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A Classification of Tracially Approximate Splitting Interval Algebras. II. Existence Theorem

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Abstract. Motivated by Huaxin Lin's axiomatization of AH-algebras, the class of TASI-algebras is introduced as the class of unital C*-algebras which can be tracially approximated by splitting interval algebras—certain sub-C*-algebras of interval algebras. It is shown that the class of simple separable nuclear TASI-algebras satisfying the UCT is classified by the Elliott invariant. As a consequence, any such TASI-algebra is then isomorphic to an inductive limit of splitting interval algebras together with certain homogeneous C*-algebras—so it is an ASH-algebra.

Résumé. Une classe de C*-algèbres qui généralisent la classe bien connue TAI de Lin est considérée—basées sur, au lieu de l'intervalle, ce qui pourrait s'appeler l'intervalle fendu ("splitting interval"), de sorte que l'on les appelle la classe TASI. On montre que la classe de C*-algèbres TASI qui sont simples, nucléaires, et à élément unité, qui vérifient le théorème à coefficients universels (UCT), peuvent se classifier d'après l'invariant d'Elliott.

1. **Introduction** This is the second part of the classification of tracially approximate splitting interval algebras (together with [14] and [15]). With the preparation of the first part ([15]), one shows the following existence theorem for TASI-algebras (see Definition 4.1 of [14]):

Theorem. *Let A and B be two simple nuclear TASI-algebras satisfying UCT. Then for any $\alpha \in \mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B))^+$ with $\alpha([1_A]) = [1_B]$, any finite subset $\mathcal{P} \in P(A)$, and any trace map $\theta : T(B) \to T(A)$ which is compatible with α, there is a sequence of completely positive linear contractions $L_n : A \to B$ such that*

- $\|L_n(ab) - L_n(a)L_n(b)\| \to 0$ *for any $a, b \in A$;*
- $[L_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}$ *for sufficient large n;*
- $|\theta(\tau)(a) - \tau(L_n(a))| \to 0$ *for any $a \in A$ and $\tau \in \mathrm{T}(B)$.*

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Together with the uniqueness theorem (proved in [15]), this existence theorem enables one to apply the Elliott intertwining arguments to obtain the classification theorem for TASI-algebras.

The main technique in the proof of the existence theorem is to construct a concrete model C*-algebra which is an inductive limit of certain homogeneous C*-algebras and splitting interval algebras such that the model C*-algebra is KK-equivalent to the given TASI-algebra. Then one proves existence theorems for two different directions: one existence theorem for K-theory maps from the given TASI-algebra to the model algebra, and the other existence theorem for K-theory maps from the concrete model algebra to the given TASI-algebra. Then the existence theorem for TASI-algebras follows from the composition of the maps obtained from the two separate existence theorems.

2. Existence Theorem

Considering a homomorphism between the Elliott invariants of two TASI-algebras, we shall show that it can be locally lifted to algebra homomorphisms. In fact, we shall show the existence theorem using a model algebra, which is a concrete TASI-algebra arising from inductive limits of certain building blocks, and show the existence theorem for homomorphisms from TASI-algebras to model algebras, and for homomorphisms from model algebras to TASI-algebras. Some of the results also hold for TAS-algebras (see [14]) .

2.1. Concrete TAS-algebras arising from inductive limits As shown in [7], any simple AH-algebra without dimension growth is a tracially approximate interval algebra. One key ingredient is the following factorization theorem:

THEOREM 2.1 (Theorem 4.35 of [7]). *Let X be a connected finite simplicial complex, and $\varepsilon > \eta > 0$. For any $\delta > 0$, there is an integer $L > 0$ such that the following holds.*

Suppose that $\mathcal{F} \subseteq \mathrm{C}(X)$ is a finite set such that $\mathrm{dist}(x, x') < 2\eta$ implies $|f(x) - f(x')| < \varepsilon/3$ for all $f \in \mathcal{F}$. If $\phi : \mathrm{C}(X) \to \mathrm{M}_k(\mathrm{C}(Y))$ is a homomorphism with the property $\mathrm{sdp}(\eta/32, \delta)$ and $\mathrm{rank}(\phi(1)) \geq 2J \cdot L^2 \cdot 2^L(\mathrm{dim}X + \mathrm{dim}Y + 1)^3$, where Y is a connected finite simplicial complex and J is any fixed positive integer, then there are mutually orthogonal projections $Q_0, Q_1 \in \mathrm{M}_k(\mathrm{C}(Y))$, and maps $\phi_0 : \mathrm{C}(X) \to Q_0\mathrm{M}_k(\mathrm{C}(Y))Q_0$ and $\phi_1 : \mathrm{C}(X) \to Q_1\mathrm{M}_k(\mathrm{C}(Y))Q_1$ such that

- $\phi(1) = Q_0 + Q_1$,
- $\|\phi(f) - \phi_0(f) \oplus \phi_1(f)\| \leq \varepsilon$ *for all* $f \in \mathcal{F}$,
- *there is a factorization*

$$\phi_1 : \mathrm{C}(X) \to \mathrm{C}([0, 1]) \to Q_1\mathrm{M}_k(\mathrm{C}(Y))Q_1,$$

- $J[Q_0] < [Q_1]$.

REMARK 2.2. The statement of the theorem is slightly different from the original statement in [7], where the map ϕ was decomposed into three components.

Note that interval algebras belong to the class of splitting tree algebras. Then, using Gong's factorization theorem, one can show that any simple unital inductive limit of splitting tree algebras together with homogeneous C*-algebras without dimension growth is in fact a TAS-algebra.

Denote by $\mathcal{SH}$ the class of C*-algebras containing finite direct sums of splitting tree algebras and homogeneous C*-algebras $p\mathrm{M}_n(\mathrm{C}(X))p$, where X is a compact metrizable space. Then, we have the following theorem.

THEOREM 2.3. *Let $A = \varinjlim(A_n, \psi_n)$ be a unital simple inductive limit with A_n in $\mathcal{SH}$ such that the dimensions of the base spaces of A_n are uniformly bounded, then A is a TAS-algebra.*

PROOF. We show that the maps between building blocks can be decomposed into the direct sum of two homomorphisms—one of them factors through a splitting tree algebra, and the image of the identity under other map is equivalent to a subprojection of any given hereditary sub-C*-algebras.

First, we assert that the order on the projections in C*-algebra A is determined by traces. In fact, for any projections p and q in A with $\tau(p) > \tau(q)$ for all $\tau \in \mathrm{T}(A)$, since A is simple, there exists $\delta > 0$ such that $\tau(p) > \tau(q)+\delta$. Since A is has no dimension growth, one may assume that $p, q \in A_n$, where $A_n = S_n \oplus H_n$ with S_n a splitting tree algebra, and H_n a homogeneous C*-algebra with base space dimension no more than d. Moreover, using an asymptotical argument, one may assume further that $\tau(p) > \tau(q) - \delta$ for all $\tau \in \mathrm{T}(A_n)$, and $m_n\delta > d/2$, where m_n is the smallest dimension of the irreducible representations of A_n. Therefore, one has that $p \succeq q$, and this proved the assertion.

Therefore, in order to prove the theorem, it is enough to show that the image of the identity under other map has a small trace.

Note that the C*-algebra A always satisfies the first two conditions of Definition 4.1 of [14] (if there is no direct summand of splitting tree algebra at any stage, then A is an AH-algebra without dimension growth, and hence is a TAI-algebra). Therefore, by Proposition 2.7 of [6], the C*-algebra A has (SP) property.

Let $\mathcal{F}$ be a finite subset of $A = \varinjlim A_n$. Let $\varepsilon > 0$, $q \in A^+$ a non-zero projection, and J be a positive integer such that $1/J < \inf\{\tau(q); \tau \in \mathrm{T}(A)\}$. Moreover, we may assume that $\mathcal{F} \subseteq A_n = S_n \oplus H_n$, where S_n is a splitting tree algebra and H_n is a homogeneous C*-algebra with the base space dimension less than d.

Then for any $m > n$, the map $\psi_{n,m} : A_n \to A_m$ can be decomposed into

$$\begin{pmatrix} \psi_{n,m}^{1,1} & \psi_{n,m}^{1,2} \\ \psi_{n,m}^{2,1} & \psi_{n,m}^{2,2} \end{pmatrix}$$

with maps $\psi_{n,m}^{1,1}$ and $\psi_{n,m}^{2,1}$ factor through the splitting tree algebra S_n, and $\psi_{n,m}^{1,2}$ factors through the splitting tree algebra S_m.

Consider the map $\psi_{n,m}^{2,2} : H_n \to H_m$. Since A is simple, one may choose m large enough so that $\psi_{n,m}^{2,2}$ satisfies Theorem 2.1 for ε and $\mathcal{F}$ so that there are projections Q_0 and Q_1 in H_m, and homomorphisms $\phi'_{n,m} : H_n \to Q_0 H_m Q_0$ and $\phi''_{n,m} : H_n \to Q_1 H_m Q_1$ such that

$$\left\| \psi^{2,2}_{n,m}(f) - \phi'_{n,m}(f) \oplus \phi''_{n,m}(f) \right\| < \varepsilon, \quad \forall f \in \mathcal{F},$$

$\phi''_{n,m}$ factor through an interval algebra, and $[\phi'_{n,m}(1)] \leq J[\phi''_{n,m}(1)]$. Hence $\tau(\phi'_{n,m}(1)) < 1/J < \tau(q)$ for all $\tau \in \mathrm{T}(A)$.

Set $\psi^0_{n,m} = \phi'_{n,m}$, and set

$$\psi^1_{n,m} = \begin{pmatrix} \psi^{1,1}_{n,m} & \psi^{1,2}_{n,m} \\ \psi^{2,1}_{n,m} & \phi''_{n,m} \end{pmatrix}.$$

The one has that

$$||\psi_{n,m}(f) - \psi^0_{n,m}(f) \oplus \psi^1_{n,m}(f)|| < \varepsilon \quad \text{for any } f \in \mathcal{F}$$

and $\psi^1_{n,m}$ factors through splitting tree algebras. Since $\tau(\phi'_{n,m}(1)) < \tau(q)$ for all $\tau \in \mathrm{T}(A)$, and the strict order on projections of A is determined by traces, one has that $\psi^1_{n,m}(1) \preceq q$, and therefore, A is a TAS-algebra. □

In [7], it was also shown that any simple AH-algebra without dimension growth has a standard inductive limit decomposition with building blocks the so called *Gong's standard homogeneous C*-algebras* (see [7] and [4]):

- $\mathrm{C}([0,1])$, which provide torsion free part of K_0-group,
- $\mathrm{C}(T_{2,k})$, which provide torsion part of K_0-group,
- $\mathrm{C}(S^1 \vee \cdots \vee S^1 \vee T_{3,k})$, which provide nontrivial K_1-group,

where $T_{2,k}$ is the 2-dimensional CW-simplex obtained by attaching a two-dimensional disk D to S^1 via a map $S^1(\cong \partial D) \to S^1$ of degree k, and $T_{3,k}$ is a three-dimensional CW simplex obtained by attaching a 3-dimensional disk B to S^2 via a map $S^2(\cong \partial B) \to S^2$ of degree k.

Let us also consider standard building blocks for tracially approximate splitting tree algebras. Consider the class of finite direct sums of the following basic building blocks:

- splitting tree algebras, which provide torsion free part of K_0-group,
- $\mathrm{C}(T_{2,k})$, which provide torsion part of K_0-group,
- $\mathrm{C}(S^1 \vee \cdots \vee S^1 \vee T_{3,k})$, which provide K_1-group.

Note that C*-algebras in (1) has torsion free K_0-groups and trivial K_1-groups; the K_0-groups of C*-algebras in (2) has form $\mathbb{Z} \oplus \mathbb{Z}/k\mathbb{Z}$, and K_1-groups are trivial; the K_0-groups of C*-algebras in (3) are isomorphic to $\mathbb{Z}$, and K_1 groups are isomorphic to $(\bigoplus \mathbb{Z}) \oplus (\mathbb{Z}/k\mathbb{Z})$.

One then has at least on the invariant level, any TAS-algebra can be realized by inductive limits of such standard building blocks.

Theorem 2.4 (Theorem A of [6]). *Let $\mathcal{S}$ be a class of splitting tree algebras containing the interval algebras, and let A be a simple separable C*-algebra in the class TA$\mathcal{S}$. There exists a simple inductive limit C*-algebra B of C*-algebras in the class $\mathcal{S}'$ consisting of $\mathcal{S}$ together with the Gong's standard homogeneous C*-algebras such that the Elliott invariant of A is isomorphic to the Elliott invariant of B.*

Remark 2.5. If one considers a C*-algebra A which can be tracially approximated by a subclass of splitting tree algebras, say, for example, by the class of splitting interval algebras, then, the theorem above shows that there is a simple inductive limit B of splitting interval algebras together with Gong's standard homogeneous C*-algebras such that the Elliott invariant of A is isomorphic to the Elliott invariant of B.

Remark 2.6. These inductive limits serve as models for TAS-algebras. And in the sequel, we shall show that the model for each TASI-algebra is indeed isomorphic to the TASI-algebra. Since one can choose a unique model for each invariant, we shall prove a classification for TASI-algebras in such a way.

2.2. Existence theorem: from basic building blocks to TAS-algebras Let us first consider splitting tree algebras. Let κ be a homomorphism from the order-unit K_0-group of a splitting tree algebra S to the order-unit K_0-group of a TAS-algebra A. We show in this section that there exists a *-homomorphism $\phi : S \to A$ such that $[\phi]_0 = \kappa$. We have the following lemma from [6].

Lemma 2.7 (Lemma 3.2 of [6]). *Let $G = \mathrm{K}_0(A)$, where A is a splitting tree algebra, and let $r : G \to \mathbb{Z}$ be the point evaluation map on a regular point. Then there exist $u \in G$ and a natural number m such that if the map $\theta : G \to G$ is defined by $g \mapsto r(g)u$, then the positive homomorphism $id + m\theta : G \to G$ factors through $\bigoplus_n \mathbb{Z}$ for some n.*

Using this lemma, one has the following decomposition theorem.

