i-1},j}-1} \lambda_{i;\mathsf{K}_{i-1},j,k} q_{i;\mathsf{K}_{i-1},j,k} + \sum_{k=1}^{\mathsf{M}_{i;\mathsf{K}_{i-1},j}} \lambda_{i;\mathsf{K}_{i-1},j,k} w_{i;\mathsf{K}_{i-1},j,k} \right), & \mathsf{K}_{i-1} \neq 1 \end{cases}.$$

It is plain to see

$$\|\mathfrak{G}_i\| < 2^{-i-2} + 8 \sum_{j'=1}^{\mathsf{K}_{i-1}} \sum_{j=1}^{\mathsf{K}_i} \sum_{k=1}^{\mathsf{M}_{i;j',j}} \lambda_{i;j',j,k} \leq 2^{-i-1} \tag{3.7}$$

so that $\mathfrak{G}$ is well-defined.

Before proving that $\mathfrak{G}$ generates $B(A,D)$, we perform some straightforward calculations which will be needed in the proof. In what follows, we drop the indices on the elements of Λ to make our equations more readable. It is immaterial which specific member of Λ is attached to which partial isometry; what is important is that for distinct partial isometries, there are distinct members of Λ attached to each. Nevertheless, for the sake of rigor, we make the following rule. Whenever an expression of the form $a\lambda u_{i;j',j,k} b$ appears below, for $a, b \in B$ and $u_{i;j',j,k} \in Q_{i;j',j} \cup W_{i;j',j}$ (for any $i \in \mathbb{N}$, $1 \leq j' \leq \mathsf{K}_{i-1}$, and $1 \leq j \leq \mathsf{K}_i$), it is to be interpreted as $a\lambda_{i;j',j,k} u_{i;j',j,k} b$ for $\lambda_{i;j',j,k} \in \Lambda$.

Define

$$S := \bigcup_{i\in\mathbb{N}} \bigcup_{j'=1}^{\mathsf{K}_{i-1}} \bigcup_{j=1}^{\mathsf{K}_i} Q_{i;j',j},$$

and let $q_{i;j',j,k}, q_{r;s',s,t} \in S$. We calculate the product $q_{i;j',j,k} q_{r;s',s,t}$ for $i+1 < r$, $i+1 = r$, and $i = r$. Indeed, for $i+1 < r$,

$$q_{i;j',j,k} q_{r;s',s,t} = \begin{cases} q_{r;s',s,t}, & j' = \mathsf{K}_{i-1},\ j = \mathsf{K}_i,\ k = \mathsf{M}_{i;\mathsf{K}_{i-1},\mathsf{K}_i} \\ 0, & \text{otherwise} \end{cases}, \tag{3.8}$$

and

$$q_{i;j',j,k}q_{i+1;s',s,t} = \begin{cases} q_{i+1;s',s,t}, & j' = \mathsf{K}_{i-1},\ j = s',\ k = \mathsf{M}_{i;\mathsf{K}_{i-1},j} \\ 0, & \text{otherwise} \end{cases}, \tag{3.9}$$

$$q_{i;j',j,k}q_{i;s',s,t} = \begin{cases} q_{i;s',s,t}, & j' = s',\ j = s,\ k = t \\ 0, & \text{otherwise} \end{cases}. \tag{3.10}$$

Taking adjoints and relabeling indices yields the products for $i-1=r$ and $i-1>r$.

Now consider the subset of S given by $R = S \setminus \{q_{i;\mathsf{K}_{i-1},j,\mathsf{M}_{i;\mathsf{K}_{i-1},j}} \mid i \in \mathbb{N},\ 1 \le j \le \mathsf{K}_i\}$, and let $q_{r;s',s,t} \in R$. Then

