Cocycles on Deaconu-Renault Groupoids and KMS States for Generalized Gauge Dynamics

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Motivation

Let (A, α) be an $\mathbb{R}$-dynamical system. Let us recall for each $\beta \in \mathbb{R}$ that a KMS$_\beta$-state for (A, α) is a state ω on A such that

$$\omega(ab) = \omega(b\alpha_{i\beta}(a))$$

for all (A, α)-analytic elements a, b of A.

The concept of KMS states originates from statistical mechanics, and has been applied in the context of rank-1 Deaconu-Renault groupoids ([Kumjian & Renault, 2006]) and higher-rank graphs ([An Huef et al, 2015] and [Farsi et al, 2018]).

The main tool used in each of the aforementioned papers is some suitably-generalized version of the Perron-Frobenius Theorem, but there are some limitations.

The commuting version of the Perron-Frobenius Theorem used in [An Huef et al, 2015] handles higher-rank graphs but treats only very specific gauge dynamics.

The Ruelle-Perron-Frobenius Theorem used in [Kumjian & Renault, 2006] is capable of handling more general gauge dynamics but only for rank-1 Deaconu-Renault groupoids.

In this talk, I shall present a new idea that at once solves these problems by exploiting more thoroughly the properties of Ruelle transfer operators.

Notation and Conventions

Throughout this talk, the following notation and conventions shall be adopted:

- $\mathbb{N}$ denotes the set of positive integers.
- $\mathbb{N}_0$ denotes the set of non-negative integers.
- k is a fixed positive integer, and $[k] \stackrel{\text{df}}{=} \mathbb{N}_{\leq k}$.
- For each $\boldsymbol{n} \in \mathbb{N}_0^k$, let $\overline{\boldsymbol{n}} \stackrel{\text{df}}{=} \left(\underbrace{1, \ldots, 1}_{\boldsymbol{n}(1)}, \ldots, \underbrace{k, \ldots, k}_{\boldsymbol{n}(k)} \right)$.
- For $\mathcal{G}$ an étale locally compact Hausdorff groupoid, $\iota_{\mathcal{G}} : \mathrm{C_c}(\mathcal{G}) \hookrightarrow C^*(\mathcal{G})$ denotes the canonical dense $*$-algebraic embedding of $\mathrm{C_c}(\mathcal{G})$ into $C^*(\mathcal{G})$.

Definition (Deaconu-Renault Groupoid)

Let X be a locally compact Hausdorff space and $\sigma = (\sigma_i)_{i \in [k]}$ a commuting k-family of local homeomorphisms on X. The Deaconu-Renault groupoid of (X, σ) is then defined as the étale locally compact Hausdorff groupoid $\mathcal{G}(X, \sigma)$ with the following properties:

- $\mathcal{G}(X, \sigma)^{(0)} = \{(x, \mathbf{0}, x) \mid x \in X\}$.
- $\mathcal{G}(X, \sigma) = \left\{ (x, \boldsymbol{l}, y) \in X \times \mathbb{Z}^k \times X \;\middle|\; \exists \boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}_0^k : \begin{array}{l} \boldsymbol{l} = \boldsymbol{m} - \boldsymbol{n} \text{ and} \\ \sigma^{\boldsymbol{m}}(x) = \sigma^{\boldsymbol{n}}(y) \end{array} \right\}$.
- If $(x, \boldsymbol{l}, y) \in \mathcal{G}(X, \sigma)$, then $s(x, \boldsymbol{l}, y) = (y, \mathbf{0}, y)$ and $r(x, \boldsymbol{l}, y) = (x, \mathbf{0}, x)$.
- If $(x, \boldsymbol{k}, y), (y, \boldsymbol{l}, z) \in \mathcal{G}(X, \sigma)$, then $(x, \boldsymbol{k}, y) \circ (y, \boldsymbol{l}, z) = (x, \boldsymbol{k} + \boldsymbol{l}, z)$.
- If $(x, \boldsymbol{l}, y) \in \mathcal{G}(X, \sigma)$, then $(x, \boldsymbol{l}, y)^{-1} = (y, -\boldsymbol{l}, x)$.

The topology on $\mathcal{G}(X, \sigma)$ is generated by basic open subsets of the form

$$\mathcal{Z}(U, V, \boldsymbol{m}, \boldsymbol{n}) \stackrel{\text{df}}{=} \left\{ (x, \boldsymbol{m} - \boldsymbol{n}, y) \in X \times \mathbb{Z}^k \times X \;\middle|\; x \in U,\; y \in V,\; \sigma^{\boldsymbol{m}}(x) = \sigma^{\boldsymbol{n}}(y) \right\},$$

where U and V are open subsets of X, and $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}_0^k$.

