REMARKS ON VILLADSEN ALGEBRAS

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Abstract. It is shown that certain unital simple C*-algebras constructed by Villadsen in [31] are classified by the K_0-group together with the radius of comparison.

1. Introduction

Villadsen algebras (of the first type) were constructed in [31] as examples of simple unital C*-algebras which have perforation in their ordered K_0-group. This class of C*-algebras lies outside the scope of the current classification theorem ([16], [17], [11], [8], [9], [28], [4]), as Villadsen algebras do not absorb the Jiang-Su algebra $\mathcal{Z}$ tensorially. Indeed, a Villadsen-type algebra was constructed in [30] which has the same value of the Elliott invariant as an AI algebra, but is not itself isomorphic to that AI algebra.

Each Villadsen algebra is an inductive limit of homogeneous C*-algebras with connecting maps induced by coordinate projections together with a small portion of point evaluations (see Section 2). In this note, we shall first show that, with different point-evaluation sets, the resulting algebras are classified by the K_0-multiplicity and the radius of comparison of Toms ([29]; Definition 3.1 below):

Theorem 1.1 (Theorem 6.1)**.** *Let X be a connected finite-dimensional solid space (see Definition 3.3). Let A_E and A_F be Villadsen algebras (see Section 2) with point-evaluation sets E and F respectively (but with the same connected space X and the same numbers and multiplicities of the coordinate projections (c_i), $(s_{i_1}, ..., s_{i,c_i})$). Then $A_E \cong A_F$ if, and only if,*

$$\rho(\mathrm{K}_0(A_E)) = \rho(\mathrm{K}_0(A_F)) \quad \textit{and} \quad \mathrm{rc}(A_E) = \mathrm{rc}(A_F),$$

where ρ_{A_E} and ρ_{A_F} are the unique states of the order-unit groups $\mathrm{K}_0(A_E)$ and $\mathrm{K}_0(A_F)$, respectively, and $\mathrm{rc}(\cdot)$ denotes the radius of comparison.

Moreover, if the fixed seed space X is further assumed to be K-contractible (i.e., $\mathrm{K}_0(\mathrm{C}(X)) = \mathbb{Z}$ and $\mathrm{K}_1(\mathrm{C}(X)) = \{0\}$), then the algebras can be classified by the K_0-group together with the radius of comparison even if the numbers and the multiplicities of the coordinate projections and the numbers of point evaluations are arbitrary:

Theorem 1.2 (Corollary 7.8)**.** *Let X be a connected finite-dimensional solid space which is K-contractible. Let*

$$A := A(X^p, (n_i^{(A)}), (k_i^{(A)}), E^{(A)}) \quad \textit{and} \quad B := B(X^q, (n_i^{(B)}), (k_i^{(B)}), F^{(B)})$$

be Villadsen algebras with non-zero radius of comparison, where $p, q = 1, 2, ...$ Then

$$A \cong B$$

if, and only if,

$$\mathrm{K}_0(A) \cong \mathrm{K}_0(B) \quad \textit{and} \quad \mathrm{rc}(A) = \mathrm{rc}(B).$$

Note that this theorem covers the example constructed in [30].

One might compare the Villadsen algebras with the UHF algebras of [14] and [6], and the present classification results with the classification of the unital UHF algebras, or, for that matter, of their non-unital hereditary subalgebras in [6]. The non-unital version of the Villadsen algebras and their classification is also an interesting question.

We hope that our result might shed some light on the possibility of classifying more general non-$\mathcal{Z}$-stable C*-algebras, for instance, general simple A(S)H algebras with diagonal maps ([10] and [1]), or general simple transformation group C*-algebras.

Acknowledgements. The research of the first named author was supported by a Natural Sciences and Engineering Research Council of Canada (NSERC) Discovery Grant, the research of the second named author was supported by a National Natural Science Foundation of China (NNSF) grant (No. 11401088), and the research of the third named author was supported by a U.S. National Science Foundation grant (DMS-1800882) and a Simon Foundation grant (MP-TSM-00002606). The third named author also thanks Ali Asadi-Vasfi, Xuanlong Fu, and Cristian Ivanescu for discussions. The authors also thank the anonymous referee for the careful reading of the manuscript and the invaluable suggestions.

2. The Villadsen algebra $A(X, (n_i), (k_i), E)$

Let X be a metrizable compact space (usually we assume X to be connected), let (c_i) and (k_i) be two sequences of non-zero natural numbers, and let

$$\begin{cases} E_1 := \{x_{1,1}, ..., x_{1,k_1}\} \subseteq X, \\ E_2 := \{x_{2,1}, ..., x_{2,k_2}\} \subseteq X^{c_1}, \\ ... \\ E_i := \{x_{i,1}, ..., x_{i,k_i}\} \subseteq X^{c_1 \cdots c_{i-1}}, \\ ... \end{cases}$$

be a sequence of finite subsets such that for each $i = 1, 2, ...,$ the set

$$\bigcup_{j=1}^{\infty} \bigcup_{s=1}^{c_i \cdots c_{i+j-1}} \pi_s(E_{i+j})$$

is dense in $X^{c_0 c_1 \cdots c_{i-1}}$, where π_s are the coordinate projections and $c_0 = 1$.

Construct the (generalized) Villadsen algebra (in [31], X was the two-sphere) as the inductive limit of the sequence

$$\mathrm{M}_{n_0}(\mathrm{C}(X^{c_0})) \longrightarrow \mathrm{M}_{n_0(n_1+k_1)}(\mathrm{C}(X^{c_0 c_1})) \longrightarrow \mathrm{M}_{n_0(n_1+k_1)(n_2+k_2)}(\mathrm{C}(X^{c_0 c_1 c_2})) \longrightarrow \cdots, \tag{2.1}$$

where the seed for the ith-stage map (recall $c_0 = 1$),

$$\phi_i : \mathrm{C}(X^{c_0 c_1 \cdots c_{i-1}}) \to \mathrm{M}_{n_i + k_i}(\mathrm{C}(X^{c_0 c_1 \cdots c_{i-1} c_i})),$$

is defined by

$$\begin{aligned} f \quad \mapsto \quad & \mathrm{diag}\{\underbrace{\underbrace{f\circ\pi_1,...,f\circ\pi_1}_{s_{i,1}},...,\underbrace{f\circ\pi_{c_i},...,f\circ\pi_{c_i}}_{s_{i,c_i}}}_{n_i},\underbrace{f(x_{i,1}),...,f(x_{i,k_i})}_{k_i}\} \\ & = \mathrm{diag}\{\underbrace{\underbrace{f\circ\pi_1,...,f\circ\pi_1}_{s_{i,1}},...,\underbrace{f\circ\pi_{c_i},...,f\circ\pi_{c_i}}_{s_{i,c_i}}}_{n_i}, f(E_i)\}, \end{aligned}$$

where $s_{i,1},...,s_{i,c_i} \geq 1$ are natural numbers, and $n_i = \sum_{j=1}^{c_i} s_{i,j}$.

A direct calculation shows that the composed map

$$\phi_{i,i+j} : \mathrm{C}(X^{c_0c_1\cdots c_{i-1}}) \to \mathrm{M}_{(n_i+k_i)\cdots(n_{i+j-1}+k_{i+j-1})}(\mathrm{C}(X^{c_0c_1\cdots c_{i-1}\cdots c_{i+j-1}}))$$

is equal (up to a permutation) to

$$f \mapsto \mathrm{diag}\{\underbrace{f\circ\pi_1,...,f\circ\pi_{c_i\cdots c_{i+j-1}}}_{n_i\cdots n_{i+j-1}}, \quad \underbrace{f(x_{i,1}),...,f(x_{i,k_i})}_{k_i[(n_{i+1}+k_{i+1})\cdots(n_{i+j-1}+k_{i+j-1})]}, \quad \underbrace{f(\cdot),...,f(\cdot)}_{\cdots},...,\},$$

i.e.,

$$\begin{aligned} f \mapsto \mathrm{diag}\{ & \underbrace{f\circ\pi_1,...,f\circ\pi_{c_i\cdots c_{i+j-1}}}_{n_i\cdots n_{i+j-1}}, f(E_i)1_{(n_{i+1}+k_{i+1})\cdots(n_{i+j-1}+k_{i+j-1})}, \\ & (f(\pi_1(E_{i+1})),...,f(\pi_{c_i}(E_{i+1})))1_{(n_{i+2}+k_{i+2})\cdots(n_{i+j-1}+k_{i+j-1})},...,\}. \end{aligned}$$

So, it can be described as

$$\mathrm{diag}\{\underbrace{f\circ\pi_1,...,f\circ\pi_{c_i\cdots c_{i+j-1}}}_{n_i\cdots n_{i+j-1}}, \text{point evaluations}\}.$$

We shall choose c_i, $s_{i,1},...,s_{i,c_i}$ (and hence the sum n_i), and k_i in such a way that

$$\lim_{j\to\infty} \frac{n_i\cdots n_{i+j}}{(n_i+k_i)\cdots(n_{i+j}+k_{i+j})} = \lim_{j\to\infty}(\frac{n_i}{n_i+k_i})\cdots(\frac{n_{i+j}}{n_{i+j}+k_{i+j}}) \neq 0.$$

Equivalently, we require

$$\lim_{i\to\infty}\lim_{j\to\infty} \frac{n_i\cdots n_{i+j}}{(n_i+k_i)\cdots(n_{i+j}+k_{i+j})} = \lim_{i\to\infty}\lim_{j\to\infty}(\frac{n_i}{n_i+k_i})\cdots(\frac{n_{i+j}}{n_{i+j}+k_{i+j}}) = 1. \tag{2.2}$$

(In other words, the numbers of point evaluations are small compared with the numbers of coordinate projections including multiplicity—the ratio is summable.) Denote the inductive limit algebra by

$$A(X,(n_i),(k_i),E),$$

or, more specifically,

$$A(X,(n_i),(c_i),(s_i),(k_i),E).$$

In what follows, we shall show, with a mild assumption on X (see Definition 3.3), that this algebra, which is always simple, is independent of the choice of points in the point-evaluation set E if the number of them at each stage is kept the same; otherwise, allowing only the number of points to vary, as we shall show, it is classified by the K_0-group together with the radius of comparison.

In the case that X is contractible, we shall show that this C*-algebra is classified by the K_0-group and the radius of comparison, and also, if the latter is zero, the trace simplex (Theorem 7.1). (Even allowing different numbers of coordinate projections and (non-zero) multiplicities, and different numbers of point evaluations.)

Remark 2.1. If $c_i = 1$, $i = 1, 2, ...$, then $A(X, (n_i), (k_i), E)$ is the C*-algebra constructed by Goodearl in [18] with real rank not equal to zero. On the other hand, if $s_{i,j} = 1$, $i = 1, 2, ...$, $j = 1, ..., c_i$, then $A(X, (n_i), (k_i), E)$ is the C*-algebra constructed by Villadsen in [31].

3. Mean dimension and radius of comparison

In this section, let us calculate the mean dimension (as formulated in [23]) and radius of comparison (as formulated in [29]) of the Goodearl–Villadsen algebras $A(X, (n_i), (k_i), E)$.

First, recall

Definition 3.1 (Definition 6.1 of [29])**.** Let A be a C*-algebra. Denote by $\mathrm{M}_n(A)$ the C*-algebra of $n \times n$ matrices over A. Regard $\mathrm{M}_n(A)$ as the upper-left corner of $\mathrm{M}_{n+1}(A)$, and consider the union,

$$\mathrm{M}_\infty(A) = \bigcup_{n=1}^{\infty} \mathrm{M}_n(A),$$

the algebra of all finite matrices over A.

The radius of comparison of a unital C*-algebra A, denoted by $\mathrm{rc}(A)$, is the infimum of the set of real numbers $r > 0$ such that if $a, b \in (\mathrm{M}_\infty(A))^+$ satisfy

$$\mathrm{d}_\tau(a) + r < \mathrm{d}_\tau(b), \quad \tau \in \mathrm{T}(A),$$

then $a \precsim b$, where $\mathrm{T}(A)$ is the simplex of tracial states. (In [29], the radius of comparison is defined in terms of quasitraces instead of traces; but since all the algebras considered in this paper are nuclear, by [19] (see also [3] in the locally finite nuclear dimension case), any quasitrace is actually a trace.)

We shall use the following remark on vector bundles:

Remark 3.2. Assume a (complex) vector bundle E over a compact metrizable space X has non-zero Chern class $\mathrm{c}_n(E) \in \mathrm{H}^n(X)$. Then the trivial sub-bundles of E have rank at most $\mathrm{rank}(E) - n/2$, as, if there is a trivial sub-bundle F of rank $r > \mathrm{rank}(E) - n/2$, then

$$\mathrm{c}(E) = \mathrm{c}(F^c \oplus F) = \mathrm{c}(F^c)\mathrm{c}(F) = \mathrm{c}(F^c),$$

but, since $\mathrm{rank}(F^c) = \mathrm{rank}(E) - \mathrm{rank}(F) < n/2$, we have $\mathrm{c}_n(F^c) = 0$ (as $\mathrm{c}_d(F^c) = 0$ for all $d > \mathrm{rank}(F^c)$), and hence $\mathrm{c}_n(E) = \mathrm{c}_n(F^c) = 0$, which contradicts the assumption.

Definition 3.3. Let us call a metrizable compact space X solid if it contains a Euclidean ball of dimension $\dim(X)$ when $\dim(X)$ is finite; when $\dim(X) = \infty$, X solid will mean that X contains a Euclidean ball of arbitrarily large dimension.

Note that all finite CW-complexes are solid. The Hawaiian earring and the Hilbert cube are also solid. Not all compact metrizable spaces are solid, as there are such X with $\dim(X \times X) < 2 \cdot \dim(X)$ which implies that X cannot be solid (see [2]).

Theorem 3.4. *Let X be a metrizable compact space. With $A = A(X, (n_i), (k_i), E)$, one has*

$$\mathrm{mdim}(A) \leq \frac{\dim(X)}{n_0} \cdot \lim_{i\to\infty} \frac{c_1 \cdots c_i}{(n_1 + k_1) \cdots (n_i + k_i)}, \tag{3.1}$$

where $\infty \cdot 0 = 0$ and $\mathrm{mdim}(\cdot)$ is the mean dimension of an AH system introduced in [23].

Moreover, if the metrizable compact space X is solid, then equality holds in (3.1)*, and the radius of comparison of A, $\mathrm{rc}(A)$, is equal to $\frac{1}{2}\mathrm{mdim}(A)$.*

Proof. Let us first prove (3.1). Consider the jth stage, $\mathrm{M}_{m_j}(\mathrm{C}(X^{d_j}))$, where $m_j = n_0(n_1 + k_1) \cdots (n_{j-1} + k_{j-1})$ and $d_j = c_1 \cdots c_{j-1}$ (note that all coordinate projections appear, i.e., $s_{j,1}, ..., s_{j,c_j} \geq 1$, $j = 1, 2, ...$), and let α be a finite open cover of X^{d_j}. Since the pull-back of α by any constant map has degree zero and $\mathcal{D}(\alpha) \leq \dim(X^{d_j}) \leq c_1 \cdots c_{j-1} \cdot \dim(X)$, we have that, for each pair $j < i$,

$$\mathcal{D}(\phi_{j,i}(\alpha)) \leq c_j \cdots c_{i-1} {\cdot} \mathcal{D}(\alpha) \leq c_1 \cdots c_{i-1} \cdot \dim(X),$$

where $\mathcal{D}(\cdot)$ denotes the degree of an open cover, and then

$$\begin{aligned} \lim_{i\to\infty} \frac{\mathcal{D}(\phi_{j,i}(\alpha))}{m_i} &\leq \lim_{i\to\infty} \frac{c_j \cdots c_{i-1} {\cdot} \mathcal{D}(\alpha)}{n_0(n_1 + k_1) \cdots (n_{i-1} + k_{i-1})} \\ &\leq \frac{\dim(X)}{n_0} \cdot \lim_{i\to\infty} \frac{c_1 \cdots c_i}{(n_1 + k_1) \cdots (n_i + k_i)}, \end{aligned}$$

where $\infty \cdot 0 = 0$ in the case that $\dim(X) = \infty$. Taking the supremum over all finite open covers α of D^{d_j} and passing to the limit as $j \to \infty$, we obtain (3.1). In particular, by [23], we have

$$\mathrm{rc}(A) \leq \frac{1}{2}\mathrm{mdim}(A) \leq \frac{1}{2} \cdot \frac{\dim(X)}{n_0} \cdot \lim_{i\to\infty} \frac{c_1 \cdots c_i}{(n_1 + k_1) \cdots (n_i + k_i)}. \tag{3.2}$$

Now, assume that X is solid (i.e., it contains a Euclidean ball of dimension $\dim(X)$, if $\dim(X) < \infty$, and of arbitrary dimension otherwise), and let us show that

$$\mathrm{rc}(A) \geq \frac{1}{2} \cdot \frac{\dim(X)}{n_0} \cdot \lim_{i\to\infty} \frac{c_1 \cdots c_i}{(n_1 + k_1) \cdots (n_i + k_i)}. \tag{3.3}$$

Together with (3.2), we will then have

$$\mathrm{rc}(A) = \frac{1}{2}\mathrm{mdim}(A) = \frac{1}{2} \cdot \frac{\dim(X)}{n_0} \cdot \lim_{i\to\infty} \frac{c_1 \cdots c_i}{(n_1 + k_1) \cdots (n_i + k_i)}.$$

Set

$$\gamma = \lim_{i\to\infty} \frac{c_1 \cdots c_i}{(n_1 + k_1) \cdots (n_i + k_i)}.$$

Since (3.3) holds trivially if $\gamma = 0$ (as $\infty \cdot 0 = 0$), let us assume that $\gamma \neq 0$ in the rest of the proof.

Suppose that $\dim(X) < \infty$. Let $\varepsilon > 0$ be arbitrary for the time being. Choose i sufficiently large that

$$\frac{c_1 \cdots c_{i-1} \cdot \dim(X) - 2}{2n_0(n_1+k_1)\cdots(n_{i-1}+k_{i-1})} > \frac{\gamma}{2} \cdot \frac{\dim(X)}{n_0} - \varepsilon$$

and

$$\frac{\dim(X)}{2n_0}\left(\frac{c_1\cdots c_{i-1}}{(n_1+k_1)\cdots(n_{i-1}+k_{i-1})} - \frac{c_1\cdots c_{j-1}}{(n_1+k_1)\cdots(n_{j-1}+k_{j-1})}\right) < \varepsilon, \quad j > i.$$

Since X contains a Euclidean ball of dimension $\dim(X)$, the space $X^{c_1\cdots c_{i-1}}$ contains a Euclidean ball of dimension $c_1 \cdots c_{i-1} \cdot \dim(X)$, and hence, if i is large enough, it contains a d-dimensional sphere S, where

$$c_1 \cdots c_{i-1} \cdot \dim(X) - 2 \leq d \leq c_1 \cdots c_{i-1} \cdot \dim(X) - 1$$

and d is non-zero and even.

Pick a (complex) vector bundle E over S such that $\mathrm{rank}(E) = d/2$ and $e := \mathrm{c}_d(E) \in \mathrm{H}^d(S)$ is non-zero, where c_d is the dth Chern class. (Recall that the total Chern class of E is $1+e$.) (Such a vector bundle exists, as, otherwise, if the d-th Chern class of every vector bundle were trivial, then the Chern character would not induce a rational isomorphism between the K-group and the cohomology group of the sphere S.) Denote by p the corresponding projection in $\mathrm{M}_\infty(\mathrm{C}(S))$, and extend p to a positive element of $\mathrm{M}_\infty(\mathrm{C}(X^{d_i}))$ such that $\mathrm{rank}(p(x)) \geq d/2$, $x \in X^{d_i}$. Denote this element still by p.

Note that, for each tracial state τ of $\mathrm{M}_{m_i}(\mathrm{C}(X^{d_i}))$,

$$\mathrm{d}_\tau(p) \geq \frac{d}{2n_0(n_1+k_1)\cdots(n_{i-1}+k_{i-1})} \geq \frac{c_1 \cdots c_{i-1} \cdot \dim(X) - 2}{2n_0(n_1+k_1)\cdots(n_{i-1}+k_{i-1})} > \frac{\gamma}{2} \cdot \frac{\dim(X)}{n_0} - \varepsilon.$$

Consider the element $\phi_{i,\infty}(p) \in A$. For each $j > i$, the restriction of $\phi_{i,j}(p) \in \mathrm{M}_{m_j}(\mathrm{C}(X^{d_j}))$ to $S \times \cdots \times S \subseteq X^{d_j}$ is a projection which corresponds to the vector bundle

$$E_j := (\bigoplus_{s_1} \pi_1^*(E)) \oplus \cdots \oplus (\bigoplus_{s_{c_i\cdots c_{j-1}}} \pi^*_{c_i\cdots c_{j-1}}(E)) \oplus \theta_j,$$

where θ_j is a trivial bundle. Then the total Chern class of E_j is

$$\begin{aligned} & \pi_1^*(1+c_d)^{s_1}\pi_2^*(1+c_d)^{s_2}\cdots\pi^*_{c_i\cdots c_{j-1}}(1+c_d)^{s_{c_i\cdots c_{j-1}}} \\ = \; & \pi_1^*(1+s_1e)\pi_2^*(1+s_2e)\cdots\pi^*_{c_i\cdots c_{j-1}}(1+s_{c_i\cdots c_{j-1}}e), \end{aligned}$$

and, by the Künneth Theorem, it is non-zero at degree $dc_i \cdots c_{j-1}$. Hence (see Remark 3.2), any trivial sub-bundle of E_j has rank at most

$$
\begin{aligned}
& \operatorname{rank}(E_j) - \frac{1}{2} dc_i \cdots c_{j-1} \\
= \ & \operatorname{rank}(E)(n_i + k_i) \cdots (n_{j-1} + k_{j-1}) - \frac{1}{2} dc_i \cdots c_{j-1} \\
= \ & \frac{d}{2}((n_i + k_i) \cdots (n_{j-1} + k_{j-1}) - c_i \cdots c_{j-1}) \\
\leq \ & \frac{\dim(X)}{2}(c_1 \cdots c_{i-1}(n_i + k_i) \cdots (n_{j-1} + k_{j-1}) - c_1 \cdots c_{j-1}) \\
= \ & \frac{\dim(X)}{2n_0}(\frac{c_1 \cdots c_{i-1}}{(n_1 + k_1) \cdots (n_{i-1} + k_{i-1})} - \frac{c_1 \cdots c_{j-1}}{(n_1 + k_1) \cdots (n_{j-1} + k_{j-1})}) n_0 (n_1 + k_1) \cdots (n_{j-1} + k_{j-1}) \\
\leq \ & \varepsilon n_0 (n_1 + k_1) \cdots (n_{j-1} + k_{j-1}).
\end{aligned}
$$

Let $r \in A$ be a trivial projection with $2\varepsilon < \mathrm{d}_\tau(r) < 3\varepsilon$, $\tau \in \mathrm{T}(A)$. Then

$$\mathrm{d}_\tau(r) + (\frac{\gamma}{2} \cdot \frac{\dim(X)}{n_0} - 4\varepsilon) < \mathrm{d}_\tau(p), \quad \tau \in \mathrm{T}(A).$$

But the rank of the (trivial) vector bundle of r at the stage j is at least

$$2\varepsilon n_0 (n_1 + k_1) \cdots (n_{j-1} + k_{j-1}) > \varepsilon n_0 (n_1 + k_1) \cdots (n_{j-1} + k_{j-1}),$$

which implies that r is not Cuntz subequivalent to p, and therefore,

$$\mathrm{rc}(A) \geq \frac{\gamma}{2} \cdot \frac{\dim(X)}{n_0} - 4\varepsilon.$$

Since ε is arbitrary, this implies $\mathrm{rc}(A) \geq \frac{\gamma}{2} \cdot \frac{\dim(X)}{n_0}$.

If X is infinite-dimensional (recall still $\lim_{i \to \infty} \frac{c_1 \cdots c_i}{(n_1+k_1) \cdots (n_i+k_i)} = \gamma \neq 0$), then the argument above (choose d arbitrarily large) shows that $\mathrm{rc}(A)$ is arbitrarily large, and hence $\mathrm{rc}(A) = \infty$. So, (3.3) always holds, as asserted. □

Corollary 3.5 (Theorem 5.1 of [29]). *For any $r \in [0, +\infty]$, there is a Villadsen algebra A such that* $\mathrm{rc}(A) = r$.