$$q_{r;s',s,t}\mathfrak{G}_i = \begin{cases} \displaystyle\sum_{j=1}^{\mathsf{K}_i}\Bigg(q_{r;s',s,t}q_{i;1,j,1}g_{i,j} \\ \qquad \displaystyle + \sum_{k=2}^{\mathsf{M}_{i;1,j}-1} q_{r;s',s,t}\lambda q_{i;1,j,k} + \sum_{k=2}^{\mathsf{M}_{i;1,j}} q_{r;s',s,t}q_{i;1,j,1}\lambda w_{i;1,j,k}\Bigg), & \mathsf{K}_{i-1} = 1 \\ \displaystyle\sum_{j=1}^{\mathsf{K}_i}\Bigg(q_{r;s',s,t}q_{i;1,j,1}g_{i,j} + \sum_{k=2}^{\mathsf{M}_{i;1,j}} q_{r;s',s,t}\lambda q_{i;1,j,k} + \sum_{k=2}^{\mathsf{M}_{i;1,j}} q_{r;s',s,t}q_{i;1,j,1}\lambda w_{i;1,j,k} \\ \qquad \displaystyle + \sum_{j'=2}^{\mathsf{K}_{i-1}-1}\sum_{k=1}^{\mathsf{M}_{i;j',j}} \left(q_{r;s',s,t}\lambda q_{i;j',j,k} + q_{r;s',s,t}q_{i;1,j,1}\lambda w_{i;j',j,k}\right) \\ \qquad\quad \displaystyle + \sum_{k=1}^{\mathsf{M}_{i;\mathsf{K}_{i-1},j}-1} q_{r;s',s,t}\lambda q_{i;\mathsf{K}_{i-1},j,k} + \sum_{k=1}^{\mathsf{M}_{i;\mathsf{K}_{i-1},j}} q_{r;s',s,t}q_{i;1,j,1}\lambda w_{i;\mathsf{K}_{i-1},j,k}\Bigg), & \mathsf{K}_{i-1} \neq 1 \end{cases}$$

so that, using Equations (3.8)–(3.10), we find $q_{r;s',s,t}\mathfrak{G}_i = 0$ when $i \neq r$ and

$$\begin{aligned} q_{r;s',s,t}\mathfrak{G} &= q_{r;s',s,t}\mathfrak{G}_r \\ &= \begin{cases} \displaystyle g_{r,s} + \sum_{k=2}^{\mathsf{M}_{r;1,s}} \lambda w_{r;1,s,k} \\ \qquad \displaystyle + \sum_{j'=2}^{\mathsf{K}_{r-1}-1}\sum_{k=1}^{\mathsf{M}_{r;j',s}} \lambda w_{r;j',s,k} + (1-\delta_{1,\mathsf{K}_{r-1}}) \sum_{k=1}^{\mathsf{M}_{r;\mathsf{K}_{r-1},s}} \lambda w_{r;\mathsf{K}_{r-1},s,k}, & s' = 1,\ t = 1 \\ \lambda q_{r;s',s,t}, & \text{otherwise} \end{cases}. \end{aligned} \tag{3.11}$$

It follows that

(3.12)

$$q_{r;s',s,t}\mathfrak{G}q_{r;s',s,t} = \begin{cases} g_{r,s}q_{i;1,s,1}q_{r;s',s,t} + \sum_{k=2}^{\mathsf{M}_{r;1,s}} \lambda w_{r;1,s,k}q_{r;1,s,k}q_{r;s',s,t} \\ \quad + \sum_{j'=2}^{\mathsf{K}_{r-1}-1} \sum_{k=1}^{\mathsf{M}_{r;j',s}} \lambda w_{r;j',s,k}q_{r;j',s,k}q_{r;s',s,t} \\ \quad\quad +(1-\delta_{1,\mathsf{K}_{r-1}}) \sum_{k=1}^{\mathsf{M}_{r;\mathsf{K}_{r-1},s}} \lambda w_{r;\mathsf{K}_{r-1},s,k}q_{r;\mathsf{K}_{r-1},s,k}q_{r;s',s,t}, & s'=1,\ t=1 \\ \lambda q_{r;s',s,t}q_{r;s',s,t}, & \text{otherwise} \end{cases}$$

$$= \begin{cases} g_{r,s}, & s'=1,\ t=1 \\ \lambda q_{r;s',s,t}, & \text{otherwise} \end{cases}.$$

