Inductive limits of C-algebras and compact quantum metric spaces*

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Main Question

Let $A = \overline{\cup_{n\in\mathbb{N}} A_n}^{\|\cdot\|_A}$ be a unital C*-algebra, where for each $n \in \mathbb{N}$ A_n is a unital C*-subalgebra and $A_n \subseteq A_{n+1}$.

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For each $n \in \mathbb{N}$, assume that (A_n, L_n) is a *compact quantum metric space* in the sense of Rieffel.

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A staple of Noncommutative Metric Geometry is the existence of noncommutative analogues of the Gromov-Hausdorff distance. The first one was introduced by Rieffel, and later came distances introduced by D. Kerr, H. Li, F. Latrémolière, and others, each of which have their advantages.

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In this talk, we will consider the *dual Gromov-Hausdorff propinquity* Λ^*, a quantum distance of Latrémolière, which is a complete metric on the quantum isometry classes of certain compact quantum metric spaces.

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and (A_n, L_n) is a compact quantum metric space.

Main question: What conditions allow us to build a quantum metric L on A from the quantum metrics on all the (A_n, L_n) such that

$$\lim_{n\to\infty} \Lambda^*((A_n, \mathsf{L}_n), (A, \mathsf{L})) = 0?$$

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Approach: The idea is to insist that $((A_n, \mathsf{L}_n))_{n\in\mathbb{N}}$ is a *Cauchy* sequence in Λ^* with further conditions such that the unital C*-algebra F given by completeness of propinquity (F, L_F) is *-isomorphic to A.

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   - Some AF algebras as quasi-Leibniz spaces
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   - All unital AF algebras

Compact Metric Spaces

Definition (Monge-Kantorovich Metric)

Let (X, d) be a compact metric space. The Lipschitz seminorm on $C(X)$ is:

$$\mathsf{L}_d(f) = \sup\{|f(x) - f(y)|/\mathsf{d}(x,y) : x \neq y \in X\}.$$

The Monge-Kantorovich metric on $\mathscr{S}(C(X))$ is:

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If (X, d) is a compact metric space, then L_d is lower semicontinuous, $\mathsf{L}_d^{-1}([0,\infty))$ is dense, and $\mathsf{L}_d^{-1}(\{0\}) = \mathbb{C}1_{C(X)}$. Furthermore:

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2. $\mathsf{mk}_{\mathsf{L}_d}$ metrizes the *weak* topology* on $\mathscr{S}(C(X))$,
3. $\mathsf{L}_d(fg) \leqslant \|f\|_{C(X)}\mathsf{L}_d(g) + \mathsf{L}_d(f)\|g\|_{C(X)}$ for all $f, g \in C(X)$.

Definition (Rieffel, 1998)

A pair $(\mathfrak{A}, \mathsf{L})$ of a unital C*-algebra $\mathfrak{A}$ and a lower semicontinuous seminorm $\mathsf{L} : \mathfrak{sa}(\mathfrak{A}) \to [0,\infty]$, where $\mathrm{dom}(\mathsf{L}) = \{a \in \mathfrak{sa}(\mathfrak{A}) : \mathsf{L}(a) < \infty\}$ is dense in $\mathfrak{sa}(\mathfrak{A})$, is a *compact quantum metric space* if:

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$$\mathsf{L}(a) = \mathsf{L}_{\mathsf{mk}_{\mathsf{L}}}(\hat{a}) = \sup_{\phi, \psi \in \mathscr{S}(\mathfrak{A}), \phi \neq \psi} \frac{|\hat{a}(\phi) - \hat{a}(\psi)|}{\mathsf{mk}_{\mathsf{L}}(\phi, \psi)},$$

where $\hat{a} \in C(\mathscr{S}(\mathfrak{A}))$ is defined by $\hat{a}(\phi) = \phi(a)$ for all $\phi \in \mathscr{S}(\mathfrak{A})$.

