Decomposing the wavelet representation for crystallographic groups

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The wavelet representation generated by dilation and translation:

Consider the well–known *translation* and *dilation* unitary operators T and D on $L^2(\mathbb{R})$ defined by:

$$T(f)(t) = f(t-1),\ f \ \in L^2(\mathbb{R})$$

$$D(f)(t) = \sqrt{2}f(2t),\ f \in L^2(\mathbb{R}),$$

and note they satisfy

$$T = DT^2D^{-1}.$$

(Note we can also define unitary translation operators $T^{\beta}(f)(x) = f(z-\beta)$ for any $\beta \in \mathbb{R}$ and dilation operator $D_{\alpha}f(x) = \sqrt{\alpha}f(\alpha \cdot x)$ for any $\alpha > 0$.)

Definition of 2–wavelet: We say that $\psi \in L^2(\mathbb{R})$ is a wavelet for dilation by 2 if

$$\{D^nT^m(\psi) :\ n, m \ \in \mathbb{Z}\}$$

is an orthonormal basis for $L^2(\mathbb{R})$.

The wavelet representation

Definition 1

Let $\mathbb{Z}[\frac{1}{2}] = \{\frac{j}{2^k} : j, k \in \mathbb{Z}\}$ denote the additive subgroup of $\mathbb{Q}$ given the discrete topology. Let $\theta : \mathbb{Z} \to \mathrm{Aut}(\mathbb{Z}[\frac{1}{2}])$ be defined by

$$\theta(n)(\frac{j}{2^k}) = \frac{1}{2^n} \cdot \frac{j}{2^k}.$$

We then form the semidirect product $\mathbb{Z}[\frac{1}{2}] \rtimes_\theta \mathbb{Z}$.
The homomorphism W from $\mathbb{Z}[\frac{1}{2}] \rtimes_\theta \mathbb{Z}$ into $\mathcal{U}(L^2(\mathbb{R}))$ given by

$$(\beta, m) \mapsto W(\beta, m) := T^\beta D^m$$

is a unitary representation of $Z[\frac{1}{2}] \rtimes_\theta \mathbb{Z}$ on $L^2(\mathbb{R})$, sometimes called the "wavelet representation" for dilation by 2.

Wavelet sets and the wavelet representation

In 1994, X. Dai and D. Larson analyzed wavelets in the frequency domain and developed the definition of **wavelet sets** [DL].
The form of the Fourier transform that we use is given by

$$\mathcal{F}(f)(\xi) \;=\; \int_{\mathbb{R}} f(t)e^{-2\pi it\xi}dt,\; f \in L^1 \cap L^2(\mathbb{R}).$$

Definition 2
(Dai–Larson[DL]) A measurable subset $E \subset \mathbb{R}$ is said to be a wavelet set of $\mathbb{R}$ for dilation by $d \geq 2$ if the function

$$\mathcal{F}^{-1}(\mathbf{1_E})(t)$$

is a **wavelet** for dilation by d in $L^2(\mathbb{R})$.

For example, $E = [-1, -\frac{1}{2}) \cup [\frac{1}{2}, 1)$ is a wavelet set for dilation by 2.

Example of a wavelet set

Example 3

Consider $E = [-1, -\frac{1}{2}) \cup [\frac{1}{2}, 1)$. Then it is known that E is a wavelet set for dilation by 2 and

$$\mathcal{F}^{-1}(\mathbf{1_E})(t) = \psi_{LP}(t),$$

the Littlewood–Paley wavelet.

Theorem 4 (DL)

(1994) Let $E \subset \mathbb{R}$. Then E is a wavelet set for dilation by 2 if and only if E tiles the real line both by **dilation by** 2 *and by* **translation by the integers.** .

Decomposing the wavelet representation as a direct integral

In 2001, L.–H. Lim, P. and K. Taylor decomposed the wavelet representation of the discrete wavelet group into a direct integral of irreducible monomial representations, using wavelet sets [LPT].

Using the wavelet set constructions of Dai-Larson-Speegle [DLS], L.-P.-T. generalized this result to to the wavelet representation on $L^2(\mathbb{R}^n)$ associated to an arbitrary $n \times n$ integral dilation matrix.

Wavelets with crystal groups as translations

In 2011, J. MacArthur and K. Taylor, building on work of K. Guo, D. Labate, W.-Q. Lim, G. Weiss and E. Wilson [GLLWW], took the point of view that it might be useful to consider area–preserving matrices as generalized *translations*. They constructed both the definition and analog of Haar wavelets for some crystallographic or "wallpaper" groups with specified dilations [MT]. MacArthur constructed analogs of Haar wavelets for all 17 crystal groups in his thesis with specific compatible dilations. These Haar wavelets are constructed using the concept of multiresolution analysis.