Moreover,

$$\mathfrak{G}_i q_{r;s',s,t} = \begin{cases} \sum_{j=1}^{\mathsf{K}_i} \Big(g_{i,j}q_{i;1,j,1}q_{r;s',s,t} \\ \quad + \sum_{k=2}^{\mathsf{M}_{i;1,j}-1} \lambda q_{i;1,j,k}q_{r;s',s,t} + \sum_{k=2}^{\mathsf{M}_{i;1,j}} \lambda w_{i;1,j,k}q_{i;1,j,k}q_{r;s',s,t} \Big), & \mathsf{K}_{i-1}=1 \\ \sum_{j=1}^{\mathsf{K}_i} \Big(g_{i,j}q_{i;1,j,1}q_{r;s',s,t} + \sum_{k=2}^{\mathsf{M}_{i;1,j}} \lambda q_{i;1,j,k}q_{r;s',s,t} + \sum_{k=2}^{\mathsf{M}_{i;1,j}} \lambda w_{i;1,j,k}q_{i;1,j,k}q_{r;s',s,t} \\ \quad + \sum_{j'=2}^{\mathsf{K}_{i-1}-1} \sum_{k=1}^{\mathsf{M}_{i;j',j}} \big(\lambda q_{i;j',j,k}q_{r;s',s,t} + \lambda w_{i;j',j,k}q_{i;j',j,k}q_{r;s',s,t}\big) \\ \quad\quad + \sum_{k=1}^{\mathsf{M}_{i;\mathsf{K}_{i-1},j}-1} \lambda q_{i;\mathsf{K}_{i-1},j,k}q_{r;s',s,t} + \sum_{k=1}^{\mathsf{M}_{i;\mathsf{K}_{i-1},j}} \lambda w_{i;\mathsf{K}_{i-1},j,k}q_{i;\mathsf{K}_{i-1},j,k}q_{r;s',s,t} \Big), & \mathsf{K}_{i-1}\neq 1 \end{cases},$$

so that, using Equations (3.8)–(3.10) again, we find $\mathfrak{G}_i q_{r;s',s,t} = 0$ when $i > r$, $\mathfrak{G}_i q_{r;s',s,t} = \lambda w_{i;\mathsf{K}_{i-1},\mathsf{K}_i,\mathsf{M}_{i;\mathsf{K}_{i-1},\mathsf{K}_i}} q_{r;s',s,t}$ when $i+1<r$, and

$$\mathfrak{G}_{r-1}q_{r;s',s,t} = \lambda w_{r-1;\mathsf{K}_{r-2},s',\mathsf{M}_{r-1;\mathsf{K}_{r-2},s'}}q_{r;s',s,t},$$

$$\mathfrak{G}_r q_{r;s',s,t} = \begin{cases} g_{r,s}, & s'=1,\ t=1 \\ \lambda q_{r;s',s,t} + \lambda w_{r;s',s,t}, & \text{otherwise} \end{cases}.$$

Hence,

$$\begin{aligned}
\mathfrak{G} q_{r;s',s,t} &= \sum_{i=1}^{r} \mathfrak{G}_i q_{r;s',s,t} \\
&= \sum_{i=1}^{r-2} \lambda w_{i;\mathsf{K}_{i-1},\mathsf{K}_i,\mathsf{M}_{i;\mathsf{K}_{i-1},\mathsf{K}_i}} q_{r;s',s,t} + \lambda w_{r-1;\mathsf{K}_{r-2},s',\mathsf{M}_{r-1;\mathsf{K}_{r-2},s'}} q_{r;s',s,t} + \mathfrak{G}_r q_{r;s',s,t} \\
&= \begin{cases} \sum_{i=1}^{r-2} \lambda w_{i;\mathsf{K}_{i-1},\mathsf{K}_i,\mathsf{M}_{i;\mathsf{K}_{i-1},\mathsf{K}_i}} q_{r;1,s,1} & \\ \quad +(1-\delta_{r,1})\lambda w_{r-1;\mathsf{K}_{r-2},1,\mathsf{M}_{r-1;\mathsf{K}_{r-2},1}} q_{r;1,s,1} + g_{r,s}, & s'=1,\, t=1 \\ \sum_{i=1}^{r-2} \lambda w_{i;\mathsf{K}_{i-1},\mathsf{K}_i,\mathsf{M}_{i;\mathsf{K}_{i-1},\mathsf{K}_i}} q_{r;s',s,t} & \\ \quad +(1-\delta_{r,1})\lambda w_{r-1;\mathsf{K}_{r-2},s',\mathsf{M}_{r-1;\mathsf{K}_{r-2},s'}} q_{r;s',s,t} & \\ \quad\quad +\lambda q_{r;s',s,t} + \lambda w_{r;s',s,t}, & \text{otherwise} \end{cases}.
\end{aligned} \tag{3.13}$$

