COMPARISON RADIUS AND MEAN TOPOLOGICAL DIMENSION: $\mathbb{Z}^d$-ACTIONS

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Abstract. Consider a minimal free topological dynamical system $(X, \mathbb{Z}^d)$. It is shown that the comparison radius of the crossed product C*-algebra $\mathrm{C}(X) \rtimes \mathbb{Z}^d$ is at most the half of the mean topological dimension of $(X, \mathbb{Z}^d)$. As a consequence, the C*-algebra $\mathrm{C}(X) \rtimes \mathbb{Z}^d$ is classified by the Elliott invariant if the mean dimension of $(X, \mathbb{Z}^d)$ is zero.

1. Introduction

Let (X, Γ) be a topological dynamical system, where X is a compact Hausdorff space and Γ is a discrete amenable group. The mean (topological) dimension of (X, Γ), denoted by $\mathrm{mdim}(X, \Gamma)$, was introduced by Gromov ([9]), and then was developed and studied systematically by Lindenstrauss and Weiss ([21]). It is a numerical invariant, taking value in $[0, +\infty]$, to measure the complexity of (X, Γ) in terms of dimension growth with respect to partial orbits. Applications of mean dimension theory can be found in topological dynamical systems ([21], [20], [10], [18], [13], [12], [14]), geometric analysis ([34], [5], [23], [35]), operator algebras ([19], [4], [26], [24], [25]), and information theory ([22]).

On the other hand, for a general unital stably finite C*-algebra A, the radius of comparison, introduced by Toms ([32]) and denoted by $\mathrm{rc}(A)$, is also a numerical invariant to measure the regularity of the C*-algebra A; and $\mathrm{rc}(A)$ can be regarded as an abstract version of the dimension growth of A. A heuristic example is $\mathrm{M}_n(\mathrm{C}(X))$, the C*-algebra of (complex) $n \times n$ matrix valued continuous functions on a finite CW-complex X; its comparison radius is around $\frac{1}{2}\frac{\dim(X)}{n}$, which is half of the dimension ratio of $\mathrm{M}_n(\mathrm{C}(X))$.

For the given topological dynamical system (X, Γ), the canonical C*-algebra to be considered is the transformation group C*-algebra $A = \mathrm{C}(X) \rtimes \Gamma$. A natural question to ask then is how the radius of comparison of the C*-algebra is connected to the mean dimension of the dynamical system. In fact, Phillips and Toms even made the following conjecture:

Conjecture (Phillips-Toms)**.** *Let (X, Γ) be a minimal and free topological dynamical system, where X is compact Hausdorff space, and Γ is a discrete amenable group. Then*

$$\mathrm{rc}(\mathrm{C}(X) \rtimes \Gamma) = \frac{1}{2}\mathrm{mdim}(X, \Gamma).$$

This conjecture is closed related to the classification of C*-algebras. In general, the C*-algebra $\mathrm{C}(X) \rtimes \Gamma$ can be wild and not to be classified by the Elliott invariant (even with $\Gamma = \mathbb{Z}$, see [6]). So, an important question in the classification program of C*-algebras

Date: July 15, 2021.

The research is supported by an NSF grant (DMS-1800882).

is to determine which transformation group C*-algebra is classifiable. Now, a special case of this conjecture is that $\mathrm{mdim}(X, \Gamma) = 0$ implies $\mathrm{rc}(\mathrm{C}(X) \rtimes \Gamma) = 0$ (strict comparison of positive elements); and by the Toms-Winter conjecture, this should imply that the C*-algebra $\mathrm{C}(X) \rtimes \Gamma$ is Jiang-Su stable and classifiable.

There have been many researches on the classifiability of transformation group C*-algebras: Under the assumption that X is finite dimensional (hence the mean dimension is automatically zero), it was shown in [33] that the algebra $\mathrm{C}(X) \rtimes \mathbb{Z}$ has finite nuclear dimension, and therefore is Jiang-Su stable. With Rokhlin dimension, this result was generalized to $\mathbb{Z}^d$-actions in [29], and then to the actions of residually finite groups with box spaces of finite asymptotic dimension ([30]); and with almost finiteness, the Jiang-Su stability is also obtained for actions by groups with comparison property ([16]).

Without the finite dimensionality assumption on X, so far the only result was [4] where $\mathbb{Z}$-actions are considered, and the zero mean dimension was shown to imply the classifiability of the C*-algebra. Note that this result particularly covers all strictly ergodic dynamical systems. Beyond the case of mean dimension zero, Phillips considered $\mathbb{Z}$-actions in [26] and showed that the radius of comparison of $\mathrm{C}(X) \rtimes \mathbb{Z}$ is at most $1 + 36\mathrm{mdim}(X, \mathbb{Z})$.

In this paper, let us consider minimal and free $\mathbb{Z}^d$-actions, and show the following:

Theorem A (Theorem 5.6)**.** *Let $(X, \mathbb{Z}^d)$ be a minimal free dynamical system. Then*

$$\mathrm{rc}(\mathrm{C}(X) \rtimes \mathbb{Z}^d) \leq \frac{1}{2}\mathrm{mdim}(X, \mathbb{Z}^d). \tag{1.1}$$

As a consequence of (1.1), one obtains the classifiability if $(X, \mathbb{Z}^d)$ has mean dimension zero:

Theorem B (Theorem 5.7)**.** *Let $(X, \mathbb{Z}^d)$ be a minimal free dynamical system with mean dimension zero, then $\mathrm{C}(X) \rtimes \mathbb{Z}^d$ is classified by its Elliott invariant. In particular, if $\dim(X) < \infty$, or $(X, \mathbb{Z}^d)$ has at most countably many ergodic measures, or $(X, \mathbb{Z}^d)$ has finite topological entropy, then $\mathrm{C}(X) \rtimes \mathbb{Z}^d$ is classified by its Elliott invariant.*