K. Merrill, on the other hand, worked on constructing wavelet set wavelets for crystallographic groups.

Crystallographic groups

Definition 5 (Ta)

A wallpaper group Γ is a discrete subgroup of the group of rigid motions on $\mathbb{R}^2$, $\mathrm{Iso}(\mathbb{R}^2)$, such that $\mathbb{R}^2/\Gamma$ is compact, having the translation subgroup isomorphic to $\mathbb{Z}^2$ as a normal subgroup of **finite** index in Γ. We write
$\mathrm{Aff}(\mathbb{R}^2) = \{[y, M] : x \in \mathbb{R}^2, M \in GL(2, \mathbb{R})\}$, and
$\mathrm{Iso}(\mathbb{R}^2) = \{[x, L] : x \in \mathbb{R}^2, L \in O(2)\}$, and note $\mathrm{Iso}(\mathbb{R}^2) \subset \mathrm{Aff}(\mathbb{R}^2)$, where following K. Taylor [Ta], the group operations in the affine group are given by

$$[x, L] \cdot [y, M] = [M^{-1}x + y, LM], \quad \text{and} \quad [x, L]^{-1} = [-Lx, L^{-1}].$$

The group Γ is a discrete subgroup of $\mathrm{Iso}(\mathbb{R}^2)$, so acts as a group of measure-preserving transformations on $\mathbb{R}^2$, determining a unitary representation T on $L^2(\mathbb{R}^2)$:
$[x, L] \cdot z = L(z + x)$ and $T[x, L](f)(z) = f(L^{-1}z - x)$, $f \in L^2(\mathbb{R}^2)$.

Wallpaper groups

Let us write $\mathcal{N}$ for the normal subgroup of Γ isomorphic to the group of translations $\mathbb{Z}^2$. Then $\mathcal{D} = \Gamma/\mathcal{N}$ is called the point group and is finite. Wallpaper groups are special cases of crystallographic groups in dimension 2.

Although wallpaper groups are subgroups of $\mathrm{Iso}(\mathbb{R}^2)$ which is a semidirect product, Γ need **NOT** be of the form $\mathcal{N} \rtimes \mathcal{D}$. This is due to glide reflections, to be described shortly. Groups that are semidirect products are called symmorphic (13 out of 17 wallpaper groups). Dilations act on $\mathbb{R}^2$ as well, and if you conjugate a wallpaper group Γ by a positive integer dilation matrix, you do not necessarily obtain a subgroup of Γ. That is, not all dilations are *compatible*.

Compatible dilations for wallpaper groups

We say more on compatible dilations for wallpaper groups Γ. Let $A \in GL(2, \mathbb{R})$. We say that A is **compatible for** Γ if $[0, A]\Gamma[0, A]^{-1}$ is a subgroup of Γ.
A calculation shows that for $d \in \mathbb{N},\ d \geq 2,\ A = d \cdot \mathrm{id}$ is compatible with each symmorphic crystal group. However in order for A to be compatible with the nonsymmorphic groups as well, d must be odd.
We will mainly concentrate on $d = 3,\ A = 3 \cdot \mathrm{id}$.
Via conjugation, compatible dilations give an increasing nested bisequence of subgroups and supergroups of Γ, i.e.

$$\Gamma_\ell := [0, A]^\ell \Gamma [0, A]^{-\ell} \subset \Gamma_{\ell+1} := [0, A]^{(\ell+1)} \Gamma [0, A]^{-(\ell+1)},\ \forall \ell \in \mathbb{Z}.$$

Note $\Gamma_0 = \Gamma$.

Wavelets for wallpaper groups in $L^2(\mathbb{R}^2)$

Viewing the crystallographic groups as the analog of "translations", we obtain MacArthur's and Taylor's definition of wallpaper wavelet in $L^2(\mathbb{R}^2)$.

Definition 6 (MT)

Let Γ be a wallpaper group in $\mathrm{Iso}(\mathbb{R}^2)$ and let D be the dilation operator on $L^2(\mathbb{R})$ corresponding the the compatible dilation matrix $A = d \cdot \mathrm{Id}$. An $d\Gamma$ wavelet is a function $\psi \in L^2(\mathbb{R}^2)$ satisfying

$$\{D^k T[x, L](\psi) : \ k \in \mathbb{Z}, \ [x, L] \in \Gamma\}$$

is an orthonormal basis for $L^2(\mathbb{R}^2)$.