Theorem 3.1. *A C^*-algebra with an AF-action is singly generated.*

Proof. We will prove that $\mathrm{C}^*(\mathfrak{G}) = B(A,D)$. For this, our goal is to show that $R \subseteq \mathrm{C}^*(\mathfrak{G})$. Once this is done, we will be able to use the elements of R to extract the finite-dimensional algebras $\bigoplus_{1\le j\le \mathsf{K}_i} \mathrm{M}_{\mathsf{N}_{i,j}}$ (and hence the AF algebra A) from $\mathrm{C}^*(\mathfrak{G})$ along with the self-adjoint generators $d_1, d_2, \ldots$ of D. Since $B = \mathrm{C}^*(A,D)$, the result will then follow.

Let $\preceq$ denote the lexicographic order on R; to be precise, $q_{i;j',j,k} \preceq q_{r;s',s,t}$ if $i<r$ or if $i=r$ and $j'<s'$ or if $i=r$, $j'=s'$, and $j<s$ or if $i=r$, $j'=s'$, $j=s$, and $k\le t$. Let $p_1 = q_{1;1,1,1}$, and for every $i\in\mathbb{N}$, define $p_{i+1}\in R$ such that $p_{i+1}\preceq q$ for every $q \in R\setminus\{p_1,\ldots,p_i\}$ (roughly speaking, p_{i+1} is the smallest element in R greater than p_i). To show that $R\subseteq \mathrm{C}^*(\mathfrak{G})$, it is sufficient to show that $\mathfrak{G}$ and the sequence $(p_i)_{i\in\mathbb{N}}$ of nonzero mutually orthogonal projections satisfy the hypotheses of Lemma 3.1.1. That $\mathfrak{G}$ and $(p_i)_{i\in\mathbb{N}}$ satisfy Conditions 3 and 4 of Lemma 3.1.1 is clear from the spectral properties of the members of $\bigcup_{r\in\mathbb{N}} G_r$ (in particular, from Conditions G2 and G4), the definition of Λ, and Equation (3.12).

We now show that Condition 1 of Lemma 3.1.1 holds; defining $P_n := \sum_{1\le i\le n} p_i$, we wish to show $(1-P_n)\mathfrak{G}P_n = 0$ for every $n\in\mathbb{N}$. Appealing to Equations (3.12) and (3.13), we have $(1-P_1)\mathfrak{G}P_1 = \mathfrak{G}q_{1;1,1,1} - q_{1;1,1,1}\mathfrak{G}q_{1;1,1,1} = g_{1,1} - g_{1,1} = 0$ so that the desired equality is true for the case $n=1$. Fix $n\in\mathbb{N}$, and suppose $(1-P_n)\mathfrak{G}P_n = 0$. Notice

$$\begin{aligned}
(1-P_{n+1})\mathfrak{G}P_{n+1} &= \big(1-(P_n+p_{n+1})\big)\mathfrak{G}(P_n+p_{n+1}) \\
&= (1-P_n)\mathfrak{G}P_n + \mathfrak{G}p_{n+1} - P_n\mathfrak{G}p_{n+1} - p_{n+1}\mathfrak{G}P_n - p_{n+1}\mathfrak{G}p_{n+1};
\end{aligned}$$

thus, to ensure $(1-P_{n+1})\mathfrak{G}P_{n+1} = 0$, we need

$$\mathfrak{G}p_{n+1} - P_n\mathfrak{G}p_{n+1} - p_{n+1}\mathfrak{G}P_n - p_{n+1}\mathfrak{G}p_{n+1} = 0. \tag{3.14}$$