(C,D)-quasi-Leibniz Compact Quantum Metric Spaces

Definition (Latrémolière, 2013, 2014)

A (C,D)-quasi-Leibniz Compact Quantum Metric Spaces, for some $C \geqslant 1$ and $D \geqslant 0$ is a Compact Quantum Metric Space $(\mathfrak{A}, \mathsf{L})$ such that $\mathrm{dom}(\mathsf{L})$ is a Jordan-Lie subalgebra of $\mathfrak{sa}(\mathfrak{A})$ and for all $a, b \in \mathrm{dom}(\mathsf{L})$:

$$\mathsf{L}\left(\frac{ab+ba}{2}\right) \leqslant C\left(\|a\|_{\mathfrak{A}}\mathsf{L}(b) + \|b\|_{\mathfrak{A}}\mathsf{L}(a)\right) + D\mathsf{L}(a)\mathsf{L}(b),$$

and

$$\mathsf{L}\left(\frac{ab-ba}{2i}\right) \leqslant C\left(\|a\|_{\mathfrak{A}}\mathsf{L}(b) + \|b\|_{\mathfrak{A}}\mathsf{L}(a)\right) + D\mathsf{L}(a)\mathsf{L}(b).$$

Quantum Isometry

Definition

Two (C,D)-quasi-Leibniz Compact Quantum Metric Spaces $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ are *quantum isometric* if there exists a *-isomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ whose dual map $\pi^* : \mathfrak{B}^* \to \mathfrak{A}^*$ is an isometry from from $(\mathscr{S}(\mathfrak{B}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{B}}})$ into $(\mathscr{S}(\mathfrak{A}), \mathsf{mk}_{\mathsf{L}_{\mathfrak{A}}})$.

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Theorem (Rieffel, 2000)

Two (C,D)-quasi-Leibniz Compact Quantum Metric Spaces $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ and $(\mathfrak{B}, \mathsf{L}_{\mathfrak{B}})$ are *quantum isometric* if and only if there exists a *-isomorphism $\pi : \mathfrak{A} \rightarrow \mathfrak{B}$ such that $\mathsf{L}_{\mathfrak{B}} \circ \pi = \mathsf{L}_{\mathfrak{A}}$.

The Dual Gromov-Hausdorff Propinquity

Definition (Latrémolière, 2013, 2014)

The *dual propinquity* $\Lambda^*_{(C,D)}((\mathfrak{A}, \mathsf{L}_\mathfrak{A}), (\mathfrak{B}, \mathsf{L}_\mathfrak{B}))$ between two (C, D)-quasi-Leibniz Compact Quantum Metric Spaces induces a complete metric on quantum isometry classes of (C, D)-quasi-Leibniz Compact Quantum Metric Spaces.

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Furthermore, if X, Y are compact metric spaces and L_X and L_Y are Lipschitz seminorms, then:

$$\Lambda^*_{(C,D)}((C(X), \mathsf{L}_X), (C(Y), \mathsf{L}_Y)) \leqslant \mathrm{GH}(X, Y).$$

Moreover, *the dual propinquity induces the same topology as the Gromov-Hausdorff distance topology on isometry classes of compact metric spaces.*

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Theorem (A-Latrémolière, 2015)

Let $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$ be a unital AF algebra with $\mathfrak{A}_0 = \mathbb{C}1_{\mathfrak{A}}$ endowed with a *faithful tracial state μ* and set $\mathscr{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$. For all $n \in \mathbb{N}$, let

$$E_n : \mathfrak{A} \to \mathfrak{A}_n$$

be the unique μ-preserving conditional expectation of $\mathfrak{A}$ onto $\mathfrak{A}_n$.

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$$\mathsf{L}^{\beta}_{\mathscr{U},\mu}(a) = \sup\left\{\frac{\|a - E_n(a)\|_{\mathfrak{A}}}{\beta(n)} : n \in \mathbb{N}\right\}$$

then $\left(\mathfrak{A}, \mathsf{L}^{\beta}_{\mathscr{U},\mu}\right)$ is a $(2,0)$–quasi-Leibniz quantum compact metric space,