Proof. Let us assume that $r \in (0, +\infty)$. Pick a natural number d such that $2r < d$, and consider $s := 2r/d \in (0, 1)$. Then pick a sequence of rational numbers $p_i/q_i \in (0, 1)$, $i = 1, 2, ...$, such that

$$\frac{p_1}{q_1} \cdot \frac{p_2}{q_2} \cdots = s.$$

Writing

$$n_i = p_i \quad \text{and} \quad k_i = q_i - p_i, \quad i = 1, 2, ...,$$

we have

$$(\frac{n_1}{n_1 + k_1})(\frac{n_2}{n_2 + k_2}) \cdots = s.$$

Let A be a Villadsen algebra associated with (n_i) and (k_i) (and $c_i = n_i$, $i = 1, 2, ..., n_0 = 1$) with the seed space $X = [0,1]^d$, which is solid. Then it follows from Theorem 3.4 that

$$\mathrm{rc} = \frac{1}{2}d(\frac{n_1}{n_1+k_1})(\frac{n_2}{n_2+k_2})\cdots = \frac{1}{2}ds = r.$$

If $r = +\infty$, then one can construct a Villadsen algebra with seed space $X = [0,1]^\infty$ and with the sequences (n_i), (k_i) as above. Then the resulting algebra has $\mathrm{rc}(A) = +\infty$.

If $r = 0$, then one can construct a Villadsen algebra with seed space X of dimension zero (e.g., a single point) and with sequences (n_i), (k_i) as above. Then the resulting AF algebra has $\mathrm{rc}(A) = 0$. □

Remark 3.6. Although the statement of Theorem 5.1 of [29] is on the range of the dimension rank ratio of a simple AH algebra, its proof actually shows that the range of the radius of comparison of a Villadsen algebra is $[0, +\infty]$ (Corollary 3.5).

Theorem 3.7. *If* $\mathrm{rc}(A) > 0$*, then*

$$\lim_{i\to\infty}\lim_{j\to\infty}(\frac{c_i}{n_i})\cdots(\frac{c_{i+j}}{n_{i+j}}) = 1 \tag{3.4}$$

and

$$\lim_{i\to\infty}\lim_{j\to\infty}\frac{|\{s_k : s_k = 1, k = 1, ..., c_i\cdots c_{i+j}\}|}{n_i\cdots n_{i+j}} = 1, \tag{3.5}$$

where

$$\phi_{i,j+1} = \mathrm{diag}\{\underbrace{\underbrace{\pi_1^*, ..., \pi_1^*}_{s_1}, ..., \underbrace{\pi^*_{c_i\cdots c_{i+j}}, ..., \pi^*_{c_i\cdots c_{i+j}}}_{s_{c_i\cdots c_{i+j}}}}_{n_i\cdots n_{i+j}}, \text{ point evaluations}\}.$$

Proof. Since $\mathrm{rc}(A) > 0$, we have

$$\lim_{i\to\infty}\frac{c_1\cdots c_i}{(n_1+k_1)\cdots(n_i+k_i)} > 0,$$

and hence

$$\lim_{i\to\infty}\lim_{j\to\infty}\frac{c_i\cdots c_{i+j}}{(n_i+k_i)\cdots(n_{i+j}+k_{i+j})} = 1.$$

Comparing this with (2.2) (and since both limits are non-zero), we have (3.4):

$$\begin{aligned} 1 &= \lim_{i\to\infty}\frac{\lim_{j\to\infty}\frac{c_i\cdots c_{i+j}}{(n_i+k_i)\cdots(n_{i+j}+k_{i+j})}}{\lim_{j\to\infty}\frac{n_i\cdots n_{i+j}}{(n_i+k_i)\cdots(n_{i+j}+k_{i+j})}} = \lim_{i\to\infty}\lim_{j\to\infty}\frac{\frac{c_i\cdots c_{i+j}}{(n_i+k_i)\cdots(n_{i+j}+k_{i+j})}}{\frac{n_i\cdots n_{i+j}}{(n_i+k_i)\cdots(n_{i+j}+k_{i+j})}} \\ &= \lim_{i\to\infty}\lim_{j\to\infty}(\frac{c_i}{n_i})\cdots(\frac{c_{i+j}}{n_{i+j}}). \end{aligned}$$

As for (3.5), note that $n_i\cdots n_{i+j} = s_1 + \cdots + s_{c_i\cdots c_{i+j}}$, and hence

$$\frac{c_i\cdots c_{i+j}}{n_i\cdots n_{i+j}} = \frac{c_i\cdots c_{i+j}}{s_1+\cdots+s_{c_i\cdots c_{i+j}}} \leq \frac{c_i\cdots c_{i+j}}{(c_i\cdots c_{i+j} - b_{i,j}) + 2b_{i,j}} = \frac{c_i\cdots c_{i+j}}{c_i\cdots c_{i+j} + b_{i,j}} \leq 1,$$

where

$$b_{i,j} := |\{s_k : s_k > 1, k = 1, ..., c_i \cdots c_{i+j}\}|.$$

Together with (3.4), this yields

$$1 = \lim_{i\to\infty} \lim_{j\to\infty} \frac{c_i \cdots c_{i+j}}{c_i \cdots c_{i+j} + b_{i,j}} = \lim_{i\to\infty} \lim_{j\to\infty} \frac{1}{1 + \frac{b_{i,j}}{c_i \cdots c_{i+j}}};$$

therefore (note that s_k, $k = 1, ..., c_i \cdots c_{i+j}$, are non-zero),

$$1 = \lim_{i\to\infty} \lim_{j\to\infty} \frac{c_i \cdots c_{i+j} - b_{i,j}}{c_i \cdots c_{i+j}} = \lim_{i\to\infty} \lim_{j\to\infty} \frac{|\{s_k : s_k = 1, k = 1, ..., c_i \cdots c_{i+j}\}|}{c_i \cdots c_{i+j}}.$$

Using (3.4) again, one obtains (3.5). □

4. Intertwinings of trace simplexes

4.1. **Trace simplex of the Villadsen algebra.** Let us first observe that, under Condition (2.2), the trace simplex of the Villadsen algebra is independent of (the number and the location of) the point evaluations.

Denote by A the (non-simple) limit of the inductive sequence

$$\mathrm{M}_{n_0}(\mathrm{C}(X)) \longrightarrow \mathrm{M}_{n_0 n_1}(\mathrm{C}(X^{c_1})) \longrightarrow \mathrm{M}_{n_0 n_1 n_2}(\mathrm{C}(X^{c_1 c_2})) \longrightarrow \cdots ,$$

where the ith-stage map,

$$\phi_i : \mathrm{C}(X^{c_1 \cdots c_{i-1}}) \to \mathrm{M}_{n_i + k_i}(\mathrm{C}(X^{c_1 \cdots c_{i-1} c_i})),$$

is defined by

$$f \quad \mapsto \quad \mathrm{diag}\{\underbrace{\underbrace{f \circ \pi_1, ..., f \circ \pi_1}_{s_{i,1}}, ..., \underbrace{f \circ \pi_{c_i}, ..., f \circ \pi_{c_i}}_{s_{i,c_i}}}_{n_i}\}.$$

Lemma 4.1. *Let A_E be a Villadsen algebra with point-evaluation set E which satisfies Condition* (2.2)*. Then* $\mathrm{T}(A_E) \cong \mathrm{T}(A)$.

Proof. Choose a decreasing sequence $\delta_1, \delta_2, ...$ of strictly positive numbers with $\sum_{n=1}^{\infty} \delta_n < 1$. Identifying $\mathrm{Aff}_{\mathbb{R}}(\mathrm{T}(\mathrm{M}_s(\mathrm{C}(Y))))$ with $\mathrm{C}_{\mathbb{R}}(Y)$ for any compact metrizable space Y, note that the map

$$(\phi_{i,i+j})^* : \mathrm{Aff}(\mathrm{M}_{n_0 \cdots n_{i-1}}(\mathrm{C}(X^{c_1 \cdots c_{i-1}}))) \to \mathrm{Aff}(\mathrm{M}_{n_0 \cdots n_{i+j-1}} \mathrm{C}(X^{c_1 \cdots c_{i+j-1}})),$$

which is induced by $\phi_{i,i+j} := \phi_{i+j-1} \circ \cdots \circ \phi_i$ (where $\phi_{i,i+1} := \phi_i$), is given by

$$\mathrm{C}_{\mathbb{R}}(X^{c_1 \cdots c_{i-1}}) \ni h \mapsto \frac{1}{n_i \cdots n_{i+j-1}} (\underbrace{h \circ \pi_1 + \cdots + h \circ \pi_{c_i \cdots c_{i+j-1}}}_{n_i \cdots n_{i+j-1}}) \in \mathrm{C}_{\mathbb{R}}(X^{c_1 \cdots c_{i+j-1}}).$$

Then a straightforward calculation shows that, for any $h \in \mathrm{C}_{\mathbb{R}}(X^{c_1\cdots c_{i-1}})$ with $\|h\|_\infty \leq 1$,

$$\begin{aligned}
&\|(\phi_{i,i+j})^*(h) - (\phi^{(E)}_{i,i+j})^*(h)\|_\infty \\
= \ &\|\frac{1}{n_i\cdots n_{i+j-1}}(\underbrace{h\circ\pi_1+\cdots+h\circ\pi_{c_i\cdots c_{i+j-1}}}_{n_i\cdots n_{i+j-1}}) \\
&-\frac{1}{(n_i+k_i)\cdots(n_{i+j-1}+k_{i+j-1})}(\underbrace{h\circ\pi_1+\cdots+h\circ\pi_{c_i\cdots c_{i+j-1}}}_{n_i\cdots n_{i+j-1}}+\text{point evaluations})\|_\infty \\
\leq \ &(\frac{1}{n_i\cdots n_{i+j-1}} - \frac{1}{(n_i+k_i)\cdots(n_{i+j-1}+k_{i+j-1})})(n_i\cdots n_{i+j-1}) \\
&+1-\frac{n_i\cdots n_{i+j-1}}{(n_i+k_i)\cdots(n_{i+j-1}+k_{i+j-1})} \\
= \ &2(1-\frac{n_i\cdots n_{i+j-1}}{(n_i+k_i)\cdots(n_{i+j-1}+k_{i+j-1})}),
\end{aligned}$$

which, by Condition (2.2), is arbitrarily small if i is sufficiently large. Therefore, there is a diagram

$$\begin{tikzcd}
\mathrm{C}_{\mathbb{R}}(X) \arrow[r, "(\phi^{(E)}_{1,i_1})^*"] \arrow[dr, "(\phi_{1,i_1})^*"] & \mathrm{C}_{\mathbb{R}}(X^{d_{i_1}}) \arrow[r, "(\phi^{(E)}_{i_1,i_2})^*"] & \mathrm{C}_{\mathbb{R}}(X^{d_{i_2}}) \arrow[r] & \cdots \arrow[r] & (\mathrm{Aff}_{\mathbb{R}}(\mathrm{T}(A_E)), \|\cdot\|_\infty) \\
\mathrm{C}_{\mathbb{R}}(X) \arrow[r, "(\phi_{1,i_1})^*"] & \mathrm{C}_{\mathbb{R}}(X^{d_{i_1}}) \arrow[r, "(\phi_{i_1,i_2})^*"] \arrow[ur, "(\phi_{i_1,i_2})^*"] & \mathrm{C}_{\mathbb{R}}(X^{d_{i_2}}) \arrow[r] & \cdots \arrow[r] & (\mathrm{Aff}_{\mathbb{R}}(\mathrm{T}(A)), \|\cdot\|_\infty)
\end{tikzcd}$$

with

$$\|(\phi_{i_{s+1},i_{s+2}})^*\circ(\phi_{i_s,i_{s+1}})^*(h) - \phi^{(E)}_{i_{s+1},i_{s+2}}\circ\phi^{(E)}_{i_s,i_{s+1}}(h)\|_\infty < \delta_s$$

for any $s = 0, 2, ...$, any $h \in \mathrm{C}_{\mathbb{R}}(X^{d_{i_s}})$ with $\|h\|_\infty \leq 1$. This implies in particular (by Theorems 2.1 and 2.2 of [7]) that $\mathrm{T}(A_E) \cong \mathrm{T}(A)$. ☐

Let us calculate the trace simplex of A. Note that $\mathrm{T}(A)$ is homeomorphic to the limit of the following affine projective system:

$$\mathcal{M}_1(X) \longleftarrow \mathcal{M}_1(X^{c_1}) \longleftarrow \mathcal{M}_1(X^{c_1c_2}) \longleftarrow \cdots$$

where $\mathcal{M}_1(\cdot)$ denotes the simplex of Borel probability measures and the connecting map θ_i : $\mathcal{M}_1(X^{c_1\cdots c_i}) \to \mathcal{M}_1(X^{c_1\cdots c_{i-1}})$ is given by

$$\theta_i(\delta_{(x_1,...,x_{c_i})}) = \frac{1}{n_i}(\underbrace{\delta_{x_1}+\cdots+\delta_{x_1}}_{s_{i,1}}+\cdots+\underbrace{\delta_{x_{c_i}}+\cdots+\delta_{x_{c_i}}}_{s_{i,c_i}}), \quad x_1,...,x_{c_i} \in X^{c_1\cdots c_{i-1}},$$

where δ_x denotes the Dirac measure concentrated at x and $X^{c_1\cdots c_{i-1}} = X$ if $i = 1$.

The following lemma is a simple observation:

Lemma 4.2. *Let $\tau = (\mu_i)$ be a tracial state on A, where μ_i, $i = 1, 2, ...$, is a probability measure on $X^{c_1\cdots c_{i-1}}$. If μ_i are Dirac measures for sufficiently large i, then τ is extreme.*

Proof. Assume

$$(\mu_i) = \alpha(\nu_i^{(1)}) + (1-\alpha)(\nu_i^{(2)})$$

for some $\alpha \in (0,1)$, where $\nu_i^{(1)}$ and $\nu_i^{(2)}$ are probability measures on $X^{c_1\cdots c_{i-1}}$. Since μ_i is extreme for sufficiently large i, we have that

$$\nu_i^{(1)} = \nu_i^{(2)} = \mu_i$$

for sufficiently large i, and hence $(\nu_i^{(1)}) = (\nu_i^{(2)}) = (\mu_i)$, as desired. □

Remark 4.3. Note that, since the multiplicities of the coordinate projections are non-zero ($s_{i,j} \neq 0$), if $\theta_i(\mu)$ is a Dirac measure, then μ must be a Dirac measure.

Pick a point $x = (x_1, ..., x_{c_1\cdots c_i}) \in X^{c_1\cdots c_i}$ (where $X^{c_1\cdots c_i} = X$ if $i=0$), and then consider the trace τ_x of A defined by

$$\tau_x = (..., \delta_x, \delta_{(x,...,x)}, ..., \delta_{(x,...,x)}, ...), \tag{4.1}$$

where, at the stage $i+k+1$ (with the product space $X^{c_1\cdots c_{i+k}}$), the Dirac measure $\delta_{(x,...,x)}$ concentates at the point $(x,...,x) \in (X^{c_1\cdots c_i})^{c_{i+1}\cdots c_{i+k}}$. It is straightforward to verify that τ_x is a trace of A as $(x,...,x)$ is a constant sequence of points in $X^{c_1\cdots c_i}$

By the lemma above, τ_x is an extreme trace. Also note that if $x \neq y$, then $\tau_x \neq \tau_y$. Hence if the seed space X is not a singleton, the trace simplex is not a singleton.

The following lemma is a direct consequence of the Krein-Milman Theorem.

Lemma 4.4. *Let $\mathcal{F} \subseteq \mathrm{C}(X)$ be a finite set, let $\mu \in \mathcal{M}_1(X)$, and let $\varepsilon > 0$. Then, there is $N \in \mathbb{N}$ such that for any $n > N$, there are $x_1, ..., x_n \in X$ such that*

$$|\mu(f) - \frac{1}{n}(f(x_1) + \cdots + f(x_n))| < \varepsilon, \quad f \in \mathcal{F}.$$

Theorem 4.5. *Assume*

$$\lim_{i\to\infty}\lim_{j\to\infty}(\frac{c_i}{n_i})\cdots(\frac{c_{i+j}}{n_{i+j}}) = 1. \tag{4.2}$$

Then the extreme points of $\mathrm{T}(A)$ *are dense, i.e.,* $\mathrm{T}(A)$ *is the Poulsen simplex (of* [25]*—see also* [22]*) if X is not a singleton.*

The trace simplex of the simple Villadsen algebra A_E with non-zero radius of comparison is the Poulsen simplex.

Proof. Let us show that the extreme traces $\tau_x \in \mathrm{T}(A)$ (see above) are dense. Let μ be a tracial state of A, and represent it as

$$\mu = (\mu_1, \mu_2, ...),$$

where μ_i is a probability measure of $X^{c_1\cdots c_{i-1}}$, and $\theta_i(\mu_{i+1}) = \mu_i$, $i = 1, 2, ...$

Let $N(\mathcal{F}; \varepsilon)$ be the fundamental neighborhood of μ

$$\{\tau \in \mathrm{T}(A) : |\mu(f) - \tau(f)| < \varepsilon,\ f \in \mathcal{F}\},$$

where $\mathcal{F} \in A$ is a finite set and $\varepsilon > 0$, and let us show that $\tau_x \in N$ for some $x \in X^{c_1\cdots c_{i-1}}$, $i \in \mathbb{N}$. This will show the first statement of the theorem (as a consequence of Lemma 4.2).

By (4.2) and the proof of Theorem 3.7, there is $i_0 > 0$ such that for all $j > 0$,

$$1 - \frac{c_{i_0} \cdots c_{i_0+j}}{n_{i_0} \cdots n_{i_0+j}} < \frac{\varepsilon}{3} \tag{4.3}$$

and

$$\frac{|\{s_k : s_k = 1, k = 1, ..., c_{i_0} \cdots c_{i_0+j}\}|}{n_{i_0} \cdots n_{i_0+j}} > 1 - \frac{\varepsilon}{3},$$

where

$$\phi_{i_0,j+1} = \mathrm{diag}\{\underbrace{\underbrace{\pi_1^*, ..., \pi_1^*}_{s_1}, ..., \underbrace{\pi^*_{c_{i_0}\cdots c_{i_0+j}}, ..., \pi^*_{c_{i_0}\cdots c_{i_0+j}}}_{s_{c_{i_0}\cdots c_{i_0+j}}}}_{n_{i_0}\cdots n_{i_0+j}}\}.$$

Therefore

$$\frac{\sum_{s_k \geq 2} s_k}{n_{i_0} \cdots n_{i_0+j}} < \frac{\varepsilon}{3}. \tag{4.4}$$

Without loss of generality, we may assume that $\mathcal{F}$ is in the unit ball of $\mathrm{M}_{n_0 \cdots n_{i_0-1}}(\mathrm{C}(X^{c_1 \cdots c_{i_0-1}}))$. Then consider the measure μ_{i_0}. By Lemma 4.4, there is a large enough j_0 that $c_{i_0} \cdots c_{i_0+j_0}$ is large enough that there are

$$x_1, ..., x_{c_{i_0}\cdots c_{i_0+j_0}} \in X^{c_1\cdots c_{i_0-1}}$$

satisfying

$$|\mu_{i_0}(f) - \frac{1}{c_{i_0} \cdots c_{i_0+j_0}}(f(x_1) + f(x_2) + \cdots + f(x_{c_{i_0}\cdots c_{i_0+j_0}}))| < \frac{\varepsilon}{3}, \quad f \in \mathcal{F}.$$

Consider the point

$$x_\mu := (x_1, x_2, ..., x_{c_{i_0}\cdots c_{i_0+j_0}}) \in (X^{c_1\cdots c_{i_0-1}})^{c_{i_0}\cdots c_{i_0+j_0}} = X^{c_1\cdots c_{i_0+j_0}},$$

and consider the trace $\tau_{x_\mu} \in \mathrm{T}(A)$ (see (4.1)). Then, for each $f \in \mathcal{F}$,

$$\begin{aligned}
& |\mu(f) - \tau_{x_\mu}(f)| \\
= \ & |\mu_{i_0}(f) - \frac{1}{n_{i_0} \cdots n_{i_0+j_0}}(\underbrace{\delta_{x_1} + \cdots + \delta_{x_1}}_{s_1} + \cdots + \underbrace{\delta_{x_{c_{i_0}\cdots c_{i_0+j_0}}} + \cdots + \delta_{x_{c_{i_0}\cdots c_{i_0+j_0}}}}_{s_{c_{i_0}\cdots c_{i_0+j_0}}})(f)| \\
< \ & |\mu_{i_0}(f) - \frac{1}{n_{i_0} \cdots n_{i_0+j_0}}(\delta_{x_1} + \cdots + \delta_{x_{c_{i_0}\cdots c_{i_0+j_0}}})(f)| + \frac{\sum_{s_k \geq 2}(s_k - 1)}{n_{i_0} \cdots n_{i_0+j_0}} \\
< \ & |\mu_{i_0}(f) - \frac{1}{n_{i_0} \cdots n_{i_0+j_0}}(\delta_{x_1} + \cdots + \delta_{x_{c_{i_0}\cdots c_{i_0+j_0}}})(f)| + \frac{\sum_{s_k \geq 2} s_k}{n_{i_0} \cdots n_{i_0+j_0}} \\
< \ & |\mu_{i_0}(f) - \frac{1}{n_{i_0} \cdots n_{i_0+j_0}}(\delta_{x_1} + \cdots + \delta_{x_{c_{i_0}\cdots c_{i_0+j_0}}})(f)| + \frac{\varepsilon}{3} \qquad \text{(by (4.4))} \\
< \ & |\mu_{i_0}(f) - \frac{1}{c_{i_0} \cdots c_{i_0+j_0}}(\delta_{x_1} + \cdots + \delta_{x_{c_{i_0}\cdots c_{i_0+j_0}}})(f)| + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \qquad \text{(by(4.3))} \\
< \ & \varepsilon,
\end{aligned}$$

and this shows the first statement of the theorem.

Now, let A_E be a simple Villadsen algebra with non-zero radius of comparison. By Theorem 3.7, Equation (4.2) holds (and X is not a singleton), and therefore, as shown above, $\mathrm{T}(A)$ is the Poulsen simplex. By Lemma 4.1, $\mathrm{T}(A_E) \cong \mathrm{T}(A)$, and so $\mathrm{T}(A_E)$ is the Poulsen simplex as well. □

Remark 4.6. N.C. Phillips also notes that the trace simplex of a Villadsen algebra is Poulsen (private communication).

Remark 4.7. Note that $\mathrm{T}(A_E)$ is always the Poulsen simplex whenever (4.2) holds and X is not a singleton. This includes the case that $\dim(X) = 0$ (and so by Theorem 3.4, or since A_E is AF, $\mathrm{rc}(A_E) = 0$).

Remark 4.8. Note that the Giol-Kerr system, a dynamical analog of the Villadsen algebra, is constructed as a small perturbation of the non-trivial two-sided shift ([13]). It is worth comparing Theorem 4.5 to the well-known fact that the simplex of invariant Borel probability measures on the non-trivial two-sided shift is isomorphic to the Poulsen simplex ([24]). Is the Giol-Kerr trace simplex also Poulsen (this seems likely), or (a stronger property) is the orbit-cutting subalgebra of the Giol-Kerr system a Villadsen algebra?

4.2. **An intertwining.** In what follows, we shall show (further to Lemma 4.1) that for two suitably close point-evaluation sets, the intertwining maps between the trace simplices actually can be chosen to be induced by C*-algebra homomorphisms between building blocks, and in such a way that the resulting diagram of building blocks commutes exactly up to point evaluations and therefore approximately at the level of traces.