Lemma 2.8. *Let $G = \mathrm{K}_0(A)$ where A is a splitting tree algebra and $H = \mathrm{K}_0(B)$ for a simple TAS-algebra B. Then for any positive homomorphism $\theta : G \to H$, one has a decomposition $\theta = \theta_1 + \theta_2$, where θ_1 and θ_2 are positive homomorphisms from G to H such that the following diagrams commute:*

where $G_1 \cong \bigoplus_n \mathbb{Z}$ for some natural number n and G_2 is the K_0-group of a sub-C-algebra of B which is a splitting tree algebra and ψ_2 is the map induced by the embedding.*

PROOF. Let m and u be as in Lemma 2.7. Since B has the (SP) property, there is an $h \in H^+$ such that h is less than the image of any generator of G^+. Define a positive homomorphism $\theta' : G \to H$ by $g \mapsto r(g)h$ where r is the point evaluation map on the regular point. Since h is sufficiently small, $\theta - \theta'$ is a positive homomorphism from G to H.

Since any matrix algebra over a TAS-algebra and any unital hereditary sub-C*-algebra of a TAS-algebra are again TAS-algebras (Lemma 2.3 of [6]), one may assume that $\theta - \theta'$ is a positive homomorphism from G to $\mathrm{K}_0(B')$, where B' is a unital hereditary sub-C*-algebra of B. Since B' is a TAS-algebra and the positive cone of G is finitely generated, one has that $\theta - \theta'$ can be decomposed into the sum of two positive homomorphisms θ'_1 and θ_2, where θ_2 factors though the K_0-group of a sub-C*-algebra of B'. Moreover, we may assume $m\theta'_1(u) < h$.

Therefore, we have $\theta = \theta' + \theta'_1 + \theta_2$. Since $m\theta'_1(u) < h$, we have a further decomposition of $\theta' + \theta'_1$:

$$\begin{aligned}\theta'(g) + \theta'_1(g) &= r(g)h + \theta'_1(g) \\ &= r(g)(h - m\theta'_1(u)) + r(g)m\theta'_1(u) + \theta'(g) \\ &= r(g)(h - m\theta'_1(u)) + \theta'_1(mr(g)u) + \theta'_1(g) \\ &= r(g)(h - m\theta'_1(u)) + \theta'_1(mr(g)u + g),\end{aligned}$$

for any $g \in G$. By Lemma 2.7, $g \to mr(g)u + g$ factors though $\bigoplus_n \mathbb{Z}$ for some n. Therefore, $\theta' + \theta'_1$ factors though $\bigoplus_{n+1} \mathbb{Z}$. Set θ_1 to be $\theta' + \theta'_1$, and it satisfies the lemma. □

With this lemma, one has the following existence theorem for splitting tree algebras.

THEOREM 2.9. *Let S be a splitting tree algebra, and let A be a simple TAS-algebra. Then, for any positive homomorphism*

$$\kappa : (\mathrm{K}_0(S), \mathrm{K}_0^+(S), [1_S]_0) \to (\mathrm{K}_0(A), \mathrm{K}_0^+(A), [1_A]_0),$$

there exists a $*$-homomorphism $\phi : S \to A$ such that $\phi_* = \kappa$.

PROOF. By Lemma 2.8, one has $\kappa = \kappa_1 + \kappa_2$ where κ_1 factors through $\bigoplus_n \mathbb{Z}$ for some natural number n, and κ_2 factors through the K_0-group of a splitting tree algebra S', which is a sub-C*-algebra of A. Then κ_2 can be lifted to a homomorphism ϕ_2 from S to A by Proposition 2.17 of [14]. Since κ_1 factors through $\bigoplus_n \mathbb{Z}$, there exists a finite-dimensional C*-algebra F such that κ has the factorization $\mathrm{K}_0(S) \to \mathrm{K}_0(F) \to \mathrm{K}_0(A)$. Therefore, there exists a $*$-homomorphism ϕ_1 from S to $A \otimes \mathcal{K}$ such that the induced K_0 map is κ_1. Therefore, the map $\phi_1 \oplus \phi_2 : S \to A \otimes \mathcal{K}$ induces $\kappa = \kappa_1 + \kappa_2$ on $\mathrm{K}_0(S)$. Let $p = (\phi_1 \oplus \phi_2)(1_S) \in A \otimes \mathcal{K}$. Then p is Murray-von Neumann equivalent to 1_A. Therefore, there is a partial isometry $v \in A \otimes \mathcal{K}$ such that $x \mapsto vxv^*$ is an *-isomorphism from $p(A \otimes \mathcal{K})p$ to A, and hence the map $\phi = v(\phi_1 \oplus \phi_2)v^*$ sends S to A, as desired. □

In fact, the existence theorem above is also valid for point-line algebras with trivial K_1-groups. First, one has the following lemma, which is a generalization of Lemma 2.7

LEMMA 2.10. *Let $G = \mathrm{K}_0(A)$, where A is a point-line algebra with $\mathrm{K}_0(A) = \{0\}$, and let $r : G \to \mathbb{Z}$ be the point evaluation map on a regular point. Then there exist an $u \in G$ and a natural number m such that if the map $\theta : G \to G$ is defined by $g \mapsto r(g)u$, then the positive homomorphism $\mathrm{id} + m\theta : G \to G$ factors through $\bigoplus_n \mathbb{Z}$ for some n.*

PROOF. Let u be an order unit of G, and define the map $\phi : G \to G$ by $\phi(g) = g + r(g)u$; that is, $\phi = \mathrm{id} + \theta$. Consider the inductive limit

$$G \xrightarrow{\phi} G \xrightarrow{\phi} \cdots \longrightarrow \varinjlim G.$$

Then the ordered group $\varinjlim G$ has the Riesz decomposition property. Since it is also unperforated, this limit group has to be a dimension group, and therefore for sufficiently large $k \in \mathbb{N}$, the map ϕ^k must factor through $\bigoplus_n \mathbb{Z}$ for some n. Note that ϕ^k has the form $\mathrm{id} + m\theta$ for some natural number m, and hence the lemma follows. □

With this lemma, one then has the following

LEMMA 2.11. *Let $G = \mathrm{K}_0(A)$, where A is a point-line algebra with $\mathrm{K}_0(A) = \{0\}$, and let $H = \mathrm{K}_0(B)$ for a simple TASI-algebra B. Let M be a given natural number. Then for any positive homomorphism $\theta : G \to H$ with multiplicity divisible by M, one has a decomposition $\theta = \theta_1 + \theta_2$, where θ_1 and θ_2 are positive homomorphisms from G to H such that the following diagrams commute:*

$$\begin{array}{ccc} G & \xrightarrow{\theta_1} & H \\ & {\scriptstyle\phi_1}\searrow \quad \nearrow{\scriptstyle\psi_1} & \\ & G_1 & \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\theta_2} & H \\ & {\scriptstyle\phi_2}\searrow \quad \nearrow{\scriptstyle\psi_2} & \\ & G_2 & \end{array},$$

where $G_1 \cong \bigoplus_n \mathbb{Z}$ for some natural number n and G_2 is the K_0-group of a sub-C-algebra S of B which is a splitting interval algebra and ψ_2 is the map induced by the embedding. Moreover, the map θ_1 has a decomposition $\theta_1 = \theta_{1,1} + \theta_{1,2}$ with $\theta_{1,1}$ induced by a *-homomorphism $A \to B$ and any element in the images of $\theta_{1,2}$ is a multiple of M, and any irreducible representation of S has dimension a multiple of M.*

PROOF. Let m and u be as in Lemma 2.10. Since B has the (SP) property, there is an $h \in H^+$ such that h is less than the image of any generator of G^+.

Define a positive homomorphism $\theta' : G \to H$ by $g \mapsto r(g)h$ where r is a faithful direct sum of point evaluation maps on points at infinity. Since B is simple, h can be choose so that $h = Mh'$ for some h'. Since the θ has multiplicity M, one has that $\theta(g) - \theta'(g)$ is divisible by M for any $g \in G$.

Writing $h = h' + Mh''$ furthermore for some positive elements h' and h'' in H^+, one has that the map θ' can be decomposed into $\theta' = \theta''_1 + M\theta''_2$, where $\theta''_1 : G \to H$ is defined by $g \mapsto r(g)h'$ and $\theta''_2 : G \to H$ is defined by $g \mapsto r(g)h''$. Since r is a point evaluation map, it is clear that θ''_1 can be induced by a *-homomorphism from A to B.

Since any matrix algebra over a TASI-algebra and any unital hereditary sub-C*-algebra of a TASI-algebra is again a TASI-algebras (Lemma 2.3 of [6]), one may assume that $\theta - \theta'$ is a positive homomorphism from G to $\mathrm{K}_0(B')$, where B' is a unital hereditary sub-C*-algebra of B. Since B' is a TASI-algebra and the positive cone of G is finite generated, one has that $\theta - \theta'$ can be decomposed as the sum of two positive homomorphisms $\tilde{\theta}_1$ and θ_2, where θ_2 factors though the K_0-group of a sub-C*-algebra S of B'. Using the the tracially approximate divisibility again for the C*-algebra B', one may assume that any irreducible representation of S is divided by M (hence the image $\theta_2(g)$ is divisible by M in H for any $g \in G$). Moreover, we may assume $mM\tilde{\theta}_1(u) < h''$.

Since $\theta(g) - \theta'(g)$ is divisible by M and any element in $\theta_2(G)$ is divisible by M, one has that any elements in $\tilde{\theta}_1(G)$ is divisible by M. Therefore, the map $\tilde{\theta}_1$ can be decomposed further as $M\theta'_1$, and one has that $\theta - \theta' = M\theta'_1 + \theta_2$.

Therefore, one has that

$$\theta = \theta' + M\theta'_1 + \theta_2 = \theta''_1 + M\theta''_2 + M\theta'_1 + \theta_2.$$

Since $m\theta'_1(u) < h''$, we have the following further decomposition of $\theta''_2 + \theta'_1$: for any $g \in G$,

$$\begin{aligned}\theta''_2(g) + \theta'_1(g) &= r(g)h'' + \theta'_1(g)\\ &= r(g)(h'' - m\theta'_1(u)) + r(g)m\theta'_1(u) + \theta'(g)\\ &= r(g)(h'' - m\theta'_1(u)) + \theta'_1(mr(g)u) + \theta'_1(g)\\ &= r(g)(h'' - m\theta'_1(u)) + \theta'_1(mr(g)u + g).\end{aligned}$$

By Lemma 2.10, $g \to mr(g)u + g$ factors though $\bigoplus_n \mathbb{Z}$ for some n. Therefore, the map $\theta''_1 + M(\theta''_2 + \theta'_1)$ factors though $\bigoplus_{nM+1} \mathbb{Z}$. Set

$$\theta_{1,1}(g) = \theta''_1(g) + Mr(g)(h'' - m\theta'_1(u)) \quad \text{and} \quad \theta_{1,2}(g) = M\theta'_1(mr(g)u + g).$$

It is clear that $\theta_{1,1}$ can be induced by a *-homomorphism from A to B. Then $\theta_{1,1}$, $\theta_{1,2}$ and θ_2 are desired maps. □

As a consequence, one has that following existence result for point-line algebras.

LEMMA 2.12. *Let S be a point-line algebra with $\mathrm{K}_1(S) = \{0\}$, and let A be a simple TASI-algebra. Then, there exists M such that for any positive homomorphism $\kappa : (\mathrm{K}_0(S), \mathrm{K}_0^+(S), [1_S]_0) \to (\mathrm{K}_0(A), \mathrm{K}_0^+(A), [1_A]_0)$ with multiplicity M, there exists a $*$-homomorphism $\phi : S \to A$ such that $[\phi] = \kappa$.*

PROOF. Denote by M the constant of Lemma 3.6 and Proposition 3.7 of [14] with respect to S. By Lemma 2.11, for any positive map κ with multiplicity M, one has $\kappa = \kappa_0 + \kappa_1 + \kappa_2$, where κ_0 can be lift to a *-homomorphism $\phi_0 : S \to A$, κ_1 factors through $\bigoplus_n \mathbb{Z}$ for some natural number n with $\kappa_1(g)$ is divisible by M for any $g \in \mathrm{K}_0(S)$, and κ_2 factors through the K_0-group of a splitting interval algebra $S' \subseteq A$ with dimension of any irreducible representation of S' divisible by M. Then κ_2 can be lifted to a homomorphism ϕ_2 from S to A by Proposition 3.7 of [14]. Since κ_1 factors through $\bigoplus_n \mathbb{Z}$, there exists a finite-dimensional C*-algebra F such that κ has the factorization $\mathrm{K}_0(S) \to \mathrm{K}_0(F) \to \mathrm{K}_0(A)$. Moreover, the image of any element of $\mathrm{K}_0(S)$ in $\mathrm{K}_0(F)$ is divisible by M. By Lemma 3.6 of [14], there is a *-homomorphism $S \to F$ which lifts the corresponding K_0-map, and therefore, there exists a *-homomorphism ϕ_1 from S to $A \otimes \mathcal{K}$ such that the induced K_0-map is κ_1.

Hence, the map $\phi_0 \oplus \phi_1 \oplus \phi_2 : S \to A \otimes \mathcal{K}$ induces the map $\kappa = \kappa_0 + \kappa_1 + \kappa_2$ on $\mathrm{K}_0(S)$. Let $p = (\phi_0 \oplus \phi_1 \oplus \phi_2)(1_S) \in A \otimes \mathcal{K}$. Then p is Murray-von Neumann equivalent to 1_A. Therefore, there is a partial isometry $v \in A \otimes \mathcal{K}$ such that $x \mapsto vxv^*$ is an *-isomorphism from $p(A \otimes \mathcal{K})p$ to A, and thus the map $\phi = v(\phi_0 \oplus \phi_1 \oplus \phi_2)v^*$ sends S to A, as desired. □

With this lemma, together with tracially approximate divisibility and Theorem 2.29, one has the following existence theorem for arbitrary positive K_0-maps from point-line algebras to a TASI-algebra. (In fact, the argument is a simplified version of the argument of Theorem 2.31.)