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then $\left(\mathfrak{A}, \mathsf{L}^{\beta}_{\mathcal{U},\mu}\right)$ is a $(2,0)$–quasi-Leibniz quantum compact metric space, and for all $n \in \mathbb{N}$:

$$\Lambda^*_{(2,0)}\left(\left(\mathfrak{A}_n, \mathsf{L}^{\beta}_{\mathcal{U},\mu}\right), \left(\mathfrak{A}, \mathsf{L}^{\beta}_{\mathcal{U},\mu}\right)\right) \leqslant \beta(n)$$

and thus $\lim_{n\to\infty} \Lambda^*_{(2,0)}\left(\left(\mathfrak{A}_n, \mathsf{L}^{\beta}_{\mathcal{U},\mu}\right), \left(\mathfrak{A}, \mathsf{L}^{\beta}_{\mathcal{U},\mu}\right)\right) = 0.$

Definition (A, 2018)

Fix $C \geqslant 1, D \geqslant 0$. Let $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$ be a unital inductive limit of C*-algebras. Let $\left((\mathfrak{A}_n, \mathsf{L}_{\mathfrak{A}_n})\right)_{n\in\mathbb{N}}$ be a sequence of (C,D)-quasi-Leibniz compact quantum metric spaces. We the call the inductive limit $\mathfrak{A}$ an $\left((\mathfrak{A}_n, \mathsf{L}_{\mathfrak{A}_n})\right)_{n\in\mathbb{N}}$*-propinquity approximable* inductive limit if the following hold for each $n \in \mathbb{N}$:

1. if $a \in \mathfrak{sa}(\mathfrak{A}_n)$, then $\mathsf{L}_{\mathfrak{A}_{n+1}}(a) \leqslant \mathsf{L}_{\mathfrak{A}_n}(a)$, and
2. there exists a sequence $(\beta(j))_{j\in\mathbb{N}} \subset (0,\infty)$ such that $\sum_{j=0}^{\infty}\beta(j) < \infty$,
3. for all $a \in \mathfrak{sa}(\mathfrak{A}_{n+1})$, $\mathsf{L}_{\mathfrak{A}_{n+1}}(a) \leqslant 1$ there exists $b \in \mathfrak{sa}(\mathfrak{A}_n), \mathsf{L}_{\mathfrak{A}_n}(b) \leqslant 1$ such that $\|a-b\|_{\mathfrak{A}} \leqslant \beta(n)$.

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We note that the above implies for each $n \in \mathbb{N}$ that

$$\Lambda^*_{(C,D)}\left((\mathfrak{A}_n, \mathsf{L}_{\mathfrak{A}_n}), (\mathfrak{A}_{n+1}, \mathsf{L}_{\mathfrak{A}_{n+1}})\right) \leqslant 4\beta(n),$$

which provides a Cauchy sequence in $\Lambda^*_{(C,D)}$.

Convergence of Inductive sequence

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Fix $C \geqslant 1, D \geqslant 0$. Let $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$ be a unital inductive limit of C*-algebras.

If $\mathfrak{A}$ is $\left((\mathfrak{A}_n, \mathsf{L}_{\mathfrak{A}_n})\right)_{n\in\mathbb{N}}$-propinquity approximable for some sequence of (C,D)-quasi-Leibniz compact quantum metric spaces and summable $(\beta(j))_{j\in\mathbb{N}} \subset (0,\infty)$, then *there exists a (C,D)-quasi-Leibniz Lip-norm $\mathsf{L}_{\mathfrak{A}}$ on $\mathfrak{A}$* such that $\cup_{n\in\mathbb{N}}\mathrm{dom}(\mathsf{L}_{\mathfrak{A}_n}) \subseteq \mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$ with for each $n \in \mathbb{N}$

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and thus $\quad \lim_{n\to\infty} \Lambda^*_{(C,D)}\left((\mathfrak{A}_n, \mathsf{L}_{\mathfrak{A}_n}), (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})\right) = 0.$

The proof uses the explicit construction of the limit of a Cauchy sequence in dual propinquity due to Latrémolière.