Let there be given two different evaluation sets

$$E_1, E_2, ..., E_i, ... \quad \text{and} \quad F_1, F_2, ..., F_i, ...,$$

with sizes (k_i^E) and (k_i^F) respectively, and both satisfying Condition (2.2) (with respect to the same (n_i)), and assume that, as supernatural numbers,

$$\prod_{i=1}^{\infty}(n_i + k_i^{(E)}) = \prod_{i=1}^{\infty}(n_i + k_i^{(F)}), \tag{4.5}$$

and as real numbers,

$$\lim_{i\to\infty} \frac{(n_1 + k_1^{(E)}) \cdots (n_i + k_i^{(E)})}{(n_1 + k_1^{(F)}) \cdots (n_i + k_i^{(F)})} = 1. \tag{4.6}$$

Lemma 4.9. *With the assumptions* (4.5) *and* (4.6) *above, let A_E and A_F denote the C*-algebras $A(X, (n_i), (k_i^{(E)}), E)$ and $A(X, (n_i), (k_i^{(F)}), F)$, respectively. Let $\delta_1, \delta_2, ...$ be a decreasing sequence of strictly positive numbers with*

$$\sum_{n=1}^{\infty} \delta_n < 1.$$

There is a diagram

$$
\begin{tikzcd}
\mathrm{M}_{n_0}(\mathrm{C}(X)) \arrow[r, "\phi^{(E)}_{1,i_1}"] \arrow[dr, "\phi^{(E,F)}_{1,i_1}"] & \mathrm{M}_{m^{(E)}_{i_1}}(\mathrm{C}(X^{d_{i_1}})) \arrow[r, "\phi^{(E)}_{i_1,i_2}"] & \mathrm{M}_{m^{(E)}_{i_2}}(\mathrm{C}(X^{d_{i_2}})) \arrow[r] \arrow[dr, "\phi^{(E,F)}_{i_2,i_3}"] & \cdots \arrow[r] & A_E \\
\mathrm{M}_{n_0}(\mathrm{C}(X)) \arrow[r, "\phi^{(F)}_{1,i_1}"] & \mathrm{M}_{m^{(F)}_{i_1}}(\mathrm{C}(X^{d_{i_1}})) \arrow[r, "\phi^{(F)}_{i_1,i_2}"] \arrow[ur, "\phi^{(F,E)}_{i_1,i_2}"] & \mathrm{M}_{m^{(F)}_{i_2}}(\mathrm{C}(X^{d_{i_2}})) \arrow[r] & \cdots \arrow[r] & A_F
\end{tikzcd}
\tag{4.7}
$$

with

$$|\tau(\phi^{(F,E)}_{i_{s+1},i_{s+2}} \circ \phi^{(E,F)}_{i_s,i_{s+1}}(h) - \phi^{(E)}_{i_{s+1},i_{s+2}} \circ \phi^{(E)}_{i_s,i_{s+1}}(h))| < \delta_s$$

for any $s = 0, 2, ...,$ any $h \in \mathrm{M}_{m^{(E)}_{i_s}}(\mathrm{C}(X^{d_{i_s}}))$ with $\|h\| \leq 1$, and any $\tau \in \mathrm{T}(\mathrm{M}_{m^{(E)}_{i_{s+2}}}(\mathrm{C}(X^{d_{i_{s+2}}})))$, and, symmetrically,

$$|\tau(\phi^{(E,F)}_{i_{s+1},i_{s+2}} \circ \phi^{(F,E)}_{i_s,i_{s+1}}(h) - \phi^{(F)}_{i_{s+1},i_{s+2}} \circ \phi^{(F)}_{i_s,i_{s+1}}(h))| < \delta_s$$

for any $s = 1, 3, ...,$ any $h \in \mathrm{M}_{m^{(F)}_{i_s}}(\mathrm{C}(X^{d_{i_s}}))$ with $\|h\| \leq 1$, and any $\tau \in \mathrm{T}(\mathrm{M}_{m^{(F)}_{i_{s+2}}}(\mathrm{C}(X^{d_{i_{s+2}}})))$, and, moreover, for each $s = 0, 2, ...,$

$$\phi^{(F,E)}_{i_{s+1},i_{s+2}} \circ \phi^{(E,F)}_{i_s,i_{s+1}} = \mathrm{diag}\{\pi_1^*, ..., \pi^*_{n_{i_s} \cdots n_{i_{s+1}}},\ \text{point evaluations}\},$$

and for each $s = 1, 3, ...,$

$$\phi^{(E,F)}_{i_{s+1},i_{s+2}} \circ \phi^{(F,E)}_{i_s,i_{s+1}} = \mathrm{diag}\{\pi_1^*, ..., \pi^*_{n_{i_s} \cdots n_{i_{s+1}}},\ \text{point evaluations}\}.$$

Definition 4.10. The sequences $(k_i^{(E)})$ and $(k_i^{(F)})$ will be said to be sufficiently close if for any $\delta > 0$, there is an arbitrarily large pair $i_1 > i_1'$ such that

$$1 - \prod_{j=0}^{\infty} \frac{n_{i_1'+j}}{n_{i_1'+j} + k^{(E)}_{i_1'+j}} < \delta,$$

$$\prod_{i=1}^{i_1-1}(n_i + k_i^{(F)}) \text{ is divisible by } \prod_{i=1}^{i_1'-1}(n_i + k_i^{(E)}),$$

and

$$\frac{(n_1 + k_1^{(F)}) \cdots (n_{i_1'-1} + k^{(F)}_{i_1'-1})}{(n_1 + k_1^{(E)}) \cdots (n_{i_1'-1} + k^{(E)}_{i_1'-1})} \cdot \frac{(n_{i_1'} + k^{(F)}_{i_1'}) \cdots (n_{i_1-1} + k^{(F)}_{i_1-1})}{n_{i_1'} \cdots n_{i_1-1}} > 1, \tag{4.8}$$

and, furthermore, there are arbitrarily large $i_2 > i_2'$ such that

$$1 - \prod_{j=0}^{\infty} \frac{n_{i_2'+j}}{n_{i_2'+j} + k^{(F)}_{i_2'+j}} < \delta,$$

$$\prod_{i=1}^{i_2-1}(n_i + k_i^{(E)}) \text{ is divisible by } \prod_{i=1}^{i_2'-1}(n_i + k_i^{(F)}),$$

and

$$\frac{(n_1+k_1^{(E)})\cdots(n_{i_2'-1}+k_{i_2'-1}^{(E)})}{(n_1+k_1^{(F)})\cdots(n_{i_2'-1}+k_{i_2'-1}^{(F)})}\cdot\frac{(n_{i_2'}+k_{i_2'}^{(E)})\cdots(n_{i_2-1}+k_{i_2-1}^{(E)})}{n_{i_2'}\cdots n_{i_2-1}}>1. \tag{4.9}$$

Lemma 4.11. *Under the assumptions* (2.2), (4.5), *and* (4.6), *the sequences* $(k_i^{(E)})$ *and* $(k_i^{(F)})$ *are sufficiently close.*

Proof. We only have to show (4.8) and (4.9). For the given $\delta>0$, choose i_1' sufficiently large that

$$1-\prod_{j=0}^{\infty}\frac{n_{i_1'+j}}{n_{i_1'+j}+k_{i_1'+j}^{(E)}}<\delta.$$

Then, with sufficiently large i_1, by (4.6), we have

$$\begin{aligned}
&\frac{(n_1+k_1^{(F)})\cdots(n_{i_1'-1}+k_{i_1'-1}^{(F)})}{(n_1+k_1^{(E)})\cdots(n_{i_1'-1}+k_{i_1'-1}^{(E)})}\cdot\frac{(n_{i_1'}+k_{i_1'}^{(F)})\cdots(n_{i_1-1}+k_{i_1-1}^{(F)})}{n_{i_1'}\cdots n_{i_1-1}}\\
=\;&\frac{(n_1+k_1^{(F)})\cdots(n_{i_1-1}+k_{i_1-1}^{(F)})}{(n_1+k_1^{(E)})\cdots(n_{i_1'-1}+k_{i_1'-1}^{(E)})}\cdot\frac{1}{n_{i_1'}\cdots n_{i_1-1}}\\
=\;&\frac{(n_1+k_1^{(F)})\cdots(n_{i_1-1}+k_{i_1-1}^{(F)})}{(n_1+k_1^{(E)})\cdots(n_{i_1'-1}+k_{i_1'-1}^{(E)})}\cdot\frac{(n_{i_1'}+k_{i_1'}^{(E)})\cdots(n_{i_1-1}+k_{i_1-1}^{(E)})}{(n_{i_1'}+k_{i_1'}^{(E)})\cdots(n_{i_1-1}+k_{i_1-1}^{(E)})}\cdot\frac{1}{n_{i_1'}\cdots n_{i_1-1}}\\
=\;&\frac{(n_1+k_1^{(F)})\cdots(n_{i_1-1}+k_{i_1-1}^{(F)})}{(n_1+k_1^{(E)})\cdots(n_{i_1-1}+k_{i_1-1}^{(E)})}\cdot\frac{(n_{i_1'}+k_{i_1'}^{(E)})\cdots(n_{i_1-1}+k_{i_1-1}^{(E)})}{n_{i_1'}\cdots n_{i_1-1}}\\
>\;&\frac{(n_1+k_1^{(F)})\cdots(n_{i_1-1}+k_{i_1-1}^{(F)})}{(n_1+k_1^{(E)})\cdots(n_{i_1-1}+k_{i_1-1}^{(E)})}\cdot\frac{n_{i_1'}+k_{i_1'}^{(E)}}{n_{i_1'}}>1.
\end{aligned}$$

So (4.8) holds. A similar argument shows that (4.9) holds. □

Proof of Lemma 4.9. Consider the inductive limit decompositions

$$\mathrm{M}_{n_0}(\mathrm{C}(X))\xrightarrow{\phi_1^{(E)}}\mathrm{M}_{n_0(n_1+k_1^{(E)})}(\mathrm{C}(X^{c_1}))\xrightarrow{\phi_2^{(E)}}\mathrm{M}_{n_0(n_1+k_1^{(E)})(n_2+k_2^{(E)})}(\mathrm{C}(X^{c_1c_2}))\longrightarrow\cdots\longrightarrow A_E,$$

$$\mathrm{M}_{n_0}(\mathrm{C}(X))\xrightarrow{\phi_1^{(F)}}\mathrm{M}_{n_0(n_1+k_1^{(F)})}(\mathrm{C}(X^{c_1}))\xrightarrow{\phi_2^{(F)}}\mathrm{M}_{n_0(n_1+k_1^{(F)})(n_2+k_2^{(F)})}(\mathrm{C}(X^{c_1c_2}))\longrightarrow\cdots\longrightarrow A_F.$$

Since the sequences $(k_i^{(E)})$ and $(k_i^{(F)})$ are sufficiently close (Lemma 4.11), there is a pair $i_1'<i_1$ sufficiently large that

$$1-\prod_{j=0}^{\infty}\frac{n_{i_1'+j}}{n_{i_1'+j}+k_{i_1'+j}^{(E)}}<\delta_1, \tag{4.10}$$

$$\prod_{i=1}^{i_1-1}(n_i+k_i^{(F)}) \text{ is divisible by } \prod_{i=1}^{i_1'-1}(n_i+k_i^{(E)}), \tag{4.11}$$

and (a slight reformulation of (4.8))

$$\frac{n_0(n_1+k_1^{(F)})\cdots(n_{i_1'-1}+k_{i_1'-1}^{(F)})}{n_0(n_1+k_1^{(E)})\cdots(n_{i_1'-1}+k_{i_1'-1}^{(E)})}\cdot\frac{(n_{i_1'}+k_{i_1'}^{(F)})\cdots(n_{i_1-1}+k_{i_1-1}^{(F)})}{n_{i_1'}\cdots n_{i_1-1}}>1. \tag{4.12}$$

Then consider the diagram

$$\begin{tikzcd}
\mathrm{M}_{n_0}(\mathrm{C}(X)) \arrow[r, "\phi_{1,i_1'}^{(E)}"] & \mathrm{M}_{m_{i_1'}^{(E)}}(\mathrm{C}(X^{d_{i_1'}})) \arrow[r, "\phi_{i_1',i_1}^{(E)}"] \arrow[dr, "\tilde{\phi}_{i_1',i_1}^{(E,F)}"] & \mathrm{M}_{m_{i_1}^{(E)}}(\mathrm{C}(X^{d_{i_1}})) \arrow[r] & \cdots \arrow[r] & A_E \\
\mathrm{M}_{n_0}(\mathrm{C}(X)) \arrow[r, "\phi_{1,i_1'}^{(F)}"] & \mathrm{M}_{m_{i_1'}^{(F)}}(\mathrm{C}(X^{d_{i_1'}})) \arrow[r, "\phi_{i_1',i_1}^{(F)}"] & \mathrm{M}_{m_{i_1}^{(F)}}(\mathrm{C}(X^{d_{i_1}})) \arrow[r] & \cdots \arrow[r] & A_F,
\end{tikzcd}$$

where

$$m_i := n_0(n_1+k_1)\cdots(n_{i-1}+k_{i-1}), \quad d_i := c_1\cdots c_{i-1}, \tag{4.13}$$

and $\tilde{\phi}_{i_1',i_1}^{(E,F)} : \mathrm{M}_{m_{i_1'}^{(E)}}(\mathrm{C}(X^{d_{i_1'}})) \to \mathrm{M}_{m_{i_1}^{(F)}}(\mathrm{C}(X^{d_{i_1}}))$ is the map

$$f \mapsto \mathrm{diag}\{\underbrace{f\circ\pi_1, ..., f\circ\pi_{c_{i_1'}\cdots c_{i_1-1}}}_{n_{i_1'}\cdots n_{i_1-1}},\ \text{certain point evaluations}\},$$

where the coordinate projections are exactly the same as for $\phi_{i_1',i_1}^{(E)}$ and the point evaluations are arbitrarily chosen to fill out the space (by (4.11) and (4.12), there is enough room for the map $\tilde{\phi}_{i_1',i_1}^{(E,F)}$ to exist). ((4.12) just says that the desired number of coordinate projections is strictly less than the ratio of the orders of the codomain and domain matrix algebras.)

Write

$$\phi_{1,i_1}^{(E,F)} = \tilde{\phi}_{i_1',i_1}^{(E,F)}\circ\phi_{1,i_1'}^{(E)},$$

and compress the diagram above as

$$\begin{tikzcd}
\mathrm{M}_{n_0}(\mathrm{C}(X)) \arrow[r, "\phi_{1,i_1}^{(E)}"] \arrow[dr, "\phi_{1,i_1}^{(E,F)}"'] & \mathrm{M}_{m_{i_1}^{(E)}}(\mathrm{C}(X^{d_{i_1}})) \arrow[r, "\phi_{i_1}^{(E)}"] & \mathrm{M}_{m_{i_1+1}^{(E)}}(\mathrm{C}(X^{d_{i_1+1}})) \arrow[r] & \cdots \arrow[r] & A_E \\
\mathrm{M}_{n_0}(\mathrm{C}(X)) \arrow[r, "\phi_{1,i_1}^{(F)}"] & \mathrm{M}_{m_{i_1}^{(F)}}(\mathrm{C}(X^{d_{i_1}})) \arrow[r, "\phi_{i_1}^{(F)}"] & \mathrm{M}_{m_{i_1+1}^{(F)}}(\mathrm{C}(X^{d_{i_1+1}})) \arrow[r] & \cdots \arrow[r] & A_F.
\end{tikzcd}$$

There are $i_2' < i_2$ sufficiently large that

$$1-\prod_{j=0}^{\infty}\frac{n_{i_2'+j}}{n_{i_2'+j}+k_{i_2'+j}^{(F)}}<\delta_2,$$

$$\prod_{i=1}^{i_2-1}(n_i+k_i^{(E)}) \text{ is divisible by } \prod_{i=1}^{i_2'-1}(n_i+k_i^{(F)}),$$

and (a slight reformulation of (4.9))

$$\frac{n_0(n_1+k_1^{(E)})\cdots(n_{i_2'-1}+k_{i_2'-1}^{(E)})}{n_0(n_1+k_1^{(F)})\cdots(n_{i_2'-1}+k_{i_2'-1}^{(F)})}\cdot\frac{(n_{i_2'}+k_{i_2'}^{(E)})\cdots(n_{i_2-1}+k_{i_2-1}^{(E)})}{n_{i_2'}\cdots n_{i_2-1}}>1. \tag{4.14}$$

In the same way as above, one obtains a unital homomorphism

$$\tilde{\phi}^{(F,E)}_{i_2',i_2}:\mathrm{M}_{m^{(F)}_{i_2'}}(\mathrm{C}(X^{d_{i_2'}}))\to \mathrm{M}_{m^{(E)}_{i_2}}(\mathrm{C}(X^{d_{i_2}}))$$

$$f\mapsto \mathrm{diag}\{\underbrace{f\circ\pi_1,...,f\circ\pi_{c_{i_2'}\cdots c_{i_2-1}}}_{n_{i_2'}\cdots n_{i_2-1}},\text{point evaluations}\},$$

such that, with

$$\phi^{(F,E)}_{i_1,i_2}=\tilde{\phi}^{(F,E)}_{i_2',i_2}\circ\phi^{(F)}_{i_1,i_2'},$$

and compressing, we have the augmented diagram

$$\begin{tikzcd}
\mathrm{M}_{n_0}(\mathrm{C}(X)) \arrow[r, "\phi^{(E)}_{1,i_1}"] \arrow[dr, "\phi^{(E,F)}_{1,i_1}"] & \mathrm{M}_{m^{(E)}_{i_1}}(\mathrm{C}(X^{d_{i_1}})) \arrow[r, "\phi^{(E)}_{i_1,i_2}"] & \mathrm{M}_{m^{(E)}_{i_2}}(\mathrm{C}(X^{d_{i_2}})) \arrow[r] & \cdots \arrow[r] & A_E \\
\mathrm{M}_{n_0}(\mathrm{C}(X)) \arrow[r, "\phi^{(F)}_{1,i_1}"] & \mathrm{M}_{m^{(F)}_{i_1}}(\mathrm{C}(X^{d_{i_1}})) \arrow[r, "\phi^{(F)}_{i_1,i_2}"] \arrow[ur, "\phi^{(F,E)}_{i_1,i_2}"] & \mathrm{M}_{m^{(F)}_{i_2}}(\mathrm{C}(X^{d_{i_2}})) \arrow[r] & \cdots \arrow[r] & A_F.
\end{tikzcd}$$

Note that, by (4.10) (which says that the point evaluations do not multiplicatively change the order of the the matrix algebra very much),

$$|\tau(\phi^{(F,E)}_{i_1,i_2}\circ\phi^{(E,F)}_{1,i_1}(h)-\phi^{(E)}_{i_1,i_2}\circ\phi^{(E)}_{1,i_1}(h))|<\delta_1,\quad \tau\in\mathrm{T}(A),\ h\in\mathrm{M}_{n_0}(\mathrm{C}(X)),\ \|h\|\le 1,$$

and, trivially, the composition $\phi^{(F,E)}_{i_1,i_2}\circ\phi^{(E,F)}_{1,i_1}$ is the map

$$f\mapsto \mathrm{diag}(\underbrace{f\circ\pi_1,...,f\circ\pi_{c_{i_1}\cdots c_{i_2-1}}}_{n_{i_1}\cdots n_{i_2-1}},\ \text{certain point evaluations}).$$

Repeating this process, we have $i_1<i_2<\cdots$ with

$$1-\prod_{j=0}^{\infty}\frac{n_{i_s+j}}{n_{i_s+j}+k^{(E)}_{i_s+j}}<\delta_s \quad\text{and}\quad 1-\prod_{j=0}^{\infty}\frac{n_{i_s+j}}{n_{i_s+j}+k^{(F)}_{i_s+j}}<\delta_s,\quad s=1,2,..., \tag{4.15}$$

and the infinite intertwining diagram

$$\begin{tikzcd}
\mathrm{M}_{n_0}(\mathrm{C}(X)) \arrow[r, "\phi^{(E)}_{1,i_1}"] \arrow[dr, "\phi^{(E,F)}_{1,i_1}"] & \mathrm{M}_{m^{(E)}_{i_1}}(\mathrm{C}(X^{d_{i_1}})) \arrow[r, "\phi^{(E)}_{i_1,i_2}"] & \mathrm{M}_{m^{(E)}_{i_2}}(\mathrm{C}(X^{d_{i_2}})) \arrow[r] \arrow[dr, "\phi^{(E,F)}_{i_2,i_3}"] & \cdots \arrow[r] & A_E \\
\mathrm{M}_{n_0}(\mathrm{C}(X)) \arrow[r, "\phi^{(F)}_{1,i_1}"] & \mathrm{M}_{m^{(F)}_{i_1}}(\mathrm{C}(X^{d_{i_1}})) \arrow[r, "\phi^{(F)}_{i_1,i_2}"] \arrow[ur, "\phi^{(F,E)}_{i_1,i_2}"] & \mathrm{M}_{m^{(F)}_{i_2}}(\mathrm{C}(X^{d_{i_2}})) \arrow[r] & \cdots \arrow[r] & A_F.
\end{tikzcd}$$

The diagram (4.7) is not commutative. But, by (4.15), we have

$$|\tau(\phi^{(F,E)}_{i_{s+1},i_{s+2}}\circ\phi^{(E,F)}_{i_s,i_{s+1}}(h)-\phi^{(E)}_{i_{s+1},i_{s+2}}\circ\phi^{(E)}_{i_s,i_{s+1}}(h))|<\delta_s$$

for any $s=0,2,...$, any $h\in \mathrm{M}_{m^{(E)}_{i_s}}(\mathrm{C}(X^{d_{i_s}}))$ with $\|h\|\leq 1$, and any $\tau\in \mathrm{T}(\mathrm{M}_{m^{(E)}_{i_{s+2}}}(\mathrm{C}(X^{d_{i_{s+2}}})))$; and similarly

$$|\tau(\phi^{(E,F)}_{i_{s+1},i_{s+2}}\circ\phi^{(F,E)}_{i_s,i_{s+1}}(h)-\phi^{(F)}_{i_{s+1},i_{s+2}}\circ\phi^{(F)}_{i_s,i_{s+1}}(h))|<\delta_s$$

for any $s=1,3,...$, any $h\in \mathrm{M}_{m^{(F)}_{i_s}}(\mathrm{C}(X^{d_{i_s}}))$ with $\|h\|\leq 1$, and any $\tau\in \mathrm{T}(\mathrm{M}_{m^{(F)}_{i_{s+2}}}(\mathrm{C}(X^{d_{i_{s+2}}})))$. That is, the diagram (4.7) is approximately commutative at the level of traces. Note incidentally that (by Theorem 2.1 and 2.2 of [7]) this implies that the simplices $\mathrm{T}(A_E)$ and $\mathrm{T}(A_F)$ are isomorphic. Moreover, as observed, the maps $\phi^{(F,E)}_{i_{s+1},i_{s+2}}\circ\phi^{(E,F)}_{i_s,i_{s+1}}$ and $\phi^{(E)}_{i_{s+1},i_{s+2}}\circ\phi^{(E)}_{i_s,i_{s+1}}$ share the same coordinate projection part, and so also do the maps $\phi^{(E,F)}_{i_{s+1},i_{s+2}}\circ\phi^{(F,E)}_{i_s,i_{s+1}}$ and $\phi^{(F)}_{i_{s+1},i_{s+2}}\circ\phi^{(F)}_{i_s,i_{s+1}}$. □

Remark 4.12. As pointed out, a direct consequence of (4.7) is that the trace simplex of A_E is isomorphic to that of A_F. In the case of a Goodearl algebra (i.e., $c_i=1$, $i=1,2,...$), the trace simplex is isomorphic to the Bauer simplex with extreme boundary X ([18]), while, as we have shown, in the case of the Villadsen algebra (i.e., $s_{i,j}=1$, $i=1,2,...$, $j=1,...,c_i$), the trace simplex is the Poulsen simplex.

5. A uniqueness theorem

Theorem 5.1. *Let X be a connected metrizable compact space, and let $\Delta:\mathrm{C}^+_1(X)\to(0,+\infty)$ be an order-preserving map. Then, for any finite set $\mathcal{F}\subseteq \mathrm{C}(X)$ and any $\varepsilon>0$, there exist finite sets $\mathcal{H}_0,\mathcal{H}_1\subseteq \mathrm{C}^+(X)$ and $\delta>0$ such that for any unital homomorphisms $\phi_0,\phi_1:\mathrm{C}(X)\to \mathrm{M}_{n+k}(\mathrm{C}(X^d))$ with*

$$\phi_0(f)=\mathrm{diag}\{f\circ\pi_1,...,f\circ\pi_n,f(x_1),...,f(x_k)\}$$

and

$$\phi_1(f)=\mathrm{diag}\{f\circ\pi_1,...,f\circ\pi_n,f(y_1),...,f(y_k)\},$$

where $x_1,...,x_k$ and $y_1,...,y_k$ are points of X and $\pi_1,...,\pi_n$ are coordinate projections (possibly with multiplicity), if

$$\tau(\phi_0(h)),\tau(\phi_1(h))>\Delta(h),\quad h\in\mathcal{H}_0,$$

and

$$|\tau(\phi_0(h)-\phi_1(h))|<\delta,\quad h\in\mathcal{H}_1,\ \tau\in \mathrm{T}(\mathrm{M}_{n+k}(\mathrm{C}(X^d))),$$

then there is a unitary $u\in \mathrm{M}_{n+k}(\mathrm{C}(X^d))$ such that

$$\|\phi_0(f)-u^*\phi_1(f)u\|<\varepsilon,\quad f\in\mathcal{F}.$$

Proof. Fix a metric for X. Since X is compact, there is $\eta>0$ such that for any $x,y\in X$ with $\mathrm{dist}(x,y)<3\eta$, one has

$$|f(x)-f(y)|<\varepsilon,\quad f\in\mathcal{F}.$$

Choose an open cover

$$\mathcal{U}=\{U_1,U_2,...,U_{|\mathcal{U}|}\}$$

with each U_i of diameter at most η. Let

$$\mathcal{O} = \{O_1, O_2, ..., O_S\}$$

denote the set of all finite unions of the sets $U_1, U_2, ..., U_{|\mathcal{U}|}$. For each $O \in \mathcal{O}$, define

$$h_O(x) = \max\{1 - \mathrm{dist}(x, O)/\eta, 0\}, \quad x \in X.$$

Also, for each $O \in \mathcal{O}$ with $O_\eta \neq X$, where O_η denotes the η-neighborhood of O (hence $O_{2\eta} \setminus O_\eta \neq \emptyset$, as otherwise O_η is a clopen set and X is assumed to be connected), choose a non-zero positive function $g_O \in \mathrm{C}(X)$ such that $g_O \leq 1$ and

$$\mathrm{supp}(g_O) \subseteq O_{2\eta} \setminus O_\eta.$$

Then

$$\mathcal{H}_0 := \{g_O : O \in \mathcal{O},\ O_\eta \neq X\}, \quad \mathcal{H}_1 := \{h_O : O \in \mathcal{O}\}, \quad \text{and} \quad \delta := \min\{\Delta(g_O) : O \in \mathcal{O}\}$$

have the properties asserted in the statement of Theorem 5.1.