To that end, assume $p_{n+1} = q_{r;s',s,t}$, and notice from the definition of P_n that

$$P_n p_i = p_i P_n = \begin{cases} p_i, & i \leq n \\ 0, & i > n \end{cases}. \tag{3.15}$$

Hence, appealing to Equation (3.13),

$$P_n \mathfrak{G} p_{n+1} = \begin{cases} \sum_{i=1}^{r-2} P_n q_{i;1,\mathsf{K}_i,1} \lambda w_{i;\mathsf{K}_{i-1},\mathsf{K}_i,\mathsf{M}_{i;\mathsf{K}_{i-1},\mathsf{K}_i}} q_{r;1,s,1} \\ \quad +(1-\delta_{r,1}) P_n q_{r-1;1,1,1} \lambda w_{r-1;\mathsf{K}_{r-2},1,\mathsf{M}_{r-1;\mathsf{K}_{r-2},1}} q_{r;1,s,1} \\ \qquad +P_n q_{r;1,s,1} g_{r,s}, & s'=1,\, t=1 \\ \sum_{i=1}^{r-2} P_n q_{i;1,\mathsf{K}_i,1} \lambda w_{i;\mathsf{K}_{i-1},\mathsf{K}_i,\mathsf{M}_{i;\mathsf{K}_{i-1},\mathsf{K}_i}} q_{r;s',s,t} \\ \quad +(1-\delta_{r,1}) P_n q_{r-1;1,s',1} \lambda w_{r-1;\mathsf{K}_{r-2},s',\mathsf{M}_{r-1;\mathsf{K}_{r-2},s'}} q_{r;s',s,t} \\ \qquad +P_n \lambda q_{r;s',s,t} + P_n q_{r;1,s,1} \lambda w_{r;s',s,t}, & \text{otherwise} \end{cases} \tag{3.16}$$

$$= \begin{cases} \sum_{i=1}^{r-2} \lambda w_{i;\mathsf{K}_{i-1},\mathsf{K}_i,\mathsf{M}_{i;\mathsf{K}_{i-1},\mathsf{K}_i}} q_{r;1,s,1} \\ \quad +(1-\delta_{r,1}) \lambda w_{r-1;\mathsf{K}_{r-2},1,\mathsf{M}_{r-1;\mathsf{K}_{r-2},1}} q_{r;1,s,1}, & s'=1,\, t=1 \\ \sum_{i=1}^{r-2} \lambda w_{i;\mathsf{K}_{i-1},\mathsf{K}_i,\mathsf{M}_{i;\mathsf{K}_{i-1},\mathsf{K}_i}} q_{r;s',s,t} \\ \quad +(1-\delta_{r,1}) \lambda w_{r-1;\mathsf{K}_{r-2},s',\mathsf{M}_{r-1;\mathsf{K}_{r-2},s'}} q_{r;s',s,t} + \lambda w_{r;s',s,t}, & \text{otherwise} \end{cases},$$

and appealing to Equation (3.11),

$$p_{n+1} \mathfrak{G} P_n = \begin{cases} g_{r,s} q_{r;1,s,1} P_n + \sum_{k=2}^{\mathsf{M}_{r;1,s}} \lambda w_{r;1,s,k} q_{r;1,s,k} P_n \\ \quad + \sum_{j'=2}^{\mathsf{K}_{r-1}-1} \sum_{k=1}^{\mathsf{M}_{r;j',s}} \lambda w_{r;j',s,k} q_{r;j',s,k} P_n \\ \qquad +(1-\delta_{1,\mathsf{K}_{r-1}}) \sum_{k=1}^{\mathsf{M}_{r;\mathsf{K}_{r-1},s}} \lambda w_{r;\mathsf{K}_{r-1},s,k} q_{r;\mathsf{K}_{r-1},s,k} P_n, & s'=1,\, t=1 \\ \lambda q_{r;s',s,t} P_n, & \text{otherwise} \end{cases} \tag{3.17}$$

$$= 0.$$

Thus, we see from Equations (3.13), (3.16), (3.17), and (3.12) that in fact Equation (3.14) holds; that is, $\mathfrak{G}$ and $(p_i)_{i\in\mathbb{N}}$ satisfy Condition 1 of Lemma 3.1.1.