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If for each $n \in \mathbb{N}\setminus\{0\}$, we define for all $a \in \mathfrak{A}_n$,

$$\mathsf{L}^{\beta}_{\mathfrak{A}_n,E}(a) = \max_{m\in\{0,\ldots,n-1\}} \left\{ \frac{\|a - E_{m+1,m}\circ E_{n-1,n-2}\circ\cdots\circ E_{n,n-1}(a)\|_{\mathfrak{A}}}{\beta(m)} \right\}$$

and $\mathsf{L}^{\beta}_{\mathfrak{A}_0,E}$ is the 0-seminorm on $\mathfrak{A}_0$, then for each $n \in \mathbb{N}$, the pair $(\mathfrak{A}_n, \mathsf{L}^{\beta}_{\mathfrak{A}_n,E})$ is a $(2,0)$-quasi-Leibniz comact quantum metric space.

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If for each $n \in \mathbb{N}$, τ_{n+1} is a faithful tracial state on $\mathfrak{A}_{n+1}$, then $\mathfrak{A}$ is $((\mathfrak{A}_n, \mathsf{L}^{\beta}_{\mathfrak{A}_n,E}))_{n\in\mathbb{N}}$-propinquity approximable with $(2,0)$-quasi-Leibniz Lip-norm $\mathsf{L}^{\beta}_{\mathscr{U},E}$ on $\mathfrak{A}$ and

$$\Lambda^*_{(2,0)}((\mathfrak{A}_n, \mathsf{L}^{\beta}_{\mathfrak{A}_n,E}), (\mathfrak{A}, \mathsf{L}^{\beta}_{\mathscr{U},E})) \leq 4\sum_{j=n}^{\infty}\beta(j) \quad \text{for each } n \in \mathbb{N}$$

and therefore

$$\lim_{n\to\infty}\Lambda^*_{(2,0)}((\mathfrak{A}_n, \mathsf{L}^{\beta}_{\mathfrak{A}_n,E}), (\mathfrak{A}, \mathsf{L}^{\beta}_{\mathscr{U},E})) = 0.$$

The above theorem is true for any unital AF algebra including the *unitization of the compact operators* since every finite-dimensional C*-algebra has a faithful tracial state.

Theorem (A, 2018)

Let $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$ be a unital AF algebra such that $\dim(\mathfrak{A}_n) < \infty$ for each $n \in \mathbb{N}$ and $\mathfrak{A}_0 = \mathbb{C}1_{\mathfrak{A}}$ equipped with a faithful tracial state μ. Denote $\mathcal{U} = (\mathfrak{A}_n)_{n\in\mathbb{N}}$, and let $(\beta(j))_{j\in\mathbb{N}} \subset (0,\infty)$ be summable. Let

$$\mathsf{L}^{\beta}_{\mathcal{U},\mu}(a) = \sup_{n\in\mathbb{N}} \frac{\|a - E_n(a)\|_{\mathfrak{A}}}{\beta(n)}$$

be the (2,0)-quasi-Leibniz Lip-norm on $\mathfrak{A}$, where $E_n : \mathfrak{A} \to \mathfrak{A}_n$ is the unique μ-preserving conditional expectation onto $\mathfrak{A}_n$.
If for each $n \in \mathbb{N}$, we define $E_{n+1,n} := E_n|_{\mathfrak{A}_{n+1}} : \mathfrak{A}_{n+1} \to \mathfrak{A}_n$ and let $\mathsf{L}^{\beta}_{\mathfrak{A}_n,E}, \mathsf{L}^{\beta}_{\mathcal{U},E}$ be the associated (2,0)-quasi-Leibniz Lip-norms on $\mathfrak{A}_n, \mathfrak{A}$, respectively, then

$$\Lambda^*_{(2,0)}\left(\left(\mathfrak{A}, \mathsf{L}^{\beta}_{\mathcal{U},\mu}\right), \left(\mathfrak{A}, \mathsf{L}^{\beta}_{\mathcal{U},E}\right)\right) = 0,$$

That is, there exists a quantum isometry from $\left(\mathfrak{A}, \mathsf{L}^{\beta}_{\mathcal{U},E}\right)$ onto $\left(\mathfrak{A}, \mathsf{L}^{\beta}_{\mathcal{U},\mu}\right)$.

Thank you!