The positive integer k is called the rank of the Deaconu-Renault groupoid.

Definition (Ruelle Transfer Operator)

Let the following objects be given:

- A non-empty compact metrizable space X.
- A surjective local homeomorphism σ on X.
- $\varphi \in \mathrm{C}(X, \mathbb{R})$.

The Ruelle transfer operator $R^X_{\sigma,\varphi}$ is then defined as a bounded linear operator on the Banach space $\mathrm{C}(X, \mathbb{R})$ by

$$\forall f \in \mathrm{C}(X,\mathbb{R}): \qquad R^X_{\sigma,\varphi}(f) \stackrel{\mathrm{df}}{=} \left\{ \begin{matrix} X & \to & \mathbb{R} \\ x & \mapsto & \displaystyle\sum_{y \in \sigma^{-1}[\{x\}]} e^{\varphi(y)} f(y) \end{matrix} \right\}.$$

Ruelle transfer operators play an important role in statistical mechanics.

They have been used in [Renault, 2003] and [Kumjian & Renault, 2006] to study KMS states associated to a surjective, positively expansive, and exact local homeomorphism on a compact metrizable space.

Definition (One-Cocycle on a Groupoid)

Let $\mathcal{G}$ be a groupoid and H a group. An H-valued one-cocycle on $\mathcal{G}$ is then a function $c : \mathcal{G} \to H$ such that

$$\forall \gamma_1, \gamma_2 \in \mathcal{G} : \qquad (\gamma_1, \gamma_2) \in \mathcal{G}^{(2)} \quad \Longrightarrow \quad c(\gamma_1 \gamma_2) = c(\gamma_1) c(\gamma_2),$$

which necessarily implies that $c\left(\gamma^{-1}\right) = c(\gamma)^{-1}$ for all $\gamma \in \mathcal{G}$.

If $\mathcal{G}$ is a topological groupoid, and H a topological group, then we denote the set of all continuous H-valued one-cocycles on $\mathcal{G}$ by $\mathcal{Z}^1_{\text{cont}}(\mathcal{G}; H)$.

In what follows, we shall only concern ourselves with continuous $\mathbb{R}$-valued one-cocycles on étale locally compact Hausdorff groupoids.

The One-Cocycle Classification Theorem

Proposition (The One-Cocycle Classification Theorem (FHKP, 2019))

Let the following objects be given:

- *A non-empty compact metrizable space* X.
- *A commuting* k*-family* $\sigma = (\sigma_i)_{i \in [k]}$ *of surjective local homeomorphisms on* X.
- *A* k*-family* $\varphi = (\varphi_i)_{i \in [k]}$ *in* $\mathrm{C}(X, \mathbb{R})$.

Then the following statements are equivalent:

1. *There exists a unique* $\mathbb{R}$*-valued one-cocycle* $c_{X,\sigma,\varphi} \in \mathcal{Z}^1_{\mathrm{cont}}(\mathcal{G}(X,\sigma); \mathbb{R})$ *such that*

$$\forall i \in [k],\ \forall x \in X: \qquad \varphi_i(x) = c_{X,\sigma,\varphi}(x, \mathbf{e}_i, \sigma_i(x)).$$

2. *The following (algebraic) identities hold:*

$$\forall i, j \in [k],\ \forall x \in X: \qquad \varphi_i(\sigma_j(x)) + \varphi_j(x) = \varphi_j(\sigma_i(x)) + \varphi_i(x).$$

   We collectively call these (essentially $\binom{k}{2}$*) identities the Cocycle Condition.*

3. *The* k*-family* $\left(R^X_{\sigma_i,\varphi_i}\right)_{i \in [k]}$ *of Ruelle transfer operators commutes.*

If any of these three statements hold, then we call (X, σ, φ) a Ruelle dynamical system.

