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A Classification of Tracially Approximate Splitting Interval Algebras. III. Uniqueness Theorem and Isomorphism Theorem

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Abstract. Motivated by Huaxin Lin's axiomatization of AH-algebras, the class of TASI-algebras is introduced as the class of unital C*-algebras which can be tracially approximated by splitting interval algebras—certain sub-C*-algebras of interval algebras. It is shown that the class of simple separable nuclear TASI-algebras satisfying the UCT is classified by the Elliott invariant. As a consequence, any such TASI-algebra is then isomorphic to an inductive limit of splitting interval algebras together with certain homogeneous C*-algebras—so it is an ASH-algebra.

Résumé. Une classe de C*-algèbres qui généralisent la classe bien connue TAI de Lin est considérée—basées sur, au lieu de l'intervalle, ce qui pourrait s'appeler l'intervalle fendu ("splitting interval"), de sorte que l'on les appelle la classe TASI. On montre que la classe de C*-algèbres TASI qui sont simples, nucléaires, et à élément unité, qui vérifient le théorème à coefficients universels (UCT), peuvent se classifier d'après l'invariant d'Elliott.

1. **Introduction** This is the third part of the classification of tracially approximate splitting interval algebras (together with [12] and [13]).

The uniqueness theorems are studied, and then together with the existence theorem of [13], one obtains the following classification theorem for the class of tracially approximate splitting interval algebras:

Theorem. *Let A and B be two simple separable nuclear TASI-algebra which satisfies UCT. Then $A \cong B$ if and only if*

$$\begin{aligned} &((\mathrm{K}_0(A), \mathrm{K}_0(A)^+, [1_A]_0), \mathrm{K}_1(A), \mathrm{T}(A), r_A) \\ \cong\ \ &((\mathrm{K}_0(B), \mathrm{K}_0(B)^+, [1_B]_0), \mathrm{K}_1(B), \mathrm{T}(B), r_B). \end{aligned}$$

*Moreover, the *-isomorphism between the C*-algebras can be chosen to induce the given isomorphism between their invariants.*

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The range of the Elliott invariant for TASI-algebras is then investigated in Section 4. The class of TASI-algebras is strictly larger than the class of AH-algebras. C*-algebras which are TASI but not AH have been constructed by several authors in [8], [4] and [2]. All of these constructions provided a TASI-algebra with the convex of the states on the K_0-group not being a simplex. We shall show further that even if the K_0-group of a TASI-algebra has the Riesz decomposition property (in particular, this implies that the convex of the states is a simplex), this algebra still might not be an AH-algebra (even not being a rationally AH-algebra). An example is constructed as a TASI-algebra with K_0-group a Riesz group, but with the canonical pairing map not preserving extreme points (Theorem 4.9).

Finally, it is also shown in Section 4.1 that, although the class of rationalized K_0-groups of TASI-algebras is much larger than the class of rationalized Riesz groups, its ordered K_0-group still does not exhaust all weakly unperforated simple ordered groups, even after tensoring by $\mathbb{Q}$ (Theorem 4.5). For instance, if an order-unit group has a pentagon as the convex of its states, then it cannot be the K_0-group of a TASI-algebra. Therefore, to obtain a classification theorem which exhausts all possible values of the Elliott invariant, one has to consider the class of C*-algebras which can be tracially approximated by arbitrary Elliott-Thomsen algebras introduced in [4] and [2]. This direction of research will be investigated in [7].

2. **Uniqueness Theorem** In this section, we shall establish a uniqueness theorem for simple separable TASI-algebras. The strategy is to get a stable uniqueness theorem for TASI-algebras first. Then, using an approximately divisibility property of TASI-algebras, one can decompose any map between two TASI-algebras into a direct sum of two maps with the image of one map is in a small corner and the other map has large multiplicity. Thus one can use the stable uniqueness to show the uniqueness of the original map. First, let us show the uniqueness theorem for maps from a splitting interval algebra to a simple TASI-algebra (assume that the splitting interval algebra is a unital sub-C*-algebra of a simple C*-algebra.)

PROPOSITION 2.1. *Let S be a splitting interval algebra inside a unital simple C*-algebra B, and let A be a simple TASI-algebra. Then, for any finite subset $\mathcal{F} \subseteq S$, any $\varepsilon > 0$, there is finite subsets $\mathcal{G} \subseteq B$, $\mathcal{G}_1 \subseteq S$, and $\delta, \sigma > 0$ such that for any two $\mathcal{G}$-σ-multiplicative maps ϕ and ψ from B to A, if*

- $[\phi] = [\psi]$ *on* $\mathrm{K}_0(S)$ *and*
- $\|\tau \circ \phi(g) - \tau \circ \psi(g)\| < \delta$ *for any* $g \in \mathcal{G}_1$ *and any* $\tau \in \mathrm{T}(A)$,

then there exists a unitary $u \in A$ such that

$$\|\phi(f) - u\psi(f)u^*\| < \varepsilon \text{ for all } f \in \mathcal{F}.$$

PROOF. Denote by k the size of the matrix algebra at each regular point of S. Without loss of generality, one may assume that $\mathcal{F}$ is a set of generators of S

containing the central elements $\{t(1-t)e_{ij};\ i,j=1,...,k\}$. Let $n=\min\{[1/\delta_0]+1, \varepsilon/4\}$, where $\delta_0>0$ is the constant of Corollary 2.23 of [12] with respect to $\mathcal{F}$ and $\varepsilon/4$.

Denote by $\mathcal{G}_0$ and $\mathcal{G}_1$ the finite subsets of Theorem 2.27 of [12] with respect to $\mathcal{F}$ and ε. Since B is simple and S has stable relation, there is a finite subset $\mathcal{G}\subset B$ and $\sigma>0$ such that for any $\mathcal{G}$-σ-multiplicative map $L: B\to A$, the restriction of L to S can be approximated by a homomorphism $L': S\to A$ with tolerance $\varepsilon/2$ on any element of $\mathcal{F}$. Moreover, there is $\delta>0$ such that $|\tau(L'(g))|>\delta$ for any $g\in\mathcal{G}_0$ and for any $\tau\in\mathrm{T}(A)$, and δ is independent of L (using the simplicity of B). By the argument above, with this choice of $\mathcal{G}$, $\mathcal{G}_1$, σ and δ, for any $\mathcal{G}$-σ-multiplicative maps $\phi,\psi: B\to A$, one may assume that ϕ and ψ are homomorphisms from S to A satisfying

$$|\tau(\phi(g))|>\delta \quad \text{and} \quad |\tau(\psi(g))|>\delta$$

for any $g\in\mathcal{G}_0$.

As the first step, let us show that the *-homomorphisms ϕ and ψ can be decomposed approximately (on $\mathcal{F}$) as a direct sum of a homomorphism and a large number of point evaluation maps. To construct one point evaluation map of S, it is enough to find a system of $k\times k$ matrix units in A on which $\phi(S)$ (or $\psi(S)$) acts as a point evaluation. For any $n\in\mathbb{Z}^+$, set $\xi_i=\frac{i}{n}$. Choose positive elements $\{s_{i,j}\}_{i=0}^n$ in S which take value 0 outside the interval $(\frac{2i-1}{2n},\frac{2i+1}{2n})$, take value e_{jj} on $[\frac{4i-1}{4n},\frac{4i+1}{4n}]$, and linear in between. Set $s_i=\sum_j s_{i,j}$. Since the elements in $\{s_{i,j}\}$ are mutually orthogonal, one has that the elements in $\{\phi(s_{i,j})\}$ are mutually orthogonal. Pick a non-zero projection $p_{i,j}$ inside each hereditary sub-C*-algebra of A generated by $\phi(s_{i,j})$. This can be done since A has the (SP) property. Moreover, since A is simple, we can assume $p_{i,j}$'s are equivalent to each others. Set $p_i=\sum_j p_{i,j}$. Note that $fs_i-f(\xi_i)s_i<(1/n)s_i$ for any $f\in\mathcal{F}$. One has that $\|\phi(f)p_i-f(\xi_i)p_i\|<1/n$. Setting $e=\sum p_i$, one then has that $\|\phi(f)e-f(\xi_i)e\|<1/n$. Hence one has the following approximate decomposition: we may write ϕ (on $\mathcal{F}$) as

$$\phi(f)=_{\mathcal{F},\varepsilon/4}(1-e)\phi(f)(1-e)+\sum f(\xi_i)p_i.$$

By the same argument, we also get

$$\psi(f)=_{\mathcal{F},\varepsilon/4}(1-e')\psi(f)(1-e')+\sum f(\xi_i)q_i,$$

where $e'=\sum q_i$. Since A is simple and has the (SP) property, for any pair of projections, there are subprojections in each one of them which are Murray-von Neumann equivalent. Thus, one may assume that $p_i=q_i$ and the p_i's are Murray-von Neumann equivalent to each other (by passing to the subprojections and composing inner automorphisms)(see, for instance, Lemma 3.5.7 of [9]). Moreover, since the positive cone of $\mathrm{K}_0(S)$ is finite generated, we may assume $(1-e)\phi(1-e)$ and $(1-e)\psi(1-e)$ induce same map on $\mathrm{K}_0(S)$.

Consider the maps $(1-e)\phi(1-e)$ and $(1-e)\psi(1-e)$. For any projections p and q in S which are not Murray-von Neumann equivalent to each other, if $[(1-e)\phi(p)(1-e)] = [(1-e)\psi(q)(1-e)]$, then there exists a partial isometry $v \in (1-e)A(1-e)$ which induces the equivalence relation. Pick one such partial isometry v for each pair of non-equivalent minimal projections which have equivalent images. Denote all such partial isometries v by V. Since there are only finite minimal projections in S, the set V is a finite subset of A. Consider $\mathcal{F}' = \phi(\mathcal{F}) \cup \psi(\mathcal{F}) \cup V$. Since $(1-e)A(1-e)$ is a TASI-algebra, there exist a projection p and a splitting interval algebra $S' \subset (1-e)A(1-e)$ with $1_{S'} = p$ such that

- $\|px - xp\| < \varepsilon_0$,
- $pxp \in_{\varepsilon_0} S'$ for all $x \in \mathcal{F}'$, and
- $1 - p \preceq p_{11}$.

Since ε_0 can be arbitrarily small and splitting interval algebra are generated by stable relations, one then may assume that the maps ϕ and ψ has the following decomposition on $\mathcal{F}$:

$$\phi(f) = \phi'(f) + (\phi''(f) + \sum f(\xi_i)p_i),$$

$$\psi(f) = \psi'(f) + (\psi''(f) + \sum f(\xi_i)p_i),$$

where $\phi', \phi'', \psi', \psi''$ are the cut-down of ϕ and ψ by p and $1-p$ respectively such that $\tau(1-p) < \delta/4$ for any $\tau \in \mathrm{T}(A)$. Moreover, since the positive cone $\mathrm{K}_0(S)$ is finitely generated, one may assume further that $[\phi']_0 = [\psi']_0$.

We assert that the sub-C*-algebra S' can be chosen so that for any $\tau \in \mathrm{T}(S')$, one has $\tau(\phi'(g)) > \delta$, $\tau(\psi'(g)) > \delta$ for any $g \in \mathcal{G}_0$, and

$$\|\tau(\phi'(f)) - \tau(\psi'(f))\| < \delta$$

for any $f \in \mathcal{G}_1$. If this were not true, for any integer m and any finite subset $\mathcal{H} \subset A$, there is a sub-C*-algebra S' in the question above and $\tau_{m,\mathcal{H}} \in \mathrm{T}(S')$ such that $\mathcal{H} \subset_{1/m} S'$, $\tau_{m,\mathcal{H}}(\phi'(g)) \leq \delta$, $\tau_{m,\mathcal{H}}(\psi'(g)) \leq \delta$ for any $g \in \mathcal{G}_0$ and

$$\|\tau_{m,\mathcal{H}}(\phi'(f)) - \tau_{m,\mathcal{H}}(\psi'(f))\| \geq \delta$$

for any $f \in \mathcal{G}_1$. Extend $\tau_{m,\mathcal{H}}$ to a positive linear functional on A, and still denote it by $\tau_{m,\mathcal{H}}$. Pick an accumulation point τ of $\{\tau_{m,\mathcal{H}}\}$, and a direct calculation shows that $\tau \in \mathrm{T}(A)$ and $\tau(\phi'(g)) \leq \delta$, $\tau(\psi'(g)) \leq \delta$ for any $g \in \mathcal{G}_0$ and

$$\left\|\tau_(\phi'(f)) - \tau(\psi'(f))\right\| \geq \delta,$$

for any $f \in \mathcal{G}_1$, which is a contradiction. This proves the assertion.

Hence, one has that

$$[\phi']_0 = [\psi']_0,$$

$$\tau(\phi'(g)) > \delta, \quad \tau(\psi'(g)) > \delta$$

for any $g \in \mathcal{G}_0$, and

$$\|\tau(\phi'(f)) - \tau(\psi'(f))\| < \delta$$

for any $f \in \mathcal{G}_1$. It follows from Theorem 2.27 of [12] that there exists a unitary $u_1 \in S' \subset pAp$ such that

$$\|\phi'(f) - u_1\phi'(f)u_1^*\| < \varepsilon$$

for any $f \in \mathcal{F}$.

Consider the map $\tilde{\phi}'' : f \mapsto \phi''(f) + \sum_{i=1}^{n} f(\xi_i)p_i$ and $\tilde{\psi}'' : f \mapsto \psi''(f) + \sum_{i=1}^{n} f(\xi_i)p_i$. Note that $[\tilde{\phi}'']_0 = [\tilde{\psi}'']_0$ By Corollary 2.23 of [12], there exists a unitary $u_2 \in (1-p)A(1-p)$ such that

$$\left\|\tilde{\phi}''(f) - u_2\tilde{\psi}''(f)u_2^*\right\| < \varepsilon, \quad \forall f \in \mathcal{F}.$$

Setting $u = u_1 + u_2$, one has

$$\|\phi(f) - u\psi(f)u^*\| < \varepsilon \quad \forall f \in \mathcal{F},$$

as desired. □

The main ingredient of the uniqueness theorem for TASI-algebra is a version of stable uniqueness theorem for approximate homomorphisms (Theorem 2.11) (see [10] and [1]).

Definition 2.2. Let A be a unital C*-algebra and $u \in \mathrm{U}_0(A)$, the connected component of the unitary group of A containing the identity. Then the exponential length of u, denoted by $\mathrm{cel}(u)$, is

$$\inf\{\sum_{k=1}^{n} ||h_k|| : n \in \mathbb{N}, h_1, \cdots, h_n \in A^{s.a.},\ u = \exp(ih_1)\cdots\exp(ih_n)\}.$$

Definition 2.3. Denote by $\mathcal{C}$ the class of unital C*-algebras A such that

- A has stable rank one,
- for any finite subset $\mathcal{F} \subseteq B$, any $\varepsilon > 0$ and any $N \in \mathbb{N}$, there exists a finite-dimensional C*-algebra $F \subseteq B$ with the size of each direct summand at least N such that if $p = 1_F$, then

– $\|[x, y]\| < \varepsilon$ for any $x \in \mathcal{F}$ and any y in the unit ball of F,
– $N[1-p] \le [p]$.