THEOREM 2.13. *Let A be a point-line algebra with trivial K_1-group, and let B be a separable simple nuclear TASI-algebra. Let $\alpha : \mathrm{K}_0(A) \to \mathrm{K}_0(B)$ be any positive homomorphism. Then, there is a *-homomorphism $\phi : A \to B$ such that $[\phi] = \alpha$.*

PROOF. Denote by M the constant of Lemma 2.12. Since A is a type-I C*-algebra, it satisfies UCT, and hence α has a lifting in $\mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B))$. With a slight abuse of notation, let us still denote it by α. Since A has a separating family of finite-dimensional representations, by Theorem 2.29, there exist two sequences of completely positive contractions $\phi_n^{(i)} : A \to B \otimes \mathcal{K}$ $(i = 1, 2)$ satisfying the following:

- $\|\phi_n^{(i)}(ab) - \phi_n^{(i)}(a)\phi_n^{(i)}(b)\| \to 0$ as $n \to \infty$,
- for any n, the images of $\phi_n^{(2)}$ are contained in a finite dimensional sub-C*-algebra of $B \otimes \mathcal{K}$ and for any finite subset $\mathcal{P} \subset \mathrm{P}(A)$, $[\phi_n^{(i)}]|_{\mathcal{P}}$ are well defined for sufficiently large n,

- for each finite subset $\mathcal{P} \subset \mathrm{P}(A)$, there exists $m > 0$ such that

$$[\phi_n^{(1)}]|_{\mathcal{P}} = \alpha + [\phi_n^{(2)}]|_{\mathcal{P}}$$

for all $n > m$,
- for each n, we may assume that $\phi_n^{(2)}$ is a homomorphism on A.

Since any point-line algebra is semiprojective and the positive cone of the K_0-group is finitely generated, one has homomorphism ϕ_0 and ϕ_1 from $A \to B \otimes \mathcal{K}$ such that

$$[\phi_0] = \alpha + [\phi_1].$$

Without lose of generality, let us assume that ϕ_0 and ϕ_1 are *-homomorphisms from A to $\mathrm{M}_r(B)$ for some r. Note that $\mathrm{M}_r(B)$ is still a simple TASI-algebra.

Denote by $\{d_1, d_2, ..., d_n\}$ the set of positive generators of $\mathrm{K}_0(A)$, and choose a projection $p_i \in A$ with $[p_i] = d_i$ for each i. Let $\mathcal{G}$ be a finite subset of $\mathrm{M}_r(B)$ which contains $\{\phi_0(p_i), \phi_1(p_i); i = 1, ..., d\}$. Also denote by

$$\mathcal{P} = \{[\phi_0(p_i)], [\phi_1(p_i)]; i = 1, ..., d\}.$$

Since any simple TASI-algebra is tracially approximately divisible (Theorem 4.13 of [14]), for some $\varepsilon > 0$ and $0 < r_0 < 1$ there is a $\mathcal{G}$-ε-multiplicative map $L : \mathrm{M}_r(B) \to \mathrm{M}_r(B)$ with the following properties:

- $[L]|_{\mathcal{P}}$ is well defined;
- $\tau \circ [L](g) \leq r_0 \tau(g)$ for all $g \in \mathcal{P}$ and $\tau \in T(B)$;
- There exist positive elements $\{f_i\} \subset \mathrm{K}_0(B)^+$ such that for $i = 1, ..., n$,

$$\alpha(d_i) - [L](\alpha(d_i)) = M f_i.$$

- the image of $\mathrm{Id} - L$ is in a splitting interval algebra $S \subset \mathrm{M}_r(B)$. (Note that by Condition (2.2), any irreducible representation of S must be divisible by M.)

Denote by $\Psi := \phi_0 \oplus \bigoplus_{M-1} \phi_1$. Since B is simple, there exists $\delta > 0$ such that $\tau(\alpha([p_i])) > \delta$ for any $\tau \in \mathrm{T}(B)$. One then choose r_0 sufficiently small such that $\tau \circ [L] \circ [\Psi]([p_i]) < \delta/2$ for all $\tau \in \mathrm{T}(B)$, and hence $\tau(\alpha([p_i]) - [L \circ \Psi]([p_i])) > 0$ for any $\tau \in \mathrm{T}(B)$. Since the strict order on $\mathrm{K}_0(B)$ is determined by traces (see Proposition 4.3 of [14]), one has that $\alpha([p_i]) - [L \circ \Psi]([p_i]) > 0$.

Moreover, one also has

$$\begin{aligned} & \alpha([p_i]) - [L \circ \Psi]([p_i]) \\ = \ & \alpha(d_i) - ([L \circ \alpha](d_i) + M[L \circ \phi_1](d_i)) \\ = \ & (\alpha(d_i) - [L \circ \alpha](d_i)) - M[L \circ \phi_1](d_i) \\ = \ & M(f_j - [L] \circ [\phi_1](d_i)) \\ = \ & M f_j', \quad \text{where } f_j' = f_j - [L] \circ [\phi_1](d_i). \end{aligned}$$

Consider the map $\beta : \alpha - [L \circ \Psi]$. Since this is a map with multiplicity M, by Lemma 2.12, there then exists a *-homomorphism $h : A \to \mathrm{M}_r(B)$ such that $[h] = \beta$. Consider the map $\phi' := L \circ \Psi \oplus h : A \to B \otimes \mathcal{K}$, one has that

$$[\phi'] = [L \circ \Psi] + \beta = \alpha.$$

Since $[\phi'(1)] = [1_B]$ and B has stable rank one, there is a unitary u in a matrix algebra of B such that the map $\phi = \mathrm{ad}(u) \circ \phi'$ satisfies $\phi(1_A) = 1_B$, as desired. □

Let us consider the existence theorem for maps from a homogeneous C*-algebra to a TASI-algebra. Our approach is similar to [12], which is for TAI algebras. First, let us introduce the following local existence property for KL-groups.

DEFINITION 2.14. A C*-algebra A is said to be KK-attainable (for TASI-algebras) if for any simple TASI-algebra B, any $\alpha \in \mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B))^+$, and any finite subset $\mathcal{P} \subset \mathrm{P}(A)$ with $[1_A] \in \mathcal{P}$, there is a sequence of completely positive linear contractions $L_n : A \to B \otimes \mathcal{K}$ such that for any $a, b \in A$,

$$\|L_n(ab) - L_n(a)L_n(b)\| \to 0,$$

and

$$[L_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}},$$

for sufficiently large n.

REMARK 2.15. If A satisfies the Universal Coefficient Theorem, and A has finitely generated and torsion free K_0-groups and trivial K_1-groups, then one has that

$$\mathrm{KL(A, B)} \cong \mathrm{Hom}(\mathrm{K}_0(A), \mathrm{K}_0(B))$$

for any C*-algebra B. Therefore, for such C*-algebras—for example, splitting tree algebras and the C*-algebra $\mathrm{C}(S^2)$, where S^2 is the two-dimensional sphere—in order to show the KK-attainability, it is enough to show that any positive homomorphisms between K-groups can be lifted. Hence, by Theorem 2.13, any point-line algebra is KK-attainable (for TASI-algebras), and by Theorem 2.9, any splitting tree algebra is KK-attainable (for TAS-algebras).

Let us consider the KK-attainability for homogeneous C*-algebras. First, we have the following Lemma, which is an analogue of Lemma 9.8 of [12] for TASI-algebras.

LEMMA 2.16. *Let A be a unital C*-algebra, let B be a unital separable simple TASI-algebra, and let S be a sub-C*-algebra of B which is a splitting interval algebra. Let G be a subgroup generated by a finite subset of* $\mathrm{P}(A)$*. If there is an $\mathcal{F}$-δ-multiplicative contractive completely positive linear map $\psi : A \to S \subseteq B$ such that $[\psi]_G$ is well defined. Then, for any $\varepsilon > 0$, there exists a splitting* *interval sub-C*-algebra $C \subseteq B$ and an $\mathcal{F}$-δ-multiplicative contractive completely positive linear map $\psi : A \to C \subseteq B$ such that*

$$[L]|_{G\cap \mathrm{K}_0(A,\mathbb{Z}/k\mathbb{Z})} = [\psi]|_{G\cap \mathrm{K}_0(A,\mathbb{Z}/k\mathbb{Z})},$$

and $\tau(1_C) < \varepsilon$ for any tracial state τ on B and any $k > 1$ with $G \cap \mathrm{K}_0(A, \mathbb{Z}/k\mathbb{Z}) \neq \{0\}$, where L and ψ are viewed as maps to B. Furthermore, if $[L]|_{G\cap \mathrm{K}_0(A,\mathbb{Z}/k\mathbb{Z})}$ is positive, so is $[\psi]|_{G\cap \mathrm{K}_0(A,\mathbb{Z}/k\mathbb{Z})}$.

PROOF. The proof is the same as that of Lemma 9.8 of [12]. Without loss of generality, let us assume that

$$S = \{f \in \mathrm{C}([0, 1], \mathrm{M}_n);\ f(0) \in \bigoplus_i M_{m_i} \text{ and } f(1) \in \bigoplus_j M_{n_j}\}.$$

Let $\mathcal{F}$ be a subset of S and ε small enough such that any $\mathcal{F} - \varepsilon$ multiplicative map from S to some C*-algebra induces a positive map on $\mathrm{K}_0(S)$. Suppose that

$$G \cap \mathrm{K}_0(A, \mathbb{Z}/k\mathbb{Z}) = \{0\}, \quad k > K.$$

By Theorem 4.13 of [14] with $\mathcal{F}$, $n = K! + 1$ and ε, one has that there is a homomorphism $S \to C \oplus \underbrace{S_0 \oplus \cdots \oplus S_0}_{n}$, $x \mapsto L_0(x) \oplus L_1(x) \oplus \cdots \oplus L_1(x)$ with each unit of S_0 is Murray-von Neumann equivalent, and $\tau(1_C) \leq \varepsilon$, where C and S_0 are splitting interval sub-C*-algebra of B. Denote by ϕ the map $S \to S_0 \oplus S_1$ by $x \mapsto L_0(x) \oplus L_1(x)$. One then has that $\phi(f) + K![L_1](f) = [f]$ for any $f \in \mathrm{K}_0(S)$. Define the map $L = \phi \circ \psi$, and denote by $j_1 : S \to B$ and $j_2 : C \to B$ be the embedding. Then, one has that $(j_1)_* = (j_2 \circ \phi)_*$ on $\mathrm{K}_0(S, \mathbb{Z}/k\mathbb{Z})$ for $k \leq K$. Since $\mathrm{K}_1(S) = \mathrm{K}_1(C) = 0$, both $[L]$ and $[\psi]$ map $\mathrm{K}_0(A, \mathbb{Z}/k\mathbb{Z})$ to $\mathrm{K}_0(B)/k\mathrm{K}_0(B)$ and factor though $\mathrm{K}_0(S, \mathbb{Z}/k\mathbb{Z})$. Therefore

$$[L]|_{G\cap \mathrm{K}_0(A,\mathbb{Z}/k\mathbb{Z})} = [\psi]|_{G\cap \mathrm{K}_0(A,\mathbb{Z}/k\mathbb{Z})}, \quad k = 1, ..., K,$$

as desired. □

Using this lemma, one has the KK-attainability for certain homogeneous C*-algebras.

LEMMA 2.17 (Lemma 9.9 of [12]). *Let $C = M_n(C(X))$, where X is a path connected compact metric space with $\mathrm{K}_0(C(X)) = \mathbb{Z} \oplus \mathrm{tor}(\mathrm{K}_0(C(X)))$, and $\mathrm{K}_1(C(X))$ is finitely generated, and $\mathrm{K}_0^+(C) \subseteq \{(z, x) : z \in \mathbb{N}, or, (z, x) = (0, 0)\}$. Then C is KK-attainable.*

PROOF. The proof is the same as that of Lemma 9.9 of [12]. Instead of using Lemma 9.8 of [12], one use Lemma 2.16, and instead considering finite C*-algebra F_1 (in the proof of Lemma 9.9 of [12]), one considers the splitting interval algebra S of Lemma 2.16, and the same argument works (only use the fact $\mathrm{K}_1(S, \mathbb{Z}/k\mathbb{Z}) = 0$). □

The proof of Lemma 2.17 does not work for B with $\inf(\mathrm{K}_0(A))$ a torsion free cyclic group. However, for $A = \mathrm{C}(S^2)$, where S^2 be the two dimensional sphere, even the infinitesimal subgroup of $\mathrm{K}_0(A)$ is isomorphic to $\mathbb{Z}$, one still has that A is KK-attainable by constructing an embedding of $\mathrm{C}(S^2)$ to an AF-algebra, which can be realized as a sub-C*-algebra of the codomain algebra under a suitable K-theory condition.

DEFINITION 2.18. An ordered group G is *weakly divisible* if for any $g \in G^+$, there exist $g_1, g_2 \in G^+$ such that $g = 2g_1 + 3g_2$, or equivalently, G is weakly divisible if for any $g \in G^+$ and any $M > 0$, there exist $g_1, ..., g_n \in G^+$ and integers $m_1, ..., m_n > M$ such that $g = m_1g_1 + \cdots + m_ng_n$.