Here

$$D(f)(x, y) \ = \ df(dx, dy), \ f \in L^2(\mathbb{R}^2).$$

K. Merrill's generalization of wavelet sets to wallpaper groups

Definition 7 (M)

Let Γ be a wallpaper group, and fix $d \in \mathbb{N}$, $d \geq 2$. A measurable set $E \subset \mathbb{R}^2$ is a Γ-**wavelet set relative to dilation by** d if the characteristic function $\mathbf{1}_E$ is the Fourier transform of a function $\psi \in L^2(\mathbb{R}^2)$ that is a wavelet for "translation" by Γ and dilation by D.

Wavelet sets for wallpaper groups, continued

By analyzing the action of the wallpaper group and the dilation in the frequency domain, Merrill proved the following analog of Theorem 4 of Dai and Larson:

Theorem 8 (K. Merrill, 2018)

Let Γ *be a wallpaper group parametrized by* $\Gamma = \{(n + c_S, S) : n \in N \cong \mathbb{Z}^2; S \in \mathcal{D}\}$, *where the point group* $\mathcal{D}$ *is a finite subgroup of* $O(2)$, *and* $c_S \in \mathbb{R}^2$ *is a constant vector for each* $S \in \mathcal{D}$, *with* $c_{Id} = \vec{0}$. *Then* E *is a* Γ*-wavelet set for dilation by* $D = d \cdot Id$ *if and only if all of the following hold:*

(i) E *tiles* $\mathbb{R}^2$ *under translation by the lattice* $\mathbb{Z}^2$.

(ii) $S(E) \cap S'(E) = \emptyset$, *unless* $S = S' \in \mathcal{D}$.

(iii) $\cup_{S \in \mathcal{D}} S(E)$ *tiles* $\mathbb{R}^2$ *under dilation by* D.

Examples of wavelets sets for specific crystallographic groups

Merrill used the theorem to construct examples of wavelet sets for every crystal group with respect to any integer dilation. In particular, she constructed wavelet sets for every crystal group corresponding to dilation by 3:

Example 9

Consider the symmorphic wallpaper group

$$\Gamma = pm = \{[n, \mathrm{Id}], [m, \sigma_y] : m, n \in \mathbb{Z}^2\}.$$

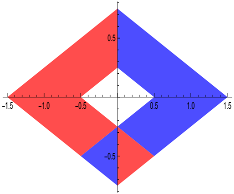
Let

$$\Gamma' = pg = \{[n, \mathrm{Id}], [m + (0, \frac{1}{2}), \sigma_y] : m, n \in \mathbb{Z}^2\}$$

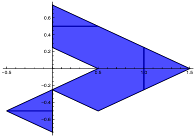
denote the non–symmorphic wallpaper group, where here σ_y is reflection in the y-axis and $C_{\sigma_y} = (0, \frac{1}{2})$. Here $\mathcal{D} = \{\mathrm{Id}, \sigma_y\}$ for both pg and pm. Merrill showed that pg and pm can be given the same wavelet sets.

Wavelet set for *pg* and *pm*:

To find a wavelet set for the nonsymmorphic group *pg*, Merrill first looked and found a set $E \sqcup \sigma_y(E)$ that tiles the plane by dilation by 3, and tiles $\mathbb{R}^2$ twice under translation by $\mathbb{Z}^2$. This is pictured below. (The red and blue colors indicate how she subdivided it later.)



The blue portion is a wavelet set for *pg* :



The wallpaper wavelet group

L. Baggett, K. Merrill, K. Taylor and I conjectured that for compatible dilations, there was an analog of the wavelet group in the wallpaper group case, and that Merrill's wavelet sets could be used to study the corresponding representation in a fashion similar to [LPT].

For $A = d\mathrm{Id},\ d \geq 3,\ d$ odd, set $\Gamma_A = \cup_{\ell\in\mathbb{Z}}\Gamma_\ell\ = \cup_{\ell\in\mathbb{Z}}[0,A]^\ell\Gamma[0,A]^{-\ell}$. Then Γ_A is a countable subgroup of $\mathrm{Aff}(\mathbb{R}^n)$ such that, if $[x,L]\in\Gamma_A$, then $L\in\mathcal{D}$. There are two subsets of Γ_A that are of particular interest to us. Let $\mathcal{N}_A = \mathrm{Trans}(\mathbb{R}^2)\cap\Gamma_A$, the pure translations in Γ_A. Notice that $\mathcal{N}_A$ is a normal subgroup of Γ_A. One can show $\Gamma_A/\mathcal{N}_A \cong \mathcal{D}$. Thus Γ_A is a type I group.