Let ϕ_0 and ϕ_1 be given as in the statement of the lemma. Let $\tilde{X} \subseteq \{x_1, x_2, ..., x_k\}$ be an arbitrary (non-empty) subset. Let $U_{i_1}, U_{i_2}, ..., U_{i_l}$ denote the elements of $\mathcal{U}$ such that $U_{i_j} \cap \tilde{X} \neq \emptyset$, and consider the union

$$O = U_{i_1} \cup \cdots \cup U_{i_l} \in \mathcal{O}.$$

Assume $O_{2\eta} \neq X$ (so that $O_\eta \neq X$), and choose

$$x'_O \in X \setminus O_{2\eta},$$

and then choose $x_O \in X^d$ (e.g., pick $x_O = (x'_O, ..., x'_O)$) such that

$$\pi_1(x_O), ..., \pi_n(x_O) \in X \setminus O_{2\eta}.$$

Then

$$\begin{aligned}
|\tilde{X}| &\leq (n+k)\mathrm{tr}_{x_O}(\phi_0(h_O)) \\
&\leq (n+k)\mathrm{tr}_{x_O}(\phi_1(h_O)) + (n+k)\delta \\
&\leq |O_\eta \cap \{y_1, y_2, ..., y_k\}| + (n+k)\delta \\
&\leq |O_\eta \cap \{y_1, y_2, ..., y_k\}| + (n+k)\Delta(g_O) \\
&\leq |O_\eta \cap \{y_1, y_2, ..., y_k\}| + (n+k)\mathrm{tr}_{x_O}(\phi_1(g_O)) \\
&\leq |O_\eta \cap \{y_1, y_2, ..., y_k\}| + |(O_{2\eta} \setminus O_\eta) \cap \{y_1, y_2, ..., y_k\}| \\
&\leq |O_{2\eta} \cap \{y_1, y_2, ..., y_k\}| \\
&\leq |\tilde{X}_{3\eta} \cap \{y_1, y_2, ..., y_k\}| \qquad (O_{2\eta} \subseteq \tilde{X}_{3\eta}),
\end{aligned}$$

where tr_{x_O} denotes the tracial state of $\mathrm{M}_{n+k}(\mathrm{C}(X^d))$ which is induced by the Dirac measure concentrated on x_O.

If $O_{2\eta} = X$, then $\tilde{X}_{3\eta} = X$. In particular, we still have

$$|\tilde{X}| \leq k = |\tilde{X}_{3\eta} \cap \{y_1, y_2, ..., y_k\}|.$$

That is, we always have

$$|\tilde{X}| \leq |\tilde{X}_{3\eta} \cap \{y_1, y_2, ..., y_k\}|. \tag{5.1}$$

The same calculation shows that, for any subset $\tilde{Y} \subseteq \{y_1, y_2, ..., y_k\}$,

$$|\tilde{Y}| \leq |\tilde{Y}_{3\eta} \cap \{x_1, x_2, ..., x_k\}|. \tag{5.2}$$

Thus, by the Marriage Lemma ([20]), there is a one-to-one correspondence

$$\sigma : \{x_1, x_2, ..., x_k\} \to \{y_1, y_2, ..., y_k\}$$

such that

$$\mathrm{dist}(x_i, \sigma(x_i)) < 3\eta, \quad i = 1, 2, ..., k.$$

Denote by $w \in \mathrm{M}_k(\mathbb{C})$ the permutation unitary that induces σ. Then

$$u = \mathrm{diag}\{1_n, w\}$$

is the desired unitary. □

6. An isomorphism theorem

Theorem 6.1. *Assume X is a solid connected metrizable compact space which is finite dimensional. Let A_E and A_F be two Villadsen algebras with point-evaluation sets E and F respectively (possibly with different numbers of points, but with the same space X and the same (c_i) and $(s_{i,1}, ..., s_{i,c_i})$). (Recall we are assuming (2.2).) Then $A_E \cong A_F$ if, and only if,*

$$\rho(\mathrm{K}_0(A_E)) = \rho(\mathrm{K}_0(A_F)) \quad \textit{and} \quad \mathrm{rc}(A_E) = \mathrm{rc}(A_F),$$

where ρ_{A_E} and ρ_{A_F} are the unique states of the order-unit groups $\mathrm{K}_0(A_E)$ and $\mathrm{K}_0(A_F)$, respectively. (The solidness condition and finite-dimensionality condition on X are not necessary when the radius of comparison is 0.)

Proof. Since a Villadsen algebra is an AH algebra with stable rank one, if $\mathrm{rc}(A_E) = \mathrm{rc}(A_F) = 0$, then A_E and A_F are $\mathcal{Z}$-stable ([27]). Since A_E and A_F are built with the same (c_i) and $(s_{i,1}, ..., s_{i,c_i})$, it follows that $\mathrm{K}_1(A_E) \cong \mathrm{K}_1(A_F)$, and by (2.2) and Lemma 4.1 that $\mathrm{T}(A_E) \cong \mathrm{T}(A_F)$. For each $d \in \mathbb{N}$, since X is connected, we have $\mathrm{K}_0(\mathrm{C}(X^d)) = \mathbb{Z} \oplus H_d$, where H_d consists of the K_0-elements vanishing on traces of $\mathrm{C}(X^d)$. Since the connecting maps are direct sums of coordinate projections and point-evaluation maps, the induced map $\mathrm{K}_0(\mathrm{C}(\mathrm{M}_{n_1}(X^{c_1}))) = \mathbb{Z} \oplus H_d \to \mathbb{Z} \oplus H_{c_1c_2} = \mathrm{K}_0(\mathrm{M}_m(\mathrm{C}(X^{c_1c_2})))$ has the form $(a, b) \mapsto (\phi_1(a), \phi_2(b))$, where ϕ_2 is independent of the point evaluation maps (hence only depends on (c_i) and $(s_{i,1}, ..., s_{i,c_i})$). Then, denoting by H the limit of $(H_{c_1 \cdots c_i})$ (which depends only on (c_i) and $(s_{i,1}, ..., s_{i,c_i})$), one has

$$\mathrm{K}_0(A_E) \cong \rho(\mathrm{K}_0(A_E)) \oplus H \quad \text{and} \quad \mathrm{K}_0(A_F) \cong \rho(\mathrm{K}_0(A_F)) \oplus H$$

as abelian groups. Since $\rho(\mathrm{K}_0(A_E)) = \rho(\mathrm{K}_0(A_F))$ and A_E and A_F are $\mathcal{Z}$-stable (so that the strict order on the K_0-group is determined by the traces), one has $\mathrm{K}_0(A_E) \cong \mathrm{K}_0(A_F)$ as order-unit groups. Also note that, since X is connected, the restrictions of all traces to the K_0-group are zero on H and induce the unique state. So, any isomorphism of the trace simplices is compatible with K_0. Therefore,

$$((\mathrm{K}_0(A_E), \mathrm{K}_0^+(A_E), [1]_0), \mathrm{K}_1(A_E), \mathrm{T}(A), \rho_A) \cong ((\mathrm{K}_0(A_F), \mathrm{K}_0^+(A_F), [1]_0), \mathrm{K}_1(A_F), \mathrm{T}(B), \rho_B),$$

and hence $A_E \cong A_F$ (see [12] and [9]).

Now, assume $\mathrm{rc}(A_E) = \mathrm{rc}(A_F) \neq 0$. Since X is solid, by Theorem 3.4, we have

$$\frac{\dim(X)}{n_0} \cdot \lim_{i\to\infty} \frac{c_1 \cdots c_i}{(n_1 + k_1^{(E)}) \cdots (n_i + k_i^{(E)})} = \frac{\dim(X)}{n_0} \cdot \lim_{i\to\infty} \frac{c_1 \cdots c_i}{(n_1 + k_1^{(F)}) \cdots (n_i + k_i^{(F)})}.$$

Since $\dim(X) < \infty$, both sides are finite non-zero numbers, and

$$\lim_{i\to\infty} \frac{c_1 \cdots c_i}{(n_1 + k_1^{(E)}) \cdots (n_i + k_i^{(E)})} = \lim_{i\to\infty} \frac{c_1 \cdots c_i}{(n_1 + k_1^{(F)}) \cdots (n_i + k_i^{(F)})}.$$

Since the limits are not 0 (otherwise, the radius of comparison is 0), the ratio of the two sequences above converges to 1, i.e.,

$$\lim_{i\to\infty} \frac{(n_1 + k_1^{(E)}) \cdots (n_i + k_i^{(E)})}{(n_1 + k_1^{(F)}) \cdots (n_i + k_i^{(F)})} = 1. \tag{6.1}$$

Also note that, since $\rho(\mathrm{K}_0(A_E)) = \rho(\mathrm{K}_0(A_F))$,

$$\prod_{i=1}^{\infty} (n_i + k_i^{(E)}) = \prod_{i=1}^{\infty} (n_i + k_i^{(F)}). \tag{6.2}$$

Consider the inductive limit constructions

$$\mathrm{M}_{n_0}(\mathrm{C}(X)) \xrightarrow{\phi_1^{(E)}} \mathrm{M}_{n_0(n_1+k_1^{(E)})}(\mathrm{C}(X^{c_1})) \xrightarrow{\phi_2^{(E)}} \mathrm{M}_{n_0(n_1+k_1^{(E)})(n_2+k_2^{(E)})}(\mathrm{C}(X^{c_1c_2})) \longrightarrow \cdots \longrightarrow A_E,$$

$$\mathrm{M}_{n_0}(\mathrm{C}(X)) \xrightarrow{\phi_1^{(F)}} \mathrm{M}_{n_0(n_1+k_1^{(F)})}(\mathrm{C}(X^{c_1})) \xrightarrow{\phi_2^{(F)}} \mathrm{M}_{n_0(n_1+k_1^{(F)})(n_2+k_2^{(F)})}(\mathrm{C}(X^{c_1c_2})) \longrightarrow \cdots \longrightarrow A_F.$$

Choose finite subsets

$$\mathcal{F}_1^{(E)} \subseteq \mathrm{M}_{n_0}(C(X)),\ \mathcal{F}_2^{(E)} \subseteq \mathrm{M}_{n_0(n_1+k_1^{(E)})}(\mathrm{C}(X^{c_1})), \ldots$$

and

$$\mathcal{F}_1^{(F)} \subseteq \mathrm{M}_{n_0}(C(X)),\ \mathcal{F}_2^{(F)} \subseteq \mathrm{M}_{n_0(n_1+k_1^{(F)})}(\mathrm{C}(X^{c_1})), \ldots$$

such that

$$\overline{\bigcup_{i=1}^{\infty} \mathcal{F}_i^{(E)}} = A_E \quad \text{and} \quad \overline{\bigcup_{i=1}^{\infty} \mathcal{F}_i^{(F)}} = A_F.$$

Also, choose $\varepsilon_1 > \varepsilon_2 > \cdots > 0$ such that

$$\sum_{i=1}^{\infty} \varepsilon_i \leq 1.$$

Since A_E and A_F are simple, we have

$$\Delta_E(h) := \inf\{\tau(h) : \tau \in \mathrm{T}(A_E)\} > 0, \quad h \in A_E^+ \setminus \{0\},$$

and

$$\Delta_F(h) := \inf\{\tau(h) : \tau \in \mathrm{T}(A_F)\} > 0, \quad h \in A_F^+ \setminus \{0\}.$$

Applying Theorem 5.1 to $(\mathcal{F}_i^{(E)}, \varepsilon_i)$, we obtain finite sets $\mathcal{H}_{i,0}^{(E)}, \mathcal{H}_{i,1}^{(E)} \subseteq \mathrm{M}_{m_i^{(E)}}(\mathrm{C}(X^{d_i}))$ and $\delta_i^{(E)} > 0$. Applying Theorem 5.1 to $(\mathcal{F}_i^{(F)}, \varepsilon_i)$, we obtain finite sets $\mathcal{H}_{i,0}^{(F)}, \mathcal{H}_{i,1}^{(F)} \subseteq \mathrm{M}_{m_i^{(F)}}(\mathrm{C}(X^{d_i}))$ and $\delta_i^{(F)} > 0$. Set $\delta_i = \min\{\delta_i^{(E)}, \delta_i^{(F)}\}$.

By (6.1) and 6.2, applying Lemma 4.9, we have a diagram

(6.3)
$$\begin{tikzcd}
\mathrm{M}_{n_0}(\mathrm{C}(X)) \arrow[r, "\phi_{1,i_1}^{(E)}"] \arrow[dr, "\phi_{1,i_1}^{(E,F)}"] & \mathrm{M}_{m_{i_1}^{(E)}}(\mathrm{C}(X^{d_{i_1}})) \arrow[r, "\phi_{i_1,i_2}^{(E)}"] & \mathrm{M}_{m_{i_2}^{(E)}}(\mathrm{C}(X^{d_{i_2}})) \arrow[r] \arrow[dr, "\phi_{i_2,i_3}^{(E,F)}"] & \cdots \arrow[r] & A_E \\
\mathrm{M}_{n_0}(\mathrm{C}(X)) \arrow[r, "\phi_{1,i_1}^{(F)}"] & \mathrm{M}_{m_{i_1}^{(F)}}(\mathrm{C}(X^{d_{i_1}})) \arrow[r, "\phi_{i_1,i_2}^{(F)}"] \arrow[ur, "\phi_{i_1,i_2}^{(F,E)}"] & \mathrm{M}_{m_{i_2}^{(F)}}(\mathrm{C}(X^{d_{i_2}})) \arrow[r] & \cdots \arrow[r] & A_F
\end{tikzcd}$$

that is approximately commutative at the level of traces: that is,

$$|\tau(\phi_{i_{s+1},i_{s+2}}^{(F,E)} \circ \phi_{i_s,i_{s+1}}^{(E,F)}(h) - \phi_{i_{s+1},i_{s+2}}^{(E)} \circ \phi_{i_s,i_{s+1}}^{(E)}(h))| < \delta_s,$$

for any $s = 0, 2, ...$, any $h \in \mathrm{M}_{m_{i_s}^{(E)}}(\mathrm{C}(X^{d_{i_s}}))$ with $\|h\| \leq 1$, and any $\tau \in \mathrm{T}(\mathrm{M}_{m_{i_{s+2}}^{(E)}}(\mathrm{C}(X^{d_{i_{s+2}}})))$; furthermore,

$$|\tau(\phi_{i_{s+1},i_{s+2}}^{(E,F)} \circ \phi_{i_s,i_{s+1}}^{(F,E)}(h) - \phi_{i_{s+1},i_{s+2}}^{(F)} \circ \phi_{i_s,i_{s+1}}^{(F)}(h))| < \delta_s,$$

for any $s = 1, 3, ...$, any $h \in \mathrm{M}_{m_{i_s}^{(F)}}(\mathrm{C}(X^{d_{i_s}}))$ with $\|h\| \leq 1$, and any $\tau \in \mathrm{T}(\mathrm{M}_{m_{i_{s+2}}^{(F)}}(\mathrm{C}(X^{d_{i_{s+2}}})))$; moreover, the maps $\phi_{i_{s+1},i_{s+2}}^{(F,E)} \circ \phi_{i_s,i_{s+1}}^{(E,F)}$ and $\phi_{i_{s+1},i_{s+2}}^{(E)} \circ \phi_{i_s,i_{s+1}}^{(E)}$ share the same coordinate projection part—more precisely, these maps satisfy the requirements of Theorem 5.1 for ϕ_0 and ϕ_1—, and so also do the maps $\phi_{i_{s+1},i_{s+2}}^{(E,F)} \circ \phi_{i_s,i_{s+1}}^{(F,E)}$ and $\phi_{i_{s+1},i_{s+2}}^{(F)} \circ \phi_{i_s,i_{s+1}}^{(F)}$.

Therefore, by Theorem 5.1, there are unitaries

$$u_2^{(E)} \in \mathrm{M}_{m_{i_2}^{(E)}}(\mathrm{C}(X^{d_{i_2}})), \quad u_4^{(E)} \in \mathrm{M}_{m_{i_4}^{(E)}}(\mathrm{C}(X^{d_{i_4}})), \ ...$$

and

$$u_3^{(F)} \in \mathrm{M}_{m_{i_3}^{(F)}}(\mathrm{C}(X^{d_{i_3}})), \quad u_5^{(F)} \in \mathrm{M}_{m_{i_5}^{(F)}}(\mathrm{C}(X^{d_{i_5}})), \ ...$$

such that

$$\|\phi_{i_{s+1},i_{s+2}}^{(F,E)} \circ \phi_{i_s,i_{s+1}}^{(E,F)}(f) - (u_{s+2}^{(E)})^*(\phi_{i_{s+1},i_{s+2}}^{(E)} \circ \phi_{i_s,i_{s+1}}^{(E)}(f))u_{s+2}^{(E)}\| < \varepsilon_s$$

for any $s = 0, 2, 4...$, any $f \in \mathcal{F}_{i_s}^{(E)} \subseteq \mathrm{M}_{m_{i_s}^{(E)}}(\mathrm{C}(X^{d_{i_s}}))$; and, furthermore,

$$\|\phi_{i_{s+1},i_{s+2}}^{(E,F)} \circ \phi_{i_s,i_{s+1}}^{(F,E)}(f) - (u_{s+2}^{(F)})^*\phi_{i_{s+1},i_{s+2}}^{(F)} \circ \phi_{i_s,i_{s+1}}^{(F)}(f)u_{s+2}^{(F)}\| < \varepsilon_s$$

for any $s = 1, 3, ...$, any $f \in \mathcal{F}_{i_s}^{(F)} \subseteq \mathrm{M}_{m_{i_s}^{(F)}}(\mathrm{C}(X^{d_{i_s}}))$.

In other words, the ith triangle of the diagram

$$
\begin{tikzcd}
\mathrm{M}_{n_0}(\mathrm{C}(X)) \arrow[r, "\phi^{(E)}_{1,i_1}"] \arrow[dr, "\phi^{(E,F)}_{1,i_1}"'] & \mathrm{M}_{m^{(E)}_{i_1}}(\mathrm{C}(X^{d_{i_1}})) \arrow[r, "\mathrm{ad}(u_2^{(E)})\circ\phi^{(E)}_{i_1,i_2}"] & \mathrm{M}_{m^{(E)}_{i_2}}(\mathrm{C}(X^{d_{i_2}})) \arrow[r] \arrow[dr, "\phi^{(E,F)}_{i_2,i_3}"] & \cdots \arrow[r] & A_E \\
\mathrm{M}_{n_0}(\mathrm{C}(X)) \arrow[r, "\phi^{(F)}_{1,i_1}"'] & \mathrm{M}_{m^{(F)}_{i_1}}(\mathrm{C}(X^{d_{i_1}})) \arrow[r, "\phi^{(F)}_{i_1,i_2}"'] \arrow[ur, "\phi^{(F,E)}_{i_1,i_2}"] & \mathrm{M}_{m^{(F)}_{i_2}}(\mathrm{C}(X^{d_{i_2}})) \arrow[r, "\mathrm{ad}(u_3^{(F)})\circ\phi^{(F)}_{i_1,i_2}"'] & \cdots \arrow[r] & A_F
\end{tikzcd}
\tag{6.4}
$$

is approximately commutative (pointwise) in norm, to within tolerance $(\mathcal{F}_s^{(E)}, \varepsilon_s)$ or $(\mathcal{F}_s^{(F)}, \varepsilon_s)$. Then, by the approximate intertwining argument (Theorems 2.1 and 2.2 of [7]), we have

$$A_E \cong A_F,$$

as desired. □

Note that the assumptions on X to be solid and to be finite dimensional are only used to get Equation (6.1). Thus, if $k_i^{(E)} = k_i^{(F)}$, $i = 1, 2, ...$, then Equation 6.1 automatically holds, and then the same argument shows that $A_E \cong A_F$ for an arbitrary compact metrizable seed space X. That is, the Villadsen algebra in this case is independent of the location of the point evaluations:

Corollary 6.2. *Let X be a connected metrizable compact space. Let A_E and A_F be two Villadsen algebras with point-evaluation sets E and F of the same size (with the same space X and the same (c_i) and $(s_{i,1}, ..., s_{i,c_i})$, and same (k_i)). (Recall we are assuming* (2.2)*.) Then $A_E \cong A_F$.*

Proof. Since E and F have the same size, Equation (6.1) holds, and then the same argument as in the proof of Theorem 6.1 shows that $A_E \cong A_F$. □

Remark 6.3. In the case of Villadsen algebras in the strict sense of [31], i.e., with coordinate projections of multiplicity one (and rapid dimension growth), both the trace simplex of $A(X, (n_i), (k_i))$ and the trace simplex of $A(X^2, (n_i), (k_i))$ are isomorphic to the Poulsen simplex (Theorem 4.5). However, the algebras $A(X, (n_i), (k_i))$ and $A(X^2, (n_i), (k_i))$ are not isomorphic in general as their radii of comparison (see [29]) are different (since by assumption $\dim(X) \neq \dim(X^2)$). In the case that X is contractible, we will show below (Corollary 7.8) that the slightly expanded class of C*-algebras based on either a given seed space X or a finite Cartesian power of X is in fact classified by the order-unit K_0-group and the radius of comparison.

Remark 6.4. In the case of a Goodearl algebra ([18]), i.e., with only one coordinate projection, $\mathcal{Z}$-stability always holds, as the mean dimension in the sense of [23] is always zero (Theorem 3.4). In this case, (2.2) may or may not hold. If it holds, then Theorem 6.1 and Corollary 6.2 are applicable. If not—by [18], this is the case for real rank zero—then in any case the trace simplex is determined as the state space of the K_0-group and the conclusions of Theorem 6.1 and Corollary 6.2 still hold. In fact, in general, the failure of (2.2) is equivalent to real rank zero, and so the hypothesis of (2.2) in Theorem 6.1 and Corollary 6.2 is unnecessary.

7. Let (n_i) vary

We shall show the following theorem in this section:

Theorem 7.1. *Let X be a K-contractible (i.e., $\mathrm{K}_0(\mathrm{C}(X)) = \mathbb{Z}$ and $\mathrm{K}_1(\mathrm{C}(X)) = \{0\}$) solid metrizable compact space which is finite-dimensional. Let*

$$A := A(X, (n_i^{(A)}), (k_i^{(A)}), E^{(A)}) \quad \textit{and} \quad B := B(X, (n_i^{(B)}), (k_i^{(B)}), F^{(B)})$$

be Villadsen algebras (with coordinate projections of arbitrary (non-zero) multiplicity). Then $A \cong B$ if, and only if,

$$\mathrm{K}_0(A) \cong \mathrm{K}_0(B), \quad \mathrm{T}(A) \cong \mathrm{T}(B), \quad \textit{and} \quad \mathrm{rc}(A) = \mathrm{rc}(B).$$

Moreover, if $\mathrm{rc}(A) \neq 0$ (or $\mathrm{rc}(B) \neq 0$), then $\mathrm{T}(A)$ (or $\mathrm{T}(B)$) is redundant in the invariant, that is, $A \cong B$ if, and only if,

$$\mathrm{K}_0(A) \cong \mathrm{K}_0(B) \quad \textit{and} \quad \mathrm{rc}(A) = \mathrm{rc}(B).$$

Remark 7.2. Since X is assumed to be K-contractible, we have

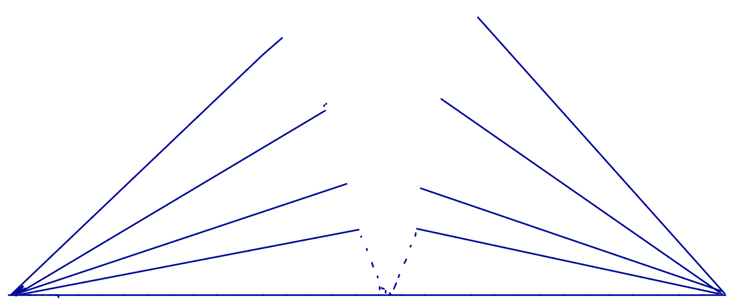
$$\mathrm{K}_0(A) \cong \mathbb{Z}[\frac{1}{n_0^{(A)}}, \frac{1}{n_1^{(A)} + k_1^{(A)}}, ...] \subseteq \mathbb{Q}$$

and

$$\mathrm{K}_0(B) \cong \mathbb{Z}[\frac{1}{n_0^{(B)}}, \frac{1}{n_1^{(B)} + k_1^{(B)}}, ...] \subseteq \mathbb{Q},$$

with the class of the unit being of course $1 \in \mathbb{Z}$.