LEMMA 2.19. *Let A be a C*-algebras with weakly divisible K_0-group. Suppose that A has the cancellation of projection and the strict order on the projections is determined by the traces. Then for any order-unit group homomorphism κ : $\mathrm{K}_0(S^2) \to \mathrm{K}_0(A)$, there is a *-homomorphism $\gamma : C(S^2) \to A$ such that $\gamma_0 = \kappa$.*

PROOF. The K_0-group of $\mathrm{C}(S^2)$ is $\mathbb{Z} \oplus \mathbb{Z}$ with the order determined by the first coordinator. Denote the positive elements $(1, 0)$ and $(1, 1)$ in $\mathrm{K}_0(\mathrm{C}(S^2))$ by p' and q' respectively. Set the image of p' under κ to be p.

Let $\{M_i\}$ be an increasing sequence of natural number tending to $+\infty$. Since $\mathrm{K}_0(A)$ is weakly divisible, there exist $p_1^{(1)}, p_1^{(2)}, ..., p_1^{(d_1)}$ in $\mathrm{K}_0(A)^+$ and

$$m_1^{(1)}, m_1^{(2)}, ..., m_1^{(d_1)}$$

such that $m_1^{(k)} > M_1$ for each k, and

$$\sum_{k=1}^{d_1} m_1^{(k)} p_1^{(k)} = p.$$

For each $p_1^{(k)}$, since $\mathrm{K}_0(A)$ is weakly divisible, there are positive elements $p_2^{(k,1)}, p_2^{(k,2)}, ..., p_2^{(k,d_{2,k})}$ in $\mathrm{K}_0(A)$ and natural numbers $m_2^{(k,1)}, m_2^{(k,2)}, ..., m_2^{(k,d_{2,k})}$ with $m_2^{(k,i)} > M_2$ for all $1 \le i \le n_{2,k}$ such that

$$\sum_{i=1}^{d_{2,k}} m_2^{(k,i)} p_2^{(k,i)} = p_1^{(k)}.$$

One can repeat this procedure for each positive elements $p_n^{(k_1,\cdots,k_n)}$ and get its further decomposition with the multiplicities larger that M_n. With these positive element, one can construct a dimension group embedding inside $\mathrm{K}_0(A)$ as following:

Set $D_1 = 1$ and set $D_n = \sum_{i=1}^{D_{n-1}} d_{n-1,i}$ for $n \ge 2$. In other words, D_n is the number of the positive elements $p_n^{(k_1,\ldots,k_n)}$ in n^{th} step. One can set

$$G'_n = \bigoplus_{D_n} \mathbb{Z},$$

and set the map ϕ_n from G'_n to G'_{n+1} by

$$\phi_n : e_n^{(k_1,\ldots,k_n)} \mapsto \sum_{j=1}^{d_{n,k}} m_{n+1}^{(k_1,\ldots,k_n,j)} e_{n+1}^{(k_1,\ldots,k_n,j)}$$

where $\{e_n^{(k_1,\ldots,k_n)}\}$ is the standard basis of G'_n and $\{e_{n+1}^{(k_1,\ldots,k_n,j)}\}$ are standard basis for G'_{n+1}. Denote the inductive limit of (G'_n, ϕ_n) by G'.

Note that there are positive maps $\iota_n : G'_n \to \mathrm{K}_0(A)$ by

$$\iota_n : e_n^{(k_1,\ldots,k_n)} \to p_n^{(k_1,\ldots,k)}.$$

It is easy to verify that $\{\iota_n\}$ are compatible with the maps $\{\phi_n\}$. Thus, there is a positive homomorphism $\iota : G' \to \mathrm{K}_0(A)$. Let $u \in G'$ be $\phi_{1,\infty}(1)$. It follows that

$$\iota(u) = p.$$

Define an ordered group

$$G = G' \oplus \mathbb{Z}$$

with the order determined by the first coordinator (in other words, $(g, n) > 0$ in G if and only if $g > 0$ in G'). One has a group homomorphism by $\theta : G \to \mathrm{K}_0(A)$ by

$$\theta : (g, n) \mapsto \iota(g) + n(q - p).$$

Since the order on the projections in A is determined by the traces and A is simple, one concludes that θ is a positive homomorphism. Note that $\theta(u, 0) = p$ and $\theta(u, 1) = p$.

On the other hand, one has a positive homomorphism $\psi : \mathrm{K}_0(\mathrm{C}(S^2)) \to G$ by

$$p' \mapsto (u, 0)$$

$$q' \mapsto (u, 1).$$

A simple calculation shows that

$$\kappa = \theta \circ \psi.$$

Since the multiplicities of building blocks G' are unbounded ($M_i \to +\infty$), there is no minimal elements in G'^{+}. Hence G satisfies the Riesz property, which implies G is a dimension group without minimal elements. Thus, there is an unital AF algebra B with

$$(\mathrm{K}_0(B), \mathrm{K}_0(B)^+, [1_B]_0) \cong (G, G^+, (u, 0)).$$

Since A has the cancellation of projections, there is a *-homomorphism $h : B \to A$ such that $h_0 = \theta$.

Consider the positive homomorphism $\psi : \mathrm{K}_0(\mathrm{C}(S^2)) \to \mathrm{K}_0(B)$. Since there is no minimal positive elements in G, there is a homomorphism $r : \mathrm{C}(S^2) \to \mathrm{K}_0(B)$ with $r_0 = \psi$ by [5]. Therefore, the *-homomorphism

$$\gamma = h \circ r$$

satisfies the theorem. □

THEOREM 2.20. *Let A be a simple TAS-algebra. Then the ordered group $G = \mathrm{K}_0(A)$ is weakly divisible.*

PROOF. By Corollary 3.4 of [6], the K_0-group of A is a simple inductive limit of K_0-groups of splitting tree algebras and ordered groups $\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}, n = 1, 2, ...$ Therefore, we may assume that $G = \varinjlim(G_i, \phi_i)$, where G_i is a direct sum of K_0-groups of splitting interval algebras and ordered groups $\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}, n = 1, 2, ...$

For any $a \in G^+$, since G is simple, we may assume that $a \in G_i$ with the multiplicity of a in each direct summand of G_i is larger than 10.

It is clear that the restriction of a to each group $\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}, n = 1, 2, ...$ can be decomposed as $2a_1 + 3a_2$ for some positive element a_1 and a_2 in $\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}, n = 1, 2, ...$

Consider the restriction of a to a direct summand of G_i which is in the K_0-group of a splitting tree algebra. Then, it can be identified with the group

$$\left\{((m_1^{(1)}, ..., m_1^{(l_1)}), ..., (m_n^{(1)}, ..., m_n^{(l_n)})) \in \bigoplus_{i=1}^{n} \mathbb{Z}^{l_i}; \sum_i m_1^{(i)} = \cdots = \sum_i m_n^{(i)}\right\}$$

for some $m_j^{(i)}$, and identify the restriction of a by

$$((a_1^{(1)}, ..., a_1^{(l_1)}), ..., (a_n^{(1)}, ..., a_n^{(l_n)}))$$

with $a_j^{(i)} > 10$. By Lemma 2.14 of [14], there are positive elements a_1' and a_2' in G_i^+ such that the restriction of a is $2a_1' + 3a_2'$. Therefore, one has that $a = 2(a_1, a_1') + 3(a_2, a_2')$, as desired. □

Together with Remark 2.15, one has

COROLLARY 2.21. *The C*-algebra $C(S^2)$ is KK-attainable for simple TAS-algebras.*

REMARK 2.22. It is clear that KK-attainability is a local property, i.e., any C*-algebra which can be locally approximated by KK-attainable C*-algebras is again KK-attainable. In particular, inductive limits of KK-attainable C*-algebras are KK-attainable. If one considers the Gong's standard homogeneous C*-algebras which appear in 2.1, all of them are KK-attainable by Lemma 2.17, and thus any concrete TAS-algebras constructed in 2.1 is KK-attainable. Moreover, for any simple TAS-algebra A, by Theorem A of [6], there is a KK-attainable C*-algebra B such that the Elliott invariant of B is isomorphic to the Elliott invariant of A. Even more general, any inductive limit of point-line algebras together with the homogeneous C*-algebras mentioned above is KK-attainable.

2.3. Existence theorem: from TASI-algebras to inductive limit C-algebras* Let A be a TASI-algebra. By Remark 2.22, there is a C*-algebra B which is an inductive limit of point-line algebras together with Gong's standard homogeneous C*-algebras and $\mathrm{M}_n(\mathrm{C}(S^2))$ such that that any element in $\mathrm{KL}(B,A)^+$ can be lifted locally approximately by maps from B to A. In this section, let us consider an element in $\mathrm{KL}(A,B)^+$, and show that it can be lifted locally approximately as well.

Our approach is a modification of Lin's proof on the local existence theorem for TAF-algebras in [11]. Let us first sketch the outline. For the C*-algebras A and B, and any positive KL-element α, since A is an inductive limit of residually finite-dimensional (RFD) C*-algebras (a C*-algebra is RFD if it has a sequence of finite-dimensional representations separating points), it follows that α can be realized as the difference of two approximate homomorphisms from A to $B\otimes\mathcal{K}$, and one of them has finite-dimensional range (Theorem 2.29). With these two maps, we then construct an approximate homomorphism Ψ from A to $B\otimes\mathcal{K}$ which agrees with α on $\mathrm{K}_1(A)$ and the K-group with non-zero coefficients of A. However, we still have to handle K_0-groups. In order to get the desired map on $\mathrm{K}_0(A)$, we compress the map Ψ into a small corner of $B\otimes\mathcal{K}$, then construct a map from A to B which factors though a splitting interval algebra (a sub-C*-algebra of A). This map together with Ψ will induce a suitable map between the K_0-groups. To find such a map, we need certain divisibility conditions on $\mathrm{K}_0(B)$. Since this map factors through the large piece of A, we also need to take care the infinitesimal elements of $\mathrm{K}_0(A)$, which vanish under any traces of A.

Let us proceed the approximation construction as in of Section 2 of [11]. By Lemma 4.4 of [14], the C*-algebra A is MF. It then follows from a result of B. Blackadar and E. Kirchberg in [1] that there is an increasing family of RFD sub-C*-algebras $\{A_n\}$ such that their union is dense in A. Let $\{x_1, x_2, ..., x_n, ...\}$ be a dense sequence of elements in the unit ball of A. For any finite subset $\mathcal{F}_1\subset A_1$ with $x_1\in\mathcal{F}_1$, $\delta_0>0$, and a homomorphism h_0 from A_1 to a finite-dimensional C*-algebra F_0 which is non-zero on $\mathcal{F}_1$, by Lemma 2.1 of [13], there is a non-zero homomorphism $h': F_0\to A$ such that

- $\|ea-ae\|<\delta$ and
- $\|h'\circ h_0(a)-eae\|<\delta$

for all $a\in\mathcal{F}_1$, with $e=h'(1)$.

Set $H=h'\circ h_0: A\to A$. Write $e_0=H(1)$. Since the hereditary sub-C*-algebra $(1-e_0)A(1-e_0)$ is a TASI algebra again, there is a projection $q_1'\leq 1-e_0$ and a splitting interval algebra S_1' with $1_{S_1'}=q_1'$ such that

- $\|q_1'x-xq_1'\|<\delta_0/8$,
- $\mathrm{dist}(q_1'xq_1', S_1')<\delta_0/32$ for any $x\in\mathcal{F}_1$, and
- $\tau(1-q_1')\leq 1/4$ for any tracial state τ on A.

Set $q_1=q_1'+e_0$ and $S_1=S_1\oplus h'(F_0)$. One has

- $\|q_1x-xq_1\|<\delta_0/4$,

- $\mathrm{dist}(q_1 x q_1, S_1) < \delta_0/16$ for any $x \in \mathcal{F}_1$, and
- $\tau(1-q_1) \leq 1/4$ for any tracial state τ on A.

Since any splitting tree algebra is a one-dimensional noncommutative CW-complex in sense of [2], the C*-algebra S_1 is finite presented and has stable relations. Denote by G_1 a finite set of generators of S_1, and let $\mathcal{F}_2$ be the union $\{x_2\} \cup \mathcal{F}_1 \cup G_1$. Since A is nuclear, there is a completely positive contraction $L'_1 : q_1 A q_1 \to S_1$ such that its restriction to S_1 is identity.

Set $L_1(a) = L'_1(q_1 a q_1)$ for any $a \in A$. Then L_1 is a completely positive contraction from A to S_1 with the restriction to S_1 the identity. Note that L_1 is $\{x_1\}$-$\delta_0/2$-multiplicative. Since S_1 is generated by stable relations, there is $\delta'_2 > 0$ such that for any G_1-δ'_2-multiplicative linear map L with domain S_1, there is a homomorphism h with domain S_1 such that

$$\|L(a) - h(a)\| < \delta_0/16 \quad \text{for any } a \in \{L_1(x_1)\} \cup G_1.$$

Set $\delta_2 = \min \delta'_2, \delta_0/4$. Then, there is a projection q_2 and a splitting interval algebra S_2 with $1_{S_2} = q_2$ such that

- $\|q_2 x - x q_2\| < \delta_2/4$,
- $\mathrm{dist}(q_2 x q_2, S_2) < \delta_2/16$ for any $x \in \mathcal{F}_2$, and
- $\tau(1-q_2) \leq 1/8$ for any tracial state τ on A.