Example (A Non-Trivial Ruelle Dynamical System on a Cantor Space)

Let $X = \mathbb{Z}_k^{\mathbb{N}}$, equipped with the product topology, and define $(\sigma_i : X \to X)_{i \in [k]}$ by

$$\forall i \in [k], \ \forall x \in X : \qquad \sigma_i(x) \stackrel{\text{df}}{=} (x_{n+1} + (i-1))_{n \in \mathbb{N}}.$$

Next, let $(a_i)_{i \in [k]} \in \mathbb{R}^k$, and define $\varphi : X \to \mathbb{R}$ by

$$\forall i \in [k], \ \forall x \in X : \qquad \varphi(x) \stackrel{\text{df}}{=} a_i \quad \Longleftrightarrow \quad \sum_{n=1}^{k} x_n = i - 1.$$

Finally, let $(c_i)_{i \in [k]} \in \mathbb{R}^k$. Then

$$\left(X, \sigma, (\varphi + c_i \cdot 1_X)_{i \in [k]} \right)$$

is a rank-k Ruelle dynamical system.

Non-Trivial Examples of Ruelle Dynamical Systems (Cont'd)

Example (A Non-Trivial Ruelle Dynamical System on the Unit Circle)

Let $(p_i)_{i \in [k]} \in (\mathbb{Z} \setminus \{0\})^k$, and define $\sigma = (\sigma_i : \mathbb{T} \to \mathbb{T})_{i \in [k]}$ by

$$\forall i \in [k], \ \forall z \in \mathbb{T} : \qquad \sigma_i(z) \stackrel{\text{df}}{=} z^{p_i}.$$

Next, define $\varphi = (\varphi_i : \mathbb{T} \to \mathbb{C})_{i \in [k]}$ by

$$\forall i \in [k], \ \forall z \in \mathbb{T} : \qquad \varphi_i(z) \stackrel{\text{df}}{=} z^{p_i} - z.$$

Then for any continuous additive map $f : \mathbb{C} \to \mathbb{R}$,

$$\left(\mathbb{T}, \sigma, (f \circ \varphi_i)_{i \in [k]} \right)$$

is a rank-k Ruelle dynamical system.

Definition (Generalized Gauge Dynamics)

Let (X,σ,φ) be a Ruelle dynamical system. Then the generalized gauge dynamics for (X,σ,φ) is the $\mathbb{R}$-dynamical system $(C^*(\mathcal{G}(X,\sigma)),\alpha_{X,\sigma,\varphi})$, where the $\mathbb{R}$-action $\alpha_{X,\sigma,\varphi}$ on $C^*(\mathcal{G}(X,\sigma))$ is uniquely determined by

$$\forall t \in \mathbb{R},\ \forall f \in \mathrm{C_c}(\mathcal{G}(X,\sigma)):$$
$$(\alpha_{X,\sigma,\varphi})_t\big(\iota_{\mathcal{G}(X,\sigma)}(f)\big) = \iota_{\mathcal{G}(X,\sigma)}\left(\begin{Bmatrix} \mathcal{G}(X,\sigma) & \to & \mathbb{C} \\ \gamma & \mapsto & e^{itc_{X,\sigma,\varphi}(\gamma)} f(\gamma) \end{Bmatrix}\right).$$

Proposition (KMS States and Ruelle Transfer Operators (FHKP, 2019))

Let (X,σ,φ) be a rank-k Ruelle dynamical system. Let μ be a probability regular Borel measure on X. Then for each $\beta \in \mathbb{R}$, the following statements are equivalent:

1. *The state ω on $C^*(\mathcal{G}(X,\sigma))$ uniquely determined by*
$$\forall f \in \mathrm{C_c}(\mathcal{G}(X,\sigma)): \qquad \omega\big(\iota_{\mathcal{G}(X,\sigma)}(f)\big) = \int_X f(x,\mathbf{0},x)\, \mathrm{d}\mu(x)$$
is a KMS_β-state for the generalized gauge dynamics for (X,σ,φ).
2. $\big(R^X_{\sigma_i,-\beta\cdot\varphi_i}\big)^*(\mu) = \mu$ *for each $i \in [k]$.*

The Walters Condition

Definition (The Walters Condition)

A rank-k Ruelle dynamical system (X, σ, φ) is said to satisfy the Walters Condition for $\boldsymbol{n} \in \mathbb{N}_0^k$ if and only if the following conditions hold:

- $\sigma^{\boldsymbol{n}}$ is positively expansive and exact.
- $\displaystyle\sum_{i=1}^{|\boldsymbol{n}|} \varphi_{\overline{\boldsymbol{n}}(i)} \circ \sigma^{\sum_{j=1}^{i-1} \boldsymbol{e}_{\overline{\boldsymbol{n}}(j)}} : X \to \mathbb{R}$ is Hölder-continuous with respect to a compatible metric on X.