The argument in [33], [4], or [26] relies on the Putnam's orbit-cutting algebra (or the large sub-algebra) A_y; and in the case of zero mean dimension, the argument in [4] also heavily depends on the small boundary property (which is equivalent to mean dimension zero in the case of $\mathbb{Z}$-actions). However, beyond the case of $\mathbb{Z}$-actions, it is not clear in general how to construct large sub-algebras; moreover, once the dynamical system does not have mean dimension zero, the small boundary property does not hold anymore. So, instead of large sub-algebra and small boundary property, the proofs of Theorem A and Theorem B depend on Uniform Rokhlin Property (URP) and Cuntz comparison of Open Sets (COS):

Definition 1.1 (Definition 3.1 and Definition 4.1 of [24])**.** A topological dynamical system (X, Γ), where Γ is a discrete amenable group, is said to have Uniform Rokhlin Property (URP) if for any $\varepsilon > 0$ and any finite set $K \subseteq \Gamma$, there exist closed sets $B_1, B_2, ..., B_S \subseteq X$ and (K, ε)-invariant sets $\Gamma_1, \Gamma_2, ..., \Gamma_S \subseteq \Gamma$ such that

$$B_s\gamma, \quad \gamma \in \Gamma_s,\ s = 1, ..., S,$$

are mutually disjoint and

$$\mathrm{ocap}(X \setminus \bigsqcup_{s=1}^{S} \bigsqcup_{\gamma \in \Gamma_s} B_s\gamma) < \varepsilon,$$

where ocap denote the orbit capacity (see, for instance, Definition 5.1 of [21]).

The dynamical system (X, Γ) is said to have (λ, m)-Cuntz-comparison of open sets, where $\lambda \in (0, 1]$ and $m \in \mathbb{N}$, if for any open sets $E, F \subseteq X$ with

$$\mu(E) < \lambda\mu(F), \quad \mu \in \mathcal{M}_1(X, \Gamma),$$

where $\mathcal{M}_1(X, \Gamma)$ is the simplex of all invariant probability measures on X, then

$$\varphi_E \precsim \underbrace{\varphi_F \oplus \cdots \oplus \varphi_F}_{m} \quad \text{in } \mathrm{C}(X) \rtimes \Gamma,$$

where φ_E and φ_F are continuous functions supporting on E and F respectively.

The dynamical system (X, Γ) is said to have Cuntz comparison of Open Sets (COS) if it has (λ, m)-Cuntz-comparison on open sets for some λ and m.

It is shown in [24] (Theorem 4.8) that the (URP) and (COS) implies

$$\mathrm{rc}(\mathrm{C}(X) \rtimes \Gamma) \leq \frac{1}{2}\mathrm{mdim}(X, \Gamma),$$

and it is also shown in [25] (Theorem 4.8) that if, in addition, (X, Γ) has mean dimension zero, then the C*-algebra $\mathrm{C}(X) \rtimes \Gamma$ is classifiable. Thus, Theorem A and Theorem B follows from the following:

Theorem (Theorem 4.2 and Theorem 5.5)**.** *Any free and minimal dynamical system $(X, \mathbb{Z}^d)$ has the (URP) and (COS).*

Remark 1.2. The adding-one-dimension and going-down argument of [11] play a crucial role in the proof of the (COS) and (URP).

Remark 1.3. In [17], it is shown that the (URP) and (COS) imply that the C*-algebra $\mathrm{C}(X) \rtimes \Gamma$ alway has stable rank one (classifiable or not), and satisfies the Toms-Winter conjecture. Thus, by the Theorem above, $\mathrm{C}(X) \rtimes \mathbb{Z}^d$ always has stable rank one (classifiable or not), and satisfies the Toms-Winter conjecture.

2. Notation and Preliminaries

2.1. **Topological Dynamical Systems.** In this paper, one only considers $\mathbb{Z}^d$-actions on a separable compact Hausdorff space X.

Definition 2.1. Consider a topological dynamical system $(X, T, \mathbb{Z}^d)$. A closed set $Y \subseteq X$ is said to be invariant if $T^n(Y) = Y$, $n \in \mathbb{Z}^d$, and $(X, T, \mathbb{Z}^d)$ is said to be minimal if $\varnothing$ and X are the only invariant closed subsets. The dynamical system $(X, T, \mathbb{Z}^d)$ is free if for any $x \in X$, $\{n \in \mathbb{Z}^d : T^n(x) = x\} = \{0\}$.

Remark 2.2. The dynamical system $(X, T, \mathbb{Z}^d)$ is induced by d commuting homeomorphisms of X, and vise versa.

Definition 2.3. A Borel measure μ on X is invariant under the action σ if $\mu(E) = \mu(T^n(E))$, for any $n \in \mathbb{Z}^d$ and any Borel set $E \subseteq X$. Denote by $\mathcal{M}_1(X, T, \mathbb{Z}^d)$ the collection of all invariant Borel probability measures on X. It is a Choquet simplex under the weak* topology.

Definition 2.4 (see [9] and [21])**.** Consider a topological dynamical system $(X, T, \mathbb{Z}^d)$, and let E be a subset of X. The orbit capacity of E is defined by

$$\mathrm{ocap}(E) := \lim_{N\to\infty} \frac{1}{N^d} \sup_{x\in X} \sum_{n\in\{0,1,\ldots,N-1\}^d} \chi_E(T^n(x)),$$

where χ_E is the characteristic function of E. The limit always exists.

Definition 2.5 (see [21])**.** Let $\mathcal{U}$ be an open cover of X. Define

$$D(\mathcal{U}) = \min\{\mathrm{ord}(\mathcal{V}) : \mathcal{V} \preceq U\},$$

where $\mathcal{V} = -1 + \sup_{x\in X} \sum_{V\in\mathcal{V}} \chi_V(x)$.

Consider a topological dynamical system $(X, T, \mathbb{Z}^d)$. Then the topological mean dimension of $(X, T, \mathbb{Z}^d)$ is defined by

$$\mathrm{mdim}(X, T, \mathbb{Z}^d) := \sup_{\mathcal{U}} \lim_{N\to\infty} \frac{1}{N^d} D(\bigvee_{n\in\{0,1,\ldots,N-1\}^d} T^{-n}(\mathcal{U})),$$

where $\mathcal{U}$ runs over all finite open covers of X.