There then exists a completely positive contraction $L'_2 : q_2 A q_2 \to S_2$ with the restriction to S_2 the identity. Set $L_2(a) = L'(q_2 a q_2)$ for any $a \in A$, and it is a completely positive contraction from A to S_2 with the restriction to S_2 the identity, and is $\mathcal{F}_2$-δ_2-multiplicative. Therefore, there is a homomorphism $h_2 : S_1 \to S_2$ such that

$$\|L_2(a) - h_2(a)\| < \delta_0/16 \quad \text{for any } a \in \{L_1(x_1)\} \cup G_1.$$

Repeating this construction, one obtains a sequence of finite subsets $\mathcal{F}_0, \mathcal{F}_1, ...$ with dense union in the unit ball of A, a sequence of decreasing positive numbers $\{\delta_n\}$, a sequence of projections $\{q_n\} \subset A$, a sequence of splitting interval sub-C*-algebras S_n with $1_{S_n} = q_n$ and a sequence of homomorphisms $h_{n+1} : S_n \to S_{n+1}$ such that

- $\|q_n x - x q_n\| < \delta_n/4$ for all $x \in \mathcal{F}_n$;
- $\mathrm{dist}(q_n x_i q_n, S_n) < \delta_n/16, i = 1, \cdots, n$;
- $\tau(1-q_n) < 1/2^{n+1}$ for all tracial states τ on A;
- $G_n \subset \mathcal{F}_{n+1}$, where G_n is a finite set of generators for S_n;
- $\|L_{n+1}(a) - h_{n+1}(a)\| < 1/16^n$ for all $a \in \{L_n(\mathcal{F}_n)\} \cup \{S_n\}$, where $L_n : A \to S_n$ is a contractive positive linear map with $L_n|_{S_n} = \mathrm{id}_{S_n}$.

As in [11], denote by $S_{n,1}, S_{n,2}, ..., S_{n,m(n)}$ the direct summands of S_n corresponding to connected components of its spectrum, and denote by $\pi_{n,i} : S_n \to S_{n,i}$ the corresponding quotient map. Let $\Psi_n : \mathcal{A} \to (1-q_n)A(1-q_n)$ denote the cut-down map sending a to $(1-q_n)a(1-q_n)$, and let $J_n : A \to A$ denote the map sending a to $\Psi_n(a) \oplus L_n(a)$. Note that Ψ_n and J_n are $\mathcal{F}_n$-$\delta_n/2$-multiplicative. Set $J_{m,n} = J_n \circ \cdots \circ J_m$ and $h_{m,n} = h_n \circ \cdots \circ h_m$. Note that $J_{m,n}$ is $\mathcal{F}_m$-δ_m-multiplicative. We also use $L_n, \Psi_n, J_n, J_{m,n}, h_m$, and $h_{m,n}$ for their extensions on a matrix algebra over A.

Using the same argument as that of Lemma 2.7 of [11], one has the following lemma.

LEMMA 2.23 (Lemma 2.7 of [11]). *Let $\mathcal{P} \subset M_k(A)$ be a finite set of projections. Assume that $\mathcal{F}_n$ is sufficiently large and δ_0 is sufficiently small such that $[L_{n+1} \circ J_{1,n}]|_{\mathcal{P}}$ and $[L_{n+1} \circ J_{1,n}]|_{G_0}$ are well defined, where G_0 is the subgroup generated by $\mathcal{P}$. Then*

$$\lim_{n\to\infty} \tau([L_{n+1} \circ J_{1,n}]([p])) = \tau([p])$$

for any $p \in \mathcal{P}, \tau \in T(A)$, with respect to uniform convergence on $T(A)$. Furthermore, for any projection q in S_1, we have

$$|\tau(h_{1+n} \circ \cdots \circ h_2(q)) - \tau(h_n \circ \cdots \circ h_2(q))| < 1/2^{n+1}$$

for all $\tau \in T(A)$, and

$$\lim_{n\to\infty} \tau(h_{1+n} \circ \cdots \circ h_2(q)) > 0$$

for all $\tau \in T(A)$.

REMARK 2.24. Since A is assume to be nuclear, therefore exact, any positive state of $\mathrm{K}_0(A)$ is the restriction of a tracial state of A (in fact, by Theorem 4.11 of [6], this is true even without exactness). Thus, the lemma above still holds if one replaces the trace τ by any positive state τ_0 on $\mathrm{K}_0(A)$.

As in 2.8 of [11], for a fixed finite subset $\mathcal{P}$ of the projections of A, and a natural number N, denote by $\tilde{\psi}_{N,i} = \pi_{N,i} \circ L_N$, $\tilde{\psi}_{N+1,i} = \pi_{N+1,i} \circ L_{N+1} \circ \Psi_N$, ..., $\tilde{\psi}_{N+n,i} = \pi_{N+n,i} \circ L_{N+n} \circ \Psi_{N+n-1}$. Then we have that

$$[L_{N+n} \circ J_{N,N+n+1}]_{\mathcal{P}} = \sum_{k,i} [\tilde{\psi}_{N+k,i}]_{\mathcal{P}}.$$

Therefore, if we rearrange $\{\tilde{\psi}_{N+m,i}\}$ as $\{\psi_j\}$, we have

$$\tau([L_{N+n} \circ J_{N,N+n+1}]([p]) = \sum_{j=1}^{s(n)} \tau([\psi_j]([p])).$$

By Lemma 2.13 of [14], the positive cone of the K_0-group of any splitting interval algebra is generated by the minimal projections with respect to finitely many relations. Therefore every projection is Murray-von Neumann equivalent to the direct sum of minimal projections $\{p_k, q_l; 1 \leq k \leq h_0, 1 < l \leq h_1\}$, where h_0 and h_1 are the numbers of splitting points at the endpoint 0 and at the endpoint 1 respectively (recall that from Section 2.4 of [14] that p_k is the constant projection with 1 at the kth splitting point at the end point 0, and q_l is the rank-one projection which is 1 at the first splitting point at 0 and is 1 at the jth splitting point at 1). Moreover, it follows from Lemma 2.12 of [14] that this decomposition is unique.

Let $\vec{p}_n = (p_1, ..., p_{h_0}, q_2, ..., q_{h_1})$ be the vector of these minimal projections. One then has that $\tau([\psi_j]([p])) = \vec{m}_n \cdot \tau([\vec{p}_n])^t$, where $\vec{m}_n$ is a vector of nonnegative integers which stand for the multiplicities of p. It follows that the entries of $\vec{p}_n$ satisfy the relations $[p_k] + [q_l] > [q_{k'}]$, where k' is the index such that $\mathrm{ev}_1^{i'}([p_i]) = 1$. And conversely, by Lemma 2.13 of [14], if there is a set of positive elements satisfying such relations in any ordered group, then there is a positive homomorphism from the K_0-group of the splitting interval algebra to the ordered group. Denote such vector of the minimal projections corresponding to ψ_i by $\vec{p}_{i,n}$. Then, by Lemma 2.23 and Remark 2.24, one has the following lemma.

Lemma 2.25. *With the notion same as above, for any $p \in \mathcal{P}$, one has that*

$$\tau([p]) = \lim_{n\to\infty} \sum_{i=1}^{s(n)} \vec{m}_i \cdot \tau(([\vec{p}_{i,n}])^t$$

uniformly for $\tau \in \mathrm{S}(\mathrm{K}_0(A))$, and the affine maps induced by each entry of $\vec{p}_{i,n}$ converge to a strictly positive element in $\mathrm{Aff}(\mathrm{S}(\mathrm{K}_0(A)))$ as $n \to \infty$.

One then has the following

Corollary 2.26. *Let $\mathcal{P}$ be a finite subset of projections in a matrix algebra over A, and let G_0 be the subgroup of $\mathrm{K}_0(A)$ generated by $\mathcal{P}$. Denote by $\tilde{\rho} : G \to \Pi\mathbb{Z}$ the map defined by*

$$[p] \mapsto (\vec{m}_1(1), ..., \vec{m}_1(k_1), \vec{m}_2(1), ..., \vec{m}_2(k_2), ...).$$

If $\tilde{\rho}(g) = 0$, then $\tau(g) = 0$ for any trace over A.

By the definition of the map $\tilde{\rho}$ and H, using the same argument as that of Lemma 2.12 of [11], one has the following lemma.

Lemma 2.27. *Let $\mathcal{P}$ be a finite subset of projections in $M_k(A_1) \subseteq M_k(A)$. Then there is a finite subset $\mathcal{F}_1 \subset A_1$ and $\delta_0 > 0$ such that if the above construction starts with them, then*

$$\ker\tilde{\rho} \subset \ker[H] \quad \textit{and} \quad \ker\tilde{\rho} \subset \ker[h_0].$$

The K_0-part of the existence theorem will almost factor through the map $\tilde{\rho}$, and this lemma will help us to handle the elements of $\mathrm{K}_0(A)$ which vanish under $\tilde{\rho}$. Moreover, to get a such K_0-homomorphism, one also needs to find a copy of the generating set of the positive cone of $\mathrm{K}_0(S)$ inside the codomain ordered group for certain splitting interval algebra S. In order to do so, one need the following technical lemma, which is a slight modification of Lemma 3.4 of [11].

LEMMA 2.28 ([11]). *Let S be a compact convex set and $\mathrm{Aff}(S)$ be the affine continuous functions on S. Let $\mathbb{D}$ be a dense ordered subgroup of $\mathrm{Aff}(S)$, and let G be an ordered group with the strict order determined by a surjective homomorphism $\rho : G \to \mathbb{D}$. Let $\{x_{ij}\}_{0<i\leq r, 0<j<\infty}$ be an $r \times \infty$ matrix having rank r and with $x_{ij} \in \mathbb{Z}$ for each i, j, and let $g_j^{(n)} \in G$ such that $\rho(g_j^{(n)}) = a_j^{(n)}$, where $\{a_j^{(n)}\}$ is a sequence of positive elements in $\mathbb{D}$ such that $a_j^{(n)} \to a_j (> 0)$ uniformly on S as $n \to \infty$. For each n, there is an $s(n)$ such that*

$$(x_{ij})_{r\times s(n)}\tilde{v_n} = \tilde{y_n}$$

where $\tilde{v_n} = (g_j^{(n)})_{s(n)\times 1}$ and $\tilde{y_n} = (\tilde{b}_j^{(n)}) \in G^r$. Set $b_i^{(n)} = \rho(\tilde{b}_i^{(n)})$ and $y_n = (b_j^{(n)})$. Suppose that $y_n \to z$ on S uniformly for some $z = (z_j)_{r\times 1}$. Then there exist $\delta > 0$ and positive integer $K > 0$ satisfying the following:

For some sufficiently large n, if $\tilde{z}' \in G^r$ and there is $\tilde{z}'' \in G^r$ such that $K^3\tilde{z}'' = \tilde{z}'$ and $\|z - Mz'\| < \delta$, where $z'_j = \rho(\tilde{z}'_j)$ if $\tilde{z}' = (\tilde{z}'_1, ..., \tilde{z}'_r)$, and M is a positive integer, then there is an $\tilde{u} = (\tilde{c}_j)_{s(n)\times 1} \in G_+^{s(n)}$ such that

$$(x_{ij})_{r\times s(n)}\tilde{u} = \tilde{z}'.$$

Moreover, we can choose $\{c_j^{(n)}\}_{j=1}^{s(n)}$ such that for each n, if $\{g_j^{(n)}\}_{j=1}^{s(n)}$ generate the K_0-group of a splitting interval algebra as the minimal projections, then $\{c_j^{(n)}\}_{j=1}^{s(n)}$ can be chosen to satisfy the same relations (so that $g_j^{(n)} \mapsto c_j^{(n)}$ induces a positive map from the K_0-group of the splitting interval algebra to G).

PROOF. The proof is repeating the argument of Lemma 3.4 of [11] with the following modification to show that $u = (\tilde{c}_j)_{s(n)\times 1}$ can be chosen to satisfy the last requirement of the statement.

Since $\{g_j^{(n)}\}_{j=1}^{s(n)}$ generate the K_0-group of a splitting interval algebra (its spectrum might have more than one connected components), it satisfies the relations in Lemma 2.13 of [14]. Thus, $(\frac{a_j^{(n)}}{M})$ satisfies the same relation. Since the strict order on G is determined by ρ, there exists $\delta'_n > 0$ such that if

$$\left\|\rho(\tilde{b}_j^{(n)}) - \frac{\rho(g_j^{(n)})}{M}\right\| < \delta'_n, \quad j = 1, ..., s(n)$$

for some $b_l^{(n)} \in G^+$, then $\{\tilde{b}_j^{(n)}\}_{j=1}^{s(n)}$ satisfies the same relations. One may assume that (δ'_n) is decreasing.

Return to the proof of Lemma 3.4 of [11]. When one chooses the vector $\xi_n = (\tilde{d}_j^{(n)})_{s(n)\times 1}$ at the first two lines of Page 108, one can further requires that

$$\left\| K^3\rho(\tilde{d}_j^{(n)}) - \frac{a_j^{(n)}}{M} \right\| < \min\{\frac{1}{s(n)^2\cdot 2^n}, \delta'_n/2\}. \tag{2.1}$$

Therefore, $\{\tilde{d}_j^{(n)}\}_{j=1}^{s(n)}$ always satisfies the relations.

Let n_r be a natural number such that $r < s(n_r)$. When ε is chosen in the proof of Lemma 3.4 of [11] (Line 16, Page 108), one can furthermore set $\varepsilon < \delta'_{n_r}$.

With such ξ_n and ε, one obtains the solution (with the notation of [11])

$$\tilde{u} = (K^2\tilde{c}_1, ..., K^2\tilde{c}_r, K^3\tilde{d}_{r+1}^{(n)}, ..., K^3\tilde{d}_{s(n)}^{(n)}).$$

Let us verify that $\tilde{u}$ satisfies the same relations as those of $\{g_j^{(n)}\}_{j=1}^{s(n)}$. One only has to verify that

$$\left\| K^2\rho(\tilde{c}_j) - \frac{\rho(g_j^{(n)})}{M} \right\| < \delta'_{n_r}, \quad j = 1, ..., r \tag{2.2}$$

and

$$\left\| K^3\rho(\tilde{d}_j^{(n)}) - \frac{\rho(g_j^{(n)})}{M} \right\| < \delta'_n, \quad j = r+1, ..., s(n). \tag{2.3}$$

Equation (2.3) follows from Equation (2.1) directly. Let us verify Equation (2.2). By the construction of $\tilde{c}_j$ in last paragraph of Page 181 of [11], one has (using the notation in [11])

$$\begin{aligned}
\left\|K^2\rho(u') - \rho(g')\right\| &= \|B'\rho(\tilde{z}') - D_n\rho(\tilde{w})_n - \rho(g')\| \\
&\leq \|B'y'_n - D_n\rho(\tilde{w})_n - \rho(g')\| + \varepsilon/2 \\
&\leq \|B'\rho(A'_n\tilde{w}_n) - D_n\rho(\tilde{w})_n - \rho(g')\| + \delta'_{n_r}/2 \\
&= \|\rho((B'A'_n - D_n)(\tilde{w}_n) - g')\| + \delta'_{n_r}/2 \\
&= \|\rho((C_n - D_n)(\tilde{w}_n) - g')\| + \delta'_{n_r}/2 \\
&= \left\|K^3\rho((\tilde{d_j^{(n)}})_{r\times 1}) - \rho(g')\right\| + \delta'_{n_r}/2 \\
&\leq \delta'_{n_r},
\end{aligned}$$

where $g' = (g_1^{(n)}/M, g_2^{(n)}/M, ..., g_r^{(n)}/M)$. This verifies Equation (2.2). Thus, $\tilde{u}$ is the desired solution. □

With the preparation above, let us turn to the existence theorem for maps from a TASI-algebra to an inductive limit of certain C*-algebras. First, one has the following theorem which holds in a general setting. It lifts any given KL-element to the formal difference of two almost multiplicative maps, and one of them has finite-dimensional range.

THEOREM 2.29 (Theorem 5.9 of [9]). *Let A be a separable C*-algebra satisfying UCT, and let B be a nuclear separable C*-algebra. Assume that A is the closure of an increasing sequence $\{A_n\}$ of RFD sub-C*-algebra. Then for any $\alpha \in \mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B))$, there exist two sequences of completely positive contractions $\phi_n^{(i)} : A \to B \otimes \mathcal{K}$ $(i = 1, 2)$ satisfying the following:*

- $\|\phi_n^{(i)}(ab) - \phi_n^{(i)}(a)\phi_n^{(i)}(b)\| \to 0$ *as* $n \to \infty$,
- *for any n, the images of $\phi_n^{(2)}$ are contained in a finite dimensional sub-C*-algebra of $B \otimes \mathcal{K}$ and for any finite subset $\mathcal{P} \subset P(A)$, $[\phi_n^{(i)}]|_{\mathcal{P}}$ are well defined for sufficiently large n,*
- *for each finite subset $\mathcal{P} \subset P(A)$, there exists $m > 0$ such that*

$$[\phi_n^{(1)}]|_{\mathcal{P}} = \alpha + [\phi_n^{(2)}]|_{\mathcal{P}}$$

for all $n > m$,
- *for each n, we may assume that $\phi_n^{(2)}$ is a homomorphism on A_n.*

By Lemma 4.4 of [14], any simple separable TAS-algebra is MF (in particular, any simple separable TASI-algebra is MF), and hence, by [1], it is a closure of an increasing sequence of RFD sub-C*-algebras. If it satisfies UCT, then it satisfies the condition for A in the theorem.

Recall that by an infinitesimal element of an ordered group (G, G^+), we mean an element of G which is vanishing under any states of (G, G^+).

For any TASI-algebra A, by [6], there exists a simple inductive limit of splitting interval algebras and Gong's standard homogeneous C*-algebras such that it shares the same Elliott invariant with A. In particular, they have the same K-theory. However, since there is no Effros-Handleman-Shen theorem for the K-theory of TASI-algebras at this moment, it is not clear that the K_0-group of a TASI-algebra modulo infinitesimals is still a K_0-group of such an inductive limit. But instead of splitting interval algebras, one is able to use point-line algebras to realize K_0-groups; and moreover, there exists a map from this inductive limit to a small corner of itself such that it induces identity maps on all parts of the invariant except the K_0-group, and the difference on the K_0-group part has an arbitrarily given multiplicity. More precisely, we have

LEMMA 2.30. *Let A be a simple separable TASI-algebra. There exists an inductive limit algebra B of point-line algebras, homogeneous C*-algebras with base spaces wedge product of the Gong standard spaces and two sphere S^2, such that A and B has the same K-theory, and the C*-algebra B satisfies the following:*

Let G_0 be a finitely generated subgroup of $\mathrm{K}_0(B)$ with decomposition $G_0 = G_{00} \oplus G_{01}$, where G_{00} is vanished by all states of $\mathrm{K}_0(A)$. Suppose $\mathcal{P} \subset \underline{\mathrm{K}}(B)$ is a finite subset which generates a subgroup G such that $G_0 \subset G \cap \mathrm{K}_0(B)$.

Then for any $\varepsilon > 0$, any finite subset $\mathcal{F} \subset B$, any $1 > r > 0$, and any positive integer K, there is an $\mathcal{F}$-ε-multiplicative map $L : B \to B$ such that:

- *$[L]|_{\mathcal{P}}$ is well defined.*
- *$[L]$ induces the identity maps on the infinitesimal part of each of $G \cap K_0(B)$, $G \cap K_1(B)$, $G \cap K_0(B, \mathbb{Z}/k\mathbb{Z})$ and $G \cap K_1(B, \mathbb{Z}/k\mathbb{Z})$ for k with $G \cap K_i(B, \mathbb{Z}/k\mathbb{Z}) \neq \{0\}$, $i = 0, 1$.*
- *$\rho_B \circ [L](g) \leq r\rho_B(g)$ for all $g \in G \cap K_0(B)$, where ρ is the canonical positive homomorphism from $\mathrm{K}_0(A)$ to $\mathrm{Aff}(\mathrm{S}(\mathrm{K}_0(A), \mathrm{K}_0(A)^+, [1_A]_0))$.*
- *For any positive element $g \in G_{01}$, we have $g - [L](g) = Kf$ for some $f \in \mathrm{K}_0(B)^+$.*

Proof. Let us construct a C*-algebra B using the given building blocks such that the infinitesimal elements of $\mathrm{K}_0(B)$ are always induced by homogeneous C*-algebras with base space two-sphere, which only appears in the finite stages of the inductive limit.

Denote by G the image $\rho(\mathrm{K}_0(A))$ in $\mathrm{Aff}(\mathrm{S}(\mathrm{K}_0(A)))$. It is clear that G is torsion free. By Theorem 5.2.2 of [3], there exists an inductive limit $B' = \varinjlim(S_i, \phi_i)$ of with S_i direct sums of point-line algebras with trivial K_1-groups such that $\mathrm{K}_0(A) = G$. Moreover, each map $\mathrm{K}_0(S_i) \to \mathrm{K}_0(S_j)$ is extendable by the construction. One asserts that the inductive system can be chosen so that S_i has copies of finite dimension C*-algebras.

Denote by G_i the K_0-group of S_i. Let $K_1 < K_2 < \cdots$ be a sequence of natural number converging to infinity, and let $r_1 > r_2 > \cdots$ be a sequence of positive number converging to 0. Consider the point-line algebra S_1. Let $\delta > 0$ such that $\tau(g) > \delta$ for any $g \in \mathrm{K}_0(S_1)^+$ and any $\tau \in \mathrm{S}_u(G)$. Denote by d_1 and m_1 the multiplicity of S_1 at regular points and the constant of Lemma 2.10 respectively, and denote by M_1 the constant of Proposition 3.7 of [14] with respect to S_1. Consider the map $\alpha_{1,\infty} : G_1 \to G$. Since the strict order on $\mathrm{K}_0(A)$ is determined by the images in G, the map $\alpha_{1,\infty}$ can be lifted to a map $G_1 \to \mathrm{K}_0(A)$. Since A is tracially approximately divisible, there is a decomposition $\alpha_{1,\infty} = \alpha'_{1,\infty} + n_1\alpha''_{1,\infty}$ with $\tau(\alpha'_{1,\infty}(u)) < r_1$ for any $\tau \in \mathrm{S}_u(G)$, where u is the order unit of G and n_1 is the integer part of $K_1M_1^2m_1d_1/\delta$.

By telescoping the inductive system, one may assume that the image of $\alpha'_{1,\infty}$ and $\alpha''_{1,\infty}$ are in G_2, so one can rewrite them as $\alpha_{1,2}$, $\alpha'_{1,2}$, and $\alpha''_{1,2}$. Consider the point-line S_2. Since S_2 has stable rank one, there then exist projections p' and $p''_1, ..., p''_{n_1}$ such that $[p'] = \alpha'(u)$, $[p''_i] = \alpha''(u)$, p''_i is unitarily equivalent to p''_j, and $p''_i \perp p''_j$ for any $i \neq j$. Consider the unital hereditary sub-C*-algebras $p'S_2p'$ and $p''_1S_2p''_1$. They are again point line-algebras. Moreover, since G is simple, one may assume that each of them is full. Note that the map $\alpha_{1,2}$ is extendable by the construction of the inductive system, one has that the map $\alpha'_{1,2}$ and $\alpha'_{1,2}$ are also extendable. Therefore, there is a *-homomorphisms $\phi'_{1,2} : S_1 \to p'S_2p'$ such that $[\phi'_{1,2}] = \alpha'_{1,2}$.

Consider the map $\theta'_{1,2} : G_1 \to G_2$ by $g \mapsto r(g)M_1K_1[p_1]$, where $r(g)$ is a point evaluation at a regular point. Note that $\tau(\alpha''_{1,2}(g) - \theta'_{1,2}(g)) > 0$ for any $g \in \mathrm{K}_0(S_1)^+$ and any $\tau \in \mathrm{S}_u(G)$. Then, by telescoping the inductive system again, one may assume that $\theta''_{1,2} := \alpha''_{1,2} - \theta'_{1,2}$ is a positive map from G_1 to G_2. It is clear that the map $\theta''_{1,2}$ has multiplicity divisible by MK_1. Therefore, by Proposition 3.7 of [14], there is a *-homomorphism $\phi''_{1,2} : S_1 \to S_2$ such that $K_1[\phi''_{1,2}] = \alpha''_{1,2} - \theta'_{1,2}$.

Noting that $\theta'_{1,2}$ factors though $\mathbb{Z}$, there is a *-homomorphism $\iota_{1,2} : S_1 \to S_2$ which factoring though a matrix algebra F_1 such that $[\iota_{1,2}] = \theta'_{1,2}$.

Consider the homomorphism $\tilde{\phi}_{1,2} = \iota_{1,2} \oplus \phi'_{1,2} \oplus (\bigoplus_{K_1} \phi''_{1,2})$. It is clear that

$$[\tilde{\phi}_{1,2}] = \theta'_{1,2} + \alpha'_{1,2} + \alpha''_{1,2} - \theta'_{1,2} = \alpha_{1,2}.$$

By conjugating a unitary, one may assume that $\tilde{\phi}_{1,2}$ is a *-homomorphism from S_1 to S_2.

Note that the map $\tilde{\phi}_{1,2}$ has the following factorization

$$S_1 \longrightarrow S_1 \oplus F_1 \oplus \mathrm{M}_{K_1}(S_1) \longrightarrow S_2 \quad ,$$

where the map $S_1 \to \mathrm{M}_{K_1}(S_1)$ is the diagonal embedding, and other maps are obvious ones. Note that the trace of the image of the unit of $S_1 \oplus F_1$ is less that r_1.

Repeating this procedure, one has an inductive system $(S_i, \tilde{\phi}_i)$ with $[\tilde{\phi}_i] = \alpha_i$, and each map $\tilde{\phi}_{i,i+1}$ has the following factorization

$$S_i \longrightarrow S_i \oplus F_i \oplus \mathrm{M}_{K_i}(S_i) \longrightarrow S_{i+1} \quad ,$$

for a matrix algebra F_i, where the map $S_i \to \mathrm{M}_{K_i}(S_i)$ is the diagonal embedding, and other maps are obvious ones. Moreover, the trace of the image of the unit of $S_i \oplus F_i$ is less that r_i. It is clear that the K_0-group of the inductive limit is isomorphic to G.

Now, let us consider the inductive system consisting of algebras $S_i \oplus F_i \oplus \mathrm{M}_{K_i}(S_i)$, and still denote by the map between the building blocks by $\tilde{\phi}_i$ and denote by the corresponding K_0-map by α_i. It is clear that this inductive system satisfies the assertion.

In the following, we shall replace the building block F_i by certain homogeneous C*-algebras so that the modified inductive system also has the desired infinitesimal subgroup and K_1-group. Since $\ker\rho$ and $\mathrm{K}_1(A)$ are countable groups, we have that $\ker\rho = \varinjlim(H_i, \beta_i)$ and $\mathrm{K}_1(A) = \varinjlim(E_i, \iota_i)$ in the category of abelian groups where H_i and F_i are finitely generated abelian groups. An argument same as that of Theorem 1.5 of [10] shows that the ordered group $(\mathrm{K}_0(A), \mathrm{K}_0(A)^+)$ has a decomposition

$$(\mathrm{K}_0(A), \mathrm{K}_0(A)^+) = \varinjlim(G_i \oplus H_i, (G_i \oplus H_i)^+, \delta_n)$$

where the order on $G_i \oplus H_i$ is determined by the first coordinator, and the map δ_i has the form

$$\left(\begin{array}{cc} \alpha_i & 0 \\ \gamma_i & \beta_i \end{array} \right).$$

Consider the stage i, and we shall produce building block with K_0-group $G_i \oplus H_i$ and K_1-group E_i. Set $X_i = Y_1 \vee \cdots \vee Y_{n_i}$ where Y_i are S^2 or the Gong standard CW-complexes described in Section 2.1 such that $\mathrm{K}_0(\mathrm{C}(X_i)) = \mathbb{Z} \oplus H_i$ with the order determined by the first coordinator, and $\mathrm{K}_1(\mathrm{C}(X_i)) = E_i$. Note that by the construction, the ordered group G_i contains a copy of $(\mathbb{Z}, \mathbb{Z}^+)$, which corresponds to F_i. One can then replace F_i by a homogeneous C*-algebra with base space X_i and keep the remaining direct summands at the stage i unchanged.

Therefore, setting $B_i = S_i \oplus (F_i \otimes (\mathrm{C}(X_i))) \oplus \mathrm{M}_{K_i}(S_i)$, one has that $\mathrm{K}_0(B_i) = G_i$ and $\mathrm{K}_1(B_i) = F_i$. We may pass to matrix algebras of B_i such that $[1_{B_i}]_0 = u_i$ where $\delta_{i,\infty}(u_i) = [1_A]_0$. Thus, we may write $B_i = B_{i,1} \oplus B_{i,2}$ where $B_{i,1}$ is a matrix algebra over $\mathrm{C}(X_i)$ and $B_{i,2}$ is a finite direct sum of point-line algebras.

We show there is a *-homomorphism from B_i to B_{i+1} which lift the map δ_i. For the restrictions of the map δ_i to $B_{i,2}$ and $B_{i+1,2}$, this is exactly induced by the restriction of the map $\tilde{\phi}_i$ to $S_i \oplus \mathrm{M}_{K_i}(S_i)$ and $S_{i+1} \oplus \mathrm{M}_{K_{i+1}}(S_{i+1})$. For the restrictions of δ_i to $B_{i,2}$ and $B_{i+1,1}$, since the K_0-group of $B_{i,2}$ does not contain infinitesimal elements, the map must factor through $\mathbb{Z}$, and thus is again induced by the restriction of the map $\tilde{\phi}_i$ to $S_i \oplus \mathrm{M}_{K_i}(S_i)$ and F_{i+1}. A same argument also apply to the restriction of δ_i to $B_{i,1}$ and $B_{i+1,2}$. The existence of a map lifting the restriction of δ_i to $B_{i,1}$ and $B_{i+1,1}$ follows an argument same as that of Theorem 1.5 of [10]. Denote the corresponding *-homomorphism from B_i to B_{i+1} by h_i. Then, one has that $[h_i]_* = \delta_i$.

Let us consider the inductive system (B_i, h_i), and denote by B the inductive limit. It is clear from the construction that

$$(\mathrm{K}_0(B), \mathrm{K}_0(B)^+, [1_B]) \cong (\mathrm{K}_0(A), \mathrm{K}_0(A)^+, [1_A])$$

and $\mathrm{K}_1(B) \cong \mathrm{K}_1(A)$, and moreover,

$$\ker \rho_{\mathrm{K}_0(A)} \cong \ker \rho_{\mathrm{K}_0(B)} = \varinjlim(\ker \rho_{\mathrm{K}_0(B_n)}, [h_n]_0).$$

Note that the restriction of h_i to $B_{i,2}$ and $B_{i+1,2}$ factors through the diagonal embedding $S_i \to \mathrm{M}_{K_i}(S_i)$, and the trace of image of the unit of $S_i \oplus F_i$ is less than r_i uniformly. Since $K_i \to \infty$ and $r_i \to 0$, it is not difficult to verify that the C*-algebra B satisfies the lemma. □

With these preparations, one has the following existence theorem. The argument is the same as the existence theorem in [11].

THEOREM 2.31. *Let A be a unital separable simple nuclear TASI-algebra which satisfies UCT. Suppose B is a inductive limit of point-line algebras and homogeneous algebras described in Lemma* 2.30 *with*

$$(\mathrm{K}_0(A), \mathrm{K}_0(A)^+, [1_A], \mathrm{K}_1(A)) \cong (\mathrm{K}_0(B), \mathrm{K}_0(B)^+, [1_B], \mathrm{K}_1(B)).$$

Let $\alpha \in \mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B))^+$ implements this isomorphism between the K-invariants. Then for any finite subset $\mathcal{P} \subset P(A)$, there is a sequence of completely positive contractions $L_n : A \to B$ such that

- $\|L_n(ab) - L_n(a)L_n(b)\| \to 0$ *for all* $a, b \in A$ *as* $n \to \infty$, *and*
- $[L_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}$ *for sufficiently large* n.

PROOF. By Lemma 4.4 of [14], A is the closure of an increasing union of RFD sub-C*-algebras $\{A_n\}$. We may assume $\mathcal{P} \subset \mathbf{P}(A_1)$. Let $\mathcal{P}_0 \subset \mathcal{P}$ such that $\mathcal{P}_0$ generate $G(\mathcal{P}) \cap \mathrm{K}_0(A)$ where $G(\mathcal{P})$ is the group generated by $\mathcal{P}$. Write $\mathcal{P}_0 = \{p_1, \cdots, p_l\}$ where p_i's are projections in a matrix algebra over A. Let $\mathcal{F}_1$ be a finite subset of A_1 and let $\delta_0 > 0$ be such that any $\mathcal{F}_1 - \delta_0$ multiplicative linear map L well-define $[L]|_{\mathcal{P}}$. Moreover, we require $\mathcal{F}_1$ and δ_0 satisfy Lemma 2.27. Let k_0 be an integer such that $G(\mathcal{P}) \cap \mathrm{K}_i(A, \mathbb{Z}/k\mathbb{Z}) = \{0\}$ for any $k \geq k_0$, $i = 0, 1$.

By Theorem 2.29, there are two $\mathcal{F}_1 - \delta_0/2$ multiplicative completely positive linear maps Φ_0, Φ_1 from A to $B \otimes \mathcal{K}$ such that

$$[\Phi_0]|_{\mathcal{P}} = \alpha|_{\mathcal{P}} + [\Phi_1]|_{\mathcal{P}}.$$

Φ_1 is a homomorphism when it is restricted on A_1, and the image is a finite-dimensional C*-algebra. With Φ_1 in the role of h_0, we can proceed with the construction as described at the beginning of this section. We will keep the same notation.

Consider the map $\tilde{\rho} : G(\mathcal{P}) \cap \mathrm{K}_0(A) \to l^\infty(\mathbb{Q})$ defined in Remark 2.26. The linear span of $\{\tilde{\rho}(p_1), ..., \tilde{\rho}(p_l)\}$ over $\mathbb{Q}$ will have finite rank, say r. So, we may assume that $\{\tilde{\rho}(p_1), ..., \tilde{\rho}(p_r)\}$ are independent and the $\mathbb{Q}$-linear span of them give us the whole subspace. Therefore, there is an integer M such that for any $g \in \tilde{\rho}(G_0)$, Mg will be inside the subgroup generated by $\{\tilde{\rho}(p_1), ..., \tilde{\rho}(p_r)\}$. Let $x_{ij} = (\tilde{\rho}(p_i))_j$, and $z_i = \rho_A([p_i]) \in \mathbb{D}$, where $\mathbb{D} = \rho_A(\mathrm{K}_0(A))$ in $\mathrm{Aff}(S(K_0(A), 1))$. By Lemma 4.6 of [14], $\mathbb{D}$ is a dense subgroup of $\mathrm{Aff}(S(\mathrm{K}_0(A), 1))$. Note that $a_j^{(n)} \in \mathbb{D}^+ \backslash \{0\}$, $\lim_{n\to\infty} a_j^{(n)} = a_j > 0$ uniformly, and $\sum_{j=1}^n x_{ij} a_j^{(n)} \to z_i$ uniformly. So, Lemma 2.28 applies. Fix K and δ obtained from Lemma 2.28. Also note that if $g \in \ker \tilde{\rho}$, then $\tau(g) = 0$ for all traces τ, $g \in \ker[H]$ and $g \in \ker[\Phi_1]$. Hence g will be in the kernel of any direct sum of $[\Phi_1]$.

Let $\Psi := \Phi_0 \oplus (\underbrace{\Phi_1 \oplus \cdots \oplus \Phi_1}_{MK^3(k_0+1)!-1 \text{ copies}})$. Since Φ_1 factors through a finite-dimensional C*-algebra, it will be zero when restricted to $\mathrm{K}_1(A) \cap G$, $\mathrm{K}_1(A_1, \mathbb{Z}/k\mathbb{Z}) \cap G$. Moreover, $(\underbrace{\Phi_1 \oplus \cdots \oplus \Phi_1}_{MK^3(k_0+1)! \text{ copies}})$ induces the zero map on $\mathrm{K}_0(A, \mathbb{Z}/k\mathbb{Z})$. Therefore we have

$$[\Psi]|_{K_1(A)\cap G} = \alpha|_{K_1(A)\cap G}, \quad [\Psi]|_{K_1(A,\mathbb{Z}/k\mathbb{Z})\cap G} = \alpha|_{K_1(A,\mathbb{Z}/k\mathbb{Z})\cap G}$$

and $[\Psi]|_{K_0(A,\mathbb{Z}/k\mathbb{Z})\cap G} = \alpha|_{K_0(A,\mathbb{Z}/k\mathbb{Z})\cap G}$. We may assume $\Psi(1_A)$ is a projection in $\mathrm{M}_r(B)$ for some r. As we can see, the map Ψ induces the desired maps on all the invariants except the K_0 group. In the following part of the proof, we are going to compress the map Ψ to a small corner of $B \otimes \mathcal{K}$, and then find a suitable map h to fix Ψ on the K_0 part.

We may assume there exist projections $\{p_1', ..., p_l'\}$ inside $B \otimes \mathcal{K}$, which are sufficiently closed to $\{\Psi(p_1), ..., \Psi(p_l)\}$ respectively. So, we get $[p_i'] = [\Psi(p_i)]$. Note that B itself is a simple TAS-algebra, and therefore the strict order on the projections of B is determined by traces. Thus we can find a subprojection q_i' inside each p_i', such that $[q_i'] = MK^3(k_0+1)![\Phi_0(p_i)]$. Set $e_i' = p_i' - q_i'$, and let $\mathcal{P}_1 = \Psi(\mathcal{P}) \cup \Phi_1(\mathcal{P}) \cup \{p_i', q_i', e_i'; i = 1, ..., l\}$. Set G_1 to be the group generated by $\mathcal{P}_1$. Remember that $G_0 = G(\mathcal{P}) \cap \mathrm{K}_0(A)$. It can be written as $G_{00} \oplus G_{01}$, where G_{00} is the infinitesimal part of G_0. Let $\{d_1, ..., d_t\}$ be positive elements which generate G_{01}.

Now applying Theorem 2.30 to $\mathrm{M}_r(B)$ with any finite subset $\mathcal{G}$, any $\varepsilon > 0$ and any $0 < r_0 < \delta < 1$, we get a $\mathcal{G} - \varepsilon$-multiplicative map $L : \mathrm{M}_r(B) \to \mathrm{M}_r(B)$ with the following properties:

- $[L]|_{\mathcal{P}_1}$ and $[L]|_{G_1}$ are well defined;
- $[L]$ induces the identity maps on the infinitesimal part of $G_1 \cap \mathrm{K}_0(B)$, $G_1 \cap \mathrm{K}_1(B)$, $G_1 \cap \mathrm{K}_0(B, \mathbb{Z}/k\mathbb{Z})$ and $G_1 \cap \mathrm{K}_1(B, \mathbb{Z}/k\mathbb{Z})$ for the k with $G_1 \cap \mathrm{K}_i(B, \mathbb{Z}/k\mathbb{Z}) \neq \{0\}$, $i = 0, 1$;
- $\tau \circ [L](g) \leq r_0 \tau(g)$ for all $g \in G_1 \cap \mathrm{K}_0(B)$ and $\tau \in T(B)$;
- There exist positive elements $\{f_i\} \subset \mathrm{K}_0(B)^+$ such that for $i = 1, ..., t$,

$$\alpha(d_i) - [L](\alpha(d_i)) = MK^3(k_0+1)! f_i.$$

We can choose r_0 is sufficient small such that $\tau \circ [L] \circ [\Psi]([p_i]) < \delta/2$ for all $\tau \in T(B)$, and $\alpha([p_i]) - [L \circ \Psi]([p_i]) > 0$. This can be done since $T(B)$ is compact and B has the fundamental comparison property of Blackadar.

Let $[p_i] = \sum m_j^{(i)} d_j + s_i$ where $m_j^{(i)} \in \mathbb{Z}$ and $s \in G_{00}$. Then we have

$$\begin{aligned}
&\alpha([p_i]) - [L \circ \Psi]([p_i]) \\
= \;& \alpha(\textstyle\sum m_j^{(i)} d_j) - ([L \circ \alpha](\sum m_j^{(i)} d_j) + MK^3(k_0+1)!\,[L \circ \Phi_1]([p_i])) \\
= \;& (\alpha(\textstyle\sum m_j^{(i)} d_j) - [L \circ \alpha](\sum m_j^{(i)} d_j)) - MK^3(k_0+1)!\,[L \circ \Phi_1]([p_i]) \\
= \;& MK^3(k_0+1)!(\textstyle\sum m_j^{(i)} f_j - [L] \circ [\Phi_1]([p_i])) \\
= \;& MK^3(k_0+1)! f_j', \quad \text{if we write} \quad f_j' = \textstyle\sum m_j^{(i)} f_j - [L] \circ [\Phi_1]([p_i]).
\end{aligned}$$

Since $\alpha([p_i]) - [L \circ \Psi])([p_i]) > 0$ and $\mathrm{K}_0(B)$ is weakly unperforated, $f_j' > 0$. Let us set $\beta : G(\mathcal{P}) \cap \mathrm{K}_0(A) \to \mathrm{K}_0(B)$ by $\beta([p_i]) = K^3(k_0+1)! f_i'$.