Remark 7.3. All contractible spaces are K-contractible, but not all K-contractible spaces are contractible. Instances of this are the 2-skeleton of the Poincaré homology 3-sphere (or the Poincaré homology 3-sphere with a small open ball removed), and the join of two infinite brooms:



7.1. **An intertwining diagram.**

Lemma 7.4. *With X a metrizable compact space, let*

$$A := A(X, (n_i^{(A)}), (k_i^{(A)}), E^{(A)}) \quad \textit{and} \quad B := B(X, (n_i^{(B)}), (k_i^{(B)}), F^{(B)})$$

be Villadsen algebras. Assume that

$$n_0^{(A)} \prod_{i=1}^{\infty} (n_i^{(A)} + k_i^{(A)}) = n_0^{(B)} \prod_{i=1}^{\infty} (n_i^{(B)} + k_i^{(B)}), \tag{7.1}$$

as supernatural numbers, and

$$\frac{1}{n_0^{(A)}}\prod_{i=1}^{\infty}\frac{c_i^{(A)}}{n_i^{(A)}+k_i^{(A)}} = \frac{1}{n_0^{(B)}}\prod_{i=1}^{\infty}\frac{c_i^{(B)}}{n_i^{(B)}+k_i^{(B)}} \neq 0, \tag{7.2}$$

as real numbers. Let $\delta_1, \delta_2, ...$ *be a decreasing sequence of strictly positive numbers with*

$$\sum_{n=1}^{\infty}\delta_n < 1.$$

Then there is a diagram

$$\begin{tikzcd}
\mathrm{M}_{n_0^{(A)}}(\mathrm{C}(X)) \arrow[r, "\phi^{(A)}_{1,i_1}"] \arrow[dr, "\phi^{(A,B)}_{1,i_1}"] & \mathrm{M}_{m_{i_1}^{(A)}}(\mathrm{C}(X^{d_{i_1}^{(A)}})) \arrow[r, "\phi^{(A)}_{i_1,i_2}"] & \mathrm{M}_{m_{i_2}^{(A)}}(\mathrm{C}(X^{d_{i_2}^{(A)}})) \arrow[r] \arrow[dr, "\phi^{(A,B)}_{i_2,i_3}"] & \cdots \arrow[r] & A \\
\mathrm{M}_{n_0^{(B)}}(\mathrm{C}(X)) \arrow[r, "\phi^{(B)}_{1,i_1}"] & \mathrm{M}_{m_{i_1}^{(B)}}(\mathrm{C}(X^{d_{i_1}^{(B)}})) \arrow[r, "\phi^{(B)}_{i_1,i_2}"] \arrow[ur, "\phi^{(B,A)}_{i_1,i_2}"] & \mathrm{M}_{m_{i_2}^{(B)}}(\mathrm{C}(X^{d_{i_2}^{(B)}})) \arrow[r] & \cdots \arrow[r] & B,
\end{tikzcd} \tag{7.3}$$

where

$$m_i := n_0(n_1+k_1)\cdots(n_{i-1}+k_{i-1}), \quad d_i := c_1\cdots c_{i-1},$$

such that

$$|\tau(\phi^{(B,A)}_{i_{s+1},i_{s+2}} \circ \phi^{(A,B)}_{i_s,i_{s+1}}(h) - \phi^{(A)}_{i_{s+1},i_{s+2}} \circ \phi^{(A)}_{i_s,i_{s+1}}(h))| < \delta_{i_s}$$

for any $s = 0, 2, ...,$ *any* $h \in \mathrm{M}_{m_{i_s}^{(A)}}(\mathrm{C}(X^{d_{i_s}^{(A)}}))$ *with* $\|h\| \leq 1$*, and any* $\tau \in \mathrm{T}(\mathrm{M}_{m_{i_{s+2}}^{(A)}}(\mathrm{C}(X^{d_{i_{s+2}}^{(A)}})))$*; and, symmetrically,*

$$|\tau(\phi^{(A,B)}_{i_{s+1},i_{s+2}} \circ \phi^{(B,A)}_{i_s,i_{s+1}}(h) - \phi^{(B)}_{i_{s+1},i_{s+2}} \circ \phi^{(B)}_{i_s,i_{s+1}}(h))| < \delta_{i_s}$$

for any $s = 1, 3, ...,$ *any* $h \in \mathrm{M}_{m_{i_s}^{(B)}}(\mathrm{C}(X^{d_{i_s}^{(B)}}))$ *with* $\|h\| \leq 1$*, and any* $\tau \in \mathrm{T}(\mathrm{M}_{m_{i_{s+2}}^{(B)}}(\mathrm{C}(X^{d_{i_{s+2}}^{(B)}})))$*; and moreover, for each* $s = 0, 2, ...,$

$$\phi^{(B,A)}_{i_{s+1},i_{s+2}} \circ \phi^{(A,B)}_{i_s,i_{s+1}} = \mathrm{diag}\{P_s, R'_s, \Theta'_s\}$$

and

$$\phi^{(A)}_{i_s,i_{s+2}} = \phi^{(A)}_{i_{s+1},i_{s+2}} \circ \phi^{(A)}_{i_s,i_{s+1}} = \mathrm{diag}\{P_s, R''_s, \Theta''_s\},$$

where P_s *is a (common) coordinate projection map, and* Θ'_s *and* Θ''_s *are point-evaluation maps with*

$$\mathrm{rank}(\Theta'_s) = \mathrm{rank}(\Theta''_s) \quad \textit{and} \quad \frac{\mathrm{rank}(R'_s)}{\mathrm{rank}(\Theta'_s)} = \frac{\mathrm{rank}(R''_s)}{\mathrm{rank}(\Theta''_s)} < \delta_{i_s},$$

and, symmetrically, for each $s = 1, 3, ...,$

$$\phi^{(A,B)}_{i_{s+1},i_{s+2}} \circ \phi^{(B,A)}_{i_s,i_{s+1}} = \mathrm{diag}\{P_s, R'_s, \Theta'_s\}$$

and

$$\phi^{(B)}_{i_s,i_{s+2}} = \phi^{(B)}_{i_{s+1},i_{s+2}} \circ \phi^{(A)}_{i_s,i_{s+1}} = \mathrm{diag}\{P_s, R''_s, \Theta''_s\},$$

where P_s (with s now odd) is a (common) coordinate projection map, and Θ'_s and Θ''_s are point evaluations with

$$\mathrm{rank}(\Theta'_s) = \mathrm{rank}(\Theta''_s) \quad \textit{and} \quad \frac{\mathrm{rank}(R'_s)}{\mathrm{rank}(\Theta'_s)} = \frac{\mathrm{rank}(R''_s)}{\mathrm{rank}(\Theta''_s)} < \delta_{i_s}.$$

Proof. Consider the inductive constructions

$$\mathrm{M}_{n_0^{(A)}}(\mathrm{C}(X)) \xrightarrow{\phi_1^{(A)}} \mathrm{M}_{m_2^{(A)}}(\mathrm{C}(X^{d_2^{(A)}})) \xrightarrow{\phi_2^{(A)}} \mathrm{M}_{m_3^{(A)}}(\mathrm{C}(X^{d_3^{(A)}})) \longrightarrow \cdots \longrightarrow A,$$

$$\mathrm{M}_{n_0^{(B)}}(\mathrm{C}(X)) \xrightarrow{\phi_1^{(B)}} \mathrm{M}_{m_2^{(B)}}(\mathrm{C}(X^{d_2^{(B)}})) \xrightarrow{\phi_2^{(B)}} \mathrm{M}_{m_3^{(B)}}(\mathrm{C}(X^{d_3^{(B)}})) \longrightarrow \cdots \longrightarrow B,$$

where

$$m_i := n_0(n_1 + k_1) \cdots (n_{i-1} + k_{i-1}), \quad d_i := c_1 \cdots c_{i-1}.$$

Set (see (7.2))

$$\gamma := \lim_{i\to\infty} \frac{c_1^{(A)} \cdots c_i^{(A)}}{n_0^{(A)}(n_1^{(A)} + k_1^{(A)}) \cdots (n_i^{(A)} + k_i^{(A)})} = \lim_{i\to\infty} \frac{c_1^{(B)} \cdots c_i^{(B)}}{n_0^{(B)}(n_1^{(B)} + k_1^{(B)}) \cdots (n_i^{(B)} + k_i^{(B)})} \in (0, 1).$$

Without loss of generality, since $k_i^{(A)} > 0$, $i = 1, 2, ...,$ we may assume

$$\delta_1 < \frac{k_1^{(A)}}{n_1^{(A)} + k_1^{(A)}} \quad \text{and} \quad \frac{\frac{3}{4}\delta_1}{1 - \frac{3}{4}\delta_1} < \delta_1 < 1. \tag{7.4}$$

There is $i'_1 > 0$ such that

$$1 - \prod_{j=0}^{\infty} \frac{n_{i'_1+j}^{(A)}}{n_{i'_1+j}^{(A)} + k_{i'_1+j}^{(A)}} < \delta_1, \tag{7.5}$$

and, by Theorem 3.7, i'_1 can be chosen sufficiently large that for all $j = 1, 2, ...,$ the ratio

$$\frac{c_{i'_1}^{(A)} \cdots c_{i'_1+j}^{(A)}}{n_{i'_1}^{(A)} \cdots n_{i'_1+j}^{(A)}}$$

is sufficiently close to 1 that

$$\frac{c_{i'_1}^{(A)} \cdots c_{i'_1+j}^{(A)}}{n_{i'_1}^{(A)} \cdots n_{i'_1+j}^{(A)}}\left(\left(\frac{n_{i'_1}^{(A)} \cdots n_{i'_1+j}^{(A)}}{c_{i'_1}^{(A)} \cdots c_{i'_1+j}^{(A)}} - 1\right) + \frac{\delta_1^2}{6}\right) < \frac{\delta_1^2}{3} \tag{7.6}$$

and

$$\frac{|\{s_k : s_k = 1, \quad k = 1, ..., c_{i'_1}^{(A)} \cdots c_{i'_1+j}^{(A)}\}|}{n_{i'_1}^{(A)} \cdots n_{i'_1+j}^{(A)}} > 1 - \frac{\delta_1^2}{12}, \tag{7.7}$$

where

$$\phi^{(A)}_{i_1',i_1'+j} = \mathrm{diag}\{\underbrace{\underbrace{\pi_1^*,...,\pi_1^*}_{s_1}, ..., \underbrace{\pi^*_{c^{(A)}_{i_1'}\cdots c^{(A)}_{i_1'+j-1}}, ..., \pi^*_{c^{(A)}_{i_1'}\cdots c^{(A)}_{i_1'+j-1}}}_{s_{c^{(A)}_{i_1'}\cdots c^{(A)}_{i_1'+j-1}}}}_{n^{(A)}_{i_1'}\cdots n^{(A)}_{i_1'+j-1}},\ \text{point evaluations}\}.$$

Then, pick $\varepsilon' > 0$ such that

$$\frac{n_0^{(A)}(n_1^{(A)}+k_1^{(A)})\cdots(n^{(A)}_{i_1'-1}+k^{(A)}_{i_1'-1})}{c_1^{(A)}\cdots c^{(A)}_{i_1'-1}} < \frac{1}{\gamma} - \varepsilon', \tag{7.8}$$

and pick $\varepsilon'' > 0$ such that

$$(\frac{1}{\gamma} - \varepsilon')(\gamma + \varepsilon'') < 1. \tag{7.9}$$

By (7.1) and (7.2), there is $i_1 > i_1'$ such that

$$n_0^{(B)}\prod_{i=1}^{i_1-1}(n_i^{(B)}+k_i^{(B)}) \text{ is divisible by } n_0^{(A)}\prod_{i=1}^{i_1'-1}(n_i^{(A)}+k_i^{(A)}), \tag{7.10}$$

$$\frac{c_1^{(B)}\cdots c^{(B)}_{i_1-1}}{n_0^{(B)}(n_1^{(B)}+k_1^{(B)})\cdots(n^{(B)}_{i_1-1}+k^{(B)}_{i_1-1})} < \gamma + \varepsilon'', \tag{7.11}$$

and (since $\prod_{i=1}^\infty c_i^{(B)} = \infty$)

$$\frac{c_1^{(A)}\cdots c^{(A)}_{i_1'-1}}{c_1^{(B)}\cdots c^{(B)}_{i_1-1}} < \frac{\delta_1^2}{12}. \tag{7.12}$$

By Theorem 3.7, one may also assume that for all $j = 1, 2, ...,$

$$\frac{n^{(B)}_{i_1}\cdots n^{(B)}_{i_1+j}}{c^{(B)}_{i_1}\cdots c^{(B)}_{i_1+j}} < 2 \tag{7.13}$$

and

$$\frac{|\{s_k : s_k = 1, \quad k = 1, ..., c^{(B)}_{i_1}\cdots c^{(B)}_{i_1+j}\}|}{n^{(B)}_{i_1}\cdots n^{(B)}_{i_1+j}} > 1 - \frac{\delta_1^2}{12}, \tag{7.14}$$

where

$$\phi^{(B)}_{i_1,i_1+j} = \mathrm{diag}\{\underbrace{\underbrace{\pi_1^*,...,\pi_1^*}_{s_1}, ..., \underbrace{\pi^*_{c^{(B)}_{i_1}\cdots c^{(B)}_{i_1+j-1}}, ..., \pi^*_{c^{(B)}_{i_1}\cdots c^{(B)}_{i_1+j-1}}}_{s_{c^{(B)}_{i_1}\cdots c^{(B)}_{i_1+j-1}}}}_{n^{(B)}_{i_1}\cdots n^{(B)}_{i_1+j-1}},\ \text{point evaluations}\}.$$

Pick $l_{i'_1,i_1} \in \mathbb{N}$ such that

$$0 \leq c_1^{(B)} \cdots c_{i_1-1}^{(B)} - (c_1^{(A)} \cdots c_{i'_1-1}^{(A)}) l_{i'_1,i_1} < c_1^{(A)} \cdots c_{i'_1-1}^{(A)} \tag{7.15}$$

(so $l_{i'_1,i_1}$ is the integer part of $c_1^{(B)} \cdots c_{i_1-1}^{(B)}/c_1^{(A)} \cdots c_{i'_1-1}^{(A)}$ and it will be used to construct the map (7.17) below).

Then

$$\begin{aligned} & \frac{n_0^{(A)}(n_1^{(A)} + k_1^{(A)}) \cdots (n_{i'_1-1}^{(A)} + k_{i'_1-1}^{(A)})}{n_0^{(B)}(n_1^{(B)} + k_1^{(B)}) \cdots (n_{i'_1-1}^{(B)} + k_{i'_1-1}^{(B)})} \cdot \frac{l_{i'_1,i_1}}{(n_{i'_1}^{(B)} + k_{i'_1}^{(B)}) \cdots (n_{i_1-1}^{(B)} + k_{i_1-1}^{(B)})} \\ \leq\ & \frac{n_0^{(A)}(n_1^{(A)} + k_1^{(A)}) \cdots (n_{i'_1-1}^{(A)} + k_{i'_1-1}^{(A)})}{n_0^{(B)}(n_1^{(B)} + k_1^{(B)}) \cdots (n_{i_1-1}^{(B)} + k_{i_1-1}^{(B)})} \cdot \frac{c_1^{(B)} \cdots c_{i_1-1}^{(B)}}{c_1^{(A)} \cdots c_{i'_1-1}^{(A)}} \qquad \text{(by (7.15))} \\ =\ & \frac{n_0^{(A)}(n_1^{(A)} + k_1^{(A)}) \cdots (n_{i'_1-1}^{(A)} + k_{i'_1-1}^{(A)})}{c_1^{(A)} \cdots c_{i'_1-1}^{(A)}} \cdot \frac{c_1^{(B)} \cdots c_{i_1-1}^{(B)}}{n_0^{(B)}(n_1^{(B)} + k_1^{(B)}) \cdots (n_{i_1-1}^{(B)} + k_{i_1-1}^{(B)})} \\ <\ & (\frac{1}{\gamma} - \varepsilon')(\gamma + \varepsilon'') < 1 \qquad \text{(by (7.8), (7.11), and (7.9)).} \end{aligned} \tag{7.16}$$

Then consider the diagram

$$\begin{tikzcd} \mathrm{M}_{n_0^{(A)}}(\mathrm{C}(X)) \arrow[r, "\phi_{1,i'_1}^{(A)}"] & \mathrm{M}_{m_{i'_1}^{(A)}}(\mathrm{C}(X^{d_{i'_1}^{(A)}})) \arrow[r, "\phi_{i'_1,i_1}^{(A)}"] \arrow[dr, "\tilde{\phi}_{i'_1,i_1}^{(A,B)}"] & \mathrm{M}_{m_{i_1}^{(A)}}(\mathrm{C}(X^{d_{i_1}^{(A)}})) \arrow[r] & \cdots \arrow[r] & A \\ \mathrm{M}_{n_0^{(B)}}(\mathrm{C}(X)) \arrow[r, "\phi_{1,i'_1}^{(B)}"] & \mathrm{M}_{m_{i'_1}^{(B)}}(\mathrm{C}(X^{d_{i'_1}^{(B)}})) \arrow[r, "\phi_{i'_1,i_1}^{(B)}"] & \mathrm{M}_{m_{i_1}^{(B)}}(\mathrm{C}(X^{d_{i_1}^{(B)}})) \arrow[r] & \cdots \arrow[r] & B, \end{tikzcd}$$

where

$$m_i := n_0(n_1 + k_1) \cdots (n_{i-1} + k_{i-1}), \quad d_i := c_1 \cdots c_{i-1},$$

and $\tilde{\phi}_{i'_1,i_1}^{(A,B)} : \mathrm{M}_{m_{i'_1}^{(A)}}(\mathrm{C}(X^{d_{i'_1}^{(A)}})) \to \mathrm{M}_{m_{i_1}^{(B)}}(\mathrm{C}(X^{d_{i_1}^{(B)}}))$ is the map

$$f \mapsto \mathrm{diag}\{\underbrace{f \circ \pi_1, ..., f \circ \pi_{l_{i'_1,i_1}}}_{l_{i'_1,i_1}}, \text{ point evaluations}\}, \tag{7.17}$$

where the point evaluations are arbitrarily chosen (the map $\tilde{\phi}_{i'_1,i_1}^{(A,B)}$, with coordinate projections as specified, exists by (7.10) and (7.16)) (the evaluation points chosen for the map $\tilde{\phi}_{i'_1,i_1}^{(A,B)}$ might not be suitably dense in $X^{d_{i'_1}^{(A)}}$, but the set of evaluation points of the map $\tilde{\phi}_{i'_1,i_1}^{(A,B)} \circ \phi_{1,i'_1}^{(A)}$, which contains the set of evaluation points of the map $\phi_{1,i'_1}^{(A)}$, is suitably dense).

Write

$$\phi_{1,i_1}^{(A,B)} = \tilde{\phi}_{i'_1,i_1}^{(A,B)} \circ \phi_{1,i'_1}^{(A)},$$

and compress the diagram above as

$$\begin{tikzcd}
\mathrm{M}_{n_0^{(A)}}(\mathrm{C}(X)) \arrow[r, "\phi_{1,i_1}^{(A)}"] \arrow[dr, "\phi_{1,i_1}^{(A,B)}"] & \mathrm{M}_{m_{i_1}^{(A)}}(\mathrm{C}(X^{d_{i_1}^{(A)}})) \arrow[r, "\phi_{i_1}^{(A)}"] & \mathrm{M}_{m_{i_1+1}^{(A)}}(\mathrm{C}(X^{d_{i_1+1}^{(A)}})) \arrow[r] & \cdots \arrow[r] & A \\
\mathrm{M}_{n_0^{(B)}}(\mathrm{C}(X)) \arrow[r, "\phi_{1,i_1}^{(B)}"] & \mathrm{M}_{m_{i_1}^{(B)}}(\mathrm{C}(X^{d_{i_1}^{(B)}})) \arrow[r, "\phi_{i_1}^{(B)}"] & \mathrm{M}_{m_{i_1+1}^{(B)}}(\mathrm{C}(X^{d_{i_1+1}^{(B)}})) \arrow[r] & \cdots \arrow[r] & B.
\end{tikzcd}$$

Note that the map $\phi_{1,i_1}^{(A,B)}$ is given by

$$f \mapsto \mathrm{diag}\{\underbrace{f\circ\pi_1, ..., f\circ\pi_{(n_1^{(A)}\cdots n_{i_1'-1}^{(A)})l_{i_1',i_1}}}_{(n_1^{(A)}\cdots n_{i_1'-1}^{(A)})l_{i_1',i_1}}, \text{ certain point evaluations}\}.$$

Without loss of generality, we may assume that

$$\delta_2 < \frac{k_{i_1}^{(B)}}{n_{i_1}^{(B)}+k_{i_1}^{(B)}} \quad \text{and} \quad \frac{\frac{3}{4}\delta_2}{1-\frac{3}{4}\delta_2} < \delta_2 < 1.$$

The same argument as above shows that there are $i_2 > i_2'$ such that:

$$1 - \prod_{j=0}^{\infty} \frac{n_{i_2'+j}^{(B)}}{n_{i_2'+j}^{(B)}+k_{i_2'+j}^{(B)}} < \delta_2, \tag{7.18}$$

$$\frac{c_{i_2'}^{(B)}\cdots c_{i_2'+j}^{(B)}}{n_{i_2'}^{(B)}\cdots n_{i_2'+j}^{(B)}}((\frac{n_{i_2'}^{(B)}\cdots n_{i_2'+j}^{(B)}}{c_{i_2'}^{(B)}\cdots c_{i_2'+j}^{(B)}}-1)+\frac{\delta_2^2}{6}) < \frac{\delta_2^2}{3}, \quad j=1,2,..., \tag{7.19}$$

and

$$\frac{|\{s_k : s_k = 1, \quad k = 1, ..., c_{i_2'}^{(B)}\cdots c_{i_2'+j}^{(B)}\}|}{n_{i_2'}^{(B)}\cdots n_{i_2'+j}^{(B)}} > 1 - \frac{\delta_2^2}{12}, \quad j=1,2,..., \tag{7.20}$$

where

$$\phi_{i_2',i_2'+j}^{(B)} = \mathrm{diag}\{\underbrace{\underbrace{\pi_1^*, ..., \pi_1^*}_{s_1}, ..., \underbrace{\pi^*_{c_{i_2'}^{(B)}\cdots c_{i_2'+j-1}^{(B)}}, ..., \pi^*_{c_{i_2'}^{(B)}\cdots c_{i_2'+j-1}^{(B)}}}_{s_{c_{i_2'}^{(B)}\cdots c_{i_2'+j-1}^{(B)}}}}_{n_{i_2'}^{(B)}\cdots n_{i_2'+j-1}^{(B)}}, \text{ point evaluations}\};$$

$$\frac{|\{s_k : s_k = 1, \quad k = 1, ..., c_{i_2}^{(A)}\cdots c_{i_2+j}^{(A)}\}|}{n_{i_2}^{(A)}\cdots n_{i_2+j}^{(A)}} > 1 - \frac{\delta_2^2}{12}, \quad j=1,2,..., \tag{7.21}$$

where

$$\phi^{(A)}_{i_2,i_2+j}=\mathrm{diag}\{\underbrace{\underbrace{\pi_1^*,...,\pi_1^*}_{s_1},...,\underbrace{\pi^*_{c^{(A)}_{i_2}\cdots c^{(A)}_{i_2+j-1}},...,\pi^*_{c^{(A)}_{i_2}\cdots c^{(A)}_{i_2+j-1}}}_{s_{c^{(A)}_{i_2}\cdots c^{(A)}_{i_2+j-1}}}}_{n^{(A)}_{i_2}\cdots n^{(A)}_{i_2+j-1}},\ \text{point evaluations}\};$$

and

$$n_0^{(A)}\prod_{i=1}^{i_2-1}(n_i^{(A)}+k_i^{(A)})\text{ is divisible by }n_0^{(B)}\prod_{i=1}^{i_2'-1}(n_i^{(B)}+k_i^{(B)}),\tag{7.22}$$

$$\frac{c_1^{(B)}\cdots c^{(B)}_{i_2'-1}}{c_1^{(A)}\cdots c^{(A)}_{i_2-1}}<\frac{\delta_2^2}{12},\tag{7.23}$$

and

$$\frac{n_0^{(B)}(n_1^{(B)}+k_1^{(B)})\cdots(n^{(B)}_{i_2'-1}+k^{(B)}_{i_2'-1})}{n_0^{(A)}(n_1^{(A)}+k_1^{(A)})\cdots(n^{(A)}_{i_2'-1}+k^{(A)}_{i_2'-1})}\cdot\frac{l_{i_2',i_2}}{(n^{(A)}_{i_1'}+k^{(A)}_{i_2'})\cdots(n^{(A)}_{i_2-1}+k^{(A)}_{i_2-1})}<1,\tag{7.24}$$

where

$$0\le c_1^{(A)}\cdots c^{(A)}_{i_2-1}-(c_1^{(B)}\cdots c^{(B)}_{i_2'-1})l_{i_2',i_2}<c_1^{(B)}\cdots c^{(B)}_{i_2'-1}.\tag{7.25}$$