Remark 2.6. It follows from the definition that if $\dim(X) < \infty$, then $\mathrm{mdim}(X, T, \mathbb{Z}^d) = 0$; By [21], if $(X, T, \mathbb{Z}^d)$ has at most countably many ergodic measures, then $\mathrm{mdim}(X, T, \mathbb{Z}^d) = 0$; and by [20], if $(X, T, \mathbb{Z}^d)$ has finite topological entropy, then $\mathrm{mdim}(X, T, \mathbb{Z}^d) = 0$.

2.2. **Crossed product C*-algebras.** Consider a topological dynamical system $(X, T, \mathbb{Z}^d)$. Then the crossed product C*-algebra $\mathrm{C}(X) \rtimes \mathbb{Z}^d$ is the universal C*-algebra

$$A = \mathrm{C}^*\{f, u_n;\ u_n f u_n^* = f \circ T^n,\ u_m u_n^* = u_{m-n},\ u_0 = 1,\ f \in \mathrm{C}(X),\ m, n \in \mathbb{Z}^d\}.$$

The C*-algebra A is nuclear, and if T is minimal, the C*-algebra A is simple. Moreover, the simplex of tracial states of $\mathrm{C}(X) \rtimes_\sigma \Gamma$ is canonically homeomorphic to the simplex of the invariant probability measures of $(X, T, \mathbb{Z}^d)$.

2.3. **Cuntz comparison of positive elements of a C*-algebra.**

Definition 2.7. Let A be a C*-algebra, and let $a, b \in A^+$. Then we say that a is Cuntz subequivalent to b, denote by $a \precsim b$, if there are x_i, y_i, $i = 1, 2, \ldots$, such that

$$\lim_{n\to\infty} x_i b y_i = a,$$

and we say that a is Cuntz equivalent to b if $a \precsim b$ and $b \precsim a$.

Let $\tau : A \to \mathbb{C}$ be a trace. Define the rank function

$$\mathrm{d}_\tau(a) := \lim_{n\to\infty} \tau(a^{\frac{1}{n}}) = \mu_\tau(\mathrm{sp}(a) \cap (0, +\infty)),$$

where μ_τ is the Borel measure induced by τ on the spectrum of a. It is well known that

$$\mathrm{d}_\tau(a) \leq \mathrm{d}_\tau(b), \quad \text{if } a \precsim b.$$

Example 2.8. Consider $h \in \mathrm{C}(X)^+$ and let μ be a probability measure on X. Then

$$\mathrm{d}_{\tau_\mu} = \mu(f^{-1}(0, +\infty)),$$

where τ_μ is the trace of $\mathrm{C}(X)$ induced by μ.

Let $f, g \in \mathrm{C}(X)$ be positive elements. Then f and g are Cuntz equivalent if and only if $f^{-1}(0, +\infty) = g^{-1}(0, +\infty)$. That is, their equivalence classes are determined by their open support. On the other hand, for each open set $E \subseteq X$, pick a continuous function

$$\varphi_E : X \to [0, +\infty) \quad \text{such that} \quad E = \varphi_E^{-1}(0, +\infty).$$

For instance, one can pick $\varphi_E(x) = d(x, X \setminus E)$, where d is a compatible metric on X. This notation will be used throughout this paper. Note that the Cuntz equivalence class of φ_E is independent of the choice of individual function φ_E.

Definition 2.9. Let $a \in A^+$, where A is a C*-algebra, and let $\varepsilon > 0$. Define

$$(a - \varepsilon)_+ = f(a) \in A,$$

where $f(t) = \max\{t - \varepsilon, 0\}$.

The following lemma is frequently used:

Lemma 2.10 (Section 2 of [27])**.** *Let a, b be positive elements of a C*-algebra A. Then $a \precsim b$ if and only if $(a - \varepsilon)_+ \precsim b$ for all $\varepsilon > 0$.*

Definition 2.11 (Definition 6.1 of [32])**.** Let A be a C*-algebra. Denote by $\mathrm{M}_n(A)$ the C*-algebra of $n \times n$ matrices over A. Regard $\mathrm{M}_n(A)$ as the upper-left conner of $\mathrm{M}_{n+1}(A)$, and denote by

$$\mathrm{M}_\infty(A) = \bigcup_{n=1}^{\infty} \mathrm{M}_n(A),$$

the algebra of all finite matrices over A.

The radius of comparison of a unital C*-algebra A, denoted by $\mathrm{rc}(A)$, is the infimum of the set of real numbers $r > 0$ such that if $a, b \in (\mathrm{M}_\infty(A))^+$ satisfy

$$\mathrm{d}_\tau(a) + r < \mathrm{d}_\tau(b), \quad \tau \in \mathrm{T}(A),$$

then $a \precsim b$, where $\mathrm{T}(A)$ is the simplex of tracial states. (In [32], the radius of comparison is defined in terms of quasitraces instead of traces; but since all the algebras considered in this note are nuclear, by [15], any quasitrace actually is a trace.)

Example 2.12. Let X be a compact Hausdorff space. Then

$$\mathrm{rc}(\mathrm{M}_n(\mathrm{C}(X))) \leq \frac{1}{2}\frac{\dim(X) - 1}{n}, \tag{2.1}$$

where dim(X) is the topological covering dimension of X (a lower bound of $\mathrm{rc}(\mathrm{C}(X))$ in terms of cohomological dimension is given in [2]).

The main result of this paper is a dynamical version of (2.1); that is,

$$\mathrm{rc}(\mathrm{C}(X) \rtimes \mathbb{Z}^d) \leq \frac{1}{2}\mathrm{mdim}(X, T, \mathbb{Z}^d)$$

if $(X, T, \mathbb{Z}^d)$ is minimal and free (Corollary 5.6).

3. Adding one dimension, going-down argument, R-boundary points, and R-interior points

Adding-one-dimension and going-down argument are introduced in [11], and they play a crucial role in this paper. Let us first take a brief review. Consider a minimal system $(X, T, \mathbb{Z}^d)$. Pick open sets $U' \subseteq U \subseteq X$ with $\overline{U'} \subseteq U$, and a continuous function $\varphi : X \to [0, 1]$ such that

$$\varphi|_{U'} = 1 \quad \text{and} \quad \varphi|_{X \setminus U} = 0.$$

Since $(X, T, \mathbb{Z}^d)$ minimal, there exists $L \in \mathbb{N}$ such that

$$\bigcup_{|n| \leq L} T^n(U') = X$$

and hence

(1) for any $x \in X$, there is $n \in \mathbb{Z}^d$ with $|n| \leq L$ such that $\varphi(T^n(x)) = 1$.