Proposition (The R-P-F Theorem for Ruelle Dynamical Systems (FHKP, 2019))

Let (X, σ, φ) be a rank-k Ruelle dynamical system that satisfies the Walters Condition for some $\boldsymbol{n} \in \mathbb{N}_0^k$. Then there exist

- *a unique probability regular Borel measure $\mu_{X,\sigma,\varphi}$ on X, and*
- *a unique k-sequence $\lambda^{X,\sigma,\varphi} = \left(\lambda_i^{X,\sigma,\varphi}\right)_{i \in [k]}$ in $\mathbb{R}_{>0}$*

such that $\left(R^X_{\sigma_i,\varphi_i}\right)^*(\mu_{X,\sigma,\varphi}) = \lambda_i^{X,\sigma,\varphi} \cdot \mu_{X,\sigma,\varphi}$ for each $i \in [k]$.

Note: *We call $\mu_{X,\sigma,\varphi}$ the R-P-F measure for (X, σ, φ).*

Let Λ be a finite, source-free, and primitive k-graph.

Let σ denote the k-family of generating shift maps on the infinite-path space Λ^∞ of Λ. If $\boldsymbol{p} \stackrel{\text{df}}{=} (1)_{i \in [k]}$, then $\sigma^{\boldsymbol{p}}$ is positively expansive and exact.

The R-P-F Theorem for Ruelle Dynamical Systems yields the following result:

Proposition (KMS States Associated To Higher-Rank Graphs (FHKP, 2019))

Let $(\Lambda^\infty, \sigma, \varphi)$ be a Ruelle dynamical system that satisfies the Walters Condition for $\boldsymbol{p}$. Then for each $\beta \in \mathbb{R} \setminus \{0\}$, the following two simplicial sets are affinely isomorphic:

1. *The state space of $C^*(\mathrm{Per}(\Lambda))$, where $\mathrm{Per}(\Lambda)$ denotes the periodicity group of Λ.*
2. *The simplex of KMS_β-states for the generalized gauge dynamics for*

$$\left(\Lambda^\infty, \sigma, \left(\frac{1}{\beta} \cdot \left(\ln\left(\lambda_i^{\Lambda^\infty, \sigma, \varphi} \right) \cdot 1_X - \varphi_i \right) \right)_{i \in [k]} \right).$$

Note: $C^*(\mathcal{G}(\Lambda^\infty, \sigma)) \cong C^*(\Lambda)$ — *the C^*-algebra of Λ.*

An Application to a Certain Topological Higher-Rank Graph

A topological higher-rank graph is like a higher-rank graph but with some differences:

- The category is small instead of just countable.
- The object space and morphism space are both second-countable locally compact Hausdorff spaces.
- Composition, the range and source maps, and the degree functor are continuous.
- The source map is a local homeomorphism, and composition is an open map.

Example (A Source-Free Compact Topological Higher-Rank Graph)

Let X be a compact metrizable space, and $\sigma : \mathbb{N}_0^k \times X \to X$ a map such that

- $\sigma(\boldsymbol{n}, \bullet) : X \to X$ is a surjective local homeomorphism on X for each $\boldsymbol{n} \in \mathbb{N}_0^k$, and
- $\sigma(\boldsymbol{m} + \boldsymbol{n}, x) = \sigma(\boldsymbol{m}, \sigma(\boldsymbol{n}, x))$ for all $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}_0^k$ and $x \in X$.

Then $\mathbb{N}_0^k \times X$ can be turned into a source-free compact topological k-graph as follows:

- For all $x \in X$ and $\boldsymbol{n} \in \mathbb{N}_0^k$, let $s(\boldsymbol{n}, x) \stackrel{\text{df}}{=} (\boldsymbol{0}, \sigma(\boldsymbol{n}, x))$ and $r(\boldsymbol{n}, x) \stackrel{\text{df}}{=} (\boldsymbol{0}, x)$.
- For all $x, y \in X$ and $\boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}_0^k$, if $\boldsymbol{x} = \sigma(\boldsymbol{n}, y)$, let $(\boldsymbol{m}, x) \circ (\boldsymbol{n}, y) \stackrel{\text{df}}{=} (\boldsymbol{m} + \boldsymbol{n}, x)$.
- For all $x \in X$ and $\boldsymbol{n} \in \mathbb{N}_0^k$, let $d(\boldsymbol{n}, x) \stackrel{\text{df}}{=} \boldsymbol{n}$.

The path groupoid of the topological k-graph above is a Deaconu-Renault groupoid of the type studied here, which allows it to be analyzed by our techniques.

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