Finally, to see Condition 2 of Lemma 3.1.1 holds for $\mathfrak{G}$ and $(p_i)_{i\in\mathbb{N}}$, we wish to show

$$\lim_{n\to\infty} \|(1-P_n)\mathfrak{G}(1-P_n)\| = \lim_{n\to\infty} \|(1-P_n)\mathfrak{G} - (1-P_n)\mathfrak{G}P_n\| = \lim_{n\to\infty} \|\mathfrak{G} - P_n\mathfrak{G}\| = 0,$$

where the second equality follows from what we just proved in the previous paragraph. Fix $n \in \mathbb{N}$, and assume $p_{n+1} = q_{r;s',s,t}$ again. Notice

$$P_n\mathfrak{G}_i = \begin{cases} \sum_{j=1}^{\mathsf{K}_i}\left(P_n q_{i;1,j,1}g_{i,j} + \sum_{k=2}^{\mathsf{M}_{i;1,j}-1} P_n\lambda q_{i;1,j,k} + \sum_{k=2}^{\mathsf{M}_{i;1,j}} P_n q_{i;1,j,1}\lambda w_{i;1,j,k}\right), & \mathsf{K}_{i-1} = 1 \\ \sum_{j=1}^{\mathsf{K}_i}\Bigg(P_n q_{i;1,j,1}g_{i,j} + \sum_{k=2}^{\mathsf{M}_{i;1,j}} P_n\lambda q_{i;1,j,k} + \sum_{k=2}^{\mathsf{M}_{i;1,j}} P_n q_{i;1,j,1}\lambda w_{i;1,j,k} & \\ \quad + \sum_{j'=2}^{\mathsf{K}_{i-1}-1}\sum_{k=1}^{\mathsf{M}_{i;j',j}}\left(P_n\lambda q_{i;j',j,k} + P_n q_{i;1,j,1}\lambda w_{i;j',j,k}\right) & \\ \quad + \sum_{k=1}^{\mathsf{M}_{i;\mathsf{K}_{i-1},j}-1} P_n\lambda q_{i;\mathsf{K}_{i-1},j,k} + \sum_{k=1}^{\mathsf{M}_{i;\mathsf{K}_{i-1},j}} P_n q_{i;1,j,1}\lambda w_{i;\mathsf{K}_{i-1},j,k}\Bigg), & \mathsf{K}_{i-1} \neq 1 \end{cases}.$$

That is, $P_n\mathfrak{G}_i$ is a sum of terms of the form P_nqb for $q \in (\bigcup_{1\le j'\le \mathsf{K}_{i-1}}\bigcup_{1\le j\le \mathsf{K}_i} Q_{i;j',j}) \cap R$ and $b \in B$; in particular, by Equation (3.15), $P_nqb = 0$ for $p_{n+1} \preceq q$ and $P_nqb = qb$ otherwise. It follows that

$$P_n\mathfrak{G}_i = \begin{cases} \mathfrak{G}_i, & 1 \le i < r \\ 0, & i > r \end{cases},$$

and subsequently, that

$$\|\mathfrak{G} - P_n\mathfrak{G}\| = \left\|\sum_{i=r+1}^{\infty}\mathfrak{G}_i + \mathfrak{G}_r - P_n\mathfrak{G}_r\right\| < \sum_{i=r+1}^{\infty} 2^{-i-1} + \|\mathfrak{G}_r - P_n\mathfrak{G}_r\| < \sum_{i=r+1}^{\infty} 2^{-i-1} + 2^{-r-1}.$$

Noticing that as n goes to infinity so does r, Condition 2 of Lemma 3.1.1 follows. We conclude that $R \subseteq \mathrm{C}^*(\mathfrak{G})$.