On the other hand, pick M such that

$$T^n(U), \quad |n| \leq M,$$

are mutually disjoint, and therefore

(2) if $\varphi(x) > 0$ for some $x \in X$, then $\varphi(T^n(x)) = 0$ for all nonzero $n \in \mathbb{Z}^k$ with $|n| \leq M$.

Note that $M \leq L$; by the freeness of $(X, T, \mathbb{Z}^d)$, the number M is arbitrarily large if U is sufficiently small.

Pick $x \in X$. Following from [11], one considers the set

$$\{(n, \frac{1}{\varphi(T^n(x))}) : n \in \mathbb{Z}^d,\ \varphi(T^n(x)) \neq 0\} \subseteq \mathbb{R}^{d+1},$$

and defines the Voronoi cell $V(x, n) \subseteq \mathbb{R}^{d+1}$ with center $(n, \frac{1}{\varphi(T^n(x))})$ by

$$V(x, n) = \left\{ \xi \in \mathbb{R}^{k+1} : \left\| \xi - (n, \frac{1}{\varphi(T^n(x))}) \right\| \leq \left\| \xi - (m, \frac{1}{\varphi(T^m(x))}) \right\|, \forall m \in \mathbb{Z}^d \right\},$$

where $\|\cdot\|$ is the ℓ^2-norm on $\mathbb{R}^{d+1}$. If $\varphi(T^n(x)) = 0$, then put

$$V(x, n) = \varnothing.$$

One then has a tiling

$$\mathbb{R}^{d+1} = \bigcup_{n \in \mathbb{Z}^d} V(x, n).$$

Pick $H > (L + \sqrt{d})^2$. For each $n \in \mathbb{Z}^d$, define

$$W_H(x, n) = V(x, n) \cap (\mathbb{R}^d \times \{-H\}),$$

and one has a tiling

$$\mathcal{W}_H : \mathbb{R}^d = \bigcup_{n \in \mathbb{Z}^d} W(x, n).$$

The following are some basic properties of this construction, and the proofs can be found in [11].

Lemma 3.1 (Lemma 4.1 of [11])**.** *With the construction above, one has*

(1) *$\mathcal{W}_H$ is continuous on x in the following sense: Suppose that $W(x,n)$ has non-empty interior. For any $\varepsilon > 0$, if $y \in X$ is sufficiently close to x, then the Hausdorff distance between $W_H(x,n)$ and $W_H(y,n)$ are smaller than ε.*
(2) *$\mathcal{W}_H$ is $\mathbb{Z}^d$-equivariant: $W_H(T^m(x), n-m) = -m + W_H(x,n)$.*
(3) *If $\varphi(T^n(x)) > 0$, then*

$$B_{\frac{M}{2}}(n, \frac{1}{\varphi(T^n(x))}) \subseteq V(x,n).$$

(4) *If $W_H(x,n)$ is non-empty, then*

$$1 \leq \frac{1}{\varphi(T^n(x))} \leq 2.$$

(5) *If $(a,-H) \in V(x,n)$, then*

$$\|a-n\| < L + \sqrt{d}.$$

Moreover, if one considers different horizontal cuts, at levels $-sH$ and $-H$ for some $s > 1$, one has the following lemma.

Lemma 3.2 (Lemma 4.1(4) of [11] and its proof)**.** *Let $s > 1$ and $r > 0$. One can choose M sufficiently large such that if $(a, -sH) \in V(x,n)$, then*

$$B_r(\frac{a}{s} + (1-\frac{1}{s})n) \subseteq W_H(x,n)$$

and

$$\left\| \frac{a}{s} + (1-\frac{1}{s})n - (a + \frac{(s-1)H}{sH+t}(n-a)) \right\| \leq \frac{4}{L+\sqrt{d}},$$

where $t = \frac{1}{\varphi(T^n(x))}$ and $\|\cdot\|$ is the ℓ^2-norm on $\mathbb{R}^d$.

Definition 3.3. Note that the point $(a + \frac{(s-1)H}{sH+t}(n-a), -H)$ is the image of $(a,-sH)$ in the plane $\mathbb{R}^d \times \{-H\}$ under the projection towards the center (n,t). Let us call $a + \frac{(s-1)H}{sH+t}(n-a)$ the H-projective image of a (with the center (n,t)).

The following is a lemma on convex bodies in $\mathbb{R}^d$, and the author is in debt to Tyrrell McAllister for the discussions.

Lemma 3.4. *Consider $\mathbb{R}^d$. For any $\varepsilon > 0$ and any $r > 0$, there is $N_0 > 0$ such that if $N \geq N_0$, then for any convex body $V \subseteq \mathbb{R}^d$, one has*

$$\frac{1}{N^d} \left| \{ n \in \mathbb{Z}^d : \mathrm{dist}(n, \partial V) \leq r,\ n \in I_N \} \right| < \varepsilon,$$

where $I_N = [0,N]^d$.

Proof. Pick N_0 sufficiently large such that

$$2\frac{\mathrm{vol}(\partial_{r+\sqrt{d}}(I_N))}{\mathrm{vol}(I_N)} < \varepsilon, \quad N > N_0,$$

where $\partial_E(K)$ denotes the E-neighbourhood of the boundary of a convex body K. Then, this N_0 satisfies the conclusion of the Lemma.