Consider the map $\tilde\phi^{(B,A)}_{i_2',i_2}:\mathrm{M}_{m^{(B)}_{i_2'}}(\mathrm{C}(X^{d^{(B)}_{i_2'}}))\to\mathrm{M}_{m^{(A)}_{i_2}}(\mathrm{C}(X^{d^{(A)}_{i_1}}))$,

$$f\mapsto\mathrm{diag}\{\underbrace{f\circ\pi_1,...,f\circ\pi_{l_{i_2',i_2}}}_{l_{i_2',i_2}},\ \text{point evaluations}\},$$

where the point evaluations are arbitrarily chosen (the map $\tilde\phi^{(B,A)}_{i_2',i_2}$ exists by (7.22) and (7.24); cf. above). Define

$$\phi^{(B,A)}_{i_1,i_2}=\tilde\phi^{(B,A)}_{i_2',i_2}\circ\phi^{(B)}_{i_1,i_2'}$$

and consider the augmented diagram

$$\begin{tikzcd}
\mathrm{M}_{n_0^{(A)}}(\mathrm{C}(X)) \arrow[r, "\phi^{(A)}_{1,i_1}"] \arrow[dr, "\phi^{(A,B)}_{1,i_1}"] & \mathrm{M}_{m^{(A)}_{i_1}}(\mathrm{C}(X^{d^{(A)}_{i_1}})) \arrow[r, "\phi^{(A)}_{i_1,i_2}"] & \mathrm{M}_{m^{(A)}_{i_2}}(\mathrm{C}(X^{d^{(A)}_{i_2}})) \arrow[r] & \cdots \arrow[r] & A \\
\mathrm{M}_{n_0^{(B)}}(\mathrm{C}(X)) \arrow[r, "\phi^{(B)}_{1,i_1}"] & \mathrm{M}_{m^{(B)}_{i_1}}(\mathrm{C}(X^{d^{(B)}_{i_1}})) \arrow[r, "\phi^{(B)}_{i_1,i_2}"] \arrow[ur, "\phi^{(B,A)}_{i_1,i_2}"] & \mathrm{M}_{m^{(B)}_{i_2}}(\mathrm{C}(X^{d^{(B)}_{i_2}})) \arrow[r] & \cdots \arrow[r] & B.
\end{tikzcd}$$

It follows from (7.5) that

$$|\tau(\phi^{(B,A)}_{i_1,i_2}\circ\phi^{(A,B)}_{1,i_1}(h)-\phi^{(A)}_{i_1,i_2}\circ\phi^{(A)}_{1,i_1}(h))|<\delta_1,\quad h\in\mathrm{M}_{n_0^{(A)}}(\mathrm{C}(X)),\ \|h\|\le1.$$

Note that

$$\phi^{(B,A)}_{i_1,i_2}\circ\phi^{(A,B)}_{1,i_1}=(\tilde\phi^{(B,A)}_{i_2',i_2}\circ\phi^{(B)}_{i_1,i_2'}\circ\tilde\phi^{(A,B)}_{i_1',i_1})\circ\phi^{(A)}_{1,i_1'}.$$

By (7.14), we have

$$\phi^{(B)}_{i_1,i'_2} = \mathrm{diag}\{\underbrace{\pi_1^*, ..., \pi^*_{c^{(B)}_{i_1}\cdots c^{(B)}_{i'_2-1}}}_{c^{(B)}_{i_1}\cdots c^{(B)}_{i'_2-1}},\ Q'_0,\ \text{point evaluations}\},$$

where Q'_0 is a coordinate projection map (possibly with multiplicity) with (in view of (7.13))

$$\mathrm{rank}(Q'_0) \leq \frac{\delta_1^2}{12}(n^{(B)}_{i_1}\cdots n^{(B)}_{i'_2-1}) < \frac{\delta_1^2}{6}(c^{(B)}_{i_1}\cdots c^{(B)}_{i'_2-1}).$$

Hence,

$$\tilde{\phi}^{(B,A)}_{i'_2,i_2}\circ\phi^{(B)}_{i_1,i'_2}\circ\tilde{\phi}^{(A,B)}_{i'_1,i_1} = \mathrm{diag}\{\underbrace{\pi_1^*, ..., \pi^*_{l_{i'_1,i_1}(c^{(B)}_{i_1}\cdots c^{(B)}_{i'_2-1})l_{i'_2,i_2}}}_{l_{i'_1,i_1}(c^{(B)}_{i_1}\cdots c^{(B)}_{i'_2-1})l_{i'_2,i_2}},\ Q_0,\ \text{point evaluations}\},$$

where Q_0 is a coordinate projection map (possibly with multiplicity) with

$$\mathrm{rank}(Q_0) \leq \frac{\delta_1^2}{6}l_{i'_1,i_1}(c^{(B)}_{i_1}\cdots c^{(B)}_{i'_2-1})l_{i'_2,i_2}, \tag{7.26}$$

and hence,

$$\phi^{(B,A)}_{i_1,i_2}\circ\phi^{(A,B)}_{1,i_1} = \mathrm{diag}\{\underbrace{\pi_1^*\circ P^{(A)}_{1,i'_1}, ..., \pi^*_{l_{i'_1,i_1}(c^{(B)}_{i_1}\cdots c^{(B)}_{i'_2-1})l_{i'_2,i_2}}\circ P^{(A)}_{1,i'_1}}_{(n^{(A)}_1\cdots n^{(A)}_{i'_1-1})l_{i'_1,i_1}(c^{(B)}_{i_1}\cdots c^{(B)}_{i'_2-1})l_{i'_2,i_2}},\ Q_0\circ P^{(A)}_{1,i'_1},\ \tilde{\Theta}'_1\}, \tag{7.27}$$

where $P^{(A)}_{1,i'_1}$ is the coordinate projection part of the map $\phi^{(A)}_{1,i'_1}$ and $\tilde{\Theta}'_1$ is a point-evaluation map. Also note that, by (7.7) (and note that the multiplicities of coordinate projections are non-zero),

$$\phi^{(A)}_{i'_1,i_2} = \mathrm{diag}\{\pi_1^*, ..., \pi^*_{c^{(A)}_{i'_1}\cdots c^{(A)}_{i_2-1}},\ Q_1,\ \text{point evaluations}\},$$

where Q_1 is a coordinate projection map (possibly with multiplicity) with

$$\mathrm{rank}(Q_1) \leq \frac{\delta_1^2}{12}(n^{(A)}_{i'_1}\cdots n^{(A)}_{i_2-1}). \tag{7.28}$$

We then have

$$\phi^{(A)}_{1,i_2} = \mathrm{diag}\{\underbrace{\pi_1^*\circ P^{(A)}_{1,i'_1}, ..., \pi^*_{c^{(A)}_{i'_1}\cdots c^{(A)}_{i_2-1}}\circ P^{(A)}_{1,i'_1}}_{(n^{(A)}_1\cdots n^{(A)}_{i'_1-1})(c^{(A)}_{i'_1}\cdots c^{(A)}_{i_2-1})},\ Q_1\circ P^{(A)}_{1,i'_1},\ \tilde{\Theta}''_1\}, \tag{7.29}$$

where $\tilde{\Theta}''_1$ is a point-evaluation map.

Let us compare $\phi^{(B,A)}_{i_1,i_2}\circ\phi^{(A,B)}_{1,i_1}$ ((7.27)) with $\phi^{(A)}_{1,i_2}$ ((7.29)). Note that

$$
\begin{aligned}
(7.30)\qquad & (n_1^{(A)}\cdots n_{i_1'-1}^{(A)})l_{i_1',i_1}(c_{i_1}^{(B)}\cdots c_{i_2'-1}^{(B)})l_{i_2',i_2} \\
\leq\ & (n_1^{(A)}\cdots n_{i_1'-1}^{(A)})\frac{c_1^{(B)}\cdots c_{i_1-1}^{(B)}}{c_1^{(A)}\cdots c_{i_1'-1}^{(A)}}(c_{i_1}^{(B)}\cdots c_{i_2'-1}^{(B)})\frac{c_1^{(A)}\cdots c_{i_2-1}^{(A)}}{c_1^{(B)}\cdots c_{i_2'-1}^{(B)}} \qquad \text{(by (7.15) and (7.25))} \\
=\ & n_1^{(A)}\cdots n_{i_1'-1}^{(A)}c_{i_1'}^{(A)}\cdots c_{i_2-1}^{(A)} \\
\leq\ & n_1^{(A)}\cdots n_{i_1'-1}^{(A)}n_{i_1'}^{(A)}\cdots n_{i_2-1}^{(A)} = n_1^{(A)}\cdots n_{i_2-1}^{(A)}.
\end{aligned}
$$

Also note that

$$
\begin{aligned}
(7.31)\qquad & c_1^{(A)}\cdots c_{i_2-1}^{(A)} - (c_1^{(A)}\cdots c_{i_1'-1}^{(A)})l_{i_1',i_1}(c_{i_1}^{(B)}\cdots c_{i_2'-1}^{(B)})l_{i_2',i_2} \\
=\ & c_1^{(A)}\cdots c_{i_2-1}^{(A)} - ((c_1^{(A)}\cdots c_{i_1'-1}^{(A)})l_{i_1',i_1})(c_{i_1}^{(B)}\cdots c_{i_2'-1}^{(B)})l_{i_2',i_2} \\
\leq\ & c_1^{(A)}\cdots c_{i_2-1}^{(A)} - (c_1^{(B)}\cdots c_{i_1-1}^{(B)})(c_{i_1}^{(B)}\cdots c_{i_2'-1}^{(B)})l_{i_2',i_2} \\
& +(c_1^{(A)}\cdots c_{i_1'-1}^{(A)})(c_{i_1}^{(B)}\cdots c_{i_2'-1}^{(B)})l_{i_2',i_2} \qquad \text{(by (7.15))} \\
=\ & c_1^{(A)}\cdots c_{i_2-1}^{(A)} - (c_1^{(B)}\cdots c_{i_2'-1}^{(B)})l_{i_2',i_2} + (c_1^{(A)}\cdots c_{i_1'-1}^{(A)})(c_{i_1}^{(B)}\cdots c_{i_2'-1}^{(B)})l_{i_2',i_2} \\
\leq\ & c_1^{(B)}\cdots c_{i_2'-1}^{(B)} + (c_1^{(A)}\cdots c_{i_1'-1}^{(A)})(c_{i_1}^{(B)}\cdots c_{i_2'-1}^{(B)})l_{i_2',i_2} \qquad \text{(by (7.25))} \\
\leq\ & c_1^{(B)}\cdots c_{i_2'-1}^{(B)} + (c_1^{(A)}\cdots c_{i_1'-1}^{(A)})(c_{i_1}^{(B)}\cdots c_{i_2'-1}^{(B)})\frac{c_1^{(A)}\cdots c_{i_2-1}^{(A)}}{c_1^{(B)}\cdots c_{i_2'-1}^{(B)}} \qquad \text{(by (7.25))} \\
=\ & c_1^{(B)}\cdots c_{i_2'-1}^{(B)} + (c_1^{(A)}\cdots c_{i_1'-1}^{(A)})\frac{c_1^{(A)}\cdots c_{i_2-1}^{(A)}}{c_1^{(B)}\cdots c_{i_1-1}^{(B)}} \\
=\ & c_1^{(B)}\cdots c_{i_2'-1}^{(B)} + \frac{c_1^{(A)}\cdots c_{i_1'-1}^{(A)}}{c_1^{(B)}\cdots c_{i_1-1}^{(B)}}(c_1^{(A)}\cdots c_{i_2-1}^{(A)}) \\
\leq\ & \frac{\delta_1^2}{6}(c_1^{(A)}\cdots c_{i_2-1}^{(A)}) \qquad \text{(by (7.23), (7.12)),}
\end{aligned}
$$

and hence,

$$
\begin{aligned}
(7.32)\qquad & \frac{1}{n_1^{(A)}\cdots n_{i_2-1}^{(A)}}(n_1^{(A)}\cdots n_{i_2-1}^{(A)}-(n_1^{(A)}\cdots n_{i_1'-1}^{(A)})l_{i_1',i_1}(c_{i_1}^{(B)}\cdots c_{i_2'-1}^{(B)})l_{i_2',i_2}) \\
= \ & 1-\frac{n_1^{(A)}\cdots n_{i_1'-1}^{(A)}}{n_1^{(A)}\cdots n_{i_2-1}^{(A)}}\cdot l_{i_1',i_1}(c_{i_1}^{(B)}\cdots c_{i_2'-1}^{(B)})l_{i_2',i_2} \\
= \ & 1-(\frac{n_1^{(A)}\cdots n_{i_1'-1}^{(A)}}{n_1^{(A)}\cdots n_{i_2-1}^{(A)}}\cdot\frac{c_1^{(A)}\cdots c_{i_2-1}^{(A)}}{c_1^{(A)}\cdots c_{i_1'-1}^{(A)}})\cdot\frac{c_1^{(A)}\cdots c_{i_1'-1}^{(A)}}{c_1^{(A)}\cdots c_{i_2-1}^{(A)}}l_{i_1',i_1}(c_{i_1}^{(B)}\cdots c_{i_2'-1}^{(B)})l_{i_2',i_2} \\
= \ & 1-(\frac{c_{i_1'}^{(A)}\cdots c_{i_2-1}^{(A)}}{n_{i_1'}^{(A)}\cdots n_{i_2-1}^{(A)}})\cdot\frac{c_1^{(A)}\cdots c_{i_1'-1}^{(A)}}{c_1^{(A)}\cdots c_{i_2-1}^{(A)}}l_{i_1',i_1}(c_{i_1}^{(B)}\cdots c_{i_2'-1}^{(B)})l_{i_2',i_2} \\
= \ & \frac{c_{i_1'}^{(A)}\cdots c_{i_2-1}^{(A)}}{n_{i_1'}^{(A)}\cdots n_{i_2-1}^{(A)}}((\frac{n_{i_1'}^{(A)}\cdots n_{i_2-1}^{(A)}}{c_{i_1'}^{(A)}\cdots c_{i_2-1}^{(A)}}-1)+(1-\frac{c_1^{(A)}\cdots c_{i_1'-1}^{(A)}}{c_1^{(A)}\cdots c_{i_2-1}^{(A)}}l_{i_1',i_1}(c_{i_1}^{(B)}\cdots c_{i_2'-1}^{(B)})l_{i_2',i_2})) \\
\leq \ & \frac{c_{i_1'}^{(A)}\cdots c_{i_2-1}^{(A)}}{n_{i_1'}^{(A)}\cdots n_{i_2-1}^{(A)}}((\frac{n_{i_1'}^{(A)}\cdots n_{i_2-1}^{(A)}}{c_{i_1'}^{(A)}\cdots c_{i_2-1}^{(A)}}-1)+\frac{\delta_1^2}{6})\qquad (\text{by } (7.31)) \\
< \ & \frac{\delta_1^2}{3}\qquad (\text{by } (7.6)).
\end{aligned}
$$

Then,

$$
\begin{aligned}
& (n_1^{(A)}\cdots n_{i_1'-1}^{(A)})(c_{i_1'}^{(A)}\cdots c_{i_2-1}^{(A)})-(n_1^{(A)}\cdots n_{i_1'-1}^{(A)})l_{i_1',i_1}(c_{i_1}^{(B)}\cdots c_{i_2'-1}^{(B)})l_{i_2',i_2} \\
\leq \ & (n_1^{(A)}\cdots n_{i_1'-1}^{(A)})(n_{i_1'}^{(A)}\cdots n_{i_2-1}^{(A)})-(n_1^{(A)}\cdots n_{i_1'-1}^{(A)})l_{i_1',i_1}(c_{i_1}^{(B)}\cdots c_{i_2'-1}^{(B)})l_{i_2',i_2} \\
\leq \ & \frac{\delta_1^2}{3}(n_1^{(A)}\cdots n_{i_2-1}^{(A)}),
\end{aligned}
$$

and hence,

$$
(7.33)\qquad (c_{i_1'}^{(A)}\cdots c_{i_2-1}^{(A)})-l_{i_1',i_1}(c_{i_1}^{(B)}\cdots c_{i_2'-1}^{(B)})l_{i_2',i_2}\leq\frac{\delta_1^2}{3}\cdot\frac{n_1^{(A)}\cdots n_{i_2-1}^{(A)}}{n_1^{(A)}\cdots n_{i_1'-1}^{(A)}}.
$$

Write

$$
P_1=\mathrm{diag}\{\underbrace{\pi_1^*\circ P_{1,i_1'}^{(A)},...,\pi^*_{l_{i_1',i_1}(c_{i_1}^{(B)}\cdots c_{i_2'-1}^{(B)})l_{i_2',i_2}}\circ P_{1,i_1'}^{(A)}}_{(n_1^{(A)}\cdots n_{i_1'-1}^{(A)})l_{i_1',i_1}(c_{i_1}^{(B)}\cdots c_{i_2'-1}^{(B)})l_{i_2',i_2}}\}.
$$

Note that (by (7.26) and (7.28) in the second step)

$$\begin{aligned}
& |\mathrm{rank}(\tilde{\Theta}_1') - \mathrm{rank}(\tilde{\Theta}_1'')| \\
\leq\ & (n_1^{(A)} \cdots n_{i_1'-1}^{(A)})((c_{i_1'}^{(A)} \cdots c_{i_2-1}^{(A)}) - l_{i_1',i_1}(c_{i_1}^{(B)} \cdots c_{i_2'-1}^{(B)})l_{i_2',i_2} + \mathrm{rank}(Q_0) + \mathrm{rank}(Q_1)) \\
\leq\ & (n_1^{(A)} \cdots n_{i_1'-1}^{(A)})(\frac{\delta_1^2}{3} \cdot \frac{n_1^{(A)} \cdots n_{i_2-1}^{(A)}}{n_1^{(A)} \cdots n_{i_1'-1}^{(A)}} + \frac{\delta_1^2}{6} l_{i_1',i_1}(c_{i_1}^{(B)} \cdots c_{i_2'-1}^{(B)})l_{i_2',i_2} + \frac{\delta_1^2}{12}(n_{i_1'}^{(A)} \cdots n_{i_2-1}^{(A)})) \\
=\ & \frac{5}{12}\delta_1^2(n_1^{(A)} \cdots n_{i_2-1}^{(A)}) + \frac{\delta_1^2}{6}(n_1^{(A)} \cdots n_{i_1'-1}^{(A)})l_{i_1',i_1}(c_{i_1}^{(B)} \cdots c_{i_2'-1}^{(B)})l_{i_2',i_2} \\
<\ & \frac{5}{12}\delta_1^2(n_1^{(A)} \cdots n_{i_2-1}^{(A)}) + \frac{\delta_1^2}{6}(n_1^{(A)} \cdots n_{i_2-1}^{(A)}) \qquad \text{by (7.30)} \\
=\ & \frac{7}{12}\delta_1^2(n_1^{(A)} \cdots n_{i_2-1}^{(A)}).
\end{aligned}$$

Then, $\tilde{\Theta}_1'$ and $\tilde{\Theta}_1''$ can be decomposed as

$$\tilde{\Theta}_1' = \tilde{R}_1' \oplus \Theta_1' \quad \text{and} \quad \tilde{\Theta}_1'' = \tilde{R}_1'' \oplus \Theta_1'',$$

with

$$\mathrm{rank}(\Theta_1') = \mathrm{rank}(\Theta_1'') \quad \text{and} \quad \max\{\mathrm{rank}(\tilde{R}_1'), \mathrm{rank}(\tilde{R}_1'')\} \leq \frac{7}{12}\delta_1^2(n_1^{(A)} \cdots n_{i_2-1}^{(A)}). \tag{7.34}$$

Define

$$R_1' = \mathrm{diag}\{Q_0 \circ P_{1,i_1'}^{(A)},\ \tilde{R}_1'\}$$

and

$$R_1'' = \mathrm{diag}\{\pi^*_{l_{i_1',i_1}(c_{i_1}^{(B)} \cdots c_{i_2'-1}^{(B)})l_{i_2',i_2}+1} \circ P_{1,i_1'}^{(A)}, ..., \pi^*_{c_{i_1'}^{(A)} \cdots c_{i_2-1}^{(A)}} \circ P_{1,i_1'}^{(A)},\ Q_1 \circ P_{1,i_1'}^{(A)},\ \tilde{R}_1''\}.$$

Then we have

$$\phi_{i_1,i_2}^{(B,A)} \circ \phi_{1,i_1}^{(A,B)} = \mathrm{diag}\{P_1, R_1', \Theta_1'\}$$

and

$$\phi_{i_1,i_2}^{(A)} \circ \phi_{1,i_1}^{(A)} = \mathrm{diag}\{P_1, R_1'', \Theta_1''\},$$

with (by (7.26), (7.34), and (7.30))

$$\begin{aligned}
\mathrm{rank}(R_1'') = \mathrm{rank}(R_1') \ \leq\ & \frac{\delta_1^2}{6}(n_1^{(A)} \cdots n_{i_1'-1}^{(A)})l_{i_1',i_1}(c_{i_1}^{(B)} \cdots c_{i_2'-1}^{(B)})l_{i_2',i_2} + \frac{7}{12}\delta_1^2(n_1^{(A)} \cdots n_{i_2-1}^{(A)}) \\
\leq\ & \frac{3\delta_1^2}{4}(n_1^{(A)} \cdots n_{i_2-1}^{(A)}). \qquad (7.35)
\end{aligned}$$

Note that $\mathrm{rank}(\tilde{\Theta}_1'')$—the number of the point evaluations appearing in $\phi_{i_1,i_2}^{(A)} \circ \phi_{1,i_1}^{(A)}$—is at least

$$\frac{k_1^{(A)}}{n_1^{(A)} + k_1^{(A)}}((n_1^{(A)} + k_1^{(A)}) \cdots (n_{i_2-1}^{(A)} + k_{i_2-1}^{(A)})),$$

and hence, by (7.4), is at least

$$\delta_1(n_1^{(A)} \cdots n_{i_2-1}^{(A)}).$$

It then follows from (7.35) that

$$\mathrm{rank}(\Theta_1') = \mathrm{rank}(\Theta_1'') \geq (\delta_1 - \frac{3\delta_1^2}{4})(n_1^{(A)} \cdots n_{i_2-1}^{(A)}),$$

and hence (by (7.35) again), that

$$\frac{\mathrm{rank}(R_1')}{\mathrm{rank}(\Theta_1')} = \frac{\mathrm{rank}(R_1'')}{\mathrm{rank}(\Theta_1'')} \leq \frac{\frac{3}{4}\delta_1^2}{\delta_1 - \frac{3}{4}\delta_1^2} < \delta_1.$$

Repeating this process, we have an intertwining diagram which is approximately commutative in the sense desired. □

7.2. **The isomorphism theorem.** First, we need the following stable uniqueness theorem, which certainly is well known to experts (see, for instance, [5], [21], and [15]). For the reader's convenience, we provide a proof.

Theorem 7.5. *Let X be a K-contractible metrizable compact space (i.e., $\mathrm{K}_0(\mathrm{C}(X)) = \mathbb{Z}$ and $\mathrm{K}_1(\mathrm{C}(X)) = \{0\}$), and let $\Delta : \mathrm{C}(X)^+ \to (0, +\infty)$ be a map. For any finite set $\mathcal{F} \subseteq \mathrm{C}(X)$ and any $\varepsilon > 0$, there exists a finite set $\mathcal{H} \subseteq \mathrm{C}(X)^+$ with $\mathrm{supp}(h) \neq X$ for each $h \in \mathcal{H}$ and there exists $M \in \mathbb{N}$ such that the following property holds: for any unital homomorphisms*

$$\phi, \psi : \mathrm{C}(X) \to \mathrm{M}_n(\mathrm{C}(Y)) \quad \textit{and} \quad \theta : \mathrm{C}(X) \to \mathrm{M}_m(\mathbb{C}) \subseteq \mathrm{M}_m(\mathrm{C}(Y)),$$

where θ is a unital point-evaluation map with $nM < m$, and such that

$$\mathrm{tr}(\theta(h)) > \Delta(h), \quad h \in \mathcal{H},$$

there is a unitary $u \in \mathrm{M}_{n+m}(\mathrm{C}(Y))$ such that

$$\|\mathrm{diag}\{\phi(a), \theta(a)\} - u^*\mathrm{diag}\{\psi(a), \theta(a)\}u\| < \varepsilon, \quad a \in \mathcal{F}.$$

The theorem follows from the following two lemmas.

Lemma 7.6. *Let X be a K-contractible metrizable compact space, and let $\Delta : \mathrm{C}(X)^+ \to (0, +\infty)$ be a map. For any finite set $\mathcal{F} \subseteq \mathrm{C}(X)$ and any $\varepsilon > 0$, there exists a finite set $\mathcal{H} \subseteq \mathrm{C}(X)^+$ with $\mathrm{supp}(h) \neq X$ and $\|h\| \leq 1$ for each $h \in \mathcal{H}$ and there exists $M \in \mathbb{N}$ such that the following property holds: for any unital homomorphisms*

$$\phi, \psi : \mathrm{C}(X) \to \mathrm{M}_n(\mathrm{C}(Y)) \quad \textit{and} \quad \theta : \mathrm{C}(X) \to \mathrm{M}_n(\mathbb{C}) \subseteq \mathrm{M}_n(\mathrm{C}(Y))$$

where θ is a unital point-evaluation map with

$$\mathrm{tr}(\theta(h)) > \Delta(h), \quad h \in \mathcal{H},$$

there is a unitary $u \in \mathrm{M}_{(1+M)n}(\mathrm{C}(Y))$ such that

$$\|\mathrm{diag}\{\phi(a), \underbrace{\theta(a), ..., \theta(a)}_{M}\} - u^*\mathrm{diag}\{\psi(a), \underbrace{\theta(a), ..., \theta(a)}_{M}\}u\| < \varepsilon, \quad a \in \mathcal{F}.$$

Proof. Assume the statement were not true. Then there would be $(\mathcal{F}_0, \varepsilon_0)$ such that for any finite set $\mathcal{H} \subseteq \mathrm{C}(X)^+$ with $\mathrm{supp}(h) \neq X$ and $\|h\| \leq 1$ for each $h \in \mathcal{H}$ and any M, there are unital homomorphisms $\phi, \psi, \theta : \mathrm{C}(X) \to \mathrm{M}_n(\mathrm{C}(Y))$ for some Y and n with θ a unital point-evaluation map with

$$\mathrm{tr}(\theta(h)) > \Delta(h), \quad h \in \mathcal{H},$$

but

$$\|\mathrm{diag}\{\phi(a), \underbrace{\theta(a), ..., \theta(a)}_{M}\} - u^*\mathrm{diag}\{\psi(a), \underbrace{\theta(a), ..., \theta(a)}_{M}\}u\| \geq \varepsilon_0, \quad a \in \mathcal{F}_0,$$

for all unitary $u \in \mathrm{M}_{(1+M)n}(\mathrm{C}(Y))$.

In particular, let $\mathcal{H}_i$, $i = 1, 2, ...$, be an increasing sequence of finite sets such that the union of $\mathcal{H}_i$ is dense in the set of positive contractions of $\mathrm{C}(X)^+$ which do not have full supports, and such that $\mathrm{supp}(h) \neq X$ for each $h \in \mathcal{H}_i$, $i = 1, 2, ...$ There are sequences of unital homomorphisms $\phi_i, \psi_i, \theta_i : \mathrm{C}(X) \to \mathrm{M}_{n_i}(\mathrm{C}(Y_i)) =: B_i$ for some Y_i and n_i with θ_i a unital point-evaluation map with

$$\mathrm{tr}(\theta_i(h)) > \Delta(h), \quad h \in \mathcal{H}_i, \tag{7.36}$$

but

$$\|\mathrm{diag}\{\phi_i(a), \underbrace{\theta_i(a), ..., \theta_i(a)}_{i}\} - u^*\mathrm{diag}\{\psi_i(a), \underbrace{\theta_i(a), ..., \theta_i(a)}_{i}\}u\| \geq \varepsilon_0, \quad a \in \mathcal{F}_0, \tag{7.37}$$

for all unitary $u \in \mathrm{M}_{1+i}(B_i) \cong \mathrm{M}_{(1+i)n_i}(\mathrm{C}(Y_i))$.

Consider the three maps $\Phi := (\phi_i), \Psi := (\psi_i), \Theta := (\theta_i) : \mathrm{C}(X) \to \prod B_i / \bigoplus B_i$. Since X is K-contractible, by the UCT ([26]), the C*-algebra $\mathrm{C}(X)$ is KK-equivalent to $\mathbb{C}$, and hence we have

$$[\Phi] = [\Psi] \quad \text{in } \mathrm{KK}(\mathrm{C}(X), \prod B_i / \bigoplus B_i). \tag{7.38}$$

By (7.36), the map Θ is a unital full embedding (see Definition 2.8 of [5]; we leave the verification to the reader), and then, by Theorem 2.22 together with Theorem 4.5 of [5], there exist $l \in \mathbb{N}$ and a unitary $u \in \mathrm{M}_{1+l}(\prod B_i / \bigoplus B_i)$ such that

$$\|\mathrm{diag}\{\Phi(a), \underbrace{\Theta(a), ..., \Theta(a)}_{l}\} - u^*\mathrm{diag}\{\Psi(a), \underbrace{\Theta(a), ..., \Theta(a)}_{l}\}u\| < \varepsilon_0, \quad a \in \mathcal{F}_0.$$

Lifting u to a unitary of $\prod B_i$ (the relations for a unitary are stable), one has a contradiction with (7.37). ∎

Lemma 7.7. *Let X be a compact metric space, and let $\Delta : \mathrm{C}(X)^+ \to (0, +\infty)$ be a map (density function). For any finite set $\mathcal{F} \subseteq \mathrm{C}(X)$, any $\varepsilon > 0$, and any $M \in \mathbb{N}$, there exist a finite set of positive contractions $\mathcal{H} \subseteq \mathrm{C}(X)^+$ and $L \in \mathbb{N}$, with $\mathrm{supp}(h) \neq X$ for each $h \in \mathcal{H}$, such that if $\theta : \mathrm{C}(X) \to \mathrm{M}_n(\mathbb{C})$, where $n > L$, is a point-evaluation map with*

$$\mathrm{tr}(\theta(h)) > \Delta(h), \quad h \in \mathcal{H},$$

then there are unital homomorphisms $\theta_0 : \mathrm{C}(X) \to \mathrm{M}_{n_0}(\mathbb{C})$ *and* $\theta_1 : \mathrm{C}(X) \to \mathrm{M}_{n_1}(\mathbb{C})$ *for some* n_0, n_1 *with* $n_0 + Mn_1 = n$ *and a (permutation) unitary* u *such that*

$$\|\theta(a) - u^*(\theta_0(a) \oplus \underbrace{\theta_1(a) \oplus \cdots \theta_1(a)}_{M})u\| < \varepsilon, \quad a \in \mathcal{F},$$

and

$$n_0 \leq n_1.$$

Proof. Pick $\delta > 0$ such that $\mathrm{dist}(x, y) < \delta$ implies $|a(x) - a(y)| < \varepsilon$ for all $a \in \mathcal{F}$. Pick a finite set $\{y_1, ..., y_s\} \subseteq X$ which is δ-dense in X. Define

$$\delta_0 := \min\{\mathrm{dist}(y_i, y_j) : i \neq j,\ i, j = 1, ..., s\},$$

and pick non-zero continuous functions $h_i : X \to [0, 1]$, $i = 1, ..., s$, such that

$$h_i(x) = 0, \quad \text{if } \mathrm{dist}(x, y_i) > \delta_0/2.$$

Set $\mathcal{H} = \{h_1, ..., h_s\}$, and pick an integer

$$L > \frac{M^2 + M}{\min\{\Delta(h_i) : i = 1, ..., s\}}.$$

Then $\mathcal{H}$ and L have the property of the lemma.

Indeed, let $\theta : \mathrm{C}(X) \to \mathrm{M}_n$, where $n > L$, be a point-evaluation map satisfying

$$\mathrm{tr}(\theta(h)) > \Delta(h), \quad h \in \mathcal{H}. \tag{7.39}$$

Write $\{x_1, ..., x_n\}$ for the evaluation points of θ. Then, choose a map $\sigma : \{x_1, ..., x_n\} \to \{y_1, ..., y_s\}$ such that

$$\mathrm{dist}(x_i, \sigma(x_i)) < \delta, \quad i = 1, ..., n, \tag{7.40}$$

and for each $j = 1, ..., s$,

$$\sigma(x_i) = y_j \quad \text{if } \mathrm{dist}(x_i, y_j) < \delta_0/2.$$

Let $\theta' : \mathrm{C}(X) \to \mathrm{M}_n(\mathbb{C})$ denote the point-evaluation map at the points $\sigma(x_1), ..., \sigma(x_n)$. Then, it follows from (7.40) and the choice of δ that

$$\|\theta(f) - \theta'(f)\| < \varepsilon, \quad f \in \mathcal{F}. \tag{7.41}$$

Up to a permutation, we have

$$\theta' = \mathrm{diag}\{\underbrace{\mathrm{ev}_{y_1}, ..., \mathrm{ev}_{y_1}}_{m_1}, ..., \underbrace{\mathrm{ev}_{y_s}, ..., \mathrm{ev}_{y_s}}_{m_s}\}.$$

Note that

$$\mathrm{tr}(\theta)(h_i) \leq \frac{m_i}{n}, \quad i = 1, ..., s.$$

By (7.39), we have

$$m_i \geq n\Delta(h_i) \geq L\Delta(h_i) > M^2 + M.$$

Then, write $m_i = Md_i + r_i$ with $0 \leq r_i \leq M-1$, so that, in particular, $d_i > r_i$. Write

$$\theta_0 = \mathrm{diag}\{\underbrace{\overbrace{\mathrm{ev}_{y_1}, ..., \mathrm{ev}_{y_1}}^{r_1}, 0, ..., 0}_{m_1}, ..., \underbrace{\overbrace{\mathrm{ev}_{y_s}, ..., \mathrm{ev}_{y_s}}^{r_s}, 0, ..., 0}_{m_s}\}$$

and

$$\theta_1 = \mathrm{diag}\{\underbrace{\overbrace{0, ..., 0}^{r_1}, \overbrace{\mathrm{ev}_{y_1}, ..., \mathrm{ev}_{y_1}}^{d_1}, 0, ..., 0}_{m_1}, ..., \underbrace{\overbrace{0, ..., 0}^{r_s}, \overbrace{\mathrm{ev}_{y_s}, ..., \mathrm{ev}_{y_s}}^{d_s}, 0, ..., 0}_{m_s}\}.$$

Regard θ_0 as unital homomorphism $\mathrm{C}(X) \to \mathrm{M}_{n_0}(\mathbb{C})$ and regard θ_1 as unital homomorphism $\mathrm{C}(X) \to \mathrm{M}_{n_1}(\mathbb{C})$, where $n_0 = r_1 + \cdots + r_s$ and $n_1 = d_1 + \cdots + d_s$. Then a straightforward calculation shows that

$$\theta' = \theta_0 \oplus \underbrace{\theta_1 \oplus \cdots \theta_1}_{M},$$

and the desired inequality is then just (7.41). □

Proof of Theorem 7.5. Applying Lemma 7.6 to $(\mathcal{F}, \varepsilon/2)$ with respect to the density function $\Delta/4$, we obtain a finite set of positive contractions $\mathcal{H}_1 \subseteq \mathrm{C}(X)^+$ with the property that $\mathrm{supp}(h) \neq X$ for all $h \in \mathcal{H}_1$ and a natural number M_1. Without loss of generality, we may assume that

$$M_1 \geq \frac{1}{\min\{\Delta(h) : h \in \mathcal{H}\}}. \tag{7.42}$$

Applying Lemma 7.7 to $(\mathcal{F} \cup \mathcal{H}_1,\ \min\{\varepsilon/4, \Delta(h)/4 : h \in \mathcal{H}_1\})$ and $2M_1$ with respect to the density function Δ, we obtain a finite set of positive contractions $\mathcal{H}_2$ and $L(\geq 2)$. Moreover, $\mathrm{supp}(h) \neq X$ for all $h \in \mathcal{H}_2$. Then $\mathcal{H} := \mathcal{H}_1 \cup \mathcal{H}_2$ and $M := 2LM_1$ satisfy the condition of the theorem.

Indeed, let $\phi, \psi : \mathrm{C}(X) \to \mathrm{M}_n(\mathrm{C}(Y))$ and $\theta : \mathrm{C}(X) \to \mathrm{M}_m(\mathrm{C}(Y))$ be unital homomorphisms such that θ is a unital point-evaluation map with $n(2LM_1) < m$ (in particular, $L < m$ and $n < m/2M_1$), and

$$\mathrm{tr}(\theta(h)) > \Delta(h), \quad h \in \mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2. \tag{7.43}$$

By Lemma 7.7, there are unital homomorphisms $\theta_0 : \mathrm{C}(X) \to \mathrm{M}_{n_0}(\mathbb{C})$ and $\theta_1 : \mathrm{C}(X) \to \mathrm{M}_{n_1}(\mathbb{C})$ for some n_0, n_1 with $n_0 + 2M_1n_1 = m$ and a permutation unitary $u \in \mathrm{M}_m(\mathbb{C})$ such that

$$\|\theta(a) - u^*(\theta_0(a) \oplus \underbrace{\theta_1(a) \oplus \cdots \theta_1(a)}_{2M_1})u\| < \min\{\varepsilon/4, \Delta(h)/4 : h \in \mathcal{H}_1\}, \quad a \in \mathcal{F} \cup \mathcal{H}_1, \tag{7.44}$$

and

$$n_0 \leq n_1.$$

In particular,

$$\|(1_n \oplus u)(\phi(f) \oplus \theta(f))(1_n \oplus u^*) - \phi(f) \oplus \theta_0(f) \oplus \underbrace{\theta_1(f) \oplus \cdots \oplus \theta_1(f)}_{2M_1}\| < \frac{\varepsilon}{4}, \quad f \in \mathcal{F} \tag{7.45}$$

and

$$\|(1_n \oplus u)(\psi(f) \oplus \theta(f))(1_n \oplus u^*) - \psi(f) \oplus \theta_0(f) \oplus \underbrace{\theta_1(f) \oplus \cdots \oplus \theta_1(f)}_{2M_1}\| < \frac{\varepsilon}{4}, \quad f \in \mathcal{F}. \tag{7.46}$$

Note that, also by (7.44), we have

$$|\mathrm{Tr}(\theta(h)) - \mathrm{Tr}(\theta_0(h)) - 2M_1\mathrm{Tr}(\theta_1(h))| < m\Delta(h)/4, \quad h \in \mathcal{H}_1,$$

Hence, by (7.43) (and note that h is a positive contraction),

$$2M_1\mathrm{Tr}(\theta_1(h)) > \mathrm{Tr}(\theta(h)) - \mathrm{Tr}(\theta_0(h)) - m\Delta(h)/4 > m\Delta(h) - n_0 - m\Delta(h)/4 = \frac{3}{4}m\Delta(h) - n_0,$$

and then, by (7.42) (and note that $n_0 \leq n_1$ and $m > 2M_1n_1$),

$$\frac{1}{n_1}\mathrm{Tr}(\theta_1(h)) > \frac{3m}{8M_1n_1}\Delta(h) - \frac{n_0}{2M_1n_1} > \frac{3}{4}(\frac{m}{2M_1n_1})\Delta(h) - \frac{1}{2}\Delta(h) > \frac{3}{4}\Delta(h) - \frac{1}{2}\Delta(h) = \frac{1}{4}\Delta(h).$$

That is,

$$\mathrm{tr}(\theta_1(h)) > \frac{1}{4}\Delta(h), \quad h \in \mathcal{H}_1.$$

Now, consider the maps

$$(\phi \oplus \theta_0) \oplus (\underbrace{\theta_1 \oplus \cdots \oplus \theta_1}_{2M_1}) \quad \text{and} \quad (\psi \oplus \theta_0) \oplus (\underbrace{\theta_1 \oplus \cdots \oplus \theta_1}_{2M_1}).$$

Since

$$n + n_0 \leq \frac{m}{2LM_1} + n_0 = \frac{n_0 + 2M_1n_1}{2LM_1} + n_0 \leq \frac{1 + 2M_1}{4M_1}n_1 + n_1 < 2n_1,$$

it follows from (the conclusion of) Lemma 7.6 that there is a unitary $v \in \mathrm{M}_{n+m}(\mathrm{C}(Y))$ such that

$$\|(\phi(f)\oplus\theta_0(f))\oplus(\underbrace{\theta_1(f) \oplus \cdots \oplus \theta_1(f)}_{2M_1}) - v^*((\psi(f)\oplus\theta_0(f))\oplus(\underbrace{\theta_1(f) \oplus \cdots \oplus \theta_1(f)}_{2M_1}))v\| < \frac{\varepsilon}{2}, \quad f \in \mathcal{F}.$$

Therefore, together with (7.45) and (7.46),

$$\|\phi(f) \oplus \theta(f) - (1_n \oplus u^*)v^*(1_n \oplus u)(\psi(f) \oplus \theta(f))(1_n \oplus u)v(1_n \oplus u)\| < \varepsilon, \quad f \in \mathcal{F},$$

as desired. □

Proof of Theorem 7.1. If $\mathrm{rc}(A) = \mathrm{rc}(B) = 0$, then A and B are $\mathcal{Z}$-stable, and hence $A \cong B$ if, and only if, $(\mathrm{K}_0(A), \mathrm{T}(A)) \cong (\mathrm{K}_0(B), \mathrm{T}(B))$ (note that $\mathrm{K}_0(A)$ and $\mathrm{K}_0(B)$ have a unique state, and hence the pairing maps are automatically isomorphic if $\mathrm{T}(A) \cong \mathrm{T}(B)$).

Now, let us assume $\mathrm{rc}(A) = \mathrm{rc}(B) \neq 0$. Since X is solid, by Theorem 3.4,

$$\frac{\dim(X)}{n_0^{(A)}}\prod_{i=1}^{\infty}\frac{c_i^{(A)}}{n_i^{(A)} + k_i^{(A)}} = \frac{\dim(X)}{n_0^{(B)}}\prod_{i=1}^{\infty}\frac{c_i^{(B)}}{n_i^{(B)} + k_i^{(B)}}.$$

Since $\dim(X) < \infty$, both sides are finite non-zero numbers, and hence

$$\frac{1}{n_0^{(A)}}\prod_{i=1}^{\infty}\frac{c_i^{(A)}}{n_i^{(A)} + k_i^{(A)}} = \frac{1}{n_0^{(B)}}\prod_{i=1}^{\infty}\frac{c_i^{(B)}}{n_i^{(B)} + k_i^{(B)}} \neq 0;$$

that is, (7.2) of Lemma 7.4 is satisfied. Note that (7.1) of Lemma 7.4 follows from the assumption $\mathrm{K}_0(A) \cong \mathrm{K}_0(B)$.

Consider the inductive limit constructions

$$\mathrm{M}_{n_0^{(A)}}(\mathrm{C}(X)) \xrightarrow{\phi_1^{(A)}} \mathrm{M}_{m_2^{(A)}}(\mathrm{C}(X^{d_2^{(A)}})) \xrightarrow{\phi_2^{(A)}} \mathrm{M}_{m_3^{(A)}}(\mathrm{C}(X^{d_3^{(A)}})) \longrightarrow \cdots \longrightarrow A,$$

$$\mathrm{M}_{n_0^{(B)}}(\mathrm{C}(X)) \xrightarrow{\phi_1^{(B)}} \mathrm{M}_{m_2^{(B)}}(\mathrm{C}(X^{d_2^{(B)}})) \xrightarrow{\phi_2^{(B)}} \mathrm{M}_{m_3^{(B)}}(\mathrm{C}(X^{d_3^{(B)}})) \longrightarrow \cdots \longrightarrow B,$$

where

$$m_i := n_0(n_1 + k_1) \cdots (n_{i-1} + k_{i-1}), \quad d_i := c_1 \cdots c_{i-1}.$$

Choose finite subsets

$$\mathcal{F}_1^{(A)} \subseteq \mathrm{M}_{n_0^{(A)}}(C(X)), \mathcal{F}_2^{(A)} \subseteq \mathrm{M}_{n_0^{(A)}(n_1^{(A)}+k_1^{(A)})}(\mathrm{C}(X^{n_1^{(A)}})), ...$$

and

$$\mathcal{F}_1^{(B)} \subseteq \mathrm{M}_{n_0^{(B)}}(C(X)), \mathcal{F}_2^{(B)} \subseteq \mathrm{M}_{n_0^{(B)}(n_1^{(B)}+k_1^{(B)})}(\mathrm{C}(X^{n_1^{(B)}})), ...$$

such that

$$\overline{\bigcup_{i=1}^{\infty} \mathcal{F}_i^{(A)}} = A \quad \text{and} \quad \overline{\bigcup_{i=1}^{\infty} \mathcal{F}_i^{(B)}} = B.$$

Also choose $\varepsilon_1 > \varepsilon_2 > \cdots > 0$ such that

$$\sum_{i=1}^{\infty} \varepsilon_i \leq 1.$$

Since A and B are simple, we have the non-zero density functions

$$\Delta_A(h) = \inf\{\tau(h) : \tau \in \mathrm{T}(A)\}, \quad h \in A^+,$$

and

$$\Delta_B(h) = \inf\{\tau(h) : \tau \in \mathrm{T}(B)\}, \quad h \in B^+.$$

For $i = 0, 2, ...$, applying Theorem 5.1 to $(\mathcal{F}_i^{(A)}, \varepsilon_i/2)$ with respect to $\Delta_A/4$, we obtain finite sets $\mathcal{H}_{i,0}^{(A)}, \mathcal{H}_{i,1}^{(A)} \subseteq \mathrm{M}_{m_i^{(A)}}(\mathrm{C}(X^{d_i^{(A)}}))$ and $\delta_i^{(A)} > 0$. Applying Theorem 7.5 to $(\mathcal{F}_i^{(A)}, \varepsilon_i/2)$ with respect to $\Delta_A/4$, we obtain $M_i^{(A)} > 0$ and a finite set $\mathcal{H}_{i,2}^{(A)} \subseteq \mathrm{M}_{m_i^{(A)}}(\mathrm{C}(X^{d_i^{(A)}}))$ with $\mathrm{supp}(h) \neq X$ for each $h \in \mathcal{H}_{i,2}^{(A)}$

For $i = 1, 3, ...$, applying Theorem 5.1 to $(\mathcal{F}_i^{(B)}, \varepsilon_i/2)$ with respect to $\Delta_B/4$, we obtain finite sets $\mathcal{H}_{i,0}^{(B)}, \mathcal{H}_{i,1}^{(B)} \subseteq \mathrm{M}_{m_i^{(B)}}(\mathrm{C}(X^{d_i^{(B)}}))$ and $\delta_i^{(B)} > 0$. Applying Theorem 7.5 to $(\mathcal{F}_i^{(B)}, \varepsilon_i/2)$ with respect to $\Delta_B/4$, we obtain $M_i^{(B)} > 0$ and a finite set $\mathcal{H}_{i,2}^{(B)} \subseteq \mathrm{M}_{m_i^{(B)}}(\mathrm{C}(X^{d_i^{(B)}}))$ with $\mathrm{supp}(h) \neq X$ for each $h \in \mathcal{H}_{i,2}^{(B)}$

For each $i = 1, 2, ...$, set

$$\mathcal{H}_i^{(A)} = \mathcal{H}_{i,0}^{(A)} \cup \mathcal{H}_{i,1}^{(A)} \cup \mathcal{H}_{i,2}^{(A)}, \quad \mathcal{H}_i^{(B)} = \mathcal{H}_{i,0}^{(B)} \cup \mathcal{H}_{i,1}^{(B)} \cup \mathcal{H}_{i,2}^{(B)},$$

and

$$\delta_i = \min\{\frac{1}{2}, \frac{\delta_i^{(A)}}{4}, \frac{\delta_i^{(B)}}{4}, \frac{1}{M_i^{(A)}}, \frac{1}{M_i^{(B)}}, \frac{1}{20}\Delta_A(h_A), \frac{1}{20}\Delta_B(h_B) : h_A \in \mathcal{H}_i^{(A)}, h_B \in \mathcal{H}_i^{(B)}\}.$$

After telescoping, we may assume that

$$\tau(\phi_{i,i+1}^{(A)}(h)) > \Delta_A(h)/2, \quad h \in \mathcal{H}_i^{(A)},\ \tau \in \mathrm{T}(\mathrm{M}_{m_{i+1}^{(A)}}(\mathrm{C}(X^{d_{i+1}^{(A)}}))),\ i = 0, 1, ..., \tag{7.47}$$

and

$$\tau(\phi_{i,i+1}^{(B)}(h)) > \Delta_B(h)/2, \quad h \in \mathcal{H}_i^{(B)},\ \tau \in \mathrm{T}(\mathrm{M}_{m_{i+1}^{(B)}}(\mathrm{C}(X^{d_{i+1}^{(B)}}))),\ i = 0, 1, ... \tag{7.48}$$

By Lemma 7.4, there is a diagram

$$\begin{tikzcd}
\mathrm{M}_{n_0^{(A)}}(\mathrm{C}(X)) \arrow[r, "\phi_{1,i_1}^{(A)}"] \arrow[dr, "\phi_{1,i_1}^{(A,B)}"] & \mathrm{M}_{m_{i_1}^{(A)}}(\mathrm{C}(X^{d_{i_1}^{(A)}})) \arrow[r, "\phi_{i_1,i_2}^{(A)}"] & \mathrm{M}_{m_{i_2}^{(A)}}(\mathrm{C}(X^{d_{i_2}^{(A)}})) \arrow[r] \arrow[dr, "\phi_{i_2,i_3}^{(A,B)}"] & \cdots \arrow[r] & A \\
\mathrm{M}_{n_0^{(B)}}(\mathrm{C}(X)) \arrow[r, "\phi_{1,i_1}^{(B)}"] & \mathrm{M}_{m_{i_1}^{(B)}}(\mathrm{C}(X^{d_{i_1}^{(B)}})) \arrow[r, "\phi_{i_1,i_2}^{(B)}"] \arrow[ur, "\phi_{i_1,i_2}^{(B,A)}"] & \mathrm{M}_{m_{i_2}^{(B)}}(\mathrm{C}(X^{d_{i_2}^{(B)}})) \arrow[r] & \cdots \arrow[r] & B
\end{tikzcd} \tag{7.49}$$

such that

$$|\tau(\phi_{i_{s+1},i_{s+2}}^{(B,A)} \circ \phi_{i_s,i_{s+1}}^{(A,B)}(h) - \phi_{i_{s+1},i_{s+2}}^{(A)} \circ \phi_{i_s,i_{s+1}}^{(A)}(h))| < \delta_{i_s} \tag{7.50}$$

for any $s = 0, 2, ...$, any $h \in \mathrm{M}_{m_{i_s}^{(A)}}(\mathrm{C}(X^{d_{i_s}^{(A)}}))$ with $\|h\| \leq 1$, and any $\tau \in \mathrm{T}(\mathrm{M}_{m_{i_{s+2}}^{(A)}}(\mathrm{C}(X^{d_{i_{s+2}}^{(A)}})))$, and, symmetrically,

$$|\tau(\phi_{i_{s+1},i_{s+2}}^{(A,B)} \circ \phi_{i_s,i_{s+1}}^{(B,A)}(h) - \phi_{i_{s+1},i_{s+2}}^{(B)} \circ \phi_{i_s,i_{s+1}}^{(B)}(h))| < \delta_{i_s}$$

for any $s = 1, 3, ...$, any $h \in \mathrm{M}_{m_{i_s}^{(B)}}(\mathrm{C}(X^{d_{i_s}^{(B)}}))$ with $\|h\| \leq 1$, and any $\tau \in \mathrm{T}(\mathrm{M}_{m_{i_{s+2}}^{(B)}}(\mathrm{C}(X^{d_{i_{s+2}}^{(B)}})))$, and such that, for each $s = 0, 2, ...$,

$$\phi_{i_{s+1},i_{s+2}}^{(B,A)} \circ \phi_{i_s,i_{s+1}}^{(A,B)} = \mathrm{diag}\{P_s, R_s', \Theta_s'\}$$

and

$$\phi_{i_s,i_{s+2}}^{(A)} = \phi_{i_{s+1},i_{s+2}}^{(A)} \circ \phi_{i_s,i_{s+1}}^{(A)} = \mathrm{diag}\{P_s, R_s'', \Theta_s''\},$$

where P_s is a coordinate projection map, and Θ_s' and Θ_s'' are point-evaluation maps with

$$\mathrm{rank}(\Theta_s') = \mathrm{rank}(\Theta_s'') \quad \text{and} \quad \frac{\mathrm{rank}(R_s')}{\mathrm{rank}(\Theta_s')} = \frac{\mathrm{rank}(R_s'')}{\mathrm{rank}(\Theta_s'')} < \delta_{i_s}, \tag{7.51}$$

and, furthermore, for each $s = 1, 3, ...$,

$$\phi_{i_{s+1},i_{s+2}}^{(A,B)} \circ \phi_{i_s,i_{s+1}}^{(B,A)} = \mathrm{diag}\{P_s, R_s', \Theta_s'\}$$

and

$$\phi_{i_s,i_{s+2}}^{(B)} = \phi_{i_{s+1},i_{s+2}}^{(B)} \circ \phi_{i_s,i_{s+1}}^{(B)} = \mathrm{diag}\{P_s, R_s'', \Theta_s''\},$$

where P_s is a coordinate projection map, and Θ'_s and Θ''_s are point-evaluation maps with

$$\mathrm{rank}(\Theta'_s) = \mathrm{rank}(\Theta''_s) \quad \text{and} \quad \frac{\mathrm{rank}(R'_s)}{\mathrm{rank}(\Theta'_s)} = \frac{\mathrm{rank}(R''_s)}{\mathrm{rank}(\Theta''_s)} < \delta_{i_s}.$$

Also note that, by (7.47) and (7.48),

$$\tau(\phi^{(A)}_{i_{s+1},i_{s+2}} \circ \phi^{(A)}_{i_s,i_{s+1}}(h)) > \Delta_A(h)/2, \quad h \in \mathcal{H}^{(A)}_{i_s},\ \tau \in \mathrm{T}(\mathrm{M}_{m^{(A)}_{i_{s+2}}}(\mathrm{C}(X^{d^{(A)}_{i_{s+2}}}))),\ s = 0, 2, ... \tag{7.52}$$

and

$$\tau(\phi^{(B)}_{i_{s+1},i_{s+2}} \circ \phi^{(B)}_{i_s,i_{s+1}}(h)) > \Delta_B(h)/2, \quad h \in \mathcal{H}^{(B)}_{i_s},\ \tau \in \mathrm{T}(\mathrm{M}_{m^{(B)}_{i_{s+2}}}(\mathrm{C}(X^{d^{(B)}_{i_{s+2}}}))),\ s = 1, 3, ... \tag{7.53}$$

For $s = 0, 2, ...$, let us show the maps $\mathrm{diag}\{P_s, R'_s, \Theta'_s\}$ (which is $\phi^{(B,A)}_{i_{s+1},i_{s+2}} \circ \phi^{(A,B)}_{i_s,i_{s+1}}$) and $\mathrm{diag}\{P_s, R''_s, \Theta''_s\}$ (which is $\phi^{(A)}_{i_{s+1},i_{s+2}} \circ \phi^{(A)}_{i_s,i_{s+1}}$) are approximately unitarily equivalent. For this purpose, by (7.51), we may assume that the $\Theta'_s(1) = \Theta''_s(1)$ (and hence $R'_s(1) = R''_s(1)$). Then denote $1_P = P_s(1)$, $1_\Theta = \Theta'_s(1)$, and $1_R = R'_s(1)$.

Consider $\mathrm{diag}\{P_s, \Theta'_s\}$ and $\mathrm{diag}\{P_s, \Theta''_s\}$ which are maps

$$\mathrm{M}_{m^{(A)}_{i_s}}(\mathrm{C}(X^{d^{(A)}_{i_s}})) \to (1_P + 1_\Theta)\mathrm{M}_{m^{(A)}_{i_{s+2}}}(\mathrm{C}(X^{d^{(A)}_{i_{s+2}}}))(1_P + 1_\Theta).$$

Then the conditions of Theorem 5.1 are satisfied for $\mathrm{diag}\{P_s, \Theta'_s\}$ and $\mathrm{diag}\{P_s, \Theta''_s\}$. Indeed, for each $h \in \mathcal{H}^{(A)}_{i_s}$ and each $x \in X^{d^{(A)}_{i_{s+2}}}$, by (7.50) and (7.51),

$$\begin{aligned}
& |\mathrm{Tr}(\mathrm{diag}\{P_s(h)(x), \Theta'_s(h)(x)\}) - \mathrm{Tr}(\mathrm{diag}\{P_s(h)(x), \Theta''_s(h)(x)\})| \\
\leq\ & |\mathrm{Tr}(\phi^{(B,A)}_{i_{s+1},i_{s+2}} \circ \phi^{(A,B)}_{i_s,i_{s+1}})(h)(x) - \mathrm{Tr}(\phi^{(A)}_{i_{s+1},i_{s+2}} \circ \phi^{(A)}_{i_s,i_{s+1}}(h)(x))| + |\mathrm{Tr}(R''_s(h)(x)) - \mathrm{Tr}(R'_s(h)(x))| \\
<\ & m^{(A)}_{i_{s+2}}\delta_{i_s} + m^{(A)}_{i_{s+2}}\delta_{i_s} = 2m^{(A)}_{i_{s+2}}\delta_{i_s}.
\end{aligned}$$

Therefore (use (7.51) again and recall that $\delta_{i_s} < 1/2$),

$$|\mathrm{tr}(\mathrm{diag}\{P_s(h)(x), \Theta'_s(h)(x)\}) - \mathrm{tr}(\mathrm{diag}\{P_s(h)(x), \Theta''_s(h)(x)\})| < (\frac{1}{1-\delta_{i_s}})2\delta_{i_s} < 4\delta_{i_s} < \delta^{(A)}_{i_s},$$

which implies

$$|\tau(\mathrm{diag}\{P_s(h), \Theta'_s(h)\}) - \tau(\mathrm{diag}\{P_s(h), \Theta''_s(h)\})| < 4\delta_{i_s} < \delta^{(A)}_{i_s}, \quad h \in \mathcal{H}^{(A)}_{i_s}, \tag{7.54}$$

where τ is a tracial state of $(1_P + 1_\Theta)\mathrm{M}_{m^{(A)}_{i_{s+2}}}(\mathrm{C}(X^{d^{(A)}_{i_{s+2}}}))(1_P + 1_\Theta)$.

Also note that for any $h \in \mathcal{H}_{i_s}^{(A)}$, by (7.51) and (7.52),

$$\begin{aligned} & \operatorname{tr}(\operatorname{diag}\{P_s(h)(x), \Theta_s''(h)(x)\}) \\ \geq\ & \frac{1}{m_{i_{s+2}}^{(A)}} \operatorname{Tr}(\operatorname{diag}\{P_s(h)(x), \Theta_s''(h)(x)\}) \\ =\ & \frac{1}{m_{i_{s+2}}^{(A)}} \operatorname{Tr}(\operatorname{diag}\{P_s(h)(x), \Theta_s''(h)(x), R_s''(h)(x)\}) - \frac{1}{m_{i_{s+2}}^{(A)}} \operatorname{Tr}(R_s''(h)(x)) \\ >\ & \frac{1}{m_{i_{s+2}}^{(A)}} \operatorname{Tr}((\phi_{i_{s+1},i_{s+2}}^{(A)} \circ \phi_{i_s,i_{s+1}}^{(A)})(h)(x)) - \delta_{i_s} \\ >\ & \Delta_A(h)/2 - \delta_{i_s}. \end{aligned}$$

Hence

$$\tau(\operatorname{diag}\{P_s(h), \Theta_s''(h)\}) > \Delta_A(h)/2 - \delta_{i_s} > \Delta_A(h)/4, \quad h \in \mathcal{H}_{i_s}^{(A)}, \tag{7.55}$$

where τ is a tracial state of $(1_P + 1_\Theta)\mathrm{M}_{m_{i_{s+2}}^{(A)}}(\mathrm{C}(X^{d_{i_{s+2}}^{(A)}}))(1_P + 1_\Theta)$.

Together with (7.54), we also have that, for all $h \in \mathcal{H}_{i_s}^{(A)}$,

$$\tau(\operatorname{diag}\{P_s(h), \Theta_s'(h)\}) > \tau(\operatorname{diag}\{P_s(h), \Theta_s''(h)\}) - 4\delta_{i_s} > \Delta_A(h)/2 - 5\delta_{i_s} > \Delta_A(h)/4, \tag{7.56}$$

where τ is a tracial state of $(1_P + 1_\Theta)\mathrm{M}_{m_{i_{s+2}}^{(A)}}(\mathrm{C}(X^{d_{i_{s+2}}^{(A)}}))(1_P + 1_\Theta)$.

Thus, by (7.54), (7.55), and (7.56), we apply Theorem 5.1 to obtain a unitary

$$u_{s+2} \in (1_P + 1_\Theta)\mathrm{M}_{m_{i_{s+2}}^{(A)}}(\mathrm{C}(X^{d_{i_{s+2}}^{(A)}}))(1_P + 1_\Theta)$$

such that

$$\|u_{s+2}^* \operatorname{diag}\{P_s, \Theta_s'\} u_{s+2} - \operatorname{diag}\{P_s, \Theta_s''\}\| < \frac{\varepsilon_{i_s}}{2} \quad \text{on } \mathcal{F}_{i_s}^{(A)}. \tag{7.57}$$

Now, consider the maps

$$\operatorname{diag}\{R_s', \Theta_s'\},\ \operatorname{diag}\{R_s'', \Theta_s''\} : \mathrm{M}_{m_{i_s}^{(A)}}(\mathrm{C}(X^{d_{i_s}^{(A)}})) \to (1_R + 1_\Theta)\mathrm{M}_{m_{i_{s+2}}^{(A)}}(\mathrm{C}(X^{d_{i_{s+2}}^{(A)}}))(1_R + 1_\Theta).$$

For each $h \in \mathcal{H}_{i_s,2}^{(A)} (\subseteq \mathcal{H}_{i_s}^{(A)})$, since $\operatorname{supp}(h) \neq X^{d_{i_s}^{(A)}}$, there is a point $x \in X^{d_{i_s}^{(A)}}$ such that $h(x) = 0$. Set $\tilde{x} = (x, ..., x) \in X^{d_{i_{s+2}}^{(A)}}$. Since P_s consists of coordinate projections, we have that

$P_s(h)(\tilde{x}) = 0$. Then

$$\begin{aligned}
\mathrm{tr}(\Theta'_s(h)) &= \frac{1}{\mathrm{rank}(1_\Theta)}\mathrm{Tr}(\Theta'_s(h)(\tilde{x})) \\
&= \frac{1}{\mathrm{rank}(1_\Theta)}\mathrm{Tr}(\mathrm{diag}\{P_s(h)(\tilde{x}), R'_s(\tilde{x}), \Theta'_s(h)(\tilde{x})\}) - \frac{1}{\mathrm{rank}(1_\Theta)}\mathrm{Tr}(R'_s(\tilde{x})) \\
&> \frac{1}{\mathrm{rank}(1_\Theta)}\mathrm{Tr}(\mathrm{diag}\{P_s(h)(\tilde{x}), R'_s(\tilde{x}), \Theta'_s(h)(\tilde{x})\}) - \delta_{i_s} \\
&> \frac{1}{m^{(A)}_{i_{i+2}}}\mathrm{Tr}(\mathrm{diag}\{P_s(h)(\tilde{x}), R'_s(\tilde{x}), \Theta'_s(h)(\tilde{x})\}) - \delta_{i_s} \\
&= \frac{1}{m^{(A)}_{i_{i+2}}}\mathrm{Tr}(\phi^{(A,B)}_{i_{s+1},i_{s+2}} \circ \phi^{(B,A)}_{i_s,i_{s+1}}(h)(\tilde{x})) - \delta_{i_s} \\
&\geq \Delta(h)/2 - 2\delta_{i_s} > \Delta(h)/4.
\end{aligned}$$

The same argument also shows that

$$\mathrm{tr}(\Theta''_s(h)) > \Delta(h)/4, \quad h \in \mathcal{H}^{(A)}_{i_s,2}.$$

Also recall that, by (7.51),

$$\frac{\mathrm{rank}(R'_s)}{\mathrm{rank}(\Theta'_s)} < \delta_{i_s} < \frac{1}{M^{(A)}_{i_s}} \quad \text{and} \quad \frac{\mathrm{rank}(R''_s)}{\mathrm{rank}(\Theta''_s)} < \delta_{i_s} < \frac{1}{M^{(A)}_{i_s}}.$$

Then, the conditions of Theorem 7.5 are satisfied, and there is a unitary

$$w_{s+2} \in (1_R + 1_\Theta)\mathrm{M}_{m^{(A)}_{i_{s+2}}}(\mathrm{C}(X^{d^{(A)}_{i_{s+2}}}))(1_R + 1_\Theta)$$

such that

$$\|w^*_{s+2}\mathrm{diag}\{R'_s, \Theta''_s\}w_{s+2} - \mathrm{diag}\{R''_s, \Theta''_s\}\| < \frac{\varepsilon_{i_s}}{2} \quad \text{on } \mathcal{F}^{(A)}_{i_s}. \tag{7.58}$$

Regarding u_{s+2} and w_{s+2} as unitaries in $\mathrm{M}_{m^{(A)}_{i_{s+2}}}(\mathrm{C}(X^{d^{(A)}_{i_{s+2}}}))$, and setting $v_{s+2} = u_{s+2}w_{s+2}$, we have

$$\|v^*_{s+2}\mathrm{diag}\{P_s, R'_s, \Theta'_s\}v_{s+2} - \mathrm{diag}\{P_s, R''_s, \Theta''_s\}\| < \varepsilon_{i_s} \quad \text{on } \mathcal{F}^{(A)}_{i_s}.$$

That is,

$$\|v^*_{s+2}(\phi^{(B,A)}_{i_{s+1},i_{s+2}} \circ \phi^{(A,B)}_{i_s,i_{s+1}})v_{s+2} - \phi^{(A)}_{i_{s+1},i_{s+2}} \circ \phi^{(A)}_{i_s,i_{s+1}}\| < \varepsilon_{i_s} \quad \text{on } \mathcal{F}^{(A)}_{i_s}.$$

A similar argument shows that for $s = 1, 3, ...$, there are unitaries v_{s+2} such that

$$\|v^*_{s+2}(\phi^{(A,B)}_{i_{s+1},i_{s+2}} \circ \phi^{(B,A)}_{i_s,i_{s+1}})v_{s+2} - \phi^{(B)}_{i_{s+1},i_{s+2}} \circ \phi^{(B)}_{i_s,i_{s+1}}\| < \varepsilon_{i_s} \quad \text{on } \mathcal{F}^{(B)}_{i_s}.$$

Therefore, in the diagram

$$\begin{tikzcd}[column sep=small]
\mathrm{M}_{n_0^{(A)}}(\mathrm{C}(X)) \arrow[r, "\phi^{(A)}_{1,i_1}"] \arrow[dr, "\phi^{(A,B)}_{1,i_1}"] & \mathrm{M}_{m^{(A)}_{i_1}}(\mathrm{C}(X^{d^{(A)}_{i_1}})) \arrow[r, "\mathrm{ad}(v_2^{(A)})\circ\phi^{(A)}_{i_1,i_2}"] & \mathrm{M}_{m^{(A)}_{i_2}}(\mathrm{C}(X^{d^{(A)}_{i_2}})) \arrow[r] \arrow[dr, "\phi^{(A,B)}_{i_2,i_3}"] & \cdots \arrow[r] & A \\
\mathrm{M}_{n_0^{(B)}}(\mathrm{C}(X)) \arrow[r, "\phi^{(B)}_{1,i_1}"'] & \mathrm{M}_{m^{(B)}_{i_1}}(\mathrm{C}(X^{d^{(B)}_{i_1}})) \arrow[r, "\phi^{(B)}_{i_1,i_2}"'] \arrow[ur, "\phi^{(B,A)}_{i_1,i_2}"] & \mathrm{M}_{m^{(B)}_{i_2}}(\mathrm{C}(X^{d^{(B)}_{i_2}})) \arrow[r, "\mathrm{ad}(v_3^{(B)})\circ\phi^{(B)}_{i_1,i_2}"'] & \cdots \arrow[r] & B,
\end{tikzcd} \tag{7.59}$$

the sth triangle is approximately commutative to within $(\mathcal{F}^{(A)}_{i_s}, \varepsilon_{i_s})$ or $(\mathcal{F}^{(B)}_{i_s}, \varepsilon_{i_s})$. By the approximate intertwining argument ([7]), we have

$$A \cong B,$$

as desired. $\square$

Corollary 7.8. *Let X be a K-contractible solid metrizable compact space which is finite dimensional. Let*

$$A := A(X^p, (n_i^{(A)}), (k_i^{(A)}), E^{(A)}) \quad \textit{and} \quad B := B(X^q, (n_i^{(B)}), (k_i^{(B)}), F^{(B)})$$

be Villadsen algebras (with coordinate projections of arbitrary (non-zero) multiplicity). Then $A \cong B$ if, and only if,

$$\mathrm{K}_0(A) \cong \mathrm{K}_0(B), \quad \mathrm{T}(A) \cong \mathrm{T}(B) \quad \textit{and} \quad \mathrm{rc}(A) = \mathrm{rc}(B).$$

Moreover, if $\mathrm{rc}(A) \neq 0$ (or $\mathrm{rc}(B) \neq 0$), then $\mathrm{T}(A)$ (or $\mathrm{T}(B)$) is redundant in the invariant; that is, $A \cong B$ if, and only if,

$$\mathrm{K}_0(A) \cong \mathrm{K}_0(B) \quad \textit{and} \quad \mathrm{rc}(A) = \mathrm{rc}(B).$$

Proof. Consider the inductive limit construction

$$\mathrm{M}_{n_0^{(A)}}(\mathrm{C}(X^p)) \longrightarrow \mathrm{M}_{m_1^{(A)}}(\mathrm{C}(X^{pd_1^{(A)}})) \longrightarrow \mathrm{M}_{m_2^{(A)}}(\mathrm{C}(X^{pd_2^{(A)}})) \longrightarrow \cdots \longrightarrow A.$$

Tensor it with $\mathrm{M}_{pq}(\mathbb{C})$ and add a new first map, which is induced by coordinate projections with multiplicity one (but no point evaluation yet), to obtain the new construction

$$\mathrm{M}_{qn_0^{(A)}}(\mathrm{C}(X)) \longrightarrow \mathrm{M}_{pq}(\mathrm{M}_{n_0^{(A)}}(\mathrm{C}(X^p))) \longrightarrow \mathrm{M}_{pq}(\mathrm{M}_{m_1^{(A)}}(\mathrm{C}(X^{pd_1^{(A)}}))) \longrightarrow \cdots \longrightarrow \mathrm{M}_{pq}(A).$$

Similarly, also consider B, and consider the new inductive limit construction

$$\mathrm{M}_{pn_0^{(B)}}(\mathrm{C}(X)) \longrightarrow \mathrm{M}_{pq}(\mathrm{M}_{n_0^{(B)}}(\mathrm{C}(X^q)) \longrightarrow \mathrm{M}_{pq}(\mathrm{M}_{m_1^{(B)}}(\mathrm{C}(X^{qd_1^{(B)}}))) \longrightarrow \cdots \longrightarrow \mathrm{M}_{pq}(B).$$

Then collapse the first two maps so that there are point evaluations in all connecting maps of the new sequences.

Since $\mathrm{rc}(A) = \mathrm{rc}(B)$, we have $\mathrm{rc}(\mathrm{M}_{pq}(A)) = \mathrm{rc}(\mathrm{M}_{pq}(B))$, and by Theorem 7.1, $\mathrm{M}_{pq}(A) \cong \mathrm{M}_{pq}(B)$; denote this algebra by C. Note that $[e_A]_0 = [e_B]_0$ in $\mathrm{K}_0(C)$, where e_A and e_B are the images of 1_A and 1_B in the upper left corner of the matrix algebras respectively. Since A and B have stable rank one ([10]), C has stable rank one, and hence cancellation of projections. This implies that e_A is Murray-von Neumann equivalent to e_B inside C, and therefore $A \cong B$, as asserted. □

Remark 7.9. Let A and B be two Villadsen algebras with seed spaces X and Y respectively. Is $X^\infty \cong Y^\infty$ sufficient for our classification to apply? Is it possible that the seed space X is completely irrelevant (when $\mathrm{rc} \neq 0$)?

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