Now we are ready to find a map h' from A to B which will carry the map β. It will be constructed by factoring through some splitting interval algebras in the construction given at the beginning of this section. Let $\tilde{z_i'} = \beta([p_i])$, and $z_i' = \rho_B(\tilde{z_i'}) \in \mathrm{Aff}(\mathrm{K}_0(B), 1)$. Recall that we identify the $\mathrm{K}_0(A)$ and $\mathrm{K}_0(B)$ and α is the identity. Then we have:

$$\begin{aligned}
\|Mz' - z\|_\infty &= \max_i\{\|\rho(\alpha([p_i]) - [L \circ \Psi]([p_i])) - \rho([p_i])\|\} \\
&= \max_i\{\sup_{\tau \in T(B)}\{\tau \circ [L] \circ [\Psi]([p_i])\}\} \\
&\leq \delta/2.
\end{aligned}$$

By Lemma 2.28, we get $\tilde{u} = \{\{u_1^{(1)}, ..., u_{r_1}^{(1)}\}, ..., \{\{u_1^{(s(n))}, ..., u_{r_{s(n)}}^{(s(n))}\}\}$ where $u_i^{(j)}$'s are positive elements of $\mathrm{K}_0(B)$ such that $\sum x_{ij}u_j = \tilde{z}_i$. Moreover, $\{u_k^{(k)}, ..., u_{r_k}^{(k)}\}$ will satisfy the same relations as the positive generators of $\mathrm{K}_0(S_k)$. Therefore, if we let $D = S_1 \oplus \cdots \oplus S_{s(n)}$, there will be an ordered homomorphism from $\mathrm{K}_0(D)$ to $\mathrm{K}_0(B)$ which sends the positive generator of $\mathrm{K}_0(S_k)$ to the correspondence $u_j^{(k)}$. Since B is a inductive limit of point-line algebras together with homogeneous C*-algebras, by Proposition 3.8 of [14], this K-theory map can be lifted to a $*$-homomorphism $h' : D \to \mathrm{M}_k(B)$ (note that the K-theory map from splitting interval algebras to homogeneous C*-algebras always factor through the group of integers). So, we have

$$[h'](\tilde{\rho}([p_i])) = \beta([p_i]), \quad i = 1, ..., r,$$

if we look at $\tilde{\rho}([p_i])$ being truncated into D. Now, we consider h' as a map from A to $\mathrm{M}_k(B)$ by composing it with the map $\psi_1 \oplus \cdots \oplus \psi_{s(n)} : A \to D$. It is $\mathcal{F} - \delta$ multiplicative.

For any $x \in \ker \tilde{\rho}$, by Lemma 2.27, $x \in \ker \tau \circ \alpha \cap \ker[H]$ and $x \in \ker[h_0] = \ker[\Phi_1]$. Therefore, we have $[\Phi_1](x) = 0$ and $[\Psi](x) = \alpha(x)$. Note that $\alpha(x)$ also vanishes under any state of $(\mathrm{K}_0(B), \mathrm{K}_0^+(B))$, we have $[L] \circ \alpha(x) = \alpha(x)$. So, we get

$$\alpha(x) - [L \circ \Psi](x) = 0.$$

Therefore $\alpha - [L \circ \Psi]$ gives us a homomorphism on $\tilde{\rho}(G_0)$.

Set h to be M copies of h'. h will also be $\mathcal{F} - \delta$ multiplicative, and

$$[h]([p_i]) = \alpha([p_i]) - [L] \circ [\Psi]([p_i]) \quad i = 1, ..., l.$$

Note that $[h]$ also has the multiplicity $MK^3(k_0+1)!$, and D is a splitting interval algebra (which has trivial K_1 groups). One concludes that h induces zero map on $G \cap \mathrm{K}_1(A)$, $G \cap \mathrm{K}_1(A, \mathbb{Z}/k\mathbb{Z})$ and $G \cap \mathrm{K}_0(A, \mathbb{Z}/k\mathbb{Z})$ for $k \leq k_0$. Therefore, we have

$$[h]|_{\mathcal{P}} = \alpha|_{\mathcal{P}} - [L] \circ [\Psi]|_{\mathcal{P}}.$$

Set $L_1 = L \circ \Psi \oplus h$. It is $\mathcal{F} - \delta$ multiplicative and

$$[L_1]|_{\mathcal{P}} = [h]|_{\mathcal{P}} + [L] \circ [\Psi]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}.$$

We may assume $L_1(1_A) = 1_B$ by taking a conjugation with a partial isometry. Then L_1 is a map from A to B, and gives us the desired K-theory map. □

2.4. Existence theorem: between TASI-algebras In this section, we shall only consider TASI-algebras. We have introduced KK-attainability in the last section, and shown that certain C*-algebras (arising concretely or abstractly) are KK-attainable. However, KK-attainability only concerns the lifting of K-theory maps. In order to lift the Elliott invariant, one also should consider the maps between the simplexes of tracial states. In the following, let us first show that for KK-attainable TASI-algebras, one can also obtain an approximate lifting for the map on traces, and thus the maps between the Elliott invariant can be lifted. Moreover, we shall also show that actually any TASI-algebra is KK-attainable, and hence one can lift maps between the Elliott invariants for arbitrary TASI-algebras.

Let us consider the lifting of the maps on traces.

PROPOSITION 2.32. *Let A be a KK-attainable TASI-algebra, and let B be a simple TASI-algebra. Then for any $\alpha \in \mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B))^+$, and any $\Gamma : T(B) \to T(A)$ which is compatible with α, there exist a sequence of completely positive linear contractions $\{L_n\}$ which satisfy the conditions in Definition* 2.14*, and they also almost induce the trace map Γ, i.e.,*

$$\sup_{\tau \in T(B)} \{|\Gamma(\tau)(a) - \tau \circ L_n(a)|\} \to 0 \quad \forall a \in A.$$

PROOF. Let $\mathcal{F}$ be any finite subset of A, $\mathcal{P}$ be any finite subset of $P(A)$ contains $[1_A]$ and any $\varepsilon > 0$. We may assume that $\alpha([1_A]) = [1_A]$ and $\mathcal{F}$ is large enough so that it contains all the entries of the projections in $\mathcal{P}$. Since A is a TASI-algebra, for any finite subset $\mathcal{F}' \subset A$ and $\delta > 0$, there is a sub-C*-algebra S which is a splitting interval algebra with $p = 1_S$ such that $\phi : a \mapsto pap$ and $\psi : a \mapsto (1-p)a(1-p)$ are $\mathcal{F}' - \delta$ multiplicative and $\tau(1_A - p) \leq \delta$. Therefore, with sufficiently large $\mathcal{F}'$ and sufficiently small δ, we may assume that $[\phi]|_\mathcal{P} + [\psi]_\mathcal{P} = \mathrm{id}|_\mathcal{P}$ and $\phi(a) \in S$ for any $a \in \mathcal{F}$.

Let $\mathcal{S}$ be a set of the generators of S and set $\mathcal{G} = \mathcal{F} \cup \mathcal{S}$. Since A is KK-attainable, there is a sequence of completely positive linear contractions $L_n : A \to B$ such that for any $a, b \in A$,

$$||L_n(ab) - L_n(a)L_n(b)|| \to 0,$$

and for sufficiently large n,

$$[L_n]|_\mathcal{P} = \alpha|_\mathcal{P}.$$

Therefore, for any $\varepsilon_0 > 0$, there is a $\mathcal{G} - \varepsilon_0$ multiplicative linear contraction L_n such that

$$[L_n \circ \phi]|_\mathcal{P} + [L_n \circ \psi]_\mathcal{P} = [L_n]|_\mathcal{P} = \alpha|_\mathcal{P}.$$

Denote by L'_n the restriction of $L_n \circ \phi$ to S. Since the generators of S satisfy stable relations, we may assume L'_n is actuary a *-homomorphism by choosing n sufficiently large. Set $\mathcal{G}'$ to be $L_n(\mathcal{G})$. Since B is also a TASI-algebra, by the same argument as for A, there exists a sub-C*-algebra S' of B which is a splitting interval algebra with identity q, such that $\tau(1-q) \leq \varepsilon$ for any trace $\tau \in B$ and

$$[qL'_nq]|_\mathcal{P} + [(1-q)L'_n(1-q)]|_\mathcal{P} = [L'_n]|_\mathcal{P}.$$

Moreover, we may assume qL'_nq is a *-homomorphism from S to S' by choosing a sufficiently large n and a small perturbation. Set κ to be the K_0-map induced by qL'_nq. Furthermore, there also exist a trace map θ' from $T(S')$ to $T(B)$ such that $|\theta'(\tau)(a) - \tau(a)| \leq \varepsilon$ for any $a \in L'_n(\mathcal{F})$. Then the map $\theta = \frac{1}{1-\varepsilon}\Gamma \circ \theta'$ is an affine map from $T(S')$ to $T(S)$, and one also has

$$\theta'(\tau)(\alpha(p)) = \theta'(\tau)([L'_n](p) + [(1-q)L_n(p)(1-q)]),$$

for any $\tau \in T(S')$ and any projection $p \in S$. On the other hand, since Γ and α are compatible, we have

$$\begin{aligned} \Gamma \circ \theta'(\tau)(p) &= \theta'(\tau)(\alpha(p)) \\ &= \theta'(\tau)([L'_n](p) + [(1-q)L_n(p)(1-q)]), \end{aligned}$$

for any $\tau \in T(S')$ and any projection $p \in S$, and hence

$$\begin{aligned} |\tau(\kappa(p)) - \theta(\tau)(p)| &\leq |\theta'(\tau)(L'_n(p)) - \tfrac{1}{1-\varepsilon}\Gamma \circ \theta'(\tau(p))| + \varepsilon \\ &\leq |\Gamma \circ \theta'(\tau)(p) - \tfrac{1}{1-\varepsilon}\Gamma \circ \theta'(\tau(p))| + 2\varepsilon \\ &\leq 3\varepsilon. \end{aligned}$$

for any K_0-class of the projections in S and any trace τ over S'. Then by the existence theorem in [8], there is a *-homomorphism $h : S \to S'$ such that $h_* = \kappa$ and $|\tau \circ h(a) - \theta(\tau)(a)| \leq \varepsilon$ for any $a \in p\mathcal{F}p$ and $\tau \in T(A)$. For the convenience, let us assume that the tolerant is still ε. Set $L = L_n \circ \psi + (1-q)L'(1-q) + h \circ \phi$. It satisfies

$$[L]|_{\mathcal{P}} = \alpha|_{\mathcal{P}},$$

and

$$|\tau \circ L(a) - \Gamma(\tau)(a)| < 6\varepsilon,$$

for any $a \in \mathcal{F}$ and any trace τ on B, as desired. □

Let us consider the KK-attainability for an arbitrary TASI-algebra. By Theorem 2.31, it is shown that for any simple separable nuclear TASI-algebra, there exists an inductive limit algebra which shares the same K-theory, and moreover, this TASI-algebra is KK-attainable for this concrete algebra. Note that in Section 2.2, all the building blocks of this concrete algebras are shown to KK-attainable for TASI-algebras, and hence this concrete algebra is KK-attainable for TASI-algebras. Thus, one can use this concrete algebra as a bridge to show that any TASI-algebra is KK-attainable for TASI-algebras. More precisely, one has the following

PROPOSITION 2.33. *Any simple separable nuclear TASI-algebra satisfying the UCT is KK-attainable.*

PROOF. Let A be a simple nuclear TASI-algebra satisfying UCT. By Lemma 2.30, there is a C*-algebra C, which is a simple inductive limit of point-line algebra, together with the Gong's standard homogeneous C*-algebras and homogeneous C*-algebras with base space S^2 such that

$$((\mathrm{K}_0(A), \mathrm{K}_0(A)^+, [1_A]), \mathrm{K}_1(A)) \cong ((\mathrm{K}_0(C), \mathrm{K}_0(C)^+, [1_C]), \mathrm{K}_1(C)).$$

Since A satisfies the UCT, there is $\beta \in \mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(C))^+$ which induces the isomorphism between the invariants of A and C. Moreover, we can choose β to be invertible. Note that all building blocks of C are KK-attainable by results in Section 2.2, and hence the C*-algebra C is KK-attainable by Remark 2.22.

For any simple TASI-algebra B, any $\alpha \in \mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B))^+$, and any finite subset $\mathcal{P} \subset P(A)$, consider the KL-element $\alpha\beta^{-1} \in \mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(C), \underline{\mathrm{K}}(B))^+$, and the subset $\mathcal{P}' \subset P(C)$ which presents $\alpha(\mathcal{P})$. Since C is KK-attainable, there exist a sequence of completely positive linear contractions (L_n) from C to B such that (L_n) are approximately multiplicative, and $[L_n]|_{\mathcal{P}'} = \alpha\beta^{-1}|_{\mathcal{P}'}$ for sufficiently large n.

By Theorem 2.31, there is a sequence of approximately multiplicative completely positive linear contractions (L'_n) from A to C such that L'_n induces β on $\mathcal{P}$ for sufficiently large n. Then the compositions $L_n \circ L'_m : A \to B$ with suitable choice of m, n will give us a sequence of completely positive linear contractions which are approximately multiplicative, and $[L_n \circ L'_m]|_{\mathcal{P}} = (\alpha \circ \beta) \circ \beta^{-1}|_{\mathcal{P}} = \alpha|_{\mathcal{P}}$ for sufficiently large m, n. Therefore, A is KK-attainable, as desired. ☐

With the propositions above, one has the main theorem of this section.

THEOREM 2.34. *Let A and B be two simple TASI-algebras satisfying UCT. Then for any $\alpha \in \mathrm{Hom}_\Lambda(\underline{\mathrm{K}}(A), \underline{\mathrm{K}}(B))^+$ with $\alpha([1_A]) = [1_B]$, any finite subset $\mathcal{P} \in P(A)$, and any trace map $\theta : T(B) \to T(A)$ which is compatible with α, there is a sequence of completely positive linear contractions $L_n : A \to B$ such that*

- $\|L_n(ab) - L_n(a)L_n(b)\| \to 0$ *for any $a, b \in A$;*
- $[L_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}$ *for sufficient large n;*
- $|\theta(\tau)(a) - \tau(L_n(a))| \to 0$ *for any $a \in A$ and $\tau \in \mathrm{T}(B)$.*

PROOF. The theorem follows directly from Proposition 2.33 and Proposition 2.32. ☐

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