Indeed, for any $N \geq N_0$, denote by $\partial^+_{r+\sqrt{d}}(V \cap I_N)$ the outer $(r+\sqrt{d})$-neighborhood of the convex body $V \cap I_N$, and it follows from Steiner formula (see, for instance, (4.1.1) of [28]) that

$$\mathrm{vol}(\partial^+_{r+\sqrt{d}}(V \cap I_N)) = \sum_{j=1}^{d} C_d^j W_j(V \cap I_N)(r+\sqrt{d})^j,$$

where $W_j(V \cap I_N)$ is the j-th quermassintegral of $V \cap I_N$. Since the quermassintegrals W_j, $j = 1, ..., d$, are monotonic (see, for instance, Page 211 of [28]), one has

$$W_j(V \cap I_N) \leq W_j(I_N), \quad j = 1, 2, ..., d,$$

and hence

$$\begin{aligned}\mathrm{vol}(\partial^+_{r+\sqrt{d}}(V \cap I_N)) &= \sum_{j=1}^{d} C_d^j W_j(V \cap I_N)(r+\sqrt{d})^j \\ &\leq \sum_{j=1}^{d} C_d^j W_j(I_N)(r+\sqrt{d})^j \\ &= \mathrm{vol}(\partial^+_{r+\sqrt{d}}(I_N)).\end{aligned}$$

Since $\mathrm{vol}(\partial_{r+\sqrt{d}}(V \cap I_N)) \leq 2\mathrm{vol}(\partial^+_{r+\sqrt{d}}(V \cap I_N))$, one has

$$\frac{\mathrm{vol}(\partial_{r+\sqrt{d}}(V \cap I_N))}{\mathrm{vol}(I_N)} \leq 2\frac{\mathrm{vol}(\partial^+_{r+\sqrt{d}}(V \cap I_N))}{\mathrm{vol}(I_N)} \leq 2\frac{\mathrm{vol}(\partial_{r+\sqrt{d}}(I_N))}{\mathrm{vol}(I_N)} < \varepsilon.$$

On the other hand, note that

$$\left|\{n \in \mathbb{Z}^d : \mathrm{dist}(n, \partial V) \leq r,\ n \in I_N\}\right| \leq \mathrm{vol}(\partial_{r+\sqrt{d}}(V \cap I_N)),$$

and hence

$$\frac{1}{N^d}\left|\{n \in \mathbb{Z}^d : \mathrm{dist}(n, \partial V) \leq r,\ n \in I_N\}\right| \leq \frac{\mathrm{vol}(\partial_{r+\sqrt{d}}(V \cap I_N))}{\mathrm{vol}(I_N)} < \varepsilon,$$

as desired. □

Definition 3.5. Consider a continuous function $X \ni x \mapsto \mathcal{W}(x)$ with $\mathcal{W}(x)$ a $\mathbb{R}^d$-tiling. For each $R \geq 0$, a point $x \in X$ is said to be an R-interior point if $\mathrm{dist}(0, \partial\mathcal{W}(x)) > R$, where $\partial\mathcal{W}(x)$ denotes the union of the boundaries of the tiles of $\mathcal{W}$. Note that, in this case, the origin $0 \in \mathbb{R}^d$ is an interior point of a (unique) tile of $\mathcal{W}(x)$. Denote this tile by $\mathcal{W}(x)_0$, and denote the set of R-interior points by $\iota_R(\mathcal{T})$.

Otherwise (if $\mathrm{dist}(0, \partial\mathcal{W}(x)) \leq R$), the point x is said to be an R-boundary point. Denote by $\beta_R(\mathcal{T})$ the set of R-boundary points.

Note that $\beta_R(\mathcal{T})$ is closed and $\iota_R(\mathcal{T})$ is open.

Lemma 3.6. *Let $(X, T, \mathbb{Z}^d)$ be a minimal free dynamical system.*

Fix $s \in (1, 2)$. Let $R_0 > 0$ and $\varepsilon > 0$ be arbitrary. Let $N > N_0$, where N_0 the constant of Lemma 3.4 with respect to ε and $2R_0 + 4 + \sqrt{d}/2$, and let $R_1 > \max\{R_0, N\sqrt{d}\}$.

Then M can be chosen large enough such that there exist a finite open cover

$$U_1 \cup U_2 \cup \cdots \cup U_K \supseteq \beta_{R_0}(\mathcal{W}_{sH}),$$

and $n_1, n_2, ..., n_K \in \mathbb{Z}^d$ *such that*

(1) $T^{n_i}(U_i) \subseteq \iota_{R_1}(\mathcal{W}_H) \subseteq \iota_0(\mathcal{W}_H),\ i = 1, 2, ..., K,$

(2) *the open sets*

$$T^{n_i}(U_i), \quad i = 1, 2, ..., K,$$

can be grouped as

$$\begin{cases} T^{n_1}(U_1), ..., T^{n_{s_1}}(U_{s_1}), \\ T^{n_{s_1+1}}(U_{s_1+1}), ..., T^{n_{s_2}}(U_{s_2}), \\ \cdots \\ T^{n_{s_{m-1}+1}}(U_{s_{m-1}+1}), ..., T^{n_{s_m}}(U_{s_m}), \end{cases}$$

with $m \leq (\lfloor 2\sqrt{d} \rfloor + 1)^d$, *such that the open sets in each group are mutually disjoint,*

(3) *for each* $x \in \iota_0(\mathcal{W}_H)$ *and each* $c \in \mathrm{int}(\mathcal{W}_H(x)_0) \cap \mathbb{Z}^d$ *with* $\mathrm{dist}(c, \partial\mathcal{W}_H) > N\sqrt{d}$, *one has*

$$\frac{1}{N^d}\left|\left\{n \in \{0, 1, ..., N-1\}^d : T^{c+n}(x) \in \bigcup_{i=1}^{K} T^{n_i}(U_i)\right\}\right| < \varepsilon.$$

Proof. By Lemma 4.1(4) of [11] (see Lemma 3.2), one can choose $U' \subseteq U$ and φ such that M is sufficiently large so that for a fixed $H > (L + \sqrt{d})^2$, if $(a, -sH) \in V(x, n)$ for some $a \in \mathbb{R}^d$, then

$$B_{R_1+2R_0+1+\frac{\sqrt{d}}{2}}(\frac{a}{s} + (1 - \frac{1}{s})n) \times \{-H\} \in V(x, n)$$

and

$$\left\|\frac{a}{s} + (1 - \frac{1}{s})n - (a + \frac{(s-1)H}{sH + t}(n - a))\right\| \leq \frac{4}{L + \sqrt{d}} < 4, \tag{3.1}$$

where $t = \frac{1}{\varphi(T^n(x))}$, and $a + \frac{(s-1)H}{sH+t}(n - a)$ is the H-projective image of a.