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Convergence of Inductive sequence

Theorem (A, 2018)

Fix $C \geqslant 1, D \geqslant 0$. Let $\mathfrak{A} = \overline{\cup_{n\in\mathbb{N}}\mathfrak{A}_n}^{\|\cdot\|_{\mathfrak{A}}}$ be a unital inductive limit of C*-algebras.

If $\mathfrak{A}$ is $\left((\mathfrak{A}_n, \mathsf{L}_{\mathfrak{A}_n})\right)_{n\in\mathbb{N}}$-propinquity approximable for some sequence of (C,D)-quasi-Leibniz compact quantum metric spaces and summable $(\beta(j))_{j\in\mathbb{N}} \subset (0,\infty)$, then there exists a (C,D)-quasi-Leibniz Lip-norm $\mathsf{L}_{\mathfrak{A}}$ on $\mathfrak{A}$ such that $\cup_{n\in\mathbb{N}}\mathrm{dom}(\mathsf{L}_{\mathfrak{A}_n}) \subseteq \mathrm{dom}(\mathsf{L}_{\mathfrak{A}})$ with for each $n \in \mathbb{N}$

$$\Lambda^*_{(C,D)}\left((\mathfrak{A}_n, \mathsf{L}_{\mathfrak{A}_n}), (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})\right) \leqslant 4\sum_{j=n}^{\infty}\beta(j)$$

and thus $\lim_{n\to\infty}\Lambda^*_{(C,D)}\left((\mathfrak{A}_n, \mathsf{L}_{\mathfrak{A}_n}), (\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})\right) = 0.$

$\mathfrak{D} = \prod_{n\in\mathbb{N}} (\mathfrak{A}_n \oplus \mathfrak{A}_{n+1})$ (bounded sequences).

$\mathfrak{K}_0 = \left\{((a_n^n, a_{n+1}^n))_{n\in\mathbb{N}} \in \mathfrak{sa}\left(\prod_{n\in\mathbb{N}} \mathfrak{A}_n \oplus \mathfrak{A}_{n+1}\right) \mid (\forall n \in \mathbb{N})\, a_{n+1}^n = a_{n+1}^{n+1}\right\}$

$S_0(d) = \sup_{n\in\mathbb{N}} \left\{\max\left\{\mathsf{L}_{\mathfrak{A}_n}\left(a_n^n\right), \frac{\|a_n^n - a_{n+1}^{n+1}\|_{\mathfrak{A}}}{2\beta(n)}\right\}\right\}$ for all $d \in \mathfrak{K}_0$.

$\mathfrak{L}_0 = \{d = (d_n)_{n\in\mathbb{N}} \in \mathfrak{K}_0 \mid S_0(d) < \infty\}$

$\mathfrak{S}_0 = \{d = (d_n)_{n\in\mathbb{N}} \in \mathfrak{D} \mid \mathfrak{Re}(d), \mathfrak{Im}(d) \in \mathfrak{L}_0\}$ and $\mathfrak{G}_0 := \overline{\mathfrak{S}_0}^{\|\cdot\|_{\mathfrak{D}}}$

$\mathfrak{I}_0 = \left\{(d_n)_{n\in\mathbb{N}} \in \mathfrak{G}_0 \mid \lim_{n\to\infty} \|d_n\|_{\mathfrak{D}_n} = 0\right\}$

$\mathfrak{F} = \mathfrak{G}_0/\mathfrak{I}_0$ is a *unital C*-algebra*.

$\mathsf{L}_{\mathfrak{F}}(a) = \inf\{S_0(d) : d \in \mathfrak{sa}(\mathfrak{G}_0)$ and $q(d) = a\}$ is a *(C, D)-quasi-Leibniz Lip-norm* on $\mathfrak{F}$.

$\exists$ *-isomorphism $\phi : \mathfrak{A} \to \mathfrak{F}$ and define $\mathsf{L}_{\mathfrak{A}} = \mathsf{L}_{\mathfrak{F}} \circ \phi$, and $(\mathfrak{A}, \mathsf{L}_{\mathfrak{A}})$ is a quasi-Liebniz compact quantum metric space.