The wallpaper wavelet group, continued

Conjugation by $[0, A]$ for compatible $A = d\mathrm{Id}$ generates an action ϑ of $\mathbb{Z}$ on Γ_A. We then form the semi-direct product group

$$\Gamma_A \rtimes_\vartheta \mathbb{Z} = \{\big([x, L], \ell\big) : [x, L] \in \Gamma, \ell \in \mathbb{Z}\},$$

equipped with group product

$$\big([x, L], \ell\big)\big([y, M], m\big) = \big([x, L](\vartheta_\ell[y, M]), \ell + m\big)$$

$$= \big([M^{-1}x + A^{-\ell}y, LM], \ell + m\big).$$

We denote the semidirect product $\Gamma_A \rtimes_\vartheta \mathbb{Z}$ by $W_{\Gamma,A}$ and call it the wallpaper wavelet group. It is the desired generalization of the wavelet group for ordinary dilations and translations.

More about the wallpaper wavelet group

Identify Γ_A with $\{([x, L], 0) : [x, L] \in \Gamma_A\}$, a normal subgroup of $W_{\Gamma,A} = \Gamma_A \rtimes_\vartheta \mathbb{Z}$. Identify $\mathcal{N}_A$ with its copy inside $W_{\Gamma,A} = \Gamma_A \rtimes_\vartheta \mathbb{Z}$.

Calculations show that $\mathcal{N}_A$ is a normal subgroup of $W_{\Gamma,A}$ and $W_{\Gamma,A}/\mathcal{N}_A$ is isomorphic to $\mathcal{D} \rtimes \mathbb{Z}$.

The following observation is stated as a lemma for future reference:

Lemma 10

For $([x, L], \ell) \in W_{\Gamma,A} = \Gamma_A \rtimes_\vartheta \mathbb{Z}$, we can view

$$([x, L], \ell) \in W_{\Gamma,A}$$

as the element $[A^\ell x, LA^{-\ell}]$ of the affine group. Therefore $W_{\Gamma,A}$ embeds as a subgroup of $Aff(\mathbb{R}^2)$.

The wavelet representation for wallpaper groups

Using the faithful representation of the affine group on the Hilbert space $L^2(\mathbb{R}^2)$, and restricting this to $W_{\Gamma,A}$ gives what we call the wallpaper wavelet representation. Extending notation used before, for $[x, C] \in \mathrm{Aff}(\mathbb{R}^2)$, let $T[x, C]$ be the unitary operator on $L^2(\mathbb{R}^2)$ defined by

$$T[x, C](g)(y) = |\det(C)|^{-1/2} g(C^{-1}y - x),$$

for all $y \in \mathbb{R}^2, g \in L^2(\mathbb{R}^2)$. Then T is a unitary representation of the affine group.

The wallpaper wavelet representation, cont.

We now note if $([x,L],\ell) \in W_{\Gamma,A} = \Gamma_A \rtimes_\vartheta \mathbb{Z}$, we have

$$W\left(([x,L],\ell)\right) g(y) = T[A^\ell x, LA^{-\ell}](g)(y)$$

$$= |\det(A)|^{\ell/2} g\left(A^\ell L^{-1} y - A^\ell x\right) = T[x,L] D_{A^\ell}(g)(y),$$

for all $y \in \mathbb{R}^2$, $g \in L^2(\mathbb{R}^2)$, where $D_A g(y) = |\det(A)|^{1/2}(g)(Ay)$.

Definition 11

The representation W of $W_{\Gamma,A} = \Gamma_A \rtimes_\vartheta \mathbb{Z}$ on $L^2(\mathbb{R}^2)$ defined above for compatible A is called the *$A\Gamma$-wavelet representation*. The remainder of the talk will be spent analyzing this representation of $W_{\Gamma,3\mathrm{Id}}$.

Taking the Fourier transform of the 3Γ-wavelet representation

As before, we want to consider the equivalent representation $\widehat{W} = \mathcal{F} \circ W \circ \mathcal{F}^{-1}$, where $\mathcal{F}$ is the Fourier transform on $L^2(\mathbb{R}^2)$:

$$\widehat{f}(\xi) = [\mathcal{F}(f)](\xi) = \int f(y)e^{-2\pi i\langle \xi, y\rangle}\, dy,\ f \in L^1(\mathbb{R}^2),$$

and then let

$$\widetilde{W} = \rho \circ \widehat{W} \circ \rho^{-1},$$

where ρ is a unitary map between the Hilbert spaces $L^2(\mathbb{R}^2)$ and $L^2(E \times \mathcal{D} \times \mathbb{Z})$, for E a Γ wavelet set for dilation by 3Id. We will decompose the representation $\widetilde{W}$ as a direct integral.

Rewriting $\widehat{W}$ as $\widetilde{W}$

Each nonsymmorphic group Γ contains an index 2 normal symmorphic subgroup we call Γ^0. Let $\mathcal{D}_0 = \{L : \exists x \in \mathbb{R}^2 \text{ with } [x, L] \in \Gamma^0\}$. For all four nonsymmorphic groups, $\mathcal{D}$ is a semi–direct product of $\mathcal{D}_0$ by $\mathbb{Z}_2$.

Let E be a Γ-wavelet set for dilation by 3Id. Define a map ρ from $L^2(\mathbb{R}^2)$ into the space of square-integrable functions on $E \times \mathcal{D} \times \mathbb{Z}$ by:

$$\rho(\phi)(\omega, M, j) = \begin{cases} 3^j e^{\frac{\pi i \langle v, \omega \rangle}{2}} \cdot \phi(3^j M(\omega)) & \text{if } M \in \mathcal{D}^0 \\ 3^j e^{\frac{-\pi i \langle v, \omega \rangle}{2}} \cdot \phi(3^j M(\omega)) & \text{if } M \notin \mathcal{D}^0 \end{cases},$$

where $v = (0, 1)$ if $\Gamma =$ pg or pmg2, and $v = (1, 1)$ if $\Gamma =$ pgg2 or p4g.

Rewriting $\widehat{W}$ as $\widetilde{W}$, continued

One obtains the formula for ρ^{-1} by taking a square integrable function on $L^2(E \times \mathcal{D} \times \mathbb{Z})$, considering its value as restricted to $E \times \{M_0\} \times \{j_0\}$, and then transferring them to obtain function values on $3^{j_0} M_0(E)$, and then pasting all these pieces together to obtain a function defined on all of $\mathbb{R}^2$.

Then if $\widetilde{W}$ is defined as $\rho \circ \widehat{W} \circ \rho^{-1}$, one computes:

$$[\widetilde{W}([x, L], \ell)(f)](\omega, M, j)$$
$$= \begin{cases} e^{-2\pi i \langle x, 3^j L^{-1} M(\omega) \rangle} f(\omega, L^{-1}M, j - \ell) & L \in \mathcal{D}_0, \\ e^{-\pi i \langle v, \omega \rangle} e^{-2\pi i \langle x, 3^j L^{-1} M(\omega) \rangle} f(\omega, L^{-1}M, j - \ell)] & L \notin \mathcal{D}_0, M \in \mathcal{D}_0 \\ e^{\pi i \langle v, \omega \rangle} e^{-2\pi i \langle x, 3^j L^{-1} M(\omega) \rangle} f(\omega, L^{-1}M, j - \ell) & L \notin \mathcal{D}_0, M \notin \mathcal{D}_0 \end{cases}$$

Inducing one-dimensional representations from the translation subgroup

We want to compare $\widetilde{W}$ to a direct integral of representations induced from characters of $\mathcal{N}_{3\mathrm{Id}} = \{[\frac{1}{3^j}k, \mathrm{Id}] : k \in \mathbb{Z}^2, j \in \mathbb{Z}\}$. For $\omega \in E$, let χ^ω be the character of $\mathcal{N}_{3\mathrm{Id}}$ defined by

$$\chi^\omega([x, \mathrm{Id}]) = e^{2\pi i \langle \omega, x \rangle}.$$

We use left cosets $(\Gamma_{3\mathrm{Id}} \rtimes_\vartheta \mathbb{Z})/\mathcal{N}_{3\mathrm{Id}}$. We let the wavelet group $W_{\Gamma,3\mathrm{Id}}$ act on these cosets on the left by multiplication by g^{-1}. The cocycle used in the inducing formula is $\gamma(s)^{-1} g \gamma(g \cdot s) \in \mathcal{N}_{3\mathrm{Id}}$, where s is a left coset, g is in the larger group, and γ is a cross section. These cosets can be parametrized by (M, j) for $M \in \mathcal{D}$ and $j \in \mathbb{Z}$.

