C^*-algebras of Artin-Tits monoids

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Session on C^*-algebras, dynamical systems and applications,
Joint mathematics meetings, Denver, Colorado, January 2019.

S - set, $M = (m_{s,t}) \in M_{S\times S}(\mathbb{N} \cup \{\infty\})$

$m_{s,s} = 1$, $m_{s,t} > 1$ if $s \neq t$, $m_{s,t} = m_{t,s}$

Artin-Tits group $\Gamma = \langle S \mid \underbrace{stst\cdots}_{m_{s,t}} = \underbrace{tsts\cdots}_{m_{t,s}} \rangle$

Artin-Tits monoid $P = \langle S \mid \underbrace{stst\cdots}_{m_{s,t}} = \underbrace{tsts\cdots}_{m_{t,s}} \rangle^{+}$

Coxeter (reflection) group $W = \langle S \mid \underbrace{stst\cdots}_{m_{s,t}} = \underbrace{tsts\cdots}_{m_{t,s}},\ s^2 = 1 \rangle$

- Γ (and P) are *irreducible* if cannot write $S = S_1 \sqcup S_2$ so that $m_{s,t} = 2$ whenever $s \in S_1$ and $t \in S_2$.

- Γ (and P) are of *finite type* if W is finite.

  (Alternatively, *right-angled* if $m_{s,t} \in \{2, \infty\}$ for all s, t)
  (studied by Crisp-Laca)

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Examples

1. $D_{2m} = \langle a, b \mid \underbrace{aba\cdots}_{m} = \underbrace{bab\cdots}_{m} \rangle$ "*dihedral type*",

   $M = \begin{pmatrix} 1 & m \\ m & 1 \end{pmatrix}$ (these are the examples with two generators).

   W is the dihedral group of order $2m$.

2. $B_4 = \langle a, b, c \mid aba = bab, bcb = cbc, ac = ca \rangle$

"*braid group on 4 strands*", $M = \begin{pmatrix} 1 & 3 & 2 \\ 3 & 1 & 3 \\ 2 & 3 & 1 \end{pmatrix}$

   $W = S_4$, the symmetric group.

Semigroup C^*-algebras

Let P be a monoid. Assume that P is

- *left cancellative*: $\alpha\beta = \alpha\gamma \Rightarrow \beta = \gamma$.
- *LCM*: $\alpha P \cap \beta P \neq \varnothing \Rightarrow \exists\gamma$ s.t. $\alpha P \cap \beta P = \gamma P$.

  (The intersection of two principal right ideals is either empty, or is another principal right ideal.)

$\lambda : P \to B(\ell^2 P)$, the left regular representation: $\lambda_\alpha(e_\beta) = e_{\alpha\beta}$.

(λ_α is an isometry by left cancellativity)

Define $C_r^*(P) := C^*(\lambda(P))$.

Abstract description

$[\alpha] := \{\beta : \alpha \in \beta P\}$ - "prefixes" of α. A subset $x \subseteq P$ is

hereditary if $\alpha \in x \Rightarrow [\alpha] \subseteq x$

directed if $\alpha, \beta \in x \Rightarrow \alpha P \cap \beta P \cap x \neq \varnothing$

$\Omega =$ set of directed hereditary subsets of P.

Topology on Ω: let $Z(\alpha) := \{x \in \Omega : \alpha \in x\}$.

$\{Z(\alpha) \setminus \cup_1^n Z(\beta_i) : \alpha, \beta_i \in P\}$ is a base of compact-open sets for a compact Hausdorff topology on Ω.

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Action by partial homeomorphisms: $P \curvearrowright \Omega$ by $\alpha \cdot Z(\beta) = Z(\alpha\beta)$.

$C^*(P) := C(\Omega) \rtimes P$ (use groupoid, partial crossed product, etc.)

Laca-Crisp showed that for right-angled Artin groups, $C^*(P)$ behaves roughly like (Toeplitz-) Cuntz-Krieger algebras (e.g. free groups).

They point out that for Artin-Tits groups of finite type, $C^*(P)$ will usually not be nuclear.

We investigate the structure of $C^*(P)$ in that case.

Let P be an Artin-Tits monoid of finite type.

1. $Z(\alpha) \setminus \cup_{\sigma \in S} Z(\alpha\sigma) = \{[\alpha]\}$.

   $\overline{P} := \{[\alpha] : \alpha \in P\}$ is a discrete open invariant subset of Ω.

   P acts freely and transitively on $\overline{P}$, so $\mathcal{K}(\ell^2 P) \lhd C^*(P)$.

2. P is a *lattice*: $\forall \alpha, \beta \in P$, $\exists \gamma \in P$ s.t. $\alpha P \cap \beta P = \gamma P$ $(\gamma =: \alpha \vee \beta)$.

   P is directed, so $P \in \Omega$. Write $\infty := P \in \Omega$ $(\partial P = \{\infty\})$.
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   $$0 \to I \to C^*(P) \to C^*(\Gamma) \to 0, \qquad I = C^*(P|_{\Omega \setminus \{\infty\}})$$

   $$0 \to \mathcal{K} \to I \to I/\mathcal{K} \to 0, \qquad I/\mathcal{K} = C^*(P|_{\Omega \setminus (\overline{P} \cup \{\infty\})})$$

We study $I/\mathcal{K}$.