Remark 2.4. By Theorem 4.13 of [12], any TASI-algebra belongs to $\mathcal{C}$.

The following lemma is due to N.C. Phillips.

LEMMA 2.5. *Let A be a unital C^*-algebra and $2 > d > 0$. Let $u_0, u_1, ..., u_n$ be $n+1$ unitaries in A such that*

$$u_n = 1 \quad \textit{and} \quad \|u_i - u_{i+1}\| \leq d \quad i = 0, 1., ..., n-1.$$

Then there exists a unitary $v \in \mathrm{M}_{2n+1}(A)$ with exponential lengh no more than 2π such that

$$\left\|(u_0 \oplus 1_{\mathrm{M}_{2n}(A)}) - v\right\| \leq d.$$

Moreover, $v \in CU(M_{2n+1}(A))$.

Let (B_n) be a sequence of C*-algebras. Define

$$\theta_1 : \mathrm{K}_1(\prod B_n) \to \prod \mathrm{K}_1(B_n)$$

to be the map induced by coordinator projections.

LEMMA 2.6. *Let (B_n) be a sequence of C^*-algebras in $\mathcal{C}$. Then the kernel of θ_1 is divisible.*

PROOF. Let $u_\infty = (u_1, u_2, ..., u_n, ...)$ be a unitary in a matrix algebra over $\prod B_n$ with $\theta_1(u_\infty) = 0$, and let $k \in \mathbb{N}$. Note that each u_n is connected to the identity by a path, say, with length l_n. Then, pick unitaries $u_n^{(1)} = u_n, u_n^{(2)}, ..., u_n^{(k_n-1)}, u_n^{(k_n)} = 1_{B_n}$ such that

$$\left\|u_n^{(i)} - u_n^{(i+1)}\right\| \leq 1/2$$

for all i. Then chose a finite-dimensional C*-algebra F_n such that the size of each direct summand is at least $N_n (> \max\{2m_n + 1, 2k\})$, and

- $\left\|[u_n^{(i)}, y]\right\| < 1/32$ for any $u_n^{(i)}$ and any y in the unit ball of F_n,
- $N_n[1 - p_n] \leq [p_n]$, where $p_n = 1_{F_n}$.

Without loss of generality, one may assume that F_n is a single matrix algebra and the rank of F_n is divided by k.

Consider the cut-down of the unitaries $u_n^{(i)}$ by p_n. One then has that

$$\left\|pu_n^{(i)}p - pu_n^{(i+1)}p\right\| \leq 1$$

for all i. Since being a unitary is a stable relation, there is a sequence of unitaries $\{v_n^i;\ 1 \leq i \leq k_n\}$ in $p_n B_n p_n$ connecting $(1-p_n)u_n(1-p_n)$ to $1 - p_n$ and

$$\left\|v_n^{(i)} - v_n^{(i+1)}\right\| < 3/2$$

for all i.

Then, by Lemma 2.5, there is a unitary $v \in B_n$ in with exponential length at most 3π such that

$$\|((1-p_n)u_n(1-p_n) \oplus p_n) - v\| < 1/16.$$

Hence, one has

$$\|u_n - v\mathrm{diag}\{1-p_n, \underbrace{w, ..., w}_{\mathrm{rank}(F)}\}\| < 1/8$$

for some unitary $w \in eB_ne$, where e is a minimal projection of F.

For each $1 \leq j \leq k$, consider

$$z_n^{(j)} = \mathrm{diag}\{1-p_n, \underbrace{e, ..., e}_{M(j-1)}, \underbrace{w, ..., w}_{M}, e, ..., e\},$$

where $M = \mathrm{rank}(F)/k$. Then

$$\mathrm{cel}(u_n^* \prod_{j=1}^{k} z_n^{(j)}) < 4\pi \tag{2.1}$$

for all n. Denote by

$$z_\infty^{(j)} = (z_1^{(j)}, z_2^{(j)}, ..., z_n^{(j)}, ...)$$

for $1 \leq j \leq k$. By (2.1), one has

$$[u_\infty]_1 = [(z_\infty^{(1)})]_1 + \cdots + [z_\infty^{(k)}]_1$$

in $\mathrm{K}_1(\prod B_n)$. On the other hand, it is a straightforward calculation that

$$[(z_\infty^{(1)})]_1 = \cdots = [z_\infty^{(k)}]_1$$

in $\mathrm{K}_1(\prod B_n)$. Therefore, $[u_\infty]_1$ is divided by k, and $\ker(\theta_1)$ is divisible, as desired. $\square$

LEMMA 2.7. *If (B_n) is a sequence of C^*-algebras in $\mathcal{C}$, the C^*-algebra $\prod B_n / \bigoplus B_n$ is also in $\mathcal{C}$.*

PROOF. It is clear that $\prod B_n / \bigoplus B_n$ has stable rank one.

Let us verify the second condition of Definition 2.3. Let $\mathcal{F} \subseteq \prod B_n / \bigoplus B_n$ be a finite subset, $\varepsilon > 0$, and $N \in \mathbb{N}$. Let $\mathcal{F}' \subseteq \prod B_n$ be a lifting of $\mathcal{F}$ in $\prod B_n$, and denote by $\mathcal{F}'_n$ the project of $\mathcal{F}'$ to B_n. Then there is a finite-dimensional sub-C*-algebra $F_n \subseteq B_n$ such that

1. $\|[x, y]\| < \varepsilon$ for any $x \in \mathcal{F}'_n$ and any y in the unit ball of F_n,
2. $N[1-p_n] \leq [p_n]$, where $p_n = 1_{F_n}$.

Moreover, each F_n can be chosen to be a single matrix algebra with the same rank.

For each n, chose an isomorphism $\phi_n : F_n \to F_{n+1}$. Consider the sub-C*-algebra

$$F' := \{(a_1, a_2, ...);\ a_{n+1} = \phi_n(a_n)\} \subseteq \prod F_n \subseteq \prod B_n,$$

and denote F the image of F' in $\prod B_n / \bigoplus B_n$. Then F is isomorphic to each F_n, and moreover, a direct calculation shows that F satisfies the second condition of Definition 2.3. □

Recall that a group G is algebraically compact if $\mathrm{Pext}(\cdot, G)$ vanishes ([5]).

LEMMA 2.8. *Let (B_i) be a sequence of C*-algebras in the class $\mathcal{C}$. Then the groups*

$$\mathrm{K}_0(\prod B_n / \bigoplus B_n) \quad \textit{and} \quad \mathrm{K}_1(\prod B_n / \bigoplus B_n)$$

are algebraically compact.

PROOF. Since each B_i has stable rank one, one has that $\mathbf{cco}(B_i) = 0$ (see 3.2 of [1] for the definition of **cco**). By the second condition of Definition 2.3 on the class $\mathcal{C}$, it is ready that $\mathbf{pfo}(B_i) \leq 1$ and $\mathbf{ipo}(B_i) \leq 1$ (again, we refer to 3.2 of [1] for the definition of **pfo** and **ipo**). It then follows from Corollary 3.6 (II) of [1] that $\mathrm{K}_0(\prod B_n / \bigoplus B_n)$ is algebraically compact.

Consider the group $\mathrm{K}_1(\prod B_n / \bigoplus B_n)$. Note that for any $w \in \mathrm{K}_1(\bigoplus B_n) \subseteq \mathrm{K}_1(\prod B_n)$, its image $\theta_1(w) \in \prod \mathrm{K}_1(B_n)$ is zero if and only if $w = 0$. Therefore, the map $\theta_1 : \mathrm{K}_1(\prod B_n) \to \prod \mathrm{K}_1(B_n)$ induces a map

$$\tilde{\theta}_1 : \mathrm{K}_1(\prod B_n) / \bigoplus \mathrm{K}_1(B_n) \to \prod \mathrm{K}_1(B_n) / \bigoplus \mathrm{K}_1(B_n),$$

and

$$\ker \tilde{\theta}_1 \cong \overline{\ker \theta_1}.$$

By Lemma 2.6, the group $\ker \theta_1$ is divisible, so is $\ker \tilde{\theta}_1$. Therefore, one has

$$\begin{aligned} \mathrm{K}_1(\prod B_n / \bigoplus B_n) &\cong \mathrm{K}_1(\prod B_n) / \bigoplus \mathrm{K}_1(B_n) \\ &\cong \ker \tilde{\theta}_1 \oplus \prod \mathrm{K}_1(B_n) / \bigoplus \mathrm{K}_1(B_n). \end{aligned}$$

By Lemma 3.5 of [1], the group $\prod \mathrm{K}_1(B_n) / \bigoplus \mathrm{K}_1(B_n)$ is algebraically compact. Since $\ker \tilde{\theta}_1$ is divisible, in particular it is also algebraically compact. Therefore, as their direct sum, the group $\mathrm{K}_1(\prod B_n / \bigoplus B_n)$ is also algebraically compact, as desired. □

LEMMA 2.9. *Let A be a C*-algebra, and let (B_n) be a sequence of C*-algebras in $\mathcal{C}$. Then the natural map*

$$\mathrm{Hom}(\mathrm{K}_0(A), \mathrm{K}_0(\prod B_n)) \to \prod \mathrm{Hom}(\mathrm{K}_0(A), \mathrm{K}_0(B_n)) \tag{2.2}$$

is injective. Moreover, let (ϕ_n) and (ψ_n) be homomorphisms from A to B_n with

$$[\phi_n] = [\psi_n] \in \mathrm{Hom}_\Lambda(\mathrm{K}(A), \mathrm{K}(B_n))$$

$$\mathrm{cel}(\phi_n(u)\psi_n(u^*)) \leq \mathbf{L}(u)$$

for some map $\mathbf{L} : \mathrm{U}(M_\infty(A)) \to \mathbb{R}^+$. Then

$$[(\phi_n)]_1 = [(\psi_n)]_1 \in \mathrm{Hom}(\mathrm{K}_1(A), \mathrm{K}_1(\prod B_n)).$$

Moreover, in this case,

$$[(\phi_n)] = [(\psi_n)] \in \mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(\prod B_n)).$$

If A satisfies the UCT and, then one has

$$\mathrm{KK}(A, \prod B_n/\bigoplus B_n) = \mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(\prod B_n/\bigoplus B_n)). \tag{2.3}$$

PROOF. Since each B_n has stable rank one, by Proposition 2.1(1) of [6],

$$\mathrm{K}_0(\prod B_n) = \prod^{\mathrm{b}} \mathrm{K}_0(B_n),$$

and hence the map of (2.2) is injective. The statement on K_1-groups follows from the fact that for any unitary u, the images $(\phi_n(u))$ and $(\psi_n(u))$ can be connected by a path of unitary which is continuous uniformly for n. And then the statement on the K-theory with coefficient follows from a diagram chase, as that Theorem 4.10 of [1].

For (2.3), since A satisfies the UCT, one only has to show

$$\mathrm{Pext}(\mathrm{K}_*(A), \mathrm{K}_{*+1}(\prod B_n/\bigoplus B_n)) = \{0\},$$

but this follows from Lemma 2.8. □

THEOREM 2.10. *Let A be a separable simple nuclear C^*-algebra satisfying the UCT, and let $\mathbf{L} : \mathrm{U}(M_\infty(A)) \to \mathbb{R}^+$ be a map. Then, for any finite subset $\mathcal{F} \subseteq A$ any $\varepsilon > 0$, there exists n such that for any C^*-algebra $B \in \mathcal{C}$, and any homomorphisms $\phi, \psi : A \to B$ and unital homomorphism $\sigma : A \to B$ with*

$$[\phi] = [\psi] \in \mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B))$$

and

$$\mathrm{cel}(\phi(u)\psi(u^*)) \leq \mathbf{L}(u) \tag{2.4}$$

for all u, then there is a unitary $u \in \mathrm{M}_{n+1}(B)$ such that

$$\|\mathrm{diag}(\phi(a), \underbrace{\sigma(a), ..., \sigma(a)}_{n}) - u^*\mathrm{diag}(\psi(a), \underbrace{\sigma(a), ..., \sigma(a)}_{n})u\| < \varepsilon$$

for any $a \in A$.

PROOF. The proof is the same as that of Theorem 4.12 of [1]. Assume that the conclusion were not true for a finite subset $\mathcal{F} \subseteq A$ and $\varepsilon > 0$. Then, for each n, there are a C*-algebra B_n in $\mathcal{C}$ and homomorphisms $\phi_n, \psi_n, \sigma_n : A \to B_n$ with σ_n unital such that

$$[\phi_n] = [\psi_n] \in \mathrm{Hom}_{\Lambda}(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B))$$

and

$$\mathrm{cel}(\phi_n(u)\psi_n(u^*)) \leq \mathbf{L}(u)$$

for all u, but

$$\begin{aligned} \inf_{u \in \mathrm{U}(\mathrm{M}_{n+1}(B))} \max_{a \in \mathcal{F}} \{ \| &\mathrm{diag}(\phi_n(a), \sigma_n(a), ..., \sigma_n(a)) \\ &- u^* \mathrm{diag}(\psi_n(a), \sigma_n(a), ..., \sigma_n(a)) u \| \} \geq \varepsilon. \end{aligned} \tag{2.5}$$

Denote by

$$\Phi, \Psi, \Sigma : A \to \prod B_n$$

the maps induced by (ϕ_n), (ψ_n) and (σ_n), and consider

$$\tilde{\Phi}, \tilde{\Psi}, \tilde{\Sigma} : A \to \prod B_n / \bigoplus B_n.$$

By Lemma 2.9, one has that

$$[\Phi] = [\Psi] \in \mathrm{Hom}_{\Lambda}(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(\prod B_n)),$$

and therefore,

$$[\tilde{\Phi}] = [\tilde{\Psi}] \in \mathrm{Hom}_{\Lambda}(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(\prod B_n / \bigoplus B_n)).$$

By (2.3), one has

$$[\tilde{\Phi}]_{\mathrm{KK}} = [\tilde{\Psi}]_{\mathrm{KK}} \in \mathrm{KK}(A, \prod B_n / \bigoplus B_n).$$

Since $\tilde{\Sigma} : A \to \prod B_n / \bigoplus B_n$ is a full embedding, it follows from Theorem 4.5 of [1] that there is N and a unitary $w \in \mathrm{M}_{N+1}(\prod B_n / \bigoplus B_n)$ such that

$$\| \mathrm{diag}(\tilde{\Phi}(a), \underbrace{\tilde{\Sigma}(a), ..., \tilde{\Sigma}(a)}_{N}) - w^* \mathrm{diag}(\tilde{\Psi}(a), \underbrace{\tilde{\Sigma}(a), ..., \tilde{\Sigma}(a)}_{N}) w \| < \varepsilon$$

for any $a \in \mathcal{F}$, which implies that

$$\limsup_{n} \max_{a \in \mathcal{F}} \| \mathrm{diag}(\phi_n(a), \underbrace{\sigma_n(a), ..., \sigma_n(a)}_{N}) - w^* \mathrm{diag}(\psi_n(a), \underbrace{\sigma_n(a), ..., \sigma_n(a)}_{N}) w \| < \varepsilon.$$

If n is sufficiently large, it reaches a contradiction to (2.5). $\square$

Using Theorem 2.10 and Lemma 2.7, one then has the following stable uniqueness theorem for approximate homomorphisms. The proof is a routine argument as in [10] or [1]. But for the readers' convenience, an argument is included below.