The formula for monomial induced representations

For a specific formula for cross–section γ, we then calculate:

$$[Ind_{\mathcal{N}_{3ld}}^{\Gamma_{3ld}\rtimes_\vartheta\mathbb{Z}}\chi^\omega]((([x,L],\ell))f((M,j))$$
$$=\chi^\omega\left(\gamma((M,j))^{-1}([x,L],\ell)\gamma((L^{-1}M,j-\ell))\right)f((L^{-1}M,j-\ell))$$
$$=\begin{cases} e^{-2\pi i\langle\omega,3^jM^{-1}Lx\rangle}f((L^{-1}M,j-\ell)) & L\in\mathcal{D}_0\\ e^{-2\pi i\langle\omega,3^jM^{-1}Lx+\frac{1}{2}v\rangle}f((L^{-1}M,j-\ell)) & L\notin\mathcal{D}_0, M\in\mathcal{D}_0\\ e^{-2\pi i\langle\omega,3^jM^{-1}Lx-\frac{1}{2}v\rangle}f((L^{-1}M,j-\ell)) & L\notin\mathcal{D}_0, M\notin\mathcal{D}_0\end{cases}$$
$$=\begin{cases} e^{-2\pi i\langle 3^jL^{-1}M\omega,x\rangle}f((L^{-1}M,j-\ell)) & L\in\mathcal{D}_0\\ e^{-\pi i\langle\omega,v\rangle}e^{-2\pi i\langle 3^jL^{-1}M\omega,x\rangle}f((L^{-1}M,j-\ell)) & L\notin\mathcal{D}_0, M\in\mathcal{D}_0\\ e^{\pi i\langle\omega,v\rangle}e^{-2\pi i\langle 3^jL^{-1}M\omega,x\rangle}f((L^{-1}M,j-\ell)) & L\notin\mathcal{D}_0, M\notin\mathcal{D}_0\end{cases}$$

Here we used that $M,\ L\in O(2)$ so that their inverses are equal to their transposes. Also, $[Ind_{\mathcal{N}_{3ld}}^{W_{\Gamma,3ld}}\chi^\omega]$ is irreducible, for each $\omega\in E$.

A decomposition of the wallpaper wavelet representation

We now compare the direct integral of induced representations and the wavelet representation W for dilation by 3 :

Theorem 12

Let Γ be a wallpaper group in $Iso(\mathbb{R}^2)$, and let $W_{\Gamma,3Id} = \Gamma_{3Id} \rtimes_\vartheta \mathbb{Z}$ be the associated wavelet group. Then the wavelet representation W of $W_{\Gamma,3Id}$ is unitarily equivalent to the direct integral of monomial irreps

$$\int_E^{\oplus} [Ind_{\mathcal{N}_{3Id}}^{W_{\Gamma,3Id}} \chi^\omega] d\omega$$

for E a Γ–wavelet set for dilation by 3 of Merrill type.

Proof: When we take the direct integral of $[Ind_{\mathcal{N}_{\mathbf{3Id}}}^{W_{\Gamma,\mathbf{3Id}}} \chi^\omega]([x, L], \ell)$ over $\omega \in E$ it is evident that we obtain exactly $\widetilde{W}([x, L], \ell)$.

Weakly equivalent representations of $W_{\Gamma,\mathbf{3Id}}$

Just as was done earlier in [LPT], the above theorem can be used to obtain a result about weak equivalence of two representations of $C^*(W_{\Gamma,3\text{Id}}) = C^*(\Gamma_{3\text{Id}} \rtimes_\vartheta \mathbb{Z})$. Note that Theorem 12 shows the wavelet representation W of $C^*(\Gamma_{3\text{Id}} \rtimes_\vartheta \mathbb{Z})$ is Type I, while the regular representation π of $C^*(\Gamma_{3\text{Id}} \rtimes_\vartheta \mathbb{Z})$is not Type I. Our Theorem and the results of [LPT] thus imply:

Theorem 13

Let Γ be a wallpaper group and consider the associated wavelet group $\Gamma_{3Id} \rtimes_\vartheta \mathbb{Z}$. Then W and π are weakly equivalent, but not equivalent, representations of $C^*(\Gamma_{3Id} \rtimes_\vartheta \mathbb{Z})$.

Proof: As in [LPT], by using induction in stages, one can decompose the regular representation π as a direct integral of monomial representations over $\widehat{\mathcal{N}_{3\text{Id}}}$ (which is a solenoid). The result on weak equivalence then follows from using a discussion similar to that given in [LPT].

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