THE C*-ALGEBRA OF A MINIMAL HOMEOMORPHISM OF ZERO MEAN DIMENSION

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Abstract. Let X be an infinite metrizable compact space, and let $\sigma : X \to X$ be a minimal homeomorphism. Suppose that (X, σ) has zero mean topological dimension. The associated C*-algebra $A = \mathrm{C}(X) \rtimes_\sigma \mathbb{Z}$ is shown to absorb the Jiang-Su algebra $\mathcal{Z}$ tensorially, i.e., $A \cong A \otimes \mathcal{Z}$. This implies that A is classifiable when (X, σ) is uniquely ergodic.

Moreover, without any assumption on the mean dimension, it is shown that $A \otimes A$ always absorbs the Jiang-Su algebra.

1. Introduction

Recently, Toms and Winter proved that a simple C*-algebra arising from an action of the group $\mathbb{Z}$ of integers on a metrizable compact space of finite dimension absorbs the Jiang-Su algebra $\mathcal{Z}$ ([28], [29]). (This definitive result followed a considerable amount of earlier work, e.g., [15].) As shown in [11], some condition on the dynamical system is necessary. Possibly, the condition of mean dimension zero, which we shall show is sufficient, is also necessary. (Phillips and Toms have conjectured that the mean dimension of a minimal dynamical system is always exactly twice the radius of comparison of the associated (crossed product) C*-algebra.)

In the present note, we shall show that the condition of mean dimension zero (reviewed in Definition 2.1 below) is sufficient: it implies that the C*-algebra of the dynamical system absorbs the Jiang-Su C*-algebra $\mathcal{Z}$. More precisely, one has

Theorem. *Let X be an infinite metrizable compact space, and let $\sigma : X \to X$ be a minimal homeomorphism. If (X, σ) has mean dimension zero, then the crossed product C*-algebra $A = \mathrm{C}(X) \rtimes_\sigma \mathbb{Z}$ absorbs the Jiang-Su algebra $\mathcal{Z}$ tensorially.*

The same classification consequences as shown in [28] and [29] (for X finite-dimensional) in the case that K_0 separates traces hold also in the present setting; in particular, the C*-algebra of any uniquely ergodic dynamical system is classifiable (in this case the mean dimension is automatically zero).

On taking into account the most recent results in the classification of C*-algebras (see [9], [12], [14], [7], [8] and [6]), this theorem implies that the C*-algebra of a minimal dynamical system with mean dimension zero is always classifiable. In particular, this includes all the dynamical systems with finite entropy or the systems with countably many ergodic measures ([19], [18]).

The proof of the main theorem also shows that the tensor product of the C*-algebras of two arbitrary minimal homeomorphisms (without any assumption on the mean dimension) is, perhaps surprisingly, always Jiang-Su stable:

Theorem. *Let (X_1, σ_1) and (X_2, σ_2) be minimal dynamical systems, where X_1 and X_2 are infinite metrizable compact spaces. Consider the C*-algebras*

$$A_1 = \mathrm{C}(X_1) \rtimes_{\sigma_1} \mathbb{Z} \quad \textit{and} \quad A_2 = \mathrm{C}(X_2) \rtimes_{\sigma_2} \mathbb{Z}.$$

Then

$$A_1 \otimes A_2 \cong (A_1 \otimes A_2) \otimes \mathcal{Z}.$$

Again, with the recent classification results referred to above, this implies that $A_1 \otimes A_2$ is always a classifiable C*-algebra.

2. Mean topological dimension and the small boundary property

Let X be a metrizable compact space, and let $\sigma : X \to X$ be a homeomorphism. (These objects will be fixed throughout the paper.)

Definition 2.1 ([19])**.** The mean (topological) dimension of (X, σ), denoted by $\mathrm{mdim}(X, \sigma)$, is defined by

$$\mathrm{mdim}(X, \sigma) = \sup_{\alpha} \lim_{N\to\infty} \frac{1}{N}\mathcal{D}(\alpha \vee \sigma(\alpha) \vee \cdots \vee \sigma^{N-1}(\alpha)),$$

where the supremum is taken over arbitrary finite open covers α, and the dimension of a finite open cover β, $\mathcal{D}(\beta)$, is the number $\min\{\mathrm{ord}(\beta');\ \beta' \preceq \beta\}$. (By the order of a cover β is meant the number $\mathrm{ord}(\beta) = -1 + \sup_x \sum_{U\in\beta} \chi_U(x)$, and by the join $\beta_1 \vee \beta_2$ of two covers β_1 and β_2 is meant the cover $\{U_1 \cap U_2 : U_1 \in \beta_1, U_2 \in \beta_2\}$.)

Definition 2.2 ([19])**.** For each set $E \subseteq X$, the orbit capacity of E, denoted by $\mathrm{ocap}(E)$, is defined to be

$$\mathrm{ocap}(E) = \lim_{N\to\infty} \frac{1}{N} \sup\{\chi_E(x) + \chi_E(\sigma(x)) + \cdots + \chi_E(\sigma^{N-1}(x));\ x \in X\}.$$

The system (X, σ) is said to have the small boundary property (SBP) if for any $x \in X$ and any open neighbourhood U of x, there is a neighbourhood V in U such that $\mathrm{ocap}(\partial V) = 0$.

Theorem 2.3 (Theorem 5.4 of [19] and Theorem 6.2 of [18])**.** *If σ is minimal, then (X, σ) has zero mean dimension if and only if it has the small boundary property.*

Proposition 2.4 (Proposition 5.3 of [19])**.** *If (X, σ) has the SBP, then for every finite open cover α of X and every $\varepsilon > 0$, there is a partition of unity $\phi_j : X \to [0, 1]$ $(j = 1, ..., |\alpha|)$ subordinate to α such that*

$$\mathrm{ocap}(\bigcup_{j=1}^{|\alpha|} \phi_j^{-1}((0, 1))) < \varepsilon.$$

3. The C*-algebra of a homeomorphism and its large subalgebras

Suppose that X as above is an infinite set and σ as above is minimal (the action is free in this case). Let us denote by σ also the automorphism of $\mathrm{C}(X)$ defined by

$$\sigma(f) = f \circ \sigma^{-1}, \quad f \in \mathrm{C}(X).$$

Consider the crossed product C*-algebra

$$A = \mathrm{C}(X) \rtimes_\sigma \mathbb{Z} = \mathrm{C}^*\{f, u;\ ufu^* = \sigma(f),\ f \in \mathrm{C}(X)\}.$$

The C*-algebra A is nuclear (Corollary 7.18 of [30]). Since σ is minimal, the C*-algebra A is simple (Theorem 5.16 of [5] and Théorème 5.15 of [32]). Fix $y \in X$, and consider the Putnam algebra ([25])

$$A_y = \mathrm{C}^*\{f, ug;\ f, g \in \mathrm{C}(X),\ g(y) = 0\} \subseteq A.$$

Let Y be a closed neighbourhood of y in X. Consider the sub-C*-algebra

$$A_Y = \mathrm{C}^*\{f, ug;\ f, g \in \mathrm{C}(X),\ g|_Y = 0\} \subseteq A_y.$$

It follows from the definition that $A_{Y_1} \subseteq A_{Y_2}$ if $Y_1 \supseteq Y_2$, and $A_y = \overline{\bigcup_{i=1}^{\infty} A_{Y_i}}$ if $\bigcap Y_i = \{y\}$.

Consider the first return times

$$\{j \in \mathbb{N} \cup \{0\};\ \sigma^j(x) \in Y,\, \sigma^i(x) \notin Y,\, 1 \leq i \leq j-1 \text{ for some } x \in Y\}.$$

Since σ is minimal, X is compact, and Y has a non-empty interior, this set of numbers is finite; let us write it as

$$J_1 < J_2 < \cdots < J_K$$

for some $K \in \mathbb{N}$. Since X is an infinite set and σ is minimal, the first return time J_1 is arbitrarily large if Y is sufficiently small.

For each $1 \leq k \leq K$, consider the (locally compact—see below) subset of X

$$Z_k = \{x \in Y;\ \sigma^{J_k}(x) \in Y \text{ but } \sigma^i(x) \notin Y \text{ for any } 1 \leq i \leq J_k - 1\}.$$

Then the sets

$$Z_1, \sigma(Z_1), ..., \sigma^{J_1-1}(Z_1), ..., Z_k, \sigma(Z_k), ..., \sigma^{J_k-1}(Z_k)$$

—which are naturally listed as shown—form a partition of X. This is often called a Rokhlin partition.

Lemma 3.1 ([16]; see also [17] or Lemma 2.15 of [24]). *In terms of the notation introduced above, one has that, for each $1 \leq k \leq K$,*

(1) *the set $Z_1 \cup \cdots \cup Z_k$ is closed (and so Z_k is locally compact),*

(2) *the set $\overline{Z_k} \cap (Z_1 \cup \cdots \cup Z_{k-1})$ is the disjoint union of the subsets*

$$W_{t_1,...,t_s} = \partial Z_k \cap Z_{t_1} \cap \sigma^{-J_{t_1}}(Z_{t_2}) \cap \cdots \cap \sigma^{-(J_{t_1}+\cdots+J_{t_{s-1}})}(Z_{t_s}),$$

where $J_{t_1} + J_{t_2} + \cdots + J_{t_s} = J_k$ for some $1 \leq t_1, t_2, ..., t_s \leq k$.

A quite explicit description of the subalgebra A_Y of the crossed product, a C*-algebra of type I, was obtained by Q. Lin ([16]). It is a subhomogeneous algebra (as defined at the beginning of Section 4), of order at most J_K. (In other words, all its irreducible representations have at most this dimension.)

Theorem 3.2 ([16]; see also [17] or Theorem 2.22 of [24]). *In terms of the notation introduced above, one has that the C*-algebra A_Y is isomorphic to the sub-C*-algebra of $\bigoplus_{k=1}^{K} \mathrm{M}_{J_k}(\mathrm{C}(\overline{Z_k}))$ consisting of the elements $(F_1, ..., F_K)$ with*

$$F_k|_{W_{t_1,...,t_s}} = \begin{pmatrix} F_{t_1}|_{W_{t_1,...,t_s}} & & & \\ & F_{t_2} \circ \sigma^{J_{t_1}}|_{W_{t_1,...,t_s}} & & \\ & & \ddots & \\ & & & F_{t_s} \circ \sigma^{J_{t_{s-1}}}|_{W_{t_1,...,t_s}} \end{pmatrix}$$

whenever

$$W_{t_1,...,t_s} = \partial Z_k \cap Z_{t_1} \cap \sigma^{-J_{t_1}}(Z_{t_2}) \cap \cdots \cap \sigma^{-(J_{t_1}+\cdots+J_{t_{s-1}})}(Z_{t_s}) \neq \emptyset,$$

where $J_{t_1} + J_{t_2} + \cdots + J_{t_s} = J_k$.

Moreover, for any $f, g \in \mathrm{C}(X)$ *with* $g|_Y = 0$*, the images of* $f, ug \in A_Y$ *in this identification are*

$$f = \bigoplus_{k=1}^{K} \begin{pmatrix} f\circ\sigma|_{\overline{Z_k}} & & & \\ & f\circ\sigma^2|_{\overline{Z_k}} & & \\ & & \ddots & \\ & & & f\circ\sigma^{J_k}|_{\overline{Z_k}} \end{pmatrix} \in \bigoplus_{k=1}^{K} \mathrm{M}_{J_k}(\mathrm{C}(\overline{Z_k})) \tag{3.1}$$

and

$$ug = \bigoplus_{k=1}^{K} \begin{pmatrix} 0 & & & \\ g\circ\sigma|_{\overline{Z_k}} & 0 & & \\ & \ddots & \ddots & \\ & & g\circ\sigma^{J_k-1}|_{\overline{Z_k}} & 0 \end{pmatrix} \in \bigoplus_{k=1}^{K} \mathrm{M}_{J_k}(\mathrm{C}(\overline{Z_k})), \tag{3.2}$$

respectively.

The sub-C*-algebra A_y of A (with $y \in X$) is a typical example of a large subalgebra:

Definition 3.3 ([23], [2])**.** Let A be a unital simple separable C*-algebra. A unital sub-C*-algebra $B \subseteq A$ is said to be large in A if for every $m \in \mathbb{Z}_{>0}$, $a_1, a_2, ..., a_m \in A$, $\varepsilon > 0$, $x \in A^+$ with $\|x\| = 1$, and $y \in B^+ \setminus \{0\}$, there are $c_1, c_2, ..., c_m \in A$ and $g \in B$ such that

(1) $0 \leq g \leq 1$,
(2) For $j = 1, 2, ..., m$ we have $\|c_j - a_j\| < \varepsilon$,
(3) For $j = 1, 2, ..., m$ we have $(1-g)c_j, c_j(1-g) \in B$,
(4) $g \precsim_B y$, and
(5) $\|(1-g)x(1-g)\| > 1 - \varepsilon$.

If, moreover, the element g can be chosen such that

(6) $\|ga_j - a_jg\| < \varepsilon$, $j = 1, 2, ..., m$,

then the sub-C*-algebra B is said to be centrally large in A.

Theorem 3.4 (Theorem 4.6 of [2])**.** *The C*-algebra* A_y *is centrally large in* A*.*

Theorem 3.5 (Proposition 3.7 of [2])**.** *Let* A_1 *and* A_2 *be simple unital C*-algebras, and let* $B_1 \subset A_1$ *and* $B_2 \subset A_2$ *be centrally large subalgebras. Assume that* $A_1 \otimes_{\min} A_2$ *is finite. Then* $B_1 \otimes_{\min} B_2$ *is a centrally large subalgebra of* $A_1 \otimes_{\min} A_2$*.*

We will use the following property of centrally large sub-C*-algebras.

Theorem 3.6 (Theorem 2.3 of [1])**.** *Let* A *be a simple separable nuclear unital C*-algebra, and let* B *be a centrally large subalgebra of* A*. If* $\mathcal{Z} \otimes B \cong B$*, then* $\mathcal{Z} \otimes A \cong A$*.*

4. The C*-algebra of a minimal homeomorphism of mean dimension zero

Recall that a C*-algebra is said to be subhomogeneous if the dimensions of its irreducible representations are finite and uniformly bounded. Let S be a subhomogeneous C*-algebra, with dimensions of irreducible representations $d_1 < d_2 < \cdots < d_n$. The dimension ratio of S is defined as

$$\mathrm{dimRatio}(S) := \max\{\frac{\dim(\mathrm{Prim}_{d_1}(S))}{d_1}, \frac{\dim(\mathrm{Prim}_{d_2}(S))}{d_2}, ..., \frac{\dim(\mathrm{Prim}_{d_n}(S))}{d_n}\},$$

where $\mathrm{Prim}_{d_i}(S)$ denotes the space of the primitive ideals of A corresponding to the irreducible representations with dimension d_i, with the (relative) Jacobson topology ($\mathrm{Prim}_{d_i}(S)$ is then locally compact and Hausdorff; see Proposition 3.6.4(i) of [4]), and $\dim(\cdot)$ denotes the topological covering dimension.

By Proposition 2.13 (together with 2.5 and 2.9) of [22], if S is separable and the primitive ideal spaces of the maximal homogeneous subalgebras of S, $\mathrm{Prim}_{d_i}(S)$, have finite dimension, then the C*-algebra S has a recursive subhomogeneous decomposition,

$$S \cong \left[\cdots\left[\left[C_0 \oplus_{C_1^{(0)}} C_1\right] \oplus_{C_2^{(0)}} C_2\right]\cdots\right] \oplus_{C_l^{(0)}} C_l,$$

with $C_k = \mathrm{C}(X_k, \mathrm{M}_{n(k)})$ for a compact Hausdorff space X_k and a positive integer $n(k)$, and with $C_k^{(0)} = \mathrm{C}(X_k^{(0)}, \mathrm{M}_{n(k)})$ for a compact subset $X_k^{(0)} \subseteq X_k$ (possibly empty), and then

$$\max\{\frac{\dim(X_k)}{n(k)} : 0 \leq k \leq l\} = \mathrm{dimRatio}(S).$$

(See [22] for more details concerning recursive subhomogeneous C*-algebras.)

In the present section, it will be shown (using Theorem 3.2 indirectly) that if (X, σ) has zero mean dimension (and, as is understood, σ is minimal), then the large subalgebra A_y can be locally approximated by subhomogeneous C*-algebras with arbitrarily small dimension ratio (see Theorem 4.5).

As a consequence of this, it follows (on applying the large subalgebra technique—see [23]) that the crossed product C*-algebra $\mathrm{C}(X) \rtimes_\sigma \mathbb{Z}$ absorbs the Jiang-Su algebra $\mathcal{Z}$, the main result of this paper (Theorem 4.7).

Of the following three lemmas (Lemmas 4.1, 4.2, and 4.3), only the first concerns dynamical systems; the other two are elementary C*-algebra results.

Lemma 4.1. *Let $Y \subseteq X$ be a closed subset with nonempty interior. Denote by $Z_1, Z_2, ..., Z_K$ the bases of the Rokhlin towers generated by Y, and by $J_1 < J_2 < \cdots < J_K$ the first return times of $Z_1, Z_2, ..., Z_K$, respectively. There is an open set $U \supseteq Y$ such that for each $1 \leq k \leq K$, one has*

$$\frac{1}{J_k}(\chi_U(x) + \chi_U(\sigma(x)) + \cdots + \chi_U(\sigma^{J_k-1}(x))) \leq \frac{1}{J_1}, \quad x \in Z_k.$$

Proof. Note that by definition the inequality holds with Y in place of U. (So, the question is to extend this in some sense by continuity to a neighbourhood—we propose to do this by induction on k.)

Since Y is closed, and the sets

$$Y, \sigma(Y), ..., \sigma^{J_1-1}(Y)$$

are pairwise disjoint, there is an open set $U \supseteq Y$ such that

$$U, \sigma(U), ..., \sigma^{J_1-1}(U)$$

are pairwise disjoint. In particular,

$$\frac{1}{J_1}(\chi_U(x) + \chi_U(\sigma(x)) + \cdots + \chi_U(\sigma^{J_1-1}(x))) \leq \frac{1}{J_1}, \quad x \in Z_1.$$

Let $2 \leq k \leq K$, and assume that we have constructed an open set $U \supseteq Y$ such that for any $1 \leq i \leq k-1$,

$$\frac{1}{J_i}(\chi_U(x) + \chi_U(\sigma(x)) + \cdots + \chi_U(\sigma^{J_i-1}(x))) \leq \frac{1}{J_1}, \quad x \in Z_i. \tag{4.1}$$

Let us construct another open neighbourhood of Y, still to be denoted by U (just shrink!), such that (4.1) holds for $i = k$.

First, pick an open neighbourhood U' of Y such that $\overline{U'} \subseteq U$. Let $x \in \overline{Z_k} \cap (Z_1 \cup \cdots \cup Z_{k-1})$. If

$$x \in W_{t_1,...,t_s} = \overline{Z_k} \cap Z_{t_1} \cap \sigma^{-J_{t_1}}(Z_{t_2}) \cap \cdots \cap \sigma^{-(J_{t_1}+\cdots+J_{t_{s-1}})}(Z_{t_s}),$$

where $J_{t_1} + J_{t_2} + \cdots + J_{t_s} = J_k$, then the orbit of x is

$$\underbrace{x, \sigma(x), ..., \sigma^{J_{t_1}-1}(x)}_{\text{in tower } Z_{t_1}}, \underbrace{\sigma^{J_{t_1}}(x), ..., \sigma^{J_{t_2}}(\sigma^{J_{t_1}}(x))}_{\text{in tower } Z_{t_2}}, ..., \underbrace{\sigma^{J_{t_1}+\cdots+J_{t_{s-1}}}(x), ..., \sigma^{J_{t_s}}(\sigma^{J_{t_1}+\cdots+J_{t_{s-1}}}(x))}_{\text{in tower } Z_{t_s}}.$$

By the induction hypothesis (4.1), one has (recall that $J_{t_1} + J_{t_2} + \cdots + J_{t_s} = J_k$)

$$\frac{1}{J_k}(\chi_U(x) + \chi_U(\sigma(x)) + \cdots + \chi_U(\sigma^{J_k-1}(x))) \leq \frac{1}{J_1},$$

and, therefore, there is a neighbourhood V_x of x such that

$$\frac{1}{J_k}(\chi_{U'}(z) + \chi_{U'}(\sigma(z)) + \cdots + \chi_{U'}(\sigma^{J_k-1}(z))) \leq \frac{1}{J_1}, \quad z \in V_x. \tag{4.2}$$

Hence, there is an open set E such that

$$\overline{Z_k} \cap (Z_1 \cup \cdots \cup Z_{k-1}) \subseteq E$$

and

$$\frac{1}{J_k}(\chi_{U'}(z) + \chi_{U'}(\sigma(z)) + \cdots + \chi_{U'}(\sigma^{J_k-1}(z))) \leq \frac{1}{J_1}, \quad z \in E. \tag{4.3}$$

Replace U by U' and still denote it by U. Since $Z_1 \cup \cdots \cup Z_{k-1} \cup Z_k$ is a closed set, one has that

$$\overline{Z_k} \setminus Z_k \subseteq \overline{Z_k} \cap (Z_1 \cup \cdots \cup Z_{k-1}),$$

and hence $\overline{Z_k} \setminus Z_k \subseteq E$. In particular,

$$\overline{Z_k} \setminus E = Z_k \setminus E,$$

and $Z_k \setminus E$ is a compact set.

For any point x in $Z_k \setminus E$, one can shrink U further so that

$$\frac{1}{J_k}(\chi_{\overline{U}}(x) + \chi_{\overline{U}}(\sigma(x)) + \cdots + \chi_{\overline{U}}(\sigma^{J_k-1}(x))) \leq \frac{1}{J_1}. \tag{4.4}$$

Since $\overline{U}$ is closed, (4.4) holds for a neighbourhood of x. Since $Z_k \setminus E$ is compact, there is an open neighbourhood U of Y such that

$$\frac{1}{J_k}(\chi_U(x) + \chi_U(\sigma(x)) + \cdots + \chi_U(\sigma^{J_k-1}(x))) \leq \frac{1}{J_1}, \quad x \in Z_k \setminus E. \tag{4.5}$$

Combining this with (4.3), one has

$$\frac{1}{J_k}(\chi_U(x) + \chi_U(\sigma(x)) + \cdots + \chi_U(\sigma^{J_k-1}(x))) \leq \frac{1}{J_1}, \quad x \in Z_k, \tag{4.6}$$

as desired. □

Lemma 4.2. *Consider the complex $n \times n$ matrices*

$$A := \mathrm{diag}(a_1, ..., a_n), \quad B := \mathrm{diag}(b_1, ..., b_n),$$

$$C := \begin{pmatrix} 0 & & & \\ c_1 & 0 & & \\ & \ddots & \ddots & \\ & & c_{n-1} & 0 \end{pmatrix} \quad \textit{and} \quad D := \begin{pmatrix} 0 & & & \\ d_1 & 0 & & \\ & \ddots & \ddots & \\ & & d_{n-1} & 0 \end{pmatrix},$$

where $0 < c_i, d_i \leq 1$. If the pair (A, C) is unitarily equivalent to the pair (B, D), then

$$a_i = b_i, \ c_j = d_j, \quad 1 \leq i \leq n, \ 1 \leq j \leq n-1.$$

Proof. Let $W \in \mathrm{M}_n(\mathbb{C})$ be a unitary such that

$$W^*AW = B \quad \text{and} \quad W^*CW = D.$$

For each $1 \leq k \leq n$, one has $W^*((C^*)^k C^k)W = (D^*)^k D^k$, and a functional calculus argument shows that

$$W^*(e_1 + \cdots + e_k)W = e_1 + \cdots + e_k, \quad 1 \leq k \leq n,$$

where e_i is the ith standard rank-one projection. This implies that

$$W^* e_i W = e_i, \quad 1 \leq i \leq n.$$

Since $W^*AW = B$, it follows that

$$W^* e_i A e_i W = e_i B e_i, \quad 1 \leq i \leq n,$$

and hence

$$a_i = b_i, \quad 1 \leq i \leq n.$$

A similar argument shows that $c_i = d_i$, $1 \leq i \leq n$. □

Consider the C*-algebra $\mathrm{M}_n(\mathrm{C}_0(Z))$, where Z is a locally compact Hausdorff space, and consider $x \in Z$. Let us use π_x to denote the evaluation map at x, i.e., $\pi_x(f) = f(x)$ for any $f \in \mathrm{M}_n(\mathrm{C}_0(Z))$.

Lemma 4.3. *Let Z be a second countable locally compact Hausdorff space, and let S be a sub-C*-algebra of $\mathrm{M}_n(\mathrm{C}_0(Z))$. Suppose that there exist a topological space Δ and a surjective continuous map $\xi : Z \to \Delta$ such that*

(1) for any $x_1, x_2 \in Z$, $\xi(x_1) = \xi(x_2)$ if and only if $\pi_{x_1}|_S$ is unitarily equivalent to $\pi_{x_2}|_S$,

(2) for any sequence $x_i, i = 1, 2, ...,$ in Z, any x in Z, and any $g \in S$, if $\xi(x_i) \to \xi(x)$ as $i \to \infty$, then $g(x_i) \to g(x)$ as $i \to \infty$, and

(3) $\pi_x(S) = \mathrm{M}_n(\mathbb{C})$, for any $x \in Z$.

Then $S \cong \mathrm{M}_n(\mathrm{C}_0(\Delta))$.

Proof. For each $f \in S$, define a function $\tilde{f} : \Delta \to \mathrm{M}_n(\mathbb{C})$ by

$$\tilde{f}(z) = f(x), \quad \text{if } \xi(x) = z.$$

By Condition (2), $\tilde{f}$ is well defined, and $\tilde{f}$ is continuous. Moreover, $\tilde{f}$ vanishes at infinity. To see this, note that, if $z_i \in \Delta$ with $z_i \to \infty$, then, since ξ is surjective, there are $x_i \in Z$ with $\xi(x_i) = z_i$. Then $x_i \to \infty$. Otherwise, there is a subsequence, say (x_{i_k}), converging to a point $x \in Z$. Since ξ is continuous, one has that $z_{i_k} = \xi(x_{i_k}) \to \xi(x)$, which contradicts the assumption $z_i \to \infty$. Hence $\tilde{f}(z_i) = f(x_i) \to 0$, and $\tilde{f} \in \mathrm{M}_n(\mathrm{C}_0(\Delta))$.

Moreover, it is clear that the map $f \to \tilde{f}$ is an injective homomorphism, and thus one can regard S as a sub-C*-algebra of $\mathrm{M}_n(\mathrm{C}_0(\Delta))$. It follows from Conditions (1) and (3) that S is a rich sub-C*-algebra of $\mathrm{M}_n(\mathrm{C}_0(\Delta))$ in the sense of Dixmier (11.1.1 of [4]), and therefore $S = \mathrm{M}_n(\mathrm{C}_0(\Delta))$ by Proposition 11.1.6 of [4] (or by Theorem 7.2 of [13]). ☐

Remark 4.4. The space Δ of Lemma 4.3 is automatically locally compact Hausdorff.

Theorem 4.5. *Let X be an infinite metrizable compact space, and let σ be a minimal homeomorphism. Suppose that (X, σ) has mean dimension zero. Let*

$$\{f_1, f_2, ..., f_n, g_1, g_2, ..., g_m\} \subseteq \mathrm{C}(X)$$

with $g_i(W) = \{0\}$, $i = 1, ..., m$, for some open set W containing some $y \in X$. Then, for any $\varepsilon > 0$, there is a closed neighbourhood Y of y contained in W such that the finite subset

$$\{f_1, f_2, ..., f_n, ug_1, ug_2, ..., ug_m\}$$

of A_Y, where u is the canonical unitary of the crossed product, is approximated to within ε by a subhomogeneous C-algebra S contained in A_Y with dimension ratio at most ε.*

Proof. Let $\varepsilon > 0$ be arbitrary. Choose a finite open cover

$$\alpha = \{U_1, U_2, ..., U_{|\alpha|}\}$$

of X such that for any $1 \leq i \leq n$ and any $1 \leq j \leq m$,

$$(4.7) \qquad |f_i(x) - f_i(x')| < \varepsilon \quad \text{and} \quad |g_j(x) - g_j(x')| < \varepsilon, \quad x, x' \in U, \ U \in \alpha.$$

Since (X, σ) is minimal and has mean dimension zero, it has SBP, and therefore by Proposition 2.4, there are a partition of unity $\{\phi_U;\ U \in \alpha\}$ subordinate to α and $T \in \mathbb{N}$ such that

$$(4.8) \qquad \frac{1}{N}(\chi_E(x) + \chi_E(\sigma(x)) + \cdots + \chi_E(\sigma^{N-1}(x))) < \frac{\varepsilon}{|\alpha| + 1}, \quad x \in X, \ N \geq T,$$

where $E = \bigcup_{U \in \alpha} \phi_U^{-1}((0, 1))$.

Choose the closed neighbourhood Y of y in W as follows: the Rokhlin partition

$$\{\{Z_1, \sigma(Z_1), ..., \sigma^{J_1 - 1}(Z_1)\}, ..., \{Z_k, \sigma(Z_k), ..., \sigma^{J_k - 1}(Z_k)\}\}$$

corresponding as in Section 3 to Y should satisfy

$$J_1 \geq \max\{\frac{|\alpha| + 1}{\varepsilon}, T\}.$$

By Lemma 4.1, there is an open set V such that $Y \subseteq V$, and for any $1 \leq k \leq K$,

$$(4.9) \qquad \frac{1}{J_k}(\chi_V(x) + \chi_V(\sigma(x)) + \cdots + \chi_V(\sigma^{J_k - 1}(x))) \leq \frac{1}{J_1} < \frac{\varepsilon}{|\alpha| + 1}, \quad x \in Z_k.$$

Choose a continuous function $H : X \to [0, 1]$ such that

$$H^{-1}(0) = Y \quad \text{and} \quad H^{-1}(1) \supseteq (X \setminus V).$$

Since $Y \subseteq W$, without loss of generality, we may assume that $V \subseteq W$, and then

$$Hg_j = g_j, \quad 1 \leq j \leq m.$$

Let us show that the sub-C*-algebra

$$S := \mathrm{C}^*\{\phi_U, uH;\ U \in \alpha\} \subseteq A_Y,$$

together with the closed set Y, provides the desired approximation, and that S has small dimension ratio as desired.