THEOREM 2.11. *Let A be a separable simple nuclear unital C*-algebra satisfying the UCT, and let $\mathbf{L} : \mathrm{U}(\mathrm{M}_\infty(A)) \to \mathbb{R}^+$ be a map. Then, for any finite subset $\mathcal{F} \subseteq A$ and any $\varepsilon > 0$, there exist finite subsets $\mathcal{G} \subseteq A$, $\mathcal{P} \subseteq \mathbf{P}(A)$, $\mathcal{H} \subseteq \mathrm{U}(\mathrm{M}_\infty(A))$, a constant $\delta > 0$, and $n \in \mathbb{N}$ such that for any C*-algebra $B \in \mathcal{C}$, any $\mathcal{G}$-δ-multiplicative maps $\phi, \psi, \sigma : A \to B$ with σ unital, if*

$$[\phi(p)] = [\psi(p)] \text{ in } \mathrm{K}(B)$$

for all $p \in \mathcal{P}$, and

$$\mathrm{cel}(\phi(u)\psi(u^*)) \leq \mathbf{L}(u) \tag{2.6}$$

for any $u \in \mathcal{H}$, then there is a unitary $w \in \mathrm{M}_{n+1}(B)$ such that

$$\|\mathrm{diag}(\phi(a), \underbrace{\sigma(a), ..., \sigma(a)}_{n}) - w^*\mathrm{diag}(\psi(a), \underbrace{\sigma(a), ..., \sigma(a)}_{n})w\| < \varepsilon$$

for any $a \in \mathcal{F}$.

PROOF. The proof is similar to that of Theorem 4.15 of [1]. Assume that the statement were not true, there would exist a finite subset $\mathcal{F} \subseteq A$, $\varepsilon > 0$, and an increasing family of finite subsets $(\mathcal{G}_i)$ of A with dense union, an increasing family of finite subsets $(\mathcal{P}_i)$ of the projections with the union exhausts all equivalent classes of projections, an increasing sequence $(\mathcal{H}_i)$ of finite subsets of unitaries with dense union, a decreasing sequence (δ_i) with $\delta_i \to 0$, a sequence (B_i) of C*-algebras in $\mathcal{C}$, and $\mathcal{G}_i$-δ_i-multiplicative maps $\phi_i, \psi_i, \sigma_i : A \to B_i$ with σ_i unital such that

$$[\phi_i(p)]_* = [\psi_i(p)]_* \quad \text{for all } p \in \mathcal{P}_i,$$

$$\mathrm{cel}(\phi(u)\psi(u^*)) \leq \mathbf{L}(u) \quad \text{for all } u \in \mathcal{H}_i, \tag{2.7}$$

but

$$\begin{aligned} \inf_{u \in \mathrm{U}(\mathrm{M}_{n+1}(B_i))} \max_{a \in \mathcal{F}} \{ \|\mathrm{diag}(\phi_i(a), \underbrace{\sigma_i(a), ..., \sigma_i(a)}_{n}) \\ -u^*\mathrm{diag}(\psi_i(a), \underbrace{\sigma_i(a), ..., \sigma_i(a)}_{n})u\| \} \geq \varepsilon, \end{aligned} \tag{2.8}$$

where $n = n(A, \mathcal{F}, \varepsilon)$ is the number specified in Theorem 2.10.

Consider the maps $\tilde{\Phi}$, $\tilde{\Psi}$, and $\tilde{\Sigma}$: $A \to \prod B_i / \bigoplus B_i$ induced by (ϕ_i), (ψ_i), and (σ_i). Since $\delta_i \to 0$, these maps are *-homomorphisms. Then, by (2.2) of Lemma 2.9, one has $[\tilde{\Phi}]_0 = [\tilde{\Psi}]_0$. Since

$$\mathrm{cel}(\tilde{\Phi}(u)\tilde{\Psi}(u^*)) < \mathbf{L}(u)$$

for any unitary u in matrix algebras over A, it also follows from Lemma 2.9 $[\tilde{\Phi}]_1 = [\tilde{\Psi}]_1$. By the same argument as that of Theorem 4.15 of [1], one has

$$[\tilde{\Phi}] = [\tilde{\Psi}] \in \mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(\prod B_n / \bigoplus B_n)).$$

By Lemma 2.7, the C*-algebra $\prod B_i / \bigoplus B_i \in \mathcal{C}$. Then, by Theorem 2.10, there is a unitary $w \in \mathrm{U}(\mathrm{M}_{n+1}(\prod B_i / \bigoplus B_i))$ such that

$$\|w^* \mathrm{diag}(\tilde{\Phi}(a), \underbrace{\tilde{\Sigma}(a), ..., \tilde{\Sigma}(a)}_{n})w - \mathrm{diag}(\tilde{\Psi}(a), \underbrace{\tilde{\Sigma}(a), ..., \tilde{\Sigma}(a)}_{n})\| < \varepsilon$$

for any $a \in \mathcal{F}$, but this contradicts to (2.8).

If $\mathrm{K}_1(A) = 0$, without assuming Condition 2.7, one always has that $[\Phi]_1 = [\Psi]_1 = 0$, and hence one still has

$$[\Phi] = [\Psi] \in \mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(\prod B_n / \bigoplus B_n)).$$

Then the argument same as above shows that the statement holds. □

In the case A is a simple separable TASI-algebra, we get a natural number n by applying the theorem above for a finite subset $\mathcal{F}$ and $\delta > 0$. Since any simple TASI-algebra is tracially approximately divisible (Theorem 4.13 of [12]), there exist mutually orthogonal projections $q, p_1, ..., p_n$ with $q + p_1 + \cdots + p_n = 1$, $q \preceq p_1$ and $p_i \sim p_1, i = 1, ..., n$, a splitting interval sub-C*-algebra S with $1_S = p_1$ and two $\mathcal{F}-\varepsilon$ multiplicative linear unital maps $\phi_0 : A \to qAq$, $\phi_1 : A \to S$, such that

$$\|x - \phi_0(x) \oplus (\underbrace{\phi_1(x) \oplus \cdots \oplus \phi_1(x)}_{n \text{ copies}})\| \leq \varepsilon$$

for any x in $\mathcal{F}$. With this n and the splitting interval algebra S, one then can use Theorem 2.1 and Theorem 2.11 to get an unitary in B rather than in the matrix algebra over B.

THEOREM 2.12. *Let A be a simple separable nuclear TASI-algebra satisfying the UCT, and let $\mathbf{L} : \mathrm{U}(\mathrm{M}_\infty(A)) \to \mathbb{R}^+$ be a map. Then for any finite subset $\mathcal{F}$ of A, and $\varepsilon > 0$, there exist a finite subset $\mathcal{P} \subset P(A)$, a finite subset $S \subset A$, $\delta_1 > 0$ and a natural number n such that there exist mutually orthogonal projections*

$q, p_1, \cdots, p_n$ with $q \preceq p_1$ and $p_1, \cdots, p_n$ mutually unitary equivalent, a sub-C-algebra S which is a splitting interval algebra with $1_S = p_1$ and unital S-$\delta_1/2$-multiplicative completely positive contractions $\phi_0 : A \to qAq$ and $\phi_1 : A \to S$ such that*

$$\|x - \phi_0(x) \oplus (\underbrace{\phi_1(x) \oplus \cdots \oplus \phi_1(x)}_{n \quad copies})\| \leq \delta_1/2$$

for all $x \in S$.

Moreover, there exist a finite subset $\mathcal{G} \subset A$, a finite subset $\mathcal{P}_0$ of projections in $\mathrm{M}_\infty(S)$, a finite subset $H \subset A_{s.a.}$, $\delta_0 > 0$ and $\sigma > 0$ such that for any simple TASI-algebra B and two $S \cup \mathcal{G}$-δ-multiplicative completely positive contractions $L_1, L_2 : A \to B$ for which ($\delta = \min\{\delta_0, \delta_1\}$)

- $[L_1]|_{\mathcal{P} \cup \mathcal{P}_0} = [L_2]|_{\mathcal{P} \cup \mathcal{P}_0}$,
- $||\tau \circ L_1(g) - \tau \circ L_2(g)|| < \delta$, *for all $g \in H$ and $\tau \in \mathrm{T}(A)$,*
- $e = L_1 \circ \phi_0(1_A) = L_2 \circ \phi_0(1_A)$ *is a projection,*
- $\mathrm{cel}((L_1 \circ \phi_0(u)^*)(L_2 \circ \phi_0(u))) \leq \mathbf{L}(u)$ *(in $\mathrm{U}(eBe)$) for any $u \in \mathrm{U}(A) \cap \mathcal{P}$,*

there is an unitary $u \in B$ such that:

$$\|u^* L_1(a) u - L_2(a)\| \leq \varepsilon \quad \forall a \in \mathcal{F}.$$

PROOF. This theorem follows from Theorem 2.1, Theorem 4.13 of [12] and Theorem 2.11. □

3. Classification Theorem In previous chapters, we have shown the existence theorem and uniqueness theorem. In this chapter, using these results, together with an approximate intertwining argument, we shall show that the class of simple separable nuclear TASI-algebras satisfying the UCT is classified by the Elliott invariant. The strategy of the argument is to construct a model algebra for any given TASI-algebra, and then to show that this concrete model is in fact *-isomorphic to the given TASI-algebra. Since the construction model algebra is only based on the Elliott invariant, this gives a classification of TASI-algebras.

3.1. Unitary groups of TASI-algebras However, in order to apply the uniqueness theorem, the two approximate homomorphisms are required not only to induce the same map on the level of the invariant, but also to satisfy the condition of the boundedness of the exponential length. Note that for the concrete TASI-algebras in the class $\mathcal{SH}$ introduced in Section 2.1 of [13] (recall that these C*-algebra are inductive limits of splitting tree (interval) algebras together with homogeneous C*-algebras), we use circle algebras to realize the torsion free part of the K_1-group. Moreover, the unitary groups of TAS-algebras share many of the properties of TAI-algebras. So one can use the same method as in [11] to control the exponential length. As in [11], we use the following notation.

For a unital C*-algebra A, let $\mathrm{CU}(A)$ denote the closure of the commutator subgroup of $\mathrm{U}(A)$. It is a normal subgroup of $\mathrm{U}(A)$, and $\mathrm{U}(A)/\mathrm{CU}(A)$ is commutative. If $\mathrm{K}_1(A)=\mathrm{U}(A)/\mathrm{U}_0(A)$ where $\mathrm{U}_0(A)$ is the component of the unitary group containing the identity, then the subgroup $\mathrm{CU}(A)$ is in fact inside $\mathrm{U}_0(A)$. For $u\in\mathrm{U}(A)$, we use $\bar{u}$ to denote the image of u in $\mathrm{U}(A)/\mathrm{CU}(A)$.

LEMMA 3.1. *Let u be a unitary in a splitting interval algebra S with generic dimension n. For any $\varepsilon>0$, there are continuous functions $s_i:[0,1]\to\mathbb{T}$, i=1,...,n, and unitaries $W\in S$ such that*

$$\|u-W^*\mathrm{diag}\{s_1,...,s_n\}W\|\leq\varepsilon.$$

PROOF. Denote by n the generic size of S, and denote by the partition of n at the points 0 and 1 by $(m_0^{(1)},...,m_0^{(l_0)})$ and $(m_1^{(1)},...,m_1^{(l_0)})$ respectively.

Let

$$W_0=\mathrm{diag}\{W_{0,1},...,W_{0,l_0}\}\text{ and }W_1=\mathrm{diag}\{W_{0,1},...,W_{0,l_0}\}$$

be two unitary matrices with the the same diagonal block size as that of S at the endpoint 0 and 1 respectively, such that $W_0^*u(0)W_0$ and $W_1^*u(1)W_1$ are diagonal matrices.

Let W' be a unitary in S with $W'(0)=W_0$ and $W'(1)=W_1$.

Consider the unitary $v:=(W')^*uW'$. It is clear that $v(0)$ and $v(1)$ are diagonal. Without loss of generality, we may assume that there exists $0<\delta<1/4$ such that the restrictions of v to $[0,\delta]$ and $[1-\delta,1]$ are constant.

Consider the restrictions of v to $[\delta,1-\delta]$, it is well known that there is a unitary $V\in\mathrm{C}([\delta,1-\delta],\mathrm{M}_n(\mathbb{C}))$ such that

$$\|V^*(t)v(t)V(t)-\mathrm{diag}\{s_1(t),...,s_n(t)\}\|<\varepsilon\quad\text{for any }t\in[\delta,1-\delta]$$

for some continuous functions $s_i:[\delta,1-\delta]\to\mathbb{T}$. Moreover, the unitary V can be chosen such that

$$V^*(\delta)v(\delta)V(\delta)=v(\delta)\quad\text{and}\quad V^*(1-\delta)v(1-\delta)V(1-\delta)=v(1-\delta).$$

Let w_0 and w_1 be one branch of natural logarithm of $V(\delta)$ and $V(1-\delta)$ respectively such that they are well-defined. Extend the unitary V to $[0,1]$ by $V(t)=\exp(\frac{t}{\delta}w_0)$ for any $t\in[0,\delta]$ and $V(t)=\exp(\frac{t-1+\delta}{\delta}w_0)$ for any $t\in[1-\delta,1]$. Since $V(\delta)$ and $V(1-\delta)$ commute with $v(\delta)$ and $v(1-\delta)$ respectively, one has that w_0 and w_1 commute with $v(\delta)$ and $v(1-\delta)$ respectively, and hence $V(t),t\in[0,\delta]$ and $V(t),t\in[1-\delta,1]$ commute with $v(\delta)$ and $v(1-\delta)$ respectively.