Normal forms

$\pi : P \twoheadrightarrow W$, $P_{\text{red}} := \{\alpha \in P : \ell(\alpha) = \ell(\pi(\alpha))\}$.

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For $\alpha \in P_{\mathrm{red}}$, put

$L(\alpha) = \{\sigma \in S : \alpha \in \sigma P\}$, *"left initial letters"* of α

$R(\alpha) = \{\sigma \in S : \alpha \in P\sigma\}$, *"right initial letters"* of α

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$\Delta := \bigvee P_{\text{red}}$ - the unique maximal element of P_{red}

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For $\alpha \in P \setminus \{1\}$, a *normal form* is $(\alpha_1, \ldots, \alpha_n)$ with

- $\alpha_i \in P_{\text{red}} \setminus \{1\}$, $1 \leq i \leq n$
- $\alpha = \alpha_1 \cdots \alpha_n$
- $R(\alpha_i) \supseteq L(\alpha_{i+1})$, $1 \leq i < n$

Every $\alpha \in P$ has a unique normal form.

Infinite Word: $(\alpha_1, \alpha_2, \ldots)$ s.t. $\forall n$, $(\alpha_1, \ldots, \alpha_n)$ is normal.

Theorem There is a bijection from the set of infinite words to $\Omega \setminus \overline{P}$ given by

$$(\alpha_1, \alpha_2, \ldots) \mapsto \bigcup_{n=1}^{\infty} [\alpha_1 \cdots \alpha_n].$$

In particular, $(\Delta, \Delta, \ldots) \mapsto \infty$

Let $X_k = \{x \in \Omega \setminus \overline{P} : \Delta^k \in x,\ \Delta^{k+1} \notin x\}$
$= \{(\alpha_1, \alpha_2, \ldots) : \alpha_i = \Delta \text{ iff } i \leq k\}$.

Put $X = \bigsqcup_{n=0}^{\infty} X_n$. Then $I/\mathcal{K} = C^*(P|_X)$.

X is invariant and relatively open in $\Omega \setminus \overline{P}$. X_0 is a relatively closed transversal in X. Then $I/\mathcal{K} \sim C^*(P|_{X_0})$ (Muhly-Renault-Williams).

X_0 is the space of *reduced infinite words* (i.e. not containing Δ).

Example D_{2m}

$$P_{\text{red}} = \{1, a, ab, aba, \ldots, \underbrace{aba\cdots}_{m-1}, b, ba, bab, \ldots, \underbrace{bab\cdots}_{m-1}, \Delta\}$$

$L(aba\cdots) = \{a\}$, $L(bab\cdots) = \{b\}$

$C^*(P|_{X_0})$ is a Cuntz-Krieger algebra, I is nuclear.

Example Γ with more than two generators (e.g. B_4)

Theorem $I/\mathcal{K} \sim C^*(P|_{X_0})$ is simple and purely infinite, but not nuclear.

Ideas of the proof: the restricted action $P \curvearrowright X_0$ is

- *minimal*: $\forall \varnothing \neq U \subseteq X_0$ open, $\forall x \in X_0$, $\exists \alpha \in P$, $\alpha x \in U$.
- *top. principal:* $\{x \in X_0 : \alpha x = x \Rightarrow \alpha = 1\}$ is dense in X_0.
- *loc. contractive:* $\forall \varnothing \neq U \subseteq X_0$ open, $\exists V \subseteq U$ open, $\exists \alpha \in P$, $\alpha \overline{V} \subsetneq V$.

A necessary move in this direction is

Lemma Let $U = Z(\alpha) \setminus \bigcup_1^n Z(\beta_i)$, suppose $U \cap X_0 \neq \varnothing$. There exists $\delta \in P \setminus \Delta P$ s.t. $Z(\delta) \subseteq U$.

(Technical) Lemma Suppose Γ is irreducible. For all s, $t \in S$ there is a normal form $(\gamma_1, \ldots, \gamma_n)$ s.t. $L(\gamma_1) = \{s\}$ and $R(\gamma_n) = S \setminus \{t\}$.

$\gamma = \gamma_1 \cdots \gamma_n$ is a *spacer*.

E.g. let $\varepsilon \in P \setminus \Delta P$ and $x = (\mu_1, \mu_2, \ldots) \in X_0$.

ε has normal form $(\varepsilon_1, \ldots, \varepsilon_m)$.

Choose $s \in R(\varepsilon_m)$ and $t \in S \setminus L(\alpha_1)$.

Let $\gamma = (\gamma_1, \ldots, \gamma_n)$ be as in the lemma. Then

$$\varepsilon\gamma x = (\varepsilon_1, \ldots, \varepsilon_m, \gamma_1, \ldots, \gamma_n, \mu_1, \mu_2, \ldots) \in Z(\varepsilon).$$

With a bit more trickery, can choose $\delta = \varepsilon\gamma$ so that $Z(\delta) \cap X_0 \subseteq U$ in the lemma.