Now, notice $W_{r;s',s} \subseteq \mathrm{C}^*(\mathfrak{G})$ for every $r \in \mathbb{N}$, $1 \le s' \le \mathsf{K}_{r-1}$ and $1 \le s \le \mathsf{K}_r$. Indeed, for any $r \in \mathbb{N}$ and $1 \le s \le \mathsf{K}_r$, $w_{r;1,s,1} \in \mathrm{C}^*(\mathfrak{G})$ since $w_{r;1,s,1} = q_{r;1,s,1} \in R$; moreover, from Equation (3.11)

$$\frac{1}{\lambda_{r;1,s,k}}(q_{r;1,s,1}\mathfrak{G} - g_{r,s})q_{r;1,s,k} = w_{r;1,s,k} \in \mathrm{C}^*(\mathfrak{G}), \quad 2 \le k \le \mathsf{M}_{r;1,s},$$

$$\frac{1}{\lambda_{r;j',s,k}}(q_{r;1,s,1}\mathfrak{G} - g_{r,s})q_{r;j',s,k} = w_{r;j',s,k} \in \mathrm{C}^*(\mathfrak{G}), \quad 1 < j' < \mathsf{K}_{r-1},\ 1 \le k \le \mathsf{M}_{r;j',s},$$

$$\frac{1}{\lambda_{r;\mathsf{K}_{r-1},s,k}}(q_{r;1,s,1}\mathfrak{G} - g_{r,s})q_{r;\mathsf{K}_{r-1},s,k} = w_{r;\mathsf{K}_{r-1},s,k} \in \mathrm{C}^*(\mathfrak{G}), \quad 1 \le k < \mathsf{M}_{r;\mathsf{K}_{r-1},s};$$

also,

$$\frac{1}{\lambda_{r;\mathsf{K}_{r-1},s,\mathsf{M}_{r;\mathsf{K}_{r-1},s}}}\Big(q_{r;1,s,1}\mathfrak{G} - g_{r,s} - \sum_{k=2}^{\mathsf{M}_{r;1,s}} \lambda w_{r;1,s,k} - \sum_{j'=2}^{\mathsf{K}_{r-1}-1}\sum_{k=1}^{\mathsf{M}_{r;j',s}} \lambda w_{r;j',s,k} - (1-\delta_{1,\mathsf{K}_{r-1}}) \sum_{k=1}^{\mathsf{M}_{r;\mathsf{K}_{r-1},s}-1} \lambda w_{r;\mathsf{K}_{r-1},s,k}\Big) = w_{r;\mathsf{K}_{r-1},s,\mathsf{M}_{r;\mathsf{K}_{r-1},s}} \in \mathrm{C}^*(\mathfrak{G}).$$

Since $U_{1,j} = W_{1;1,j}$ for each $1 \leq j \leq \mathsf{K}_1$, we see $\bigoplus_{1\leq j\leq \mathsf{K}_1} \mathrm{M}_{\mathsf{N}_{1,j}} \subseteq \mathrm{C}^*(\mathfrak{G})$ (see the discussion following Lemma 3.2.2); hence, by Lemma 3.3.1, $\bigoplus_{1\leq j\leq \mathsf{K}_i} \mathrm{M}_{\mathsf{N}_{i,j}} \subseteq \mathrm{C}^*(\mathfrak{G})$ for each $i \in \mathbb{N}$, and we see $A \subseteq \mathrm{C}^*(\mathfrak{G})$. Furthermore, it is clear from Equation (3.12) that $G_i \subseteq \mathrm{C}^*(\mathfrak{G})$ for each $i \in \mathbb{N}$; but $d_i \in \mathrm{C}^*(\bigoplus_{1\leq j\leq \mathsf{K}_i} \mathrm{M}_{\mathsf{N}_{i,j}}, G_i)$ by Condition G3 so that $\{d_1, d_2, \dots\}$ and hence D is contained in $\mathrm{C}^*(\mathfrak{G})$. □

The following corollaries now follow from the discussion at the end of Section 2.

Corollary 3.1. *A simple AH algebra with diagonal maps is singly generated.*

Corollary 3.2. *A Villadsen algebra is singly generated.*

Corollary 3.3. *Let $B = B(A, D)$ have an AF-action, and let C be a separable unital simple C*-algebrabe. Then $B \otimes C$ is singly generated. In particular, if B is a Villadsen algebra, then $B \otimes C$ is singly generated.*

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