For each $n \in \mathbb{Z}^d$, define

$$U_n = \{x \in X : \mathrm{dist}(0, \partial W_{sH}(x, n)) < 2R_0,\ \mathrm{int} W_{sH}(x, n) \neq \varnothing\}.$$

Note that U_n is open. For the same n, one also picks $h_n \in \mathbb{Z}^d$ such that

$$\left\|(1 - \frac{1}{s})n - h_n\right\| \leq \frac{\sqrt{d}}{2}. \tag{3.2}$$

For each $x \in U_n$, there is $a \in \partial W_{sH}(x, n) \subseteq \mathbb{R}^d$ with

$$\|a\| < 2R_0.$$

By the choice of M (hence H), one has

$$B_{R_1+2R_0+1+\frac{\sqrt{d}}{2}}(\frac{a}{s} + (1 - \frac{1}{s})n) \subseteq W_H(x, n). \tag{3.3}$$

Since

$$\left\|h_n - (\frac{a}{s} + (1 - \frac{1}{s})n)\right\| \leq \left\|\frac{a}{s}\right\| + \left\|(1 - \frac{1}{s})n - h_n\right\| < 2R_0 + \frac{\sqrt{d}}{2}, \tag{3.4}$$

by (3.3), one has

$$B_{R_1+1}(h_n) \subseteq W_H(x, n),$$

which implies

$$B_{R_1}(0) \subset B_{R_1+1}(0) \subseteq -h_n + W_H(x,n) = W_H(T^{h_n}(x), n - h_n). \tag{3.5}$$

In particular, $T^{h_n}(x) \in \iota_{R_1}(\mathcal{W}_H)$, which implies

$$T^{h_n}(U_n) \subseteq \iota_{R_1}(\mathcal{W}_H),$$

and this shows Property (1).

Note that by (3.1) and (3.4),

$$\left\| h_n - (a + \frac{(s-1)H}{sH+t}(n-a)) \right\| < 2R_0 + 4 + \frac{\sqrt{d}}{2}. \tag{3.6}$$

Since $a \in \partial W_{sH}(x,n)$, this implies that h_n is in the $(2R_0 + 4 + \frac{\sqrt{d}}{2})$-neighbourhood of the the H-projective image of $\partial W_{sH}(x,n)$ (with respect to (n,t)).

On the other hand, if $x \in \beta_{R_0}(\mathcal{W}_{sH})$, then $\mathrm{dist}(0, \partial W_{sH}(x,n)) \leq R_0$ for some $n \in \mathbb{Z}^d$ with $\mathrm{int}(W_{sH}(x,n)) \neq \varnothing$, which implies that $x \in U_n$. Therefore, $\{U_n : n \in \mathbb{Z}^d\}$ form an open cover of $\beta_{R_0}(\mathcal{W}_{sH})$. Since $\beta_{R_0}(\mathcal{W}_{sH})$ is a compact set, there is a finite subcover

$$U_{n_1}, U_{n_2}, ..., U_{n_K}.$$

(In fact, $\{U_n : \|n\| < L + \sqrt{d} + 2R_0\}$ already covers $\beta_{R_0}(\mathcal{W}_{sH})$ by (5) of Lemma 3.1.)

Assume that n_i and n_j satisfy

$$T^{h_{n_i}}(U_{n_i}) \cap T^{h_{n_j}}(U_{n_j}) \neq \varnothing.$$

Then there are $x_i \in U_{n_i}$ and $x_j \in U_{n_j}$ with

$$T^{h_{n_i}}(x_i) = T^{h_{n_j}}(x_j).$$

Since $x_i \in U_{n_i}$ and $x_j \in U_{n_j}$, by (3.5), one has that

$$B_R(0) \subseteq W_H(T^{h_{n_i}}(x_i), n_i - h_{n_i})$$

and

$$\begin{aligned} B_R(0) &\subseteq W_H(T^{h_{n_j}}(x_j), n_j - h_{n_j}) \\ &= W_H(T^{h_{n_i}}(x_i), n_j - h_{n_j}). \end{aligned}$$

Therefore, $n_i - h_{n_i} = n_j - h_{n_j}$, and

$$n_i - n_j = h_{n_i} - h_{n_j}.$$

Together with (3.2), one has

$$\begin{aligned} \|n_i - n_j\| &= \left\| h_{n_j} - h_{n_j} \right\| \\ &\leq (1 - \frac{1}{s}) \|n_i - n_j\| + \sqrt{d} \\ &< \frac{1}{2} \|n_i - n_j\| + \sqrt{d}, \end{aligned}$$

and hence

$$\|n_i - n_j\| < 2\sqrt{d}.$$

Note that the set $\mathbb{Z}^d$ can be divided into $(\lfloor 2\sqrt{d}\rfloor+1)^d$ groups $(\mathbb{Z}^d)_1$, ..., $(\mathbb{Z}^d)_{(\lfloor 2\sqrt{d}\rfloor+1)^d}$ such that any pair of elements inside each group has distance at least $2\sqrt{d}$, and therefore

$$T^{h_n}(U_n)\cap T^{h_{n'}}(U_{n'})=\varnothing,\quad n,n'\in(\mathbb{Z}^d)_m,\ m=1,...,(\lfloor 2\sqrt{d}\rfloor+1)^d.$$

Then group $U_{n_1},...,U_{n_K}$ as

$$\{U_{n_i}: i=1,...,K,\ n_i\in(\mathbb{Z}^d)_1\},...,\{U_{n_i}: i=1,...,K,\ n_i\in(\mathbb{Z}^d)_{(\lfloor 2\sqrt{d}\rfloor+1)^d}\},$$

and this shows Property (2).

Let $x\in\iota_0(\mathcal{W}_H)$ (so that $\mathcal{W}_H(x)_0$ is well defined). Write

$$\mathcal{W}_H(x)_0=W_H(x,n(x))=V(x,n(x))\cap(\mathbb{R}^d\times\{-H\}),\quad \text{where } n(x)\in\mathbb{Z}^d.$$

Assume there is $m\in\mathrm{int}(\mathcal{W}_H(x)_0)\cap\mathbb{Z}^d$ such that

$$T^m(x)\in T^{h_{n_k}}(U_{n_k}) \tag{3.7}$$

for some n_k.