Extend each function $s_i(t)$ to the interval $[0,1]$ such that the restrictions of s_i to $[0,\delta]$ and $[\delta,1]$ are constant. Since that $V(0)$ and $V(1)$ are diagonal, one has that $V\in S$. Then, a direct calculation shows that

$$\|V^*(t)v(t)V(t)-\mathrm{diag}\{s_1(t),...,s_n(t)\}\|\leq\varepsilon\quad\text{for any }t\in[0,1].$$

Therefore, set $W:=W'V$, one has that

$$\|W^*uW-\mathrm{diag}\{s_1,...,s_n\}\|\leq\varepsilon,$$

as desired. □

COROLLARY 3.2. *Let u be a unitary in a splitting interval algebra S. For any $\varepsilon > 0$, there is an self-adjoint element $h \in S$ such that*

$$\|u - \exp(ih)\| \leq \varepsilon.$$

PROOF. Since for any continuous function $s : [0,1] \to \mathbb{T}$, there is a function $h : [0,1] \to \mathbb{R}$ such that $s = \exp(ih)$, the corollary follows from Lemma 3.1. □

LEMMA 3.3. *Let u be a unitary in a splitting interval algebra S. If $\det_t(u) = 1$ for any $t \in \mathrm{Sp}(S)$, then $u \in \mathrm{CU}(S)$.*

PROOF. Denote by n the generic size of S, and denote by the partition of n at the points 0 and 1 by $(m_0^{(1)}, ..., m_0^{(l_0)})$ and $(m_1^{(1)}, ..., m_1^{(l_0)})$ respectively.

By Lemma 3.1, we may assume that $u = \mathrm{diag}\{s_1(t), ..., s_n(t)\} \in S$. Since $\det_t(u) = 1$ for any $t \in \mathrm{Sp}(S)$, one has that

$$\prod_{i \in m_0^{(k)}} s_i(0) = 1 \quad \text{and} \quad \prod_{i \in m_1^{(k)}} s_i(1) = 1 \tag{3.1}$$

for each k.

Since $\mathrm{CU}(S)$ is closed, without loss of generality, let us assume that each function $s_i(t)$ is constant if $t \in [0, \delta]$ and $t \in [1-\delta, 1]$ for some $\delta > 0$. Denote by

$$u_1 = \mathrm{diag}(1, s_1, s_2 s_1, ..., s_{n-1}s_{n-2}\cdots s_1) \in S,$$

and consider

$$uu_1 = \mathrm{diag}(s_1, s_2 s_1, s_3 s_2 s_1, ..., s_n s_{n-1}\cdots s_1).$$

Using (3.1), one can find a unitaries $W_0 = \mathrm{diag}(W_{1,0}, ..., W_{l_0,0})$ and $W_1 = \mathrm{diag}(W_{1,1}, ..., W_{l_1,1})$ such that each $W_{i,j}$ is an $|m_j^{(i)}| \times |m_j^{(i)}|$ matrix, and

$$uu_1(0) = W_0^* u_1(0) W_0 \quad \text{and} \quad uu_1(1) = W_1^* u_1(1) W_1.$$

Moreover, there is a $n \times n$ matrix V such that

$$uu_1(t) = V^* u_1(t) V \quad \text{for any } t \in [\delta, 1-\delta].$$

Consider the path of unitaries

$$W_0(t) = W_0 \exp(t \ln(W_0^* V)) \quad \text{and} \quad W_1(t) = V \exp(t \ln(V^* W_1)).$$

Then, $W_0(0) = W_0$ and $W_0(1) = V$. Note that

$$V^* W_0 uu_1(0) W_0^* V = V^* u_1(0) V = V^* u_1(\delta) V = uu_1(0).$$

We have that for any $t \in [0,1]$,

$$\exp(t \ln(V^* W_0)) uu_1(0) \exp(t \ln(W_0^* V)) = uu_1(0),$$

and hence

$$\begin{aligned} W_0^*(t)u_1(0)W_0(t) &= \exp(t\ln(V^*W_0))W_0^*u_1(0)W_0\exp(t\ln(W_0^*V)) \\ &= \exp(t\ln(V^*W_0))uu_1(0)\exp(t\ln(W_0^*V)) \\ &= uu_1(0). \end{aligned}$$

With the same argument, we have a path of unitaries $W_1(t)$ such that $W_1(0) = V$, $W_1(1) = W_1$, and

$$W_1^*(t)u_1(1)W_1(t) = uu_1(1).$$

Denote by

$$W(t) = \begin{cases} W_0(\frac{t}{\delta}) & \text{if } t \in [0, \delta] \\ V & \text{if } t \in [\delta, 1-\delta] \\ W_1(\frac{1-\delta-t}{\delta}) & \text{if } t \in [1-\delta, 1] \end{cases}$$

It is clear that $W(t) \in S$ and $uu_1 = W^*u_1W$. Hence one has that $u = u_1^*W^*uW \in \mathrm{CU}(S)$. $\square$

LEMMA 3.4. *For any $\varepsilon > 0$, there is a constant K such that for any splitting interval algebra S and a unitary $u \in S$, if for any irreducible representation π_t of S, $\dim(\pi_t) > K$ and $\det(\pi_t(u)) = 1$, then*

$$\|u - \exp(ih_1)\exp(ih_2)\exp(ih_3)\| \leq \varepsilon$$

for some self-adjoint element h_i with $\|h_i\| \leq 2\pi$, $i = 1, ..., 3$.

PROOF. Without loss of generality, one may assume that there exists $0 < \delta < 1/4$ such that the restrictions of u to $[0, \delta]$ and $[1-\delta, 1]$ are constant. Denote by the constant matrices by u_0 and u_1, and assume that u and v can be diagonalized by unitary matrices W_0' and W_1'. Noting that W_0' and W_1 have the same block diagonal form as u in 0 and 1, there are unitaries $W_0 \in S$ and $W_1 \in S$ such that $W_0(t) = W_0'$ and $W_1(t) = I$ for $t \in [0, \delta]$, and $W_0(t) = I$ and $W_1(t) = W_1'$ for $t \in [1-\delta, 1]$. Thus, by consider $(W_0W_1)^*u(W_0W_1)$, we may assume that $u(t)$ is diagonal for $t \in [0, \delta] \cup [1-\delta, 1]$.

Denote by K the constant of Theorem 3.3 of [14] corresponding to $X = [\delta, 1-\delta]$ and ε, and consider the restriction of u to the interval $[\delta, 1-\delta]$. It follows from Theorem 3.3 of [14] that there exist self-adjoint functions $h_1, h_2, h_3 : [\delta, 1-\delta] \to \mathrm{M}_n(\mathbb{C})$ such that

$$\|u(t) - \exp(ih_1(t))\exp(ih_2(t))\exp(ih_3(t))\| \leq \varepsilon, \quad \forall t \in [\delta, 1-\delta],$$

and $\|h_i\| \leq 2\pi$, $i = 1, 2, 3$. (The restriction $\|h_i\| \leq 2\pi$, $i = 1, 2, 3$ is not in the statement of Theorem 3.3 of [14]. However, it follows from the construction of h_1, h_2, and h_3 in the proof. For more details, see the proof of Lemma 3.4 of [15].)

In order to proof the lemma, one has to show that for each h_i, the matrices $h_i(\delta)$ and $h_i(1-\delta)$ has the right diagonal form to fit into the splitting points of S.

By checking the proof of Theorem 3.3 of [14], one has that the matrices $h_1(\delta)$ and $h_1(1-\delta)$ are diagonal (since the unitary u is diagonal at the points δ and $1-\delta$, the unitary u_4 in the step 3 of the proof can be chose to be diagonal at these two points, and hence in the step 4 of the proof, $h_1(x) = g^{(x)}(u_4(x))$ is diagonal if $x = \delta$ or $x = 1-\delta$).

By checking the step 5 and the step 6 of proof of Theorem 3.3 of [14], one has that the unitary v_5 and v_6 can be chose in such a way that their restrictions to δ and $1-\delta$ are inside one of the diagonal blocks (which has size at least K). Hence there is a projection p in $\mathrm{M}_n(\mathrm{C}([\delta, 1-\delta]))$ such that p has rank at least K, $p(\delta)$ and $p(1-\delta)$ are inside the corresponding diagonal blocks, and v_5 and v_6 are in the hereditary sub-C*-algebra generated by p.

Consider the unital hereditary sub-C*-algebra generated by p and the element $v_6 \oplus 1_K$ inside this sub-C*-algebra, and applying step 7 of the proof of Theorem 3.3 of [14]. There are elements h_2 and h_3 such that

$$\|v_6 \oplus (p - v_6^* v_6) - \exp(ih_2')\exp(ih_3')\| < 2\varepsilon/5.$$

Since h_2' and h_3' are in the hereditary sub-C*-algebra generated by p, the elements $h_2 = h_2' \oplus (1-p)$ and $h_3 = h_3' \oplus (1-p)$ has the right form of diagonal blocks at δ and $1-\delta$, and

$$\|u_6 - \exp(ih_2)\exp(ih_3)\| < 2\varepsilon/5.$$

Therefore,

$$\|u(t) - \exp(ih_1(t))\exp(ih_2(t))\exp(ih_3(t))\| \leq \varepsilon, \quad \forall t \in [\delta, 1-\delta],$$

and the matrices $h_i(\delta)$ and $h_i(1-\delta)$ has the right diagonal form to fit into the splitting points of S respectively. Extend h_i to the whole interval $[0,1]$ constantly on $[0,\delta]$ and $[1-\delta, 1]$. Then, each h_i induces an element of S. Using the assumption that u is constant on $[0,\delta]$ and $[\delta, 1]$, one has that

$$\|u(t) - \exp(ih_1(t))\exp(ih_2(t))\exp(ih_3(t))\| \leq \varepsilon, \quad \forall t \in [0,1],$$

as desired. □

If $\phi : A \to B$ is a unital *-homomorphism, it will induce a homomorphism $\phi^\ddagger : \mathrm{U}(A)/\mathrm{CU}(A) \to \mathrm{U}(B)/\mathrm{CU}(B)$. Moreover, for any finite subset $\mathcal{U} \subset \mathrm{U}(A)$ and any $\varepsilon > 0$, denote by $\bar{F}$ the subgroup of $\mathrm{U}(A)/\mathrm{CU}(A)$ generated by $\mathcal{U}$. We can choose a finite subset $\mathcal{G} \subset A$ and $\delta > 0$, such that for any $L : A \to B$ which is a $\mathcal{G}-\delta$-multiplicative completely positive linear contraction, there is a homomorphism $L^\ddagger : \bar{F} \to \mathrm{U}(B)/\mathrm{CU}(B)$ with $||\overline{L(u)} - L^\ddagger(\bar{u})|| < \varepsilon$ for any $u \in \mathcal{U}$. We say $L^\ddagger$ is induced by L.

THEOREM 3.5 (See Theorem 6.5 of [11]). *Let A be a simple TASI-algebra, and let $u \in \mathrm{U}_0(A)$. Then, for any $\varepsilon > 0$, there are unitaries u_1 and u_2 such that u_1 has exponential length no more than 2π, u_2 is an exponential, and*

$$\|u - u_1 u_2\| < \varepsilon.$$

Moreover, $\mathrm{cer}(A) \leq 3 + \varepsilon$.

PROOF. Using the fact that the exponential rank of splitting interval algebra is $1 + \varepsilon$ (Corollary 3.2), one can repeat the proof of Theorem 6.5 of [11]. □

LEMMA 3.6. *Let A be a simple TASI-algebra, and let $u \in \mathrm{CU}(A)$. Then, $u \in \mathrm{U}_0(A)$ and* $\mathrm{cel}(u) \leq 8\pi$.

PROOF. The proof is a repeating of that of Lemma 6.9 of [11]. Instead appealing to 3.4 of [15], one uses Lemma 3.4 in the proof. □

THEOREM 3.7 (See Theorem 6.10 of [11]). *Let A be a simple TASI-algebra. Let $u, v \in \mathrm{U}(A)$ such that $[u] = [v]$ in $\mathrm{K}_1(A)$ and*

$$u^k, v^k \in \mathrm{U}_0(A) \quad \textit{and} \quad \mathrm{cel}((u^*)^k u^k) \leq L.$$

Then,

$$\mathrm{cel}(u^* v) \leq 8\pi + L/k.$$

Moreover, there is $y \in \mathrm{U}_0(A)$ with $\mathrm{cel}(y) \leq L/k$ such that $\overline{u^* v} = \overline{y}$ in $\mathrm{U}(A)/\mathrm{CU}(A)$.

PROOF. The proof is similar to that of Theorem 6.10 of [11]. Write

$$u^* v = \prod_j \exp(i a_i) \quad \text{and} \quad (u^k)^* v^k = \prod_m \exp(i b_m),$$

where a_j and b_m are self-adjoint. We may assume that $\sum \|b_m\| < L$, since $\mathrm{cel}((u^*)^k u^k) \leq L$. Write $M = \sum \|b_m\|$. Since A is a TASI-algebra, for any $\delta > 0$ with $\delta/(1-\delta) < \varepsilon/2(M + L + 1)$ and sufficiently small $\eta > 0$ and sufficiently large finite subset $\mathcal{G}$, there is a projection $p \in A$ and a sub-C*-algebra $S \subset A$ with $1_S = p$ such that S is a direct sum of splitting interval algebras, and

1. $pxp \in_\eta S$ for any $x \in \mathcal{G}$,
2. $\|u - u_0 \oplus u_1\| \leq \eta$ and $\|v - v_0 \oplus v_1\| \leq \eta$ for some unitaries u_0, v_0 in $(1-p)A(1-p)$ and u_1, v_1 in $S \subseteq pAp$,
3. $\mathrm{cel}(u_0^* v_0) \leq M + 1$ in $(1-p)A(1-p)$ and $\mathrm{cel}((u_1^k)^* v_1^k) < L$ in S,
4. $\tau(1-p) \leq \delta$ for any $\tau \in \mathrm{T}(A)$.

Without loss of generality, we may assume that the dimension of any non-zero irreducible representation of S is greater that $M := \max(2\pi^2/\varepsilon, K)$, where K is the constant of Lemma 3.4.

Let us assume that S is consisted of one direct summand and has generic dimension n. Without loss of generality, we may assume that there exists $0 < \delta < 1/4$ such that $(u_1^*)^k v_1^k$ is constant on $[0, \delta]$ and on $[1-\delta, 1]$. Consider the restriction of S to $[\delta, 1-\delta]$. By Lemma 3.3 (1) of [15], there exists $a \in (\mathrm{M}_n(\mathrm{C}([\delta, 1-\delta])))_{\mathrm{s.a.}}$ such that

$$\det(\exp(ia)(u_1^*)^k v_1^k) = 1 \quad \text{for any } t \in [\delta, 1-\delta].$$

Using the connectivity of the unitary subgroup of a matrix algebras, one may assume further that

$$a(\delta) = \mathrm{diag}\{\underbrace{r_1, ..., r_{m_0}}_{m_0^1}, \underbrace{r_{m_0^{(1)}}, ..., r_{m_0^{(1)}+m_0^{(2)}}}_{m_0^1}, ..., \underbrace{r_{n-m_0^{(l_n)}}, ..., r_n}_{m_0^{l_0}}\}$$

and

$$a(1-\delta) = \mathrm{diag}\{\underbrace{s_1, ..., s_{m_0}}_{m_1^1}, \underbrace{s_{m_1^{(1)}}, ..., s_{m_1^{(1)}+m_1^{(2)}}}_{m_1^1}, ..., \underbrace{s_{n-m_1^{(l_n)}}, ..., s_n}_{m_1^{l_1}}\}$$

with products of each group of r's or s's equal to one. Therefore, one can extend a constantly to $[0, 1]$ to get an element in S. Since the restriction of $(u_1^*)^k v_1^k$ is constant on $[0, \delta]$ and on $[1-\delta, 1]$, one has that

$$\det(\exp(ia)(u_1^*)^k v_1^k) = 1 \quad \text{for any } t \in \mathrm{Sp}(S).$$

Therefore

$$\det((\exp(ia/k)u_1^* v_1)^k) = 1 \quad \text{for any } t \in \mathrm{Sp}(S),$$

and

$$\det(\exp(ia/k)u_1^* v_1) = \exp(i2l\pi/k) \quad \text{for any } t \in (0, 1),$$

for some $l \in 0, 1, ..., k-1$. Define the function f in the following way: On $[\delta, 1-\delta]$, set $f = -2l\pi/(nk)1_n$; on points 0 and 1, define $a_i = -\sum_{j=1}^{m_0^{(i)}} r_j/m_0^{(i)}$ and $b_i = -\sum_{j=1}^{m_1^{(i)}} r_j/m_1^{(i)}$

Set

$$f(0) = \mathrm{diag}\{\underbrace{a_1, ..., a_1}_{m_0^1}, \underbrace{a_2, ..., a_2}_{m_0^1}, ..., \underbrace{a_{l_0}, ..., a_{l_0}}_{m_0^{l_0}}\},$$

and

$$f(1) = \mathrm{diag}\{\underbrace{b_1, ..., b_1}_{m_1^1}, \underbrace{b_2, ..., b_2}_{m_1^1}, ..., \underbrace{b_{l_1}, ..., b_{l_1}}_{m_1^{l_1}}\},$$

and define

$$f(t) = (1 - \frac{t}{\delta})f(0) + \frac{t}{\delta}f(\delta), \quad t \in [0, \delta],$$

$$f(t) = (\frac{1-t}{\delta})f(\delta) + (1 + \frac{t-1}{\delta})f(1), \quad t \in [1-\delta, 1].$$

We then have $f \in S$ with $\|f\| \leq \|a\|/M + 2\pi \leq L/M + 2\pi$ and

$$\det(\exp(if)\exp(ia/k)u_1^*v_1) = 1 \quad \text{for any } t \in \mathrm{Sp}(S).$$

By Lemma 3.3, $z_1 := \exp(if)\exp(ia/k)u_1^*v_1 \in \mathrm{CU}(S)$. Moreover, by Lemma 3.4, there are unitaries w_1, w_2, and w_3 with $\|w_i\| \leq 2\pi$ such that

$$\|\exp(if)\exp(ia/k)u_1^*v_1 - \exp(iw_1)\exp(iw_2)\exp(iw_3)\| \leq \varepsilon,$$

and hence

$$\|u_1^*v_1 - \exp(if)\exp(ia/k)\exp(iw_1)\exp(iw_2)\exp(iw_3)\| \leq \varepsilon.$$

Therefore,

$$\mathrm{cel}(u_1^*v_1) \leq L/M + L/k + 6\pi.$$

Then, by Lemma 6.4 of [11], there is $y' \in \mathrm{CU}(A)$ and $y'' \in \mathrm{U}_0(A)$ such that $(u_0 \oplus p)^*(v_0 \oplus p) = y'y''$ and $\mathrm{cel}(y'') < \varepsilon/2$. Note that $((1-p) \oplus z_1)y' \in \mathrm{CU}(A)$. Therefore

$$\overline{u^*v} = \overline{\exp(if)\exp(ia)w}$$

for some $w_0 \in \mathrm{U}_0(A)$ with $\mathrm{cel}(w) \leq \varepsilon/2$ for sufficiently small η. By Lemma 3.6, one has that

$$\mathrm{cel}(u^*v) \leq 2\pi/M + L/k + 8\pi + \varepsilon/2 < 8\pi + L/k + \varepsilon,$$

as desired. □

THEOREM 3.8 (See Theorem 6.11 of [11]). *Let A be a simple TASI-algebra. Then the group $\mathrm{U}_0(A)/\mathrm{CU}(A)$ is torsion free.*

PROOF. The proof is the same as that of Theorem 6.11 of [11], and also essentially the same as that of Theorem 3.7. □

COROLLARY 3.9 (See Corollary 6.12 of [11]). *Let B_n be a sequence of unital simple TASI-algebras. Let $\prod_n^b \mathrm{K}_1(B_n)$ be the set of sequence $z = \{z_n\}$, where $z_n \in \mathrm{K}_1(B_n)$ and each z_n can be represented by a unitary in a matrix algebra over B_n. Then the kernel of the map*

$$\mathrm{K}_1(\prod_n B_n) \to \prod_n \mathrm{K}_1(B_n) \to 0$$

is a divisible and torsion free subgroup of $\mathrm{K}_1(\prod_n B_n)$.

PROOF. Using Theorem 3.5 and Theorem 3.7 instead of 6.5 and 6.10 of [11], one can repeat the argument of 6.12 of [11]. □

Let us consider concrete model algebras in the class $\mathcal{SH}$, and consider homogeneous C*-algebras in the building blocks.

DEFINITION 3.10. Put $C' = P\mathrm{M}_n(\mathrm{C}(X))P$, where $X = S^1 \vee \cdots \vee S^1 \vee Y$ for some finite CW complex Y with torsion K_1-group and dimension no more than 3, and P is a projection in $\mathrm{M}_n(\mathrm{C}(X))$ with rank $r \geq 6$. Then $K_1(P\mathrm{M}_n(\mathrm{C}(X))P) = \mathrm{Tor}(K_1(C')) \oplus G_1$ for some torsion free group $G_1 \cong \mathbb{Z}^s$. Denote by $D' = \bigoplus_{i=1}^{s} \mathrm{M}_r(\mathrm{C}(\mathbb{T}))$. Then, there is an obvious map $\Pi : P\mathrm{M}_n(\mathrm{C}(X))P \to D'$ induced by the restriction to each circle. We have that $K_1(D') \cong G_1$ and the map Π is not surjective if $s \geq 2$. Denote by $\Pi_i : P\mathrm{M}_n(\mathrm{C}(X))P \to E_i$ to be the composition of Π with the projection from D' to E_i.

Denote by C a finite direct sum of the C*-algebras of the form C' above, matrix algebras, splitting interval algebras, and $p\mathrm{M}_m(\mathrm{C}(Y))p$ with Y a finite CW-complex with dimension at most 3, p a projection with rank at least 6 and $\mathrm{K}^1(Y)$ finite. Write D by the direct sum of D' corresponding to the C*-algebras in the form C'. Then, one has that

$$\mathrm{U}(C)/\mathrm{CU}(C) = \mathrm{U}_0(C)/\mathrm{CU}(C) \oplus K_1(D) \oplus \mathrm{Tor}(K_1(C)).$$

We will set π_0, π_1, π_2 to be the projection maps from $\mathrm{U}(C)/\mathrm{CU}(C)$ to each component according to the decomposition above.

As in [11], we have the following lemmas to control the exponential length in the approximate intertwining argument. The proofs are the repeatings of the corresponding arguments in [11].

LEMMA 3.11 (See Lemma 7.2 of [11]). *Let $C = \bigoplus_{i=1}^{l+l_1} C_i$ be as above, let $\mathcal{U} \subset \mathrm{U}(C)$ be a finite subset, and let F be the group generated by $\mathcal{U}$. Suppose that G is a subgroup of $\mathrm{U}(C)/\mathrm{CU}(C)$ which contains $\bar{F}$, $\pi_1(\mathrm{U}(C)/\mathrm{CU}(C))$, and $\pi_2(\mathrm{U}(C)/\mathrm{CU}(C))$. Suppose that the composition map $\gamma : \bar{F} \to \mathrm{U}(D)/\mathrm{CU}(D) \to \mathrm{U}(D)/\mathrm{U}_0(D)$ is injective and $\gamma(\bar{F})$ is free. Let B be a unital C*-algebra and $\Lambda : G \to \mathrm{U}(B)/\mathrm{CU}(B)$ be a homomorphism such that $\Lambda(G \cap \mathrm{U}_0(C))/\mathrm{CU}(C) \subset \mathrm{U}_0(B)/\mathrm{CU}(B)$. Then there are homomorphism $\beta : \mathrm{U}(D)/\mathrm{CU}(D) \to \mathrm{U}(B)/\mathrm{CU}(B)$ with $\beta(\mathrm{U}_0(D)/\mathrm{CU}(D)) \subset \mathrm{U}(B)/\mathrm{CU}(B)$ and $\theta : \pi_2(\mathrm{U}(C)/\mathrm{CU}(C)) \to \mathrm{U}(B)/\mathrm{CU}(B)$ such that*

$$\beta \circ \Pi^{\ddagger} \circ \pi_1(\bar{w}) = \Lambda(\bar{w})(\theta \circ \pi_2(\bar{w_2}))$$

for any $w \in F$ and such that $\theta(g) = \Lambda|_{\pi_2(\mathrm{U}(C)/\mathrm{CU}(C))}(g^{-1})$ for any $g \in \pi_2(\mathrm{U}(C)/\mathrm{CU}(C))$. Moreover, $\beta(\mathrm{U}_0(D)/\mathrm{CU}(D)) \subset \mathrm{U}_0(B)/\mathrm{CU}(B)$.

If furthermore B is a simple TASI-algebra and

$$\Lambda(\mathrm{U}(C)/\mathrm{CU}(C)) \subset \mathrm{U}_0(B)/\mathrm{CU}(B),$$

then $\beta \circ \Pi^{\ddagger} \circ (\pi_1)|_{\bar{F}} = \Lambda|_{\bar{F}}$.

PROOF. The first part of the statement is exactly the same as that of Lemma 7.2 of [11]. Noting that $\mathrm{U}_0(B)/\mathrm{CU}(B)$ is torsion free by Theorem 3.8, then the second part of the statement follows. □

LEMMA 3.12 (See Lemma 7.3 of [11]). *Let B be a separable simple TASI-algebra, and let C be as above. Let $\mathcal{U} \subset U(B)$ be a finite subset, and let F be the subgroup generated by $\mathcal{U}$ such that $\kappa_1(\bar{F})$ is free, where $\kappa_1 : U(B)/CU(B) \to \mathrm{K}_1(B)$ is the quotient map. Suppose that $\alpha : \mathrm{K}_1(C) \to \mathrm{K}_1(B)$ is a one-to-one homomorphism and $L : \bar{F} \to U(C)/CU(C)$ is a ono-to-one homomorphism with $L(\bar{F} \cap U_0(C)/CU(C)) \subset U_0(B)/CU(B)$ such that $\pi_1 \circ L$ is one-to-one and*

$$\alpha \circ \kappa_1' \circ L(g) = \kappa_1(g) \quad \text{for all} \quad g \in \bar{F},$$

where $\kappa_1' : U(C)/CU(C) \to \mathrm{K}_1(C)$ is the quotient map. Then there exists a homomorphism $\beta : U(C)/CU(C) \to U(B)/CU(B)$ with $\beta(U_0(C)/CU(C)) \subset U_0(B)/CU(B)$ such that

$$\beta \circ L(f) = f$$

for all $f \in \bar{F}$.

PROOF. The proof is exactly a repeating the that of Lemma 7.3 of [11]. □

LEMMA 3.13 (See Lemma 7.4 of [11]). *Let B be a simple separable TASI-algebra, and let C be as above. Let F be a group generated by a finite subset $\mathcal{U} \subset U(C)$ such that $(\pi_1)|_{\bar{F}}$ is one-to-one. Let G be a subgroup containing $\bar{F}$, $\pi_1(U(C)/CU(C))$ and $\pi_2(U(C)/CU(C))$. Suppose that $\alpha : U(C)/CU(C) \to U(B)/CU(B)$ is a homomorphism ($\alpha(U_0(C)/CU(C)) \subset U_0(B)/CU(B)$). Then for any $\varepsilon > 0$, there is $\delta > 0$ satisfying the following: if $\phi = \phi_0 \oplus \phi_1 : C \to B$ is a $\mathcal{G} - \eta$-multiplicative completely positive linear contraction such that*

- *both ϕ_0 and ϕ_1 are $\mathcal{G} - \eta$-multiplicative,*
- *$\mathcal{G}$ is sufficiently large and η is sufficiently small depending only on F and C (such that $\phi^{\ddagger}$ is well defined on a subgroup of $U(C)/CU(C)$ containing all of $\bar{F}$, $\pi_0(\bar{F})$, $\pi_1(U(C)/CU(C))$, and $\pi_2(U(C)/CU(C))$),*
- *ϕ_0 is homotopically trivial (homotopic to a point evaluation), $(\phi_0)_{*0}$ is well-defined and $[\phi]|_{\mathrm{K}_1(C)} = \alpha_*$,*
- *$\tau(\phi_0(1_C)) < \delta$ for all $\tau \in T(B)$ (assume $e_0 = \phi_0(1_C)$),*

then there is a homomorphism $\Phi : C \to e_0 B e_0$ such that

- *Φ is homotopically trivial and $(\Phi)_{*0} = (\phi)_{*0}$ and*
- *$\alpha(\bar{w})^{-1}(\Phi \oplus \phi_1)^{\ddagger}(\bar{w}) = \bar{g_w}$ where $g_w \in U_0(B)$ and $cel(g_w) < \varepsilon$ for any $w \in \mathcal{U}$.*

PROOF. By Theorem 3.8, the group $\mathrm{U}(B)/\mathrm{CU}(B)$ is torsion free. One then can repeat the argument of Lemma 7.4 of [11]. □

LEMMA 3.14 (See Lemma 7.5 of [11]). *Let B be a separable simple TASI-algebra and C as above. Let $\mathcal{U} \subset U(B)$ be a finite subset and F be the subgroup generated by $\mathcal{U}$ such that $\kappa_1(\bar{F})$ is free, where $\kappa : U(B)/CU(B) \to \mathrm{K}_1(B)$ is the quotient map. Let $\phi : C \to B$ be a homomorphism such that $(\phi)_{*1}$ is one-to-one. Suppose that $j, L : \bar{F} \to U(C)/CU(C)$ are two one-to-one homomorphisms with $j(\overline{F \cap U_0(B)}), L(\overline{F \cap U_0(B)}) \subset U_0(C)/CU(C)$ such that $\kappa_1 \circ \phi^{\ddagger} \circ L = \kappa_1 \circ \phi^{\ddagger} \circ j = \kappa_1|_{\bar{F}}$, and they are one-to-one.*

Then for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $\phi = \phi_0 \oplus \phi_1 : C \to B$, where ϕ_0 and ϕ_1 are homomorphisms satisfying the following:

- *$\tau(\phi_0(1_C)) < \delta$ for all $\tau \in T(B)$ and*
- *ϕ_0 is homotopically trivial,*

then there is a homomorphism $\psi : C \to e_0Be_0$ ($e_0 = \phi_0(1_C)$) such that

- *$[\psi] = [\phi_0]$ in $\mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(C), \underline{\mathrm{K}}(B))$ and*
- *$(\phi^\ddagger \circ j(\bar{w}))^{-1}(\psi \oplus \phi_1)^\ddagger(L(\bar{w})) = \bar{g_w}$ where $g_w \in U_0(B)$ and $cel(g_w) < \varepsilon$ for any $w \in \mathcal{U}$.*

PROOF. The argument of Lemma 7.5 of [11] can be duplicated in the following way: instead of using Lemma 7.4 of [11], one uses Lemma 3.13. □

3.2. A classification of TASI-algebras With the preparation above, we shall prove the classification theorem for TASI-algebras. Using an approximately intertwining argument, the proof is exactly the same as that of Theorem 10.4 of [11] of Lin for the classification theorem of TAI-algebras. First, we have the existence of a model algebra for any given TASI-algebra.

THEOREM 3.15 ([3]). *Let A be a simple separable TASI-algebra. Then there exists a simple inductive limit $B = \varinjlim(B_n, \phi_n)$ with $B_n \cong C \oplus S$ for some homogeneous C*-algebra C as described in 3.10 and S a direct sum of splitting interval algebras such that*

- $\mathrm{Ell}(A) = \mathrm{Ell}(B)$,
- *$\phi_n = \phi_n^{(0)} \oplus \phi_n^{(1)} \oplus \phi_n^{(2)}$, where $\phi_n^{(0)}$ factors through a point evaluation map, and $\phi_n^{(1)}$ factors through a splitting interval algebra (in particular, $\phi_n^{(0)}$ is homotopically trivial),*
- *$\tau(\phi_{n+1,\infty} \circ \phi_n^{(0)}(1_{B_n})) \to 0$ and $\tau(\phi_{n+1,\infty} \circ \phi_n^{(2)}(1_{B_n})) \to 0$ uniformly on $\mathrm{T}(B)$, and*
- *$[\phi_n]_1$ is injective for any n.*

REMARK 3.16. The model algebra B is automatically a TASI-algebra.

REMARK 3.17. The statement of Theorem 3.15 is stronger than the statement of Theorem A of [3], which only states the first property of Theorem 3.15. However, one can easily obtains the rest of properties in its proof in [3].

THEOREM 3.18. *Let A and B be two simple separable nuclear TASI-algebra which satisfies UCT. Then $A \cong B$ if and only if*

$$\begin{aligned} &((\mathrm{K}_0(A), \mathrm{K}_0(A)^+, [1_A]_0), \mathrm{K}_1(A), \mathrm{T}(A), r_A) \\ \cong\quad &((\mathrm{K}_0(B), \mathrm{K}_0(B)^+, [1_B]_0), \mathrm{K}_1(B), \mathrm{T}(B), r_B). \end{aligned}$$

*Moreover, the *-isomorphism between the C*-algebras can be chosen to induce the given isomorphism between their invariants.*

PROOF. By Theorem 3.15, there is a inductive limit algebra B' satisfying all the properties of Theorem 3.15. In particular, $\mathrm{Ell}(A) \cong \mathrm{Ell}(B') \cong \mathrm{Ell}(B)$, and B' is a simple TASI-algebra. Fix the algebra B', and let us prove the theorem for the TASI-algebras A and B', and TASI-algebras B and B' respectively. Once it is done, one has that $A \cong B'$ and $B \cong B'$, and hence $A \cong B'$, as desired. Thus, one may assume that the C*-algebra B is one of the concrete algebra described in Theorem 3.15.

Denote by κ the isomorphism

$$\kappa : (\mathrm{K}_0(A), \mathrm{K}_0(A)^+, [1_A]_0; \mathrm{K}_1(A)) \to (\mathrm{K}_0(B), \mathrm{K}_0(B)^+, [1_B]_0; \mathrm{K}_1(B)),$$

and let θ be the isomorphism from $T(B)$ to $T(A)$ compatible with κ. Since A and B satisfy the UCT, there is $\alpha \in \mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B))^+$ which induces κ. Moreover, α can be chosen to be invertible.

Define the function $\mathbb{L} : \mathrm{U}(B) \to \mathbb{R}^+$ as follows:

$$\mathbb{L}(u) = \begin{cases} 2\mathrm{cel}(u) + 8\pi + \pi/16, & \text{if } u \in \mathrm{U}_0(B), \\ 16\pi + \pi/16, & \text{if } u \notin \mathrm{U}_0(B) \text{ and } [u]_1 \text{ is torsion-free}, \\ (2\mathrm{cel}(u^k))/k + 16\pi + \pi/16, & \text{if } u \notin \mathrm{U}_0(B) \text{ and } [u]_1 \text{ has order } k. \end{cases}$$

Fix $\varepsilon > 0$ and finite subset $\mathcal{F} \subset B$. Let $\delta' > 0$, the natural number n, finite subset $\mathcal{P} \subset P(B)$, finite subsets $\mathcal{S} \subseteq B$ be as required in Theorem 2.12. Then there exist mutually orthogonal projections $q, p_1, ..., p_n$ with $q \preceq p_1$ and $p_1, ..., p_n$ mutually unitary equivalent, a sub-C*-algebra S_1, which is a splitting interval algebra with $1_{S_1} = p_1$ and unital $\mathcal{S}$-$\delta_1'/2$-multiplicative completely positive contractions $h_0 : B \to qBq$ with $h_0(x) = qxq$, and $h_1 : B \to S_1$ such that

$$||x - h_0(x) \oplus \underbrace{(h_1(x) \oplus \cdots \oplus h_1(x))}_{n \quad \text{copies}}|| \leq \delta_1'/16$$

for all $x \in S$. Put $S = \mathrm{M}_n(S_1) \subset (1-q)B(1-q)$. Let $\mathcal{P}_0, \mathcal{G}_0, H, \delta_0$ and σ_1 be required by Theorem 2.12. Set $\delta = \min\{\delta_0, \delta'\}$. We may assume that $\mathcal{P}_0$ contains the minimal projections of S which present a generating set of the positive cone of $\mathrm{K}_0(S)$.

Without loss of generality, we may assume that for each $u \in \mathrm{U}(B) \cap \mathcal{P}_0$ has the form $quq \oplus (1-q)u(1-q)$, where $quq \in \mathrm{U}(qBq)$ and $(1-q)u(1-q) \in \mathrm{U}(C)$. Since B is the inductive limit of B_i, we also assume $q \in B_1$ and $quq \in \mathrm{U}(qB_1q)$. Let $\mathcal{U}' = \{quq, u \in \mathrm{U}(B) \cap \mathcal{P}\}$ and let F be the subgroup of $\mathrm{U}(qBq)$ generated by $\mathcal{U}'$. Let $\bar{F}$ be the image of F in $\mathrm{U}(qBq)/\mathrm{CU}(qBq)$ where $\mathrm{CU}(qBq)$ is the commutator subgroup of $\mathrm{U}(qBq)$. By 6.6(3) of [11], we have $\bar{F} = (\bar{F} \cap \mathrm{U}_0(qBq)/\mathrm{CU}(qBq)) \oplus \bar{F}_0 \oplus \bar{F}_1$, where $\bar{F}_0$ is torsion and $\bar{F}_1$ is torsion free. Furthermore, we can assume $\mathcal{U}' = \mathcal{U}_0 \cup \mathcal{U}_1$ with $\bar{\mathcal{U}}_0$ generating $(\bar{F} \cap \mathrm{U}_0(qBq)/\mathrm{CU}(qBq)) \oplus \bar{F}_0$ and $\bar{\mathcal{U}}_1$ generating $\bar{F}_1$. We also assume $q \in B_1$ and $\mathcal{U}_0, \mathcal{U}_1 \subset qB_1q$. Note that $\mathrm{K}_1(B_m) \to \mathrm{K}_1(B_{m+1}) \to \mathrm{K}_1(B)$ is one-to-one for all m.

Let $\mathcal{G}_1$ be a finite subset of B containing S, $\mathcal{G}_0$, H, $\mathcal{U}'$, $\{q, p_1, ..., p_n\}$ and a finite set of generators of S. We assume $\mathcal{G}_1 \subset B_1$. By Theorem 2.34 of [13], there exists a $\mathcal{G}_1$-$\delta/4$ multiplicative completely positive linear contraction $L_1 : B \to A$ such that

$$[L_1]|_{\mathcal{P}\cup\mathcal{P}_0} = \alpha^{-1}|_{\mathcal{P}\cup\mathcal{P}_0}$$

and

$$|\theta^{-1}(\tau)(a) - \tau(L_1(a))| \le \sigma/2 \;\text{ for all } a \in H, \tau \in T(A).$$

We assume that $L_1^{\ddagger}$ is well defined on $\bar{F}$. Define $\mathbb{L}_1 : \mathrm{U}(A) \to \mathbb{R}^+$ in the same manner as the $\mathbb{L}$. Let $\mathcal{F}_1$ be a finite subset of A. Let $\delta_1' > 0$, the natural number n_1, finite subset $\mathcal{P}_1 \subset P(A)$, finite subsets $\mathcal{S}_1 \subset A$ as required in Theorem 2.12 (for A, $\mathbb{L}_1$, $\mathcal{F}_1$ and $\varepsilon/4$). Then there exist mutually orthogonal projections $q', p_1', ..., p_n'$ with $q' \preceq p_1'$ and $p_1', ..., p_n'$ mutually unitary equivalent, a sub-C*-algebra S_2' which is a splitting interval algebra with $1_{S_2'} = p_1'$ and unital $\mathcal{S}_1$-$\delta_1'/4$-multiplicative completely positive contractions $h_0' : A \to q'Aq'$ with $h_0'(x) = q'xq'$, and $h_1 : A \to S_1'$ such that

$$||x - h_0'(x) \oplus \underbrace{(h_1'(x) \oplus \cdots \oplus h_1'(x))}_{n_1 \;\; \text{copies}}|| \le \delta_1'/16$$

for all $x \in S_2'$. We assume $L_1(\mathcal{S}) \subset \mathcal{S}_1$. Set $S_2 = M_n(S_2') \subset (1-q')A(1-q')$, and let $\mathcal{P}_{01}$, $\mathcal{G}_{01}$, H_1, $\delta_{0,1}$ and σ_{01} also be as required by Theorem 2.12. Let $\delta_1 = \min\{\delta_1', \delta_{01}\}$. We may assume that $\delta_1 < \delta/2$, $\sigma_1 < \sigma/4$ and $\mathcal{P}_{01}$ contains the minimal projections of S_2 which present the generating set of the positive cone of $\mathrm{K}_0(S_1)$. We also assume that q' commutes with each elements of H_1 and $\mathcal{S}_1$, and $[L_1](\mathcal{P} \cup \mathcal{P}_0) \subseteq [\mathcal{P}_1]$.

Again, we assume for each $u \in \mathrm{U}(A) \cap \mathcal{P}_1$ has the form $q'uq' \oplus (1-q')u(1-q')$, where $q'uq' \in \mathrm{U}(q'Aq')$ and $(1-q')u(1-q') \in \mathrm{U}(S_1)$. Let $\mathcal{V}' = \{q'uq', u \in \mathrm{U}(A) \cap \mathcal{P}_1\}$ and let F' be the subgroup of $\mathrm{U}(q'Aq')$ generated by $\mathcal{V}'$. Let $\bar{F}'$ be the image of F' in $\mathrm{U}(q'Aq')/\mathrm{CU}(q'Aq')$. By (3) of Lemma 6.6 of [11], one has that $\bar{F}' = (\bar{F}' \cap \mathrm{U}_0(q'Aq')/\mathrm{CU}(q'Aq')) \oplus \bar{F}_0' \oplus \bar{F}_1'$, where $\bar{F}_0'$ is torsion and $\bar{F}_1'$ is torsion free. Furthermore, we may assume that $\mathcal{V}' = \mathcal{V}_0 \cup \mathcal{V}_1$ with $\bar{\mathcal{V}}_0$ generates $(\bar{F}' \cap \mathrm{U}_0(q'Aq')/\mathrm{CU}(q'Aq')) \oplus \bar{F}_0'$ and $\bar{\mathcal{V}}_1$ generates $\bar{F}_1'$.

Let $\mathcal{G}_2'$ be a finite subset of A which contains $\mathcal{S}_1$, $\mathcal{G}_{01}$, $L_1(\mathcal{G}_1')$, H_1, $\mathcal{V}'$, $\{q', p_1', ..., p_n'\}$ and a finite generating set of S_2. By Theorem 2.34 of [13], there exists a $\mathcal{G}_2'$-$\delta_1/4$ multiplicative completely positive linear contraction $\Phi_1' : A \to B$ such that

$$[\Phi_1]|_{\mathcal{P}_1\cup\mathcal{P}_{01}} = \alpha|_{\mathcal{P}_1\cup\mathcal{P}_{01}}$$

and

$$|\theta(\tau)(a) - \tau(L_1(a))| \le \sigma/2 \;\text{ for all } a \in L_1 H \cup H_1, \tau \in T(B).$$

We also assume $(\Phi_1')^{\ddagger}$ is well defined on $\bar{F}'$, $(\Phi_1' \circ L_1)^{\ddagger}$ is well defined on $\bar{F}$ and the image of Φ_1' is contained in B_n.

Let $B'_n = qB_nq$. Since B is simple, we may assume the rank of q is sufficiently large (> 6). By the construction, we have that $[\Phi'_1 \circ L_1](q)$ is equivalent to q. Therefore we may assume

$$||\Phi'_1 \circ L_1(q) - q|| < \delta/4$$

by adjoining a unitary.

Write $B_n = \bigoplus_{j=1}^m B_n(j)$, where each $B_n(j)$ is a splitting interval algebra or the homogeneous algebra with dimension less than 3. Therefore, we can write $q = q_1 \oplus q_2 \oplus \cdots \oplus q_l$ with $0 \leq l \leq m$ and $q_j \neq 0$. Choose an integer $N_1 > 0$ such that $N_1[q_j] \geq 3[1_{B_n(j)}]$. Note that we assume q_j has rank at least 6. By applying an inner automorphism, we may assume that $\bigoplus_{j=1}^l B_n(j)$ is a hereditary C*-subalgebra of $M_{N_1}(B'_n)$. Since F_1 is finite generated, with sufficiently large n, we obtain a homomorphism $j : \bar{F_1} \to \mathrm{U}(qB'_nq)/\mathrm{CU}(qB'_nq)$ such that $\phi_n^\ddagger \circ j = \mathrm{id}_{\bar{F}_1}$. Then

$$\kappa_1 \circ \phi_n^\ddagger \circ (\Phi'_1 \circ L_1)^\ddagger|_{\bar{F}_1} = \kappa_1 \circ \phi_n^\ddagger \circ j = \kappa_1|_{\bar{F}_1},$$

where $\kappa_1 : \mathrm{U}(qBq)/\mathrm{CU}(qBq) \to \mathrm{K}_1(qBq)$ is the quotient map. Note that $\mathrm{K}_1(qBq) = \mathrm{K}_1(B)$. Let Δ_1 and δ be as in Lemma 3.14. We may assume that $\Delta_1 < \sigma_1/4$. To simplify notation, we assume that $\phi_n(q) = q$. By the assumption on B, we may write that $\phi_n|_{B'_n} = (\phi_n)_0 \oplus (\phi_n)_1$, where

- $\tau((\phi_n)_0(1_{B'_n}) < \delta_1/2(N_1+1)^2$ for all $\tau \in T(B)$ and
- $(\phi_n)_0$ is homotopically trivial (but non-zero).

It follows from Lemma 3.14 that there is a homomorphism $h : B'_n \to e_0Be_0$ such that

- $[h] = [(\phi_n)_0]$ in $\mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(B'_n), \underline{\mathrm{K}}(B))$ and
- $(\phi_n^\ddagger \circ j(\overline{(w)})^{-1}(h \oplus (\phi_n)_1)^\ddagger(\Lambda^\ddagger\overline{(w)}) = \bar{g}_w$, where $g_w \in \mathrm{U}_0(qBq)$ and $\mathrm{cel}(g_w) < \varepsilon/4$ (in $\mathrm{U}(qBq)$) for all $w \in \mathcal{U}_1$.

Define (we assume that $B_n \subset M_{N_1}(B'_n)$)

$$h' = (h \oplus (\phi_n)_1 \otimes \mathrm{id}_{M_{N_1}})|_{\bigoplus_{j=1}^l B_n(j)},$$

and define $\Phi' = h' \oplus (\phi_n)|_{\oplus_{j=l+1}^m B_n(j)}$. Let $\Phi_1 = \Phi' \circ \Phi'_1$, we have

$$[\Phi_1]|_{\mathcal{P}_1 \cup \mathcal{P}_{01}} = [\Phi'_1]|_{\mathcal{P}_1 \cup \mathcal{P}_{01}} \quad \text{and} \quad |\tau \circ \Phi_1(a) - \tau \circ \Phi'_1(a)| < \sigma_1/2$$

for all $a \in A_{s.a.}$ and $\tau \in \mathrm{T}(B)$. For all $w \in \mathcal{U}_1$, we have

$$\mathrm{cel}(w^*(\Phi_1 \circ L_1(w))) < 8\pi + \varepsilon/4 \text{ in } \mathrm{U}(qBq).$$

For any $w \in \mathcal{U}_0$, we also have

$$\mathrm{cel}(w^*(\Phi \circ L_1(w))) < 2\mathrm{cel}(w) + \pi/64 \text{ or } < 8\pi + 2\mathrm{cel}(w^k)/k + \pi/16$$

in $\mathrm{U}(qBq)$, depending in $[w]=0$ or $[w]$ has order k in $\mathrm{K}_1(B)$. Therefore

$$\mathrm{cel}(\mathrm{id}_B(h_0(u))^{-1}(\Phi_1\circ L_1(h_0(u)))<\mathbb{L}(u) \text{ in } \mathrm{U}(qBq)$$

for all $u\in \mathrm{U}(B)\cap\mathcal{P}_1$. Since we also have

$$[\mathrm{id}]|_{\mathcal{P}\cup\mathcal{P}_0}=[\Phi_1\circ L_1]|_{\mathcal{P}\cup\mathcal{P}_0} \text{ and } \sup_{\tau\in T(B)}|\tau(a)-\tau(\Phi_1\circ L_1(a))|<\delta$$

for all $a\in H$, by Theorem 2.12, there is a unitary $W\in \mathrm{U}(B)$ such that

$$||W(\Phi_1\circ L_1(x))W^*-x||<\varepsilon/2 \quad \text{for any } x\in\mathcal{F}.$$

Let $\mathcal{F}_2\subseteq B$ be a finite subset. We may assume $\mathcal{F}_2\subseteq B_{m_1'}$ $(m_1'>n)$. Let $\delta_2'>0$, the natural number n_2, finite subsets $\mathcal{P}_2\subseteq P(B)$, $\mathcal{S}_2\subseteq B$ as required in Theorem 2.12. Then there exist mutually orthogonal projections $q'',p_1'',...,p_n''$ with $q''\preceq p_1''$ and $p_1'',...,p_n''$ mutually unitary equivalent, a sub-C*-algebra S_3' which is a splitting interval algebra with $1_{S_3'}=p_1''$ and unital $\mathcal{S}$-$\delta_2'/4$-multiplicative completely positive contractions $h_0'':B\to q''Bq''$ with $h_0''(x)=q''xq''$, and $h_1'':B\to S_3'$ such that

$$||x-h_0''(x)\oplus(\underbrace{h_1''(x)\oplus\cdots\oplus h_1''(x)}_{n\ \ \text{copies}})||\leq\delta_2'/16$$

for all $x\in\mathcal{S}_2$. Put $S_2=M_{n_2}(S_3')\subseteq(1-q'')B(1-q'')$. Let $\mathcal{P}_{02}$, $\mathcal{G}_{02}$, H_2, δ_{02} and $\sigma_2>0$ be required by Theorem 2.12. Set $\delta_2=\min\{\delta_2',\delta_{02}\}$. We may assume that $\sigma_2<\sigma_1/4$, $\delta_2<\delta_1/4$, $[\Phi(\mathcal{P}_1\cup\mathcal{P}_{01})\subset[\mathcal{P}_2]$ and $\mathcal{P}_{02}$ contains the minimal projections of S_2 which present a generating set of the positive cone of $\mathrm{K}_0(S_2)$. Furthermore, we may assume that each $u\in \mathrm{U}(B)\cap\mathcal{P}_2$ has the form $q''uq''\oplus(1-q'')u(1-q'')$, where $q''uq''\in \mathrm{U}(q''Bq'')$ and $(1-q'')u(1-q'')\in \mathrm{U}(S_2)$. Put $\mathcal{W}=\{q''uq'':u\in \mathrm{U}(B)\cap\mathcal{P}_2\}$. Let F'' be the subgroup generated by $\mathcal{W}$. Write $\bar{F}''=(\bar{F}''\cap \mathrm{U}_0(q''Bq'')/\mathrm{CU}(q''Bq''))\oplus\bar{F}_0''\oplus\bar{F}_1''$, where $\bar{F}_0''$ is torsion and $\bar{F}_1''$ is torsion free. We may also assume that $\Phi_1^{\ddagger}(\bar{F}')\subset\bar{F}''$. Furthermore, we also assume that $\mathcal{W}=\mathcal{W}_0\cup\mathcal{W}_1$ where $\mathcal{W}$, generates $\bar{F}''\cap \mathrm{U}_0(q''Bq'')/\mathrm{CU}(q''Bq'')\oplus\bar{F}_0''$ and $\mathcal{W}_1$ generates $\bar{F}_1''$.

Let $\mathcal{G}_3'$ be a finite subset which contains $\mathcal{S}_2$, $\mathcal{G}_{02}$, q'', $p_1'',...,p_n''$, H_2, $\Phi_1(\mathcal{G}_2)$, a generating set of S_2 and $\mathcal{W}$. Without lose of generality, we can assume $\Phi_1(A)\subseteq B_m$ for some $m>m'$, and there is a completely positive linear map $J:B\to B_m$ such that

$$||J(a)-\mathrm{id}(a)||<\delta_2/8.$$

Then we can find a projection $\tilde{p}'\in B_m$ such that

$$||\phi(q')-\tilde{q}'||<\delta_2/2.$$

We may write $B_m=\bigoplus_{j=1}^{s}B_m(j)$. By choosing large m, we may also assume that $\tilde{q}'$ has at least rank 6. We write $\tilde{q}'=q_1'\oplus\cdots\oplus q_l'$ according to the direct sum decomposition ($q_i' \neq 0$ for each i). Let $N_2 > 0$ be an integer such that $N_2[q_j'] > 3[1_{m(j)}]$ for any j. Set $B_m' = \tilde{q}' B_m \tilde{q}'$. Note $\Phi_1^\ddagger$ is one-to-one on $\bar{F}_1'$. We may further assume that $\mathcal{G}_3'$ contains $q_1', ..., q_l'$ and a generating set of B_m' and B_m.

Now, let $L_2' : B \to A$ be a $\mathcal{G}_3' - \delta_2/16(N_2+1)^2$-multiplicative completely positive linear contraction such that

$$[L_2']|_{\mathcal{P}_2} = \alpha^{-1}|_{\mathcal{P} \cup \mathcal{P}_{02}} \quad \text{and}$$

$$\sup_{\tau \in TA} \{|\tau(L_2'(a)) - \theta^{-1}(\tau)(a)|\} < \delta_2/4 \quad \text{for any } a \in H_2 \cup \Phi_1(H_1).$$

We may assume $(L_2')^\ddagger$ is well defined on $\bar{F}''$. Let $e \in A$ is a projection such that

$$||L_2' \circ \Phi_1(q') - e|| < \delta_2/4.$$

Since $q' \in \mathcal{P}_1$, $[e] = [q']$ in $\mathrm{K}_0(A)$. Therefore, we can assume $e = q' = L_2' \circ \Phi_1(q')$ by adjoining some unitary in A. Note that $\bar{F}_1'$ is free and $(\phi_{m,M})_{*1}$ is one-to-one. We get that

$$\alpha^{-1} \circ (\phi_m)_{*1} \circ \kappa_1' \circ (\Phi_1)^\ddagger(g) = \kappa_1(g)$$

for all $g \in \bar{F}_1'$, where $\kappa_1' : \mathrm{U}(B_m')/\mathrm{CU}(B_m') \to \mathrm{K}_1(B_m')$ and $\kappa_1 : \mathrm{U}(\tilde{q}' B \tilde{q}')/\mathrm{CU}(\tilde{q}' B \tilde{q}') \to \mathrm{K}_1(\tilde{q}' B \tilde{q}')$ are the quotient maps. Note that we have $\mathrm{K}_1(\tilde{q}' B \tilde{q}') = \mathrm{K}_1(B)$. By Lemma 3.12, there exists a homomorphism

$$\beta : \mathrm{U}(B_m')/\mathrm{CU}(B_m') \to \mathrm{U}(q'Aq')/\mathrm{CU}(q'Aq')$$

with $\beta(\mathrm{U}_0(B_m')/\mathrm{CU}(B_m')) \subset \mathrm{U}_0(q'Aq')/\mathrm{CU}(q'Aq')$ such that

$$\beta \circ (\Phi_1^\ddagger)(\bar{w}) = \bar{w}$$

for all $\bar{w} \in \bar{F}_1'$. Let $\delta_2' = \delta(\varepsilon/16)$. By the assumption on B, there is $M > m$ such that $\phi_{m,M} = \phi_{m,M}^{(0)} \oplus \phi_{m,M}^{(1)} : B_m \to B_M$ such that $\phi_{m,M}^{(0)}$ is homotopically trivial and $\tau(\phi_M \circ \phi_{m,M}^{(0)}(1_{B_m'})) < \Delta_2'/4(N_2+1)^2$ for all $\tau \in T(B)$. To simplify the notation, we assume that $e_0' = L_2' \circ \phi_M \circ \phi_{m,M}^{(0)}(1_{B_m'})$ and $e_1' = L_2' \circ \phi_M \circ \phi_{m,M}^{(1)}(1_{B_m'})$ are mutually orthogonal projections. It follows from Lemma 3.13 that there is a homomorphism $\Psi' : B_m' \to e_0' A e_0'$ such that

- Φ' is homotopically trivial, $\Phi_{*0}' = [L_2'] \circ (\phi_M \circ \phi_{m,M}^{(0)})_{*0}|_{\mathrm{K}_0(B_m')}$ and
- $(\beta(\Phi_1^\ddagger(\bar{w})^{-1})(\Phi' \oplus (L_2' \circ \phi_M \circ \phi_{m,M}^{(1)}))^\ddagger(\Phi_1^\ddagger(\bar{w})) = \bar{g}_w$ where $g_w \in \mathrm{U}_0(q'Aq')$ and $\mathrm{cel}(g_w) < \varepsilon/4$ (in $\mathrm{U}(q'Aq')$) for all $w \in \mathcal{V}_1$.

As in the construction of the map from A to B, we have a homomorphism $\tilde{\Phi}' : B_m \to q'Aq'$ such that $\tilde{\Phi}'$ is homotopically trivial, $\tilde{\Phi}_{*0}' = [L_2'] \circ [\phi_M \circ \phi_{m,M}]_0$ and $\tilde{\Phi}'|_{B_m'} = \Phi'$. Define $L_2 = (\tilde{\Phi}' \oplus L_2' \circ \phi_M \circ \phi_{n,M}^{(1)}) \circ J$. One can verify that

$$[L_2]|_{\mathcal{P}_2 \cup \mathcal{P}_{02}} = [L_2']|_{\mathcal{P}_2 \cup \mathcal{P}_{02}} = \alpha^{-1}|_{\mathcal{P}_2 \cup \mathcal{P}_{02}} \quad \text{and}$$

$$|\tau \circ L_2(a) - \tau \circ L_2'(a)| < \sigma_2/4$$

for all $a \in A_{s.a}$ with norm 1 and $\tau \in T(A)$. In particular

$$\sup_{\tau \in T(A)} \{|\tau \circ L_2 \circ \Phi_1(a) - \tau(a)|\} < \sigma_1/2$$

for all $a \in H_1$. Since $\beta \circ \Phi_1^{\ddagger}(\bar{w}) = \bar{w}$ for all $w \in \mathcal{V}_1$, we have

$$\mathrm{cel}(\mathrm{id}_A(h_0'(w^*))L_2(\Phi_1(h_0'(w)))) < 8\pi + \mathrm{cel}(g_w) + \varepsilon/4 < 8\pi + \varepsilon/2$$

in $\mathrm{U}(q'Aq')$ for all $w \in \mathcal{V}_1$. We also have

$$\mathrm{cel}(\mathrm{id}_A(h_0'(w^*))L_2(\Phi_1(h_0'(w))) < 2\mathrm{cel}((w) + \pi/16$$

or

$$\mathrm{cel}(\mathrm{id}_A(h_0'(w^*))L_2(\Phi_1(h_0'(w))) < 8\pi + 2\mathrm{cel}(w^k)/k + \pi/16$$

in $\mathrm{U}(q'Aq')$ for all $w \in \mathcal{V}_0$ (depends on $[w] = 0$ or has torsion k in $\mathrm{K}_1(A)$). Therefore, we have

$$\mathrm{cel}(\mathrm{id}(h_0'(u^*))L_2(\Phi_1(h_0'(u)))) < \mathbb{L}(u)$$

for all $u \in \mathrm{U}(A) \cap \mathcal{P}_2$ in $\mathrm{U}(q'Aq')$. By Theorem 2.12, we have a unitary $Z \in \mathrm{U}(A)$ such that

$$||Z(L_2 \circ \Phi(a))Z^* - a|| < \varepsilon/16 \quad \text{for all} \quad x \in \mathcal{F}_1.$$

Therefore, by replacing L_2 by $\mathrm{ad} \circ L_2$, we obtain the approximate intertwining diagram. By applying Elliott's intertwining argument, one has that A is isomorphic to B, and the isomorphism induces the given isomorphism between the invariants. □

4. **Two remarks on the range of the invariant** It is known that there exists simple inductive limit A of splitting interval algebras such that $\mathrm{K}_0(A)$ does not satisfy the Riesz decomposition theorem (see Section 6 of [8], where the authors constructed such a C*-algebra A with $\mathrm{S}_u(\mathrm{K}_0(A))$ a square rather than a simplex). Then it is natural to ask the following two questions: Do the invariants of inductive limits of splitting interval algebras exhaust all countable simple torsion free unperforated ordered groups. And is A automatically an AH-algebra if $\mathrm{K}_0(A)$ is a Riesz group; in other words, does the pairing map preserves extrema point automatically once the K_0-group is Riesz. In the following, we shall give negative answers to both questions.

4.1. A remark on ordered K_0*-groups* The ordered K_0-group of any simple TASI-algebra is always simple and weakly unperforated (see Proposition 4.3 of [12]), but it cannot exhaust all such ordered group. Let us consider the convex set consisting of the states of the K_0-group. The following remark shows that although this convex might not be a simplex, the defect is actually very small in certain sense, and it cannot be an arbitrary convex.

DEFINITION 4.1. A compact convex set Δ is called a pseudo-simplex of order n if there is a simplex E and a continuous surjective affine map $r : E \to \Delta$ such that for any $x \in \Delta$, the pre-image $r^{-1}(x)$ is a simplex with dimension at most n.

LEMMA 4.2. *Let S be a splitting tree algebra with n edges. Then the convex $\mathrm{S}(\mathrm{K}_0(S))$ is a pseudo-simplex of order n.*

PROOF. Denote by m the number of vertices of S. Then, there is a positive embedding $\mathrm{K}_0(S) \to \mathbb{Z}^m$. The dual map r sends $E_m := \mathrm{S}(\mathbb{Z}^m)$, which is a simplex, onto $\mathrm{S}(\mathrm{K}_0(S))$, as desired. □

LEMMA 4.3. *Let S_1 and S_2 be two splitting tree algebras with number of vertices m_1 and m_2 respectively. Denote by $\iota_1 : \mathrm{K}_0(S_1) \to \mathbb{Z}^{m_1}$ and $\iota_2 : \mathrm{K}_0(S_2) \to \mathbb{Z}^{m_2}$ the canonical embeddings respectively. Then, any homomorphism $\kappa : \mathrm{K}_0(S_1) \to \mathrm{K}_0(S_2)$ can be extended to a positive homomorphism $\mathbb{Z}^{m_1} \to \mathbb{Z}^{m_2}$.*

PROOF. Let $\mathrm{ev} : \mathrm{K}_0(S_2) \to \mathbb{Z}$ be the evaluation map on a splitting point of S_2. It is clear that ev can be naturally extended to $\mathbb{Z}^{m_2}$. Thus, in order to prove the lemma, it is enough to show that $\mathrm{ev} \circ \kappa$ can be extended to a positive map from $\mathbb{Z}^{m_1}$ to $\mathbb{Z}$.

By Lemma 2.16 of [12], the map $\mathrm{ev} \circ \kappa$ is a sum of point evaluations, and hence it can be extended to a positive map on $\mathbb{Z}^{m_1}$ naturally. □

LEMMA 4.4. *Let $\Delta_i, i = 1, 2, ...$ be pseudo-simplexes of order n with simplexes E_i and affine maps $r_i : E_i \to \Delta_i$ such that $r_i^{-1}(x)$ is a simplex with dimension at most n for any $x \in \Delta_i$. Let $\varphi_i : \Delta_{i+1} \to \Delta_i$ and $\tilde{\varphi}_i : E_{i+1} \to E_i$ be affine maps such that the diagram*

$$\begin{tikzcd} E_i \arrow[d, "r_i"'] & E_{i+1} \arrow[l, "\tilde{\varphi}_i"'] \arrow[d, "r_{i+1}"] \\ \Delta_i & \Delta_{i+1} \arrow[l, "\varphi_i"'] \end{tikzcd}$$

commutes. Then the inverse limit of the system (Δ_i, φ_i) is a pseudo-simplex of order n.

PROOF. Denote by $E = \varprojlim E_i$, $\Delta = \varprojlim \Delta_i$, and $r : E \to \Delta$ the canonical affine map. Since each E_i is a compact simplex, E is a compact simplex. Consider

$x = (x_i) \in \varprojlim \Delta_i$. It follows that $\tilde{\varphi}_i(r_{i+1}^{-1}(x_{i+1})) \subseteq r_i^{-1}(x_i)$, and $r^{-1}(x)$ is the inverse limit of $(r_i^{-1}(x_i), \tilde{\varphi}_i)$. Since each $r_i^{-1}(x_i)$ is a simplex of dimension at most n, their inverse limit $r^{-1}(x)$ is a simplex with dimension at most n, as desired. □

Applying this lemma to K_0-groups of splitting tree algebras, one has that

Theorem 4.5. *Let G be an inductive limit of order-unit K_0-groups of splitting interval algebras. Then, the convex set $\mathrm{S}(G)$ is a pseudo-simplex of order 1.*

Proof. Write $G = \varinjlim(G_i, \varphi_i)$ where each G_i is the K_0-group of a splitting interval algebra. Then each convex set $\mathrm{S}(G_i)$ is a pseudo-simplex of order 1. By Lemma 4.3, the maps φ_i lift to a map $\tilde{\varphi}_i : \mathbb{Z}^{m_i} \to \mathbb{Z}^{m_{i+1}}$ such that

$$\begin{tikzcd} \mathbb{Z}^{m_i} \arrow[r, "\tilde{\varphi}_i"] & \mathbb{Z}^{m_{i+1}} \\ G_i \arrow[u, "\iota_i"] \arrow[r, "\varphi_i"] & G_{i+1} \arrow[u, "\iota_{i+1}"'] \end{tikzcd}$$

commutes, where ι_i and ι_{i+1} are canonical embeddings. Consider the dual, one has the commutative diagram of Lemma 4.4, and hence the $\mathrm{S}(G)$ is the pseudo-simplex of order 1. □

Since the K_0-group of any simple TASI-algebra is an inductive limit of direct sums of K_0-group of splitting interval algebras and $\mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z})$, one has the following corollary.

Corollary 4.6. *Let A be a simple TASI-algebras. Then $\mathrm{S}(\mathrm{K}_0(S))$ is a pseudo-simplex of order 1.*

Remark 4.7. It is then clear that the K_0-group of TASI-algebras cannot exhaust all simple weakly unperforated ordered groups. For instance, let G be an order-unit group with $\mathrm{S}(G)$ a pentagon. Then G cannot be realized as the K_0-group of a TASI-algebra, since pentagon is not a pseudo-simplex of order 1 (in fact, it is a pseudo-simplex of order 2).

Remark 4.8. Note that in [2], the author showed that the class of simple inductive limits of point-line algebras is able to exhaust all simple unperforated ordered group. Thus, in order to have a classification for a class of simple C*-algebras with the most general unperforated K_0-groups, it is reasonable to consider the class of C*-algebras which can be tracially approximated by certain point-line algebras. This will be the topic of forthcoming paper(s).

4.2. A remark on pairing maps It is known that the range of the ordered K_0-groups of TASI-algebras is strictly bigger than that of AH-algebras. In this section, we shall show further that even if the K_0-group of a given TASI-algebra is a dimension group, it still might not be an AH-algebra. We shall consider the pairing map from tracial simplex to the convex set consisting of the states of K_0-group (it is in fact a simplex in this case), and show that it does not necessarily preserve extreme points (note that for AH-algebras, the pairing map always preserves extreme points).

THEOREM 4.9. *There exists a TASI-algebra such that the ordered* K_0*-group has the Riesz decomposition property, but the canonical pairing map from trace simplex to the convex set of the states on the* K_0*-group does not preserve extreme points.*

PROOF. Let (k_n) be a sequence of natural numbers such that

$$\sum_{n=1}^{\infty}\frac{1}{k_n+1}<\infty,$$

and

$$\frac{k_1}{k_1+2}\cdot\frac{k_2}{k_2+2}\cdots\frac{k_n}{k_n+2}>c\quad\text{for any } n$$

for some strictly positive number c.

Set $m_n=(k_1+2)(k_2+2)\cdots(k_{n-1}+2)$ (assume $m_1=1$). Consider the algebra

$$A_n=\{f\in\mathrm{M}_{2m_n}([0,1]); f(0)\in\mathrm{M}_{m_n}\oplus\mathrm{M}_{m_n}\}.$$

(In fact, $A_{n+1}=A_n\otimes\mathrm{M}_{k_n+2}$.)

A simple calculation shows that $\mathrm{K}_0(A_n)=\mathbb{Z}\oplus\mathbb{Z}$ with

$$\mathrm{K}_0^+(A_n)=\{(a,b);\ a,b\in\mathbb{Z}^+\cup\{0\}\}.$$

It is a Riesz group, so does any inductive limit.

However, the pairing map does not preserve extreme points. For example, $\tau_{\{1/2\}}$, the Dirac measure on $\{1/2\}$, is an extreme trace. But the induced state is

$$[\tau_{\{1/2\}}]:(a,b)\mapsto(a+b)/2,$$

which is not extremal.

Consider the following inductive limit of A_n: Let $\{x_n\}$ be a dense sequence in $[0,1]$. Define $\phi_n: A_n\to A_{n+1}$ by

$$f\mapsto f(x_n)p_0\oplus(f\otimes 1_{\mathrm{M}_{k_n}})p_1\oplus f(x_n)p_2\in A_{n+1},$$

where

$$p_0=\mathrm{diag}\{\underbrace{1,...,1}_{m_n},0,...,0\},$$

$$p_2 = \mathrm{diag}\{0, ..., 0, \underbrace{1, ..., 1}_{m_n}\},$$

and $p_1 = 1 - p_0 - p_2$. Note that

$$[\phi_n] : (a, b) \mapsto (k_n a + (a + b), k_n b + (a + b)).$$

Then $[\phi_n]$ is injective as a map between abelian groups. Indeed, if

$$k_n a + (a + b) = 0 \quad \text{and} \quad k_n b + (a + b) = 0$$

for some $a, b \in \mathbb{Z}$, one then has

$$(k_n + 1)^2 a = a \quad \text{and} \quad (k_n + 1)^2 b = b.$$

Since $k_n \geq 1$, one has that $a = b = 0$.

Then, $A = \varinjlim A_n$ is simple, and $\mathrm{K}_0(A)$ is a Riesz group. Note that

$$[\phi_n]^{-1} = \frac{1}{(k_n + 1)^2 - 1} \begin{pmatrix} k_n + 1 & -1 \\ -1 & k_n + 1 \end{pmatrix}.$$

Since

$$\sum_{n=1}^{\infty} \frac{1}{k_n + 1} < \infty,$$

then $\mathrm{K}_0(A)$ has two extreme states, denoted by ρ_0 and ρ_1.

Denote by p_n and q_n two orthogonal projections in A_n with

$$[p_n] = (m_n, 0) \quad \text{and} \quad [q_n] = (0, m_n).$$

Denote by $\rho_0^{(n)}$ and $\rho_1^{(n)}$ the standard extreme states on $\mathrm{K}_0(A_n)$, one has

$$\left|\rho_0^{(n)}(p_n) - \rho_0(p_n)\right| < \frac{1}{k_n + 1} + \frac{1}{k_{n+1} + 1} + \cdots$$

and

$$\left|\rho_0^{(n)}(q_n) - \rho_1(q_n)\right| < \frac{1}{k_n + 1} + \frac{1}{k_{n+1} + 1} + \cdots.$$

(Note that $\rho_0^{(n)}(q_n) = 0$.)

Since

$$\frac{k_1}{k_1 + 2} \cdot \frac{k_2}{k_2 + 2} \cdots \frac{k_n}{k_n + 2} > c \quad \text{for any } n$$

for some prescribed $c > 0$, then an asymptotical argument shows that for any $x_0 \in [0, 1]$ (as spectrum of A_n), there is a trace τ on A such that $\tau(\{x_0\}) > c$. In particular, for any A_n, there is a trace τ on A such that the restriction of τ to A_n has mass at least c on $\{1/2\}$.

Let us show that the pairing map of A does not preserve extreme points. Let τ be any tracial state on A with restriction to projections an extreme state, say $[\tau] = \rho_0$. We assert that for any $\varepsilon > 0$, there is N such that for any $n > N$, one has that $\tau(\{x\}) < \varepsilon$ for any $x \in (0, 1)$ (in the spectrum of A_n). In fact, for any n sufficiently large such that

$$\frac{1}{k_n + 1} + \frac{1}{k_{n+1} + 1} + \cdots < \varepsilon,$$

one has

$$\tau(q_n) = \rho_0(q_n) =_\varepsilon \rho_0^{(n)}(q_n) = 0.$$

If $\tau(\{x\}) \geq \varepsilon$ for some $x \in (0, 1)$, one has that $\tau(q_n) \geq \varepsilon$, and this is a contradiction. The same argument shows that if $[\tau] = \rho_1$, then for any $\varepsilon > 0$, there is N such that $\tau(\{x\}) < \varepsilon$ for any $n > N$. Thus, for any trace τ with $[\tau]$ an extreme state, for any $\varepsilon > 0$, there is N such that for any $n > N$, one has

$$\tau(\{x\}) < \varepsilon \quad \text{for any } x \in (0, 1).$$

Assume that the pairing map of A preserves extreme points, and applying the statement above to extreme traces, one then has that for any tracial state τ on A and for any $\varepsilon > 0$, there is N such that

$$\tau(\{x\}) < \varepsilon \quad \text{for any } x \in (0, 1).$$

But this contradicts to the construction of A which guarantees the existence of a tracial state with mass at least c on $\{1/2\}$ on any A_n. Thus A is the desired TASI-algebra. ☐

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