For each $U \in \alpha$, pick a point $x_U \in U$. Then, by (4.7), for each f_i, $1 \leq i \leq n$,

$$\|f_i - \sum_{U\in\alpha} f_i(x_U)\phi_U\| \leq \sup_{x\in X} \sum_{U\in\alpha} |f(x) - f_i(x_U)|\, \phi_U(x) < \varepsilon$$

and for each g_j, $1 \leq j \leq m$,

$$\begin{aligned} \|ug_j - uH\sum_{U\in\alpha} g_j(x_U)\phi_U\| &= \|uHg_j - uH\sum_{U\in\alpha} g_j(x_U)\phi_U\| \\ &\leq \|g_j - \sum_{U\in\alpha} g_j(x_U)\phi_U\| \\ &< \varepsilon. \end{aligned}$$

This shows the approximate inclusion

$$\{f_1, f_2, ..., f_n, ug_1, ug_2, ..., ug_m\} \subseteq_\varepsilon S.$$

Finally, let us show that $\mathrm{dimRatio}(S) < \varepsilon$. For each $1 \leq k \leq K$, consider the algebra

$$\mathrm{M}_{J_k}(\mathrm{C}(\overline{Z_k}))$$

of Theorem 3.2, and consider the map

$$\xi_k : \overline{Z_k} \to \mathbb{R}^{(|\alpha|+1)J_k - 1}$$

defined by

$$\xi_k(x) \mapsto ((\Phi\circ\sigma(x), \Phi\circ\sigma^2(x), ..., \Phi\circ\sigma^{J_k}), (H\circ\sigma(x), ..., H\circ\sigma^{J_k-1}(x))), \tag{4.10}$$

where the map $\Phi : \overline{Z_k} \to \mathbb{R}^{|\alpha|}$ is defined by

$$\Phi = \bigoplus_{U\in\alpha} \phi_U.$$

By (4.8) and (4.9), the image of Z_k under ξ_k is contained in the set

$$\{(t_1, t_2, ..., t_{(|\alpha|+1)J_k-1}) \in [0,1]^{(|\alpha|+1)J_k-1};\ \text{at most } \varepsilon J_k \text{ of the } t_i \text{ are not 0 or 1}\},$$

which has dimension at most $\varepsilon J_k - 1$ (as it is a union of simplices with at most εJ_k vertices). Therefore, $\xi_k(Z_k)$ has dimension at most εJ_k. For convenience, write $\xi_k(Z_k) = \Delta_k$. We have

$$\dim(\Delta_k) < \varepsilon J_k.$$

For each $x \in Z_k$, the evaluation map π_x on A_Y (regard A_Y as a subalgebra of $\bigoplus_{k=1}^K \mathrm{M}_{J_k}(\mathrm{C}(\overline{Z_K}))$ as in Theorem 3.2) is an irreducible representation of A_Y with dimension J_k.

Consider the restriction of π_x to S. Then the representation $\pi_x|_S$ is irreducible on S (and hence has dimension J_k). Indeed, let us consider the image of uH under π_x, which is

$$w := \begin{pmatrix} 0 & & & \\ H(\sigma(x)) & 0 & & \\ & \ddots & \ddots & \\ & & H(\sigma^{J_k-1}(x)) & 0 \end{pmatrix} \in \pi_x(S).$$

Noting that $H^{-1}(0) = Y$ and $x \in Z_k$, we have

$$H(\sigma^i(x)) \neq 0, \quad 1 \leq i \leq J_k - 1. \tag{4.11}$$

It follows that the C*-algebra generated by w is the full matrix algebra $\mathrm{M}_{J_k}(\mathbb{C})$, and the restriction of π_x to S is irreducible. In particular,

$$\pi_x(S) = \mathrm{M}_{J_k}(\mathbb{C}). \tag{4.12}$$

This shows that the dimension of any irreducible representation of S must be J_k for some k, and each irreducible representation of S with dimension J_k is the restriction of π_x for some $x \in Z_k$.

Moreover, for any $g \in S$, $x \in Z_k$, and any sequence (x_n) in Z_k, if $\xi_k(x_n) \to \xi_k(x)$, then

$$\pi_{x_n}(g) \to \pi_x(g). \tag{4.13}$$

Let $x_1, x_2 \in Z_k$. Let us show that

$$\pi_{x_1}|_S \text{ and } \pi_{x_2}|_S \text{ are unitarily equivalent if and only if } \xi_k(x_1) = \xi_k(x_2). \tag{4.14}$$

If

$$\xi_k(x_1) = \xi_k(x_2),$$

then, by the definition of ξ_k, one has

$$\phi_U \circ \sigma^i(x_1) = \phi_U \circ \sigma^i(x_2) \quad \text{and} \quad H \circ \sigma^j(x_1) = H \circ \sigma^j(x_2),$$

where $U \in \alpha, 1 \leq i \leq J_k, 1 \leq j \leq J_k - 1$. By (3.1) and (3.2) of Theorem 3.2, one has that

$$\pi_{x_1}(\phi_U) = \pi_{x_2}(\phi_U) \quad \text{and} \quad \pi_{x_1}(uH) = \pi_{x_2}(uH), \quad U \in \alpha.$$

Since S is the sub-C*-algebra generated by ϕ_U, $U \in \alpha$, and uH, this shows that

$$\pi_{x_1}|_S = \pi_{x_2}|_S. \tag{4.15}$$

In particular, $\pi_{x_1}|_S$ and $\pi_{x_2}|_S$ are unitarily equivalent.

Now, assume that $\pi_{x_1}|_S$ and $\pi_{x_2}|_S$ are unitarily equivalent. Pick ϕ_U, and consider the pair (ϕ_U, uH). Again, by (3.1) and (3.2) of Theorem 3.2, we have

$$\pi_{x_1}(\phi_U) = \begin{pmatrix} \phi_U(\sigma(x_1)) & & \\ & \ddots & \\ & & \phi_U(\sigma^{J_k}(x_1)) \end{pmatrix}, \quad \pi_{x_1}(uH) = \begin{pmatrix} 0 & & & \\ H(\sigma(x_1)) & 0 & & \\ & \ddots & \ddots & \\ & & H(\sigma^{J_k-1}(x_1)) & 0 \end{pmatrix},$$

and

$$\pi_{x_2}(\phi_U) = \begin{pmatrix} \phi_U(\sigma(x_2)) & & \\ & \ddots & \\ & & \phi_U(\sigma^{J_k}(x_2)) \end{pmatrix}, \quad \pi_{x_2}(uH) = \begin{pmatrix} 0 & & & \\ H(\sigma(x_2)) & 0 & & \\ & \ddots & \ddots & \\ & & H(\sigma^{J_k-1}(x_2)) & 0 \end{pmatrix}.$$

Since π_{x_1} and π_{x_2} are assumed to be unitarily equivalent, the pair of matrices $(\pi_{x_1}(\phi_U), \pi_{x_1}(uH))$ is unitarily equivalent to the pair of matrices $(\pi_{x_2}(\phi_U), \pi_{x_2}(uH))$. By (4.11), we may apply Lemma 4.2 to obtain

$$\phi_U(\sigma^i(x_1)) = \phi_U(\sigma^i(x_2)) \quad \text{and} \quad H(\sigma^j(x_1)) = H(\sigma^j(x_2)), \quad 1 \leq i \leq J_k,\ 1 \leq j \leq J_k - 1.$$

Applying this argument for all $U \in \alpha$, we have that

$$\phi_U(\sigma^i(x_1)) = \phi_U(\sigma^i(x_2)) \quad \text{and} \quad H(\sigma^j(x_1)) = H(\sigma^j(x_2)), \quad U \in \alpha,\ 1 \leq i \leq J_k,\ 1 \leq j \leq J_k - 1,$$

and this implies (by the construction of the map ξ_k; see (4.10))

$$\xi_k(x_1) = \xi_k(x_2).$$

This proves the assertion (4.14).

Since any irreducible representation of S extends to an irreducible representation of A_Y, and $\{\pi_x;\ x \in Y\}$ are all of the irreducible representations of A_Y, by (4.12), the dimensions of the irreducible representations of S have to be $J_1, J_2, ..., J_K$, and the map ξ induces a bijection between $\mathrm{Prim}_{J_k}(S)$ and Δ_k for each $1 \leq k \leq K$. Then the subquotient with J_k-dimensional representations of S, denoted (say) by S_k, is a sub-C*-algebra of the subquotient with J_k-dimensional representations of A_Y, which is canonically isomorphic to $\mathrm{M}_{J_k}(\mathrm{C}_0(Z_k))$. By (4.14), for any $x_1, x_2 \in Z_k$,

$$\pi_{x_1}|_{S_k} \text{ is unitarily equivalent to } \pi_{x_2}|_{S_k} \text{ if and only if } \xi_k(x_1) = \xi_k(x_2). \tag{4.16}$$

By (4.13), one has that for any $g \in S_k$, any $x \in Z_k$, and any sequence (x_n) in Z_k, if $\xi_k(x_n) \to \xi_k(x)$, then

$$\pi_{x_n}(g) \to \pi_x(g). \tag{4.17}$$

Therefore, the conditions of Lemma 4.3 are satisfied for the sub-C*-algebra S_k of $\mathrm{M}_{J_k}(\mathrm{C}_0(Z_k))$, and it follows that

$$S_k \cong \mathrm{M}_{J_k}(\mathrm{C}_0(\Delta_k)).$$

This implies that

$$\mathrm{Prim}_{J_k}(S) = \mathrm{Prim}(S_k) = \Delta_k,$$

and so

$$\dim(\mathrm{Prim}_{J_k}(S)) = \dim(\Delta_k) < \varepsilon J_k.$$

In other words,

$$\mathrm{dimRatio}(S) < \varepsilon,$$

as desired. □

Theorem 4.6. *Let X be an infinite metrizable compact space, and let $\sigma : X \to X$ be a minimal homeomorphism. If (X, σ) has mean dimension zero, then the C*-algebra A_y can be locally approximated by subhomogeneous sub-C*-algebras with arbitrarily small dimension ratio.*

Proof. This follows directly from Theorem 4.5. □

Theorem 4.7. *Let X be an infinite metrizable compact space, and let $\sigma : X \to X$ be a minimal homeomorphism. If (X, σ) has mean dimension zero, then the C*-algebra $A = \mathrm{C}(X) \rtimes_\sigma \mathbb{Z}$ absorbs the Jiang-Su algebra $\mathcal{Z}$ tensorially.*

Proof. By Theorem 4.6, the C*-algebra A_y is locally approximated by subhomogeneous C*-algebras with arbitrarily small dimension ratio. By Lemma 5.8 of [21], the C*-algebra A_y has strict comparison of positive elements. On using Lemma 5.10 of [21] in place of Theorem 3.4 of [27], the same argument as that in the proof of Theorem 1.2 of [27] shows that the Cuntz semigroups of A_y and $A_y \otimes \mathcal{Z}$ are isomorphic, and therefore $A_y \cong A_y \otimes \mathcal{Z}$ by Corollary 7.4 of [31]. (Note that by [20] the subhomogeneous C*-algebras of Theorem 4.6 have finite nuclear dimension, as required in Corollary 7.4 of [31].) Since, by Theorem 3.4 (see [2]), A_y is centrally large in A in the sense of D. Archey and N. C. Phillips, by Theorem 3.6 (see [1]) the nuclear C*-algebra A also satisfies $A \cong A \otimes \mathcal{Z}$. □

Remark 4.8. Since this paper was first posted on arXiv, it was shown in [9] (Theorem 4.1) that the C*-algebra $\mathrm{C}(X) \rtimes_\sigma \mathbb{Z}$ is always rationally locally approximately subhomogeneous for a minimal dynamical system (X, σ), and the Jiang-Su stable rationally approximately subhomogeneous C*-algebras have also been classified (they were shown to have finite nuclear dimension in [9] (Corollary 3.20), and hence to be classified in [7] (Theorem 5.9) (using [12]); this classification result was vastly generalized in [8] and [6]). (The special case of crossed products was also considered in [14].) In other words, one has the following corollary.

Corollary 4.9. *Let X be an infinite metrizable compact space, and let $\sigma : X \to X$ be a minimal homeomorphism. If (X, σ) has mean dimension zero, then $\mathrm{C}(X) \rtimes_\sigma \mathbb{Z}$ is classifiable and in fact is an ASH algebra. In particular, if the minimal system (X, σ) has finite entropy or countably many ergodic measures, the C*-algebra $\mathrm{C}(X) \rtimes_\sigma \mathbb{Z}$ is classifiable and is an ASH algebra.*

5. Tensor products

In this section, let us show that the tensor product of the crossed product C*-algebras of two or more minimal homeomorphisms is $\mathcal{Z}$-stable (Theorem 5.6). In particular, this implies that the Toms growth rank ([26]) of any crossed product C*-algebra $\mathrm{C}(X) \rtimes_\sigma \mathbb{Z}$ with (X, σ) minimal is at most two. It also shows that the examples of Giol and Kerr ([11]) are prime among the C*-algebras of minimal homeomorphisms.

Theorem 5.1. *Let X be an infinite metrizable compact space, and let $y \in X$. Let σ be a minimal homeomorphism. Let*

$$\{f_1, f_2, ..., f_n, g_1, g_2, ..., g_m\} \subseteq \mathrm{C}(X),$$

with $g_i(W) = \{0\}$, $i = 1, ..., m$, for some open set W containing y. Then, for any $\varepsilon > 0$, there is $R > 0$ such that for any $J \in \mathbb{N}$, there is a closed neighbourhood Y of y contained in W such that the finite subset

$$\{f_1, f_2, ..., f_n, ug_1, ug_2, ..., ug_m\}$$

of A_Y, where u is the canonical unitary of the crossed product, is approximated to within ε by a subhomogeneous C-algebra S in A_Y with dimension ratio at most R, and with the dimension of each irreducible representation at least J.*

Proof. The proof is a slight modification of the proof of Theorem 4.5.

Let $\varepsilon > 0$ be arbitrary. Choose a finite open cover

$$\alpha = \{U_1, U_2, ..., U_{|\alpha|}\}$$

of X such that

$$|f_i(x) - f_i(y)| < \varepsilon \quad \text{and} \quad |g_j(x) - g_j(y)| < \varepsilon, \quad x, y \in U_i,\ 1 \leq i \leq |\alpha|\,. \tag{5.1}$$

Then

$$R = |\alpha| + 1$$

is the desired constant.

Let $J \in \mathbb{N}$ be arbitrary. Choose the closed neighbourhood Y of y in W as follows: the Rokhlin partition

$$\{\{Z_1, \sigma(Z_1), ..., \sigma^{J_1-1}(Z_1)\}, ..., \{Z_k, \sigma(Z_k), ..., \sigma^{J_k-1}(Z_k)\}\}$$

corresponding as in Section 3 to Y should satisfy

$$J_1 \geq J.$$

Pick an open set V such that $Y \subseteq V \subseteq W$, and pick a continuous function $H : X \to [0,1]$ such that

$$H^{-1}(0) = Y \quad \text{and} \quad H^{-1}(1) \supseteq (X \setminus V).$$

Since $V \subseteq W$, one has

$$Hg_j = g_j, \quad 1 \leq j \leq m.$$

Choose a partition of unity $\{\phi_U : U \in \alpha\}$ subordinate to α.

Let us show that the sub-C*-algebra

$$S := \mathrm{C}^*\{\phi_U, uH;\ U \in \alpha\} \subseteq A_Y,$$

together with the closed set Y, satisfies the conditions of the theorem (for R and J).

For each $U \in \alpha$, pick a point $x_U \in U$. Then, by (5.1), for each f_i, $1 \leq i \leq n$,

$$\|f_i - \sum_{U\in\alpha} f_i(x_U)\phi_U\| \leq \sup_{x\in X} \sum_{U\in\alpha} |f(x) - f_i(x_U)|\, \phi_U(x) < \varepsilon;$$

and for each g_j, $1 \leq j \leq m$,

$$\begin{aligned} \|ug_j - uH\sum_{U\in\alpha} g_j(x_U)\phi_U\| &= \|uHg_j - uH\sum_{U\in\alpha} g_j(x_U)\phi_U\| \\ &\leq \|g_j - \sum_{U\in\alpha} g_j(x_U)\phi_U\| \\ &< \varepsilon. \end{aligned}$$

This shows that

$$\{f_1, f_2, ..., f_n, ug_1, ug_2, ..., ug_m\} \subseteq_\varepsilon S.$$

Let us show that $\mathrm{dimRatio}(S) \leq R$. For each $1 \leq k \leq K$, consider the algebra

$$\mathrm{M}_{J_k}(\mathrm{C}(\overline{Z_k}))$$

of Theorem 3.2, and consider the map

$$\xi_k : \overline{Z_k} \to \mathbb{R}^{(|\alpha|+1)J_k - 1}$$

defined by

$$\xi_k(x) \mapsto ((\Phi\circ\sigma(x), \Phi\circ\sigma^2(x), ..., \Phi\circ\sigma^{J_k}), (H\circ\sigma(x), ..., H\circ\sigma^{J_k-1}(x))), \tag{5.2}$$

where the map $\Phi : \overline{Z_k} \to \mathbb{R}^{|\alpha|}$ is defined by

$$\Phi = \bigoplus_{U\in\alpha} \phi_U.$$

It is clear that image of $\xi_k(Z_k)$ has dimension at most $(|\alpha|+1)J_k - 1$. Very much as in the proof of Theorem 4.5 one now verifies that the irreducible representations of S have dimensions

$$J_1 < J_2 < \cdots < J_K,$$

and that

$$\dim(\mathrm{Prim}_{J_k}(S)) < (|\alpha|+1)J_k.$$

In other words,

$$\mathrm{dimRatio}(S) < |\alpha| + 1 = R,$$

and the dimension of each irreducible representation of S is at least J (recall that $J \leq J_1$). $\square$

Consider separable subhomogeneous C*-algebras C and S, and consider $\pi_0 \in \mathrm{Prim}_m(C)$ and $\pi_1 \in \mathrm{Prim}_n(C)$. Then $\pi_0 \otimes \pi_1$ is an mn-dimensional irreducible representation of $C \otimes S$. By Theorem 3.3 of [3], this map induces a homeomorphism between $\mathrm{Prim}(C) \times \mathrm{Prim}(S)$ and $\mathrm{Prim}(C \otimes S)$. In particular, for any $d \in \mathbb{N}$, this yields a homeomorphism

$$\mathrm{Prim}_d(C \otimes S) \cong \bigsqcup_{mn=d} (\mathrm{Prim}_m(C) \times \mathrm{Prim}_n(S)).$$

Lemma 5.2. *Let C and S be subhomogeneous C*-algebras. Let $m, n, d \in \mathbb{N}$ satisfy $mn = d$. Then the subset $\mathrm{Prim}_m(C) \times \mathrm{Prim}_n(S)$ is relatively closed in $\mathrm{Prim}_d(C \otimes S)$.*

Proof. Let $m, n \in \mathbb{N}$ satisfying $mn = d$. Let $(\pi^0_k \otimes \pi^1_k)$ converge to $\pi^0_\infty \otimes \pi^1_\infty$ in $\mathrm{Prim}_d(C \otimes S)$, where

$$\pi^0_k \in \mathrm{Prim}_m(C),\ \ \pi^1_k \in \mathrm{Prim}_n(S), \quad k = 1, 2, ...$$

and

$$\pi^0_\infty \in \mathrm{Prim}_{m'}(C),\ \ \pi^1_\infty \in \mathrm{Prim}_{n'}(S), \quad k = 1, 2, ...$$

for some $m', n' \in \mathbb{N}$ with $m'n' = d$. Since $\mathrm{Prim}_d(C \otimes S)$ is homeomorphic to $\bigsqcup_{mn=d}(\mathrm{Prim}_m(C) \times \mathrm{Prim}_n(S))$, one has that π^0_k converges to π^0_∞ and π^1_k converges to π^1_∞, and hence (by Proposition 3.6.3 of [4]) $m \leq m'$ and $n \leq n'$. Since $mn = m'n' = d$, one has $m = m'$ and $n = n'$, i.e., $\pi^0_\infty \otimes \pi^1_\infty \in \mathrm{Prim}_m(C) \times \mathrm{Prim}_n(S)$, as desired. □

Lemma 5.3. *Let C and S be unital subhomogeneous C*-algebras, and let J be a natural number such that each irreducible representation of C or S has dimension at least J. Then*

$$\mathrm{dimRatio}(C \otimes S) \leq \frac{\mathrm{dimRatio}(C) + \mathrm{dimRatio}(S)}{J}.$$

Proof. Let d be any natural number. Then

$$\mathrm{Prim}_d(C \otimes S) = \bigsqcup_{mn=d} (\mathrm{Prim}_m(C) \times \mathrm{Prim}_n(S)).$$

By Lemma 5.2, each $\mathrm{Prim}_m(C) \times \mathrm{Prim}_n(S)$ is relatively closed (and hence clopen) in $\mathrm{Prim}_d(C \otimes S)$, and then (see, for instance, Theorem 3.2.13 of [10]) we have

$$\begin{aligned} \dim(\mathrm{Prim}_d(C \otimes S)) &= \max_{mn=d}\{\dim(\mathrm{Prim}_m(C) \times \mathrm{Prim}_n(S))\} \\ &\leq \max_{mn=d}\{\dim(\mathrm{Prim}_m(C)) + \dim(\mathrm{Prim}_n(S))\}. \end{aligned}$$

This implies

$$\begin{aligned} \frac{\dim(\mathrm{Prim}_d(C \otimes S))}{d} &\leq \max_{d=mn}\{\frac{\dim(\mathrm{Prim}_m(C)) + \dim(\mathrm{Prim}_n(S))}{d}\} \\ &= \max_{d=mn}\{\frac{\dim(\mathrm{Prim}_m(C))}{m} \cdot \frac{1}{n} + \frac{\dim(\mathrm{Prim}_n(S))}{n} \cdot \frac{1}{m}\} \\ &\leq \max_{d=mn}\{\mathrm{dimRatio}(C) \cdot \frac{1}{n} + \mathrm{dimRatio}(S) \cdot \frac{1}{m}\} \\ &\leq \frac{\mathrm{dimRatio}(C) + \mathrm{dimRatio}(S)}{J}, \end{aligned}$$

as desired. □

Lemma 5.4. *Let A and B be two C*-algebras each with the following property: For any finite subset $\mathcal{F}$ of A (or B) and any $\varepsilon > 0$, there are $R > 0$ (which depends on $\mathcal{F}$ and ε) and a sequence of unital sub-C*-algebras (S_n) such that*

(1) each S_n is a subhomogeneous C-algebra with $\mathrm{dimRatio}(S_n) \leq R$,*

(2) each S_n approximately contains $\mathcal{F}$ up to ε, and

(3) $d_n \to \infty$ as $n \to \infty$, where d_n is the smallest dimension of the irreducible representations of S_n.

Then $A \otimes B$ can be locally approximated by subhomogeneous C-algebras with arbitrarily small dimension ratio.*

Proof. It is enough to show that for any finite subsets $\mathcal{F} \subseteq A$, $\mathcal{G} \subseteq B$, and any $\varepsilon \in (0,1)$, there is a subhomogeneous C*-algebra D in $A \otimes B$ such that $\mathcal{F} \otimes \mathcal{G} \subseteq_\varepsilon D$ and $\mathrm{dimRatio}(D) < \varepsilon$.

Without loss of generality, we may assume that each element of $\mathcal{F}$ and $\mathcal{G}$ has norm one. By the assumptions, there are subhomogeneous C*-algebras $C \subseteq A$ and $S \subseteq B$ such that

$$\mathrm{dimRatio}(C) \leq R \quad \text{and} \quad \mathcal{F} \subseteq_{\frac{\varepsilon}{4}} C,$$

and

$$\mathrm{dimRatio}(S) \leq R \quad \text{and} \quad \mathcal{G} \subseteq_{\frac{\varepsilon}{4}} S,$$

and, furthermore, the dimension of each irreducible representation of C or S is at least $2R/\varepsilon$. Then consider the C*-algebra

$$D := C \otimes S.$$

By Lemma 5.3,

$$\mathrm{dimRatio}(D) \leq \varepsilon.$$

A straightforward calculation also shows that

$$\mathcal{F} \otimes \mathcal{G} \subseteq_\varepsilon D,$$

and this completes the proof. □

Proposition 5.5. *Let (X_1, σ_1) and (X_2, σ_2) be minimal dynamical systems, where X_1 and X_2 are infinite. Fix $y_1 \in X_1$ and $y_2 \in X_2$, and consider the large sub-C*-algebras*

$$A_{y_1} \subseteq \mathrm{C}(X_1) \rtimes_{\sigma_1} \mathbb{Z} \quad \textit{and} \quad A_{y_2} \subseteq \mathrm{C}(X_2) \rtimes_{\sigma_2} \mathbb{Z}.$$

Then

$$A_{y_1} \otimes A_{y_2} \cong (A_{y_1} \otimes A_{y_2}) \otimes \mathcal{Z}.$$

Proof. By Theorem 5.1 and Lemma 5.4, the C*-algebra $A_{y_1} \otimes A_{y_2}$ can be locally approximated by subhomogeneous C*-algebras with arbitrarily small dimension ratio, and therefore (as in the proof of Theorem 4.7) it absorbs the Jiang-Su algebra tensorially. □

Theorem 5.6. *Let (X_1, σ_1) and (X_2, σ_2) be minimal dynamical systems, where X_1 and X_2 are infinite metrizable compact spaces. Consider the crossed-product C*-algebras*

$$A_1 = \mathrm{C}(X_1) \rtimes_{\sigma_1} \mathbb{Z} \quad \textit{and} \quad A_2 = \mathrm{C}(X_2) \rtimes_{\sigma_2} \mathbb{Z}.$$

Then

$$A_1 \otimes A_2 \cong (A_1 \otimes A_2) \otimes \mathcal{Z}.$$

In particular, the (crossed product) C-algebra of a single minimal homeomorphism has Toms growth rank ([26]) at most two.*

Proof. By Proposition 5.5, the C*-algebra $A_{y_1} \otimes A_{y_2}$ absorbs the Jiang-Su algebra $\mathcal{Z}$. By Lemma 3.5, $A_{y_1} \otimes A_{y_2}$ is centrally large in $A_1 \otimes A_2$. By Theorem 3.6 (2.3 of [1]), the nuclear C*-algebra $A_1 \otimes A_2$ absorbs the Jiang-Su algebra. □

Corollary 5.7. *Let (X_1, σ_1) and (X_2, σ_2) be minimal dynamical systems, where X_1 and X_2 are infinite metrizable compact spaces. Consider the crossed-product C*-algebras*

$$A_1 = \mathrm{C}(X_1) \rtimes_{\sigma_1} \mathbb{Z} \quad \textit{and} \quad A_2 = \mathrm{C}(X_2) \rtimes_{\sigma_2} \mathbb{Z}.$$

Then $A_1 \otimes A_2$ is classifiable and is an ASH algebra.

Proof. By [9], the algebras A_1 and A_2 are rationally locally approximately subhomogeneous, and therefore $A_1 \otimes A_2$ is rationally locally approximately subhomogeneous. By Theorem 5.6, $A_1 \otimes A_2$ is Jiang-Su stable, and hence has finite decomposition rank by [9] again and is classifiable by [8] and [6] (alternatively, the classifiability of $A_1 \otimes A_2$ follows from [7]—which of course also uses [9]). □

Remark 5.8. Note that, with A_1 and A_2 as in 5.6,

$$A_1 \otimes A_2 \cong \mathrm{C}(X_1 \times X_2) \rtimes_\sigma \mathbb{Z}^2,$$

where the action σ of $\mathbb{Z}^2$ on $X_1 \times X_2$ is the product action:

$$\sigma_{(m,n)}(x_1, x_2) = (\sigma_1^m(x_1), \sigma_2^n(x_2)).$$

Thus, Theorem 5.6 states that for minimal actions of $\mathbb{Z}$ on X_1 and X_2 the crossed product C*-algebra $\mathrm{C}(X_1 \times X_2) \rtimes_\sigma \mathbb{Z}^2$ always absorbs the Jiang-Su algebra.

On the other hand, the minimal $\mathbb{Z}^2$-system $(X_1 \times X_2, \sigma)$ has mean dimension zero, as shown below. Therefore, Theorem 5.6 could be evidence that the C*-algebra of a minimal free action of $\mathbb{Z}^n$ (or an arbitrary discrete amenable group) with mean dimension zero is classifiable. But it is not clear how to generalize the argument of this paper to the case of a minimal free action of an arbitrary discrete amenable group (even of $\mathbb{Z}^n, n \geq 2$), if it is possible.

Let us show that $(X_1 \times X_2, \sigma)$ has mean dimension zero. Let α be an open cover of $X_1 \times X_2$. Pick open covers β_1, β_2 of X_1, X_2, respectively, such that

$$\beta := \{U \times V : U \in \beta_1,\ V \in \beta_2\} = \pi_1^{-1}(\beta_1) \vee \pi_2^{-1}(\beta_2)$$

is a refinement of α, where $\pi_i : X_1 \times X_2 \to X_i$, $i = 1, 2$, is the projection map.

For each $M, N \in \mathbb{N}$, consider

$$F_{M,N} := \{(m, n) : 0 \leq m \leq M - 1, 0 \leq n \leq N - 1\}.$$

Note that for any $0 \leq m < M$ and $0 \leq n < N$, one has

$$\sigma_{(m,n)}(\beta) = \{\sigma_1^m(U) \times \sigma_2^n(V) : U \in \beta_1,\ V \in \beta_2\} = \pi_1^{-1}(\sigma_1^m(\beta_1)) \vee \pi_2^{-1}(\sigma_2^n(\beta_2)),$$

and

$$\begin{aligned} \bigvee_{(m,n)\in F_{M,N}} \sigma^{(m,n)}(\beta) &= \bigvee_{m=0}^{M-1} \bigvee_{n=0}^{N-1} (\pi_1^{-1}(\sigma_1^m(\beta_1)) \vee \pi_2^{-1}(\sigma_2^n(\beta_2))) \\ &= (\bigvee_{m=0}^{M-1} \pi_1^{-1}(\sigma_1^m(\beta_1))) \vee (\bigvee_{n=0}^{N-1} \pi_2^{-1}(\sigma_2^n(\beta_2))). \end{aligned}$$

Therefore,

$$\begin{aligned}\mathcal{D}(\bigvee_{(m,n)\in F_{M,N}}\sigma^{(m,n)}(\beta)) &= \mathcal{D}((\bigvee_{m=1}^{M}\pi_1^{-1}(\sigma_1^m(\beta_1)))\vee(\bigvee_{n=1}^{N}\pi_2^{-1}(\sigma_2^n(\beta_2))))\\ &\leq \sum_{m=0}^{M-1}\mathcal{D}(\pi_1^{-1}(\sigma_1^m(\beta_1)))+\sum_{n=0}^{N-1}\mathcal{D}(\pi_2^{-1}(\sigma_1^n(\beta_2)))\\ &\leq M\mathcal{D}(\beta_1)+N\mathcal{D}(\beta_2),\end{aligned}$$

and

$$\lim_{M,N\to\infty}\frac{1}{|F_{M,N}|}\mathcal{D}(\bigvee_{(m,n)\in F_{M,N}}\sigma^{(m,n)}(\beta))\leq\lim_{M,N\to\infty}\frac{M\mathcal{D}(\beta_1)+N\mathcal{D}(\beta_2)}{MN}=0.$$

Since β is a refinement of α, one has

$$\mathcal{D}(\bigvee_{(m,n)\in F_{M,N}}\sigma^{(m,n)}(\alpha))\leq\mathcal{D}(\bigvee_{(m,n)\in F_{M,N}}\sigma^{(m,n)}(\beta))$$

and

$$\lim_{M,N\to\infty}\frac{1}{|F_{M,N}|}\mathcal{D}(\bigvee_{(m,n)\in F_{M,N}}\sigma^{(m,n)}(\alpha))\leq\lim_{M,N\to\infty}\frac{1}{|F_{M,N}|}\mathcal{D}(\bigvee_{(m,n)\in F_{M,N}}\sigma^{(m,n)}(\beta))=0.$$

Hence,

$$\mathrm{mdim}(X_1\times X_2,\sigma)=0.$$

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