Since $m\in\mathrm{int}(\mathcal{W}_H(x)_0)\cap\mathbb{Z}^d$, one has that

$$0\in\mathrm{int}(-m+W_H(x,n(x)))=\mathrm{int}W_H(T^m(x),n(x)-m).$$

Hence $T^m(x)\in\iota_0(\mathcal{W}_H)$ and

$$\mathcal{W}_H(T^m(x))_0=W_H(T^m(x),n(x)-m). \tag{3.8}$$

By the assumption (3.7), there is $x_{n_k}\in U_{n_k}$ such that

$$T^m(x)=T^{h_{n_k}}(x_{n_k}).$$

Then, with (3.5), one has

$$B_{R_1}(0)\subseteq W_H(T^{h_{n_k}}(x_{n_k}),n_k-h_{n_k})=W_H(T^m(x),n_k-h_{n_k}),$$

and therefore (with (3.8)),

$$W_H(T^m(x),n(x)-m)=W_H(T^{h_{n_k}}(x_{n_k}),n_k-h_{n_k})$$

and

$$V(T^m(x),n(x)-m)=V(T^{h_{n_k}}(x_{n_k}),n_k-h_{n_k}).$$

Hence, at the $-sH$ level, one also has

$$W_{sH}(T^m(x),n(x)-m)=W_{sH}(T^{h_{n_k}}(x_{n_k}),n_k-h_{n_k})=-h_{n_k}+W_{sH}(x_{n_k},n_k). \tag{3.9}$$

By (3.6), h_{n_k} is in the $(2R_0+4+\sqrt{d}/2)$-neighbourhood of the H-projective image of $\partial W_{sH}(x_{n_k},n_k)$, and therefore 0 is in the $(2R_0+4+\sqrt{d}/2)$-neighbourhood of the H-projective image of

$$-h_{n_k}+\partial W_{sH}(x_{n_k},n_k)=W_{sH}(T^{h_{n_k}}(x_{n_k}),n_k-h_{n_k})$$

Thus, by (3.9), the origin 0 is in the $(2R_0+4+\sqrt{d}/2)$-neighbourhood of the H-projective image of $\partial W_{sH}(T^m(x),n(x)-m)$, and hence m is in the $(2R_0+4+\sqrt{d}/2)$-neighbourhood of the H-projective image of $\partial W_{sH}(x,n(x))$, which is denoted by $\partial W^H_{sH}(x,n(x))$.

Therefore, for any $c\in\mathrm{int}(\mathcal{W}_H(x)_0)\cap\mathbb{Z}^d$ with $\mathrm{dist}(c,\partial\mathcal{W}_H)>N\sqrt{d}$, since

$$c+n\in\mathrm{int}(\mathcal{W}_H(x)_0),\quad n\in\{0,1,...,N-1\}^d,$$

one has

$$\begin{aligned} & \left\{n \in \{0,1,...,N-1\}^d : T^{c+n}(x) \in \bigcup_{i=1}^{K} h_i(U_i)\right\} \\ \subseteq\ & \left\{n \in \{0,1,...,N-1\}^d : \mathrm{dist}(c+n, \partial W^H_{sH}(x, n(x))) < 2R_0 + 4 + \sqrt{d}/2\right\}. \end{aligned}$$

Hence, by the choice of N and Lemma 3.4,

$$\begin{aligned} & \frac{1}{N^d}\left|\left\{n \in \{0,1,...,N-1\}^d : T^{c+n}(x) \in \bigcup_{i=1}^{K} h_i(U_i)\right\}\right| \\ \leq\ & \frac{1}{N^d}\left|\left\{n \in c+\{0,1,...,N-1\}^d : \mathrm{dist}(n, \partial W^H_{sH}(x, n(x))) < 2R_0 + 4 + \sqrt{d}/2\right\}\right| \\ <\ & \varepsilon. \end{aligned}$$

This proves Property (3). □

4. Two towers

4.1. **Rokhlin towers.** Let $x \mapsto \mathcal{W}(x) = \bigcup_{n \in \mathbb{Z}^d} W(x,n)$ be a map with $\mathcal{W}(x)$ a tiling of $\mathbb{R}^d$ and $W(x,n)$ is the cell with label n. Assume that the map $x \mapsto \mathcal{W}(x)$ is continuous in the sense that for any $\varepsilon > 0$ and any $W(x,n)$ with non-empty interior, if $y \in X$ is sufficiently close to x then the Hausdorff distance between $W(x,n)$ and $W(y,n)$ are smaller than ε. One also assumes that the map $x \mapsto \mathcal{W}(x)$ is equivariant in the sense that

$$W(T^{-m}(x), n+m) = m + W(x,n), \quad x \in X,\ m,n \in \mathbb{Z}^d.$$

The tiling functions $\mathcal{W}_H$ and $\mathcal{W}_{sH}$ constructed in the previous section clearly satisfy the assumptions above. With a such tiling function, one actually can build a Rokhlin tower as the following:

Let $N \in \mathbb{N}$ be arbitrary. Put

$$\Omega = \{x \in X : \mathrm{dist}(0, \partial\mathcal{W}(x)) > N\sqrt{d} \text{ and } \mathcal{W}(x)_0 = W(x,n) \text{ for some } n = 0 \mod N\},$$

where by $n = 0 \mod N$, one means $n_i = 0 \mod N$, $i = 1, 2, ..., d$, if $n = (n_1, n_2, ..., n_d) \in \mathbb{Z}^d$. Note that Ω is open.

Let $m \in \{0,1,...,N-1\}^d$. Pick arbitrary $x \in \Omega$ and consider $T^{-m}(x)$. Note that $0 \in W(x,n)$ for some $n = 0 \mod N$ and $\mathrm{dist}(0, \partial W(x,n)) > N\sqrt{d}$. Since

$$W(T^{-m}(x), n+m) = m + W(x,n),$$

one has that

$$0 \in \mathrm{int}W(T^{-m}(x), n+m) \quad \text{and} \quad n+m = m \mod N.$$

Hence

$$T^{-m}(\Omega) \subseteq \Omega'_m. \tag{4.1}$$

where

$$\Omega'_m := \{x \in X : 0 \notin \partial\mathcal{W}(x) \text{ and } \mathcal{W}(x)_0 = W(x,n),\ n = m \mod N\}.$$

For the same reason, if one defines

$$\Omega''_m := \{x \in X : \mathrm{dist}(0, \partial\mathcal{W}(x)) > 2N\sqrt{d} \text{ and } \mathcal{W}(x)_0 = W(x, n),\ n = m \mod N\},$$

then

$$\Omega''_m \subseteq T^{-m}(\Omega). \tag{4.2}$$

Since the sets

$$\Omega'_m, \quad m \in \{0, 1, ..., N-1\}^d,$$

are mutually disjoint, it follows from (4.1) that

$$T^{-m}(\Omega), \quad m \in \{0, 1, ..., N-1\}^d$$

are mutually disjoint. That is, it forms a Rokhlin tower for $(X, T, \mathbb{Z}^d)$.

On the other hand, by (4.2) and the construction of Ω''_m, one has

$$\bigsqcup_{m\in\{0,1,...,N-1\}^d} T^{-m}(\Omega) \supseteq \bigsqcup_{m\in\{0,1,...,N-1\}^d} \Omega''_m = \{x \in X : \mathrm{dist}(0, \partial\mathcal{W}(x)) > 2N\sqrt{d}\}. \tag{4.3}$$

In particular, one has

$$\mathrm{ocap}\left(X \setminus \bigsqcup_{m\in\{0,1,...,N-1\}^d} T^{-m}(\Omega)\right) \leq \mathrm{ocap}(\{x \in X : \mathrm{dist}(0, \partial\mathcal{W}(x)) \leq 2N\sqrt{d}\}). \tag{4.4}$$

Lemma 4.1. *For any $E > 0$, one has*

$$\mathrm{ocap}(\{x \in X : \mathrm{dist}(0, \partial\mathcal{W}(x)) \leq E\}) \leq \limsup_{R\to\infty} \frac{1}{\mathrm{vol}(B_R)} \sup_{x\in X} \mathrm{vol}(\partial_E\mathcal{W}(x) \cap B_R),$$

where $\partial_E\mathcal{W}(x) = \{\xi \in \mathbb{R}^d : \mathrm{dist}(\xi, \partial W(x)) \leq E\}$.

Proof. Pick an arbitrary $x \in X$ and an arbitrary positive number R, and consider the partial orbit

$$T^m(x), \quad \|m\| < R.$$

Note that if $\mathrm{dist}(0, \partial\mathcal{W}(T^m(x))) \leq E$ (i.e., $0 \in \partial_E\mathcal{W}(T^m(x))$) for some m, then

$$-m \in \partial_E\mathcal{W}(x).$$

Therefore

$$\{\|m\| < R : 0 \in \partial_E\mathcal{W}(T^m(x))\} \subseteq \{\|m\| < R : m \in \partial_E\mathcal{W}(x)\}.$$

As $N \to \infty$, one has

$$\begin{aligned} & \frac{1}{|B_R \cap \mathbb{Z}^d|} \left|\{\|m\| < R : 0 \in \partial_E\mathcal{W}(T^m(x))\}\right| \\ \leq\ & \frac{1}{|B_R \cap \mathbb{Z}^d|} \left|\{\|m\| < R : m \in \partial_E\mathcal{W}(x)\}\right| \\ \approx\ & \frac{1}{\mathrm{vol}(B_R)} \mathrm{vol}(\partial_E\mathcal{W}(x) \cap B_R), \quad \text{(if } R \text{ is sufficiently large).} \end{aligned}$$

Hence

$$\limsup_{R\to\infty} \frac{1}{|B_R \cap \mathbb{Z}^d|} \left|\{|m| < R : 0 \in \partial_E\mathcal{W}(T^m(x))\}\right| \leq \limsup_{R\to\infty} \frac{1}{\mathrm{vol}(B_R)} \sup_{x\in X} \mathrm{vol}(\partial_E\mathcal{W}(x) \cap B_R).$$

Since x is arbitrary, this proves the desired conclusion. □

Theorem 4.2. *Consider the minimal free dynamical system $(X, T, \mathbb{Z}^d)$. Then, for any $\varepsilon > 0$ and $N \in \mathbb{N}$, there is an open set $\Omega \subseteq X$ such that*

$$T^{-n}(\Omega), \quad n \in \{0, 1, ..., N-1\}^d$$

are mutually disjoint (hence form a Rokhlin tower), and

$$\operatorname{ocap}\left(X \setminus \bigcup_{n \in \{0,1,...,N-1\}^d} T^{-n}(\Omega)\right) < \varepsilon.$$

In other words, the system $(X, T, \mathbb{Z}^d)$ has the Uniform Rohklin Property (see Definition 1.1 and Lemma 3.2 of [24]*).*

Proof. By Lemma 4.2 of [11], there is an equivariant $\mathbb{R}^d$-tiling $x \mapsto \mathcal{W}(x)$ such that

$$\limsup_{R \to \infty} \frac{1}{\operatorname{vol}(B_R)} \sup_{x \in X} \operatorname{vol}(\partial_{2N\sqrt{d}} \mathcal{W}(x) \cap B_R) < \varepsilon.$$

Then, the statement follows from (4.4) and Lemma 4.1 (with $E = 2N\sqrt{d}$). □

4.2. **The two towers.** The Rokhlin tower constructed above in general cannot cover the whole space X. Consider the two continuous tiling functions $\mathcal{W}_{sH}$ and $\mathcal{W}_H$, and consider the Rokhlin towers $\mathcal{T}_0$ and $\mathcal{T}_1$ constructed from them respectively. It is still possible that $\mathcal{T}_0$ together with $\mathcal{T}_1$ do not cover the whole space X. However, in the following theorem, one can show that the complement of the tower $\mathcal{T}_0$ can be cut into pieces and then each piece can be translated into the tower $\mathcal{T}_1$ in a way that the order of the overlaps of the translations are universally bounded, and the intersection of the translations with each $\mathcal{T}_1$-orbit is uniformly small. This eventually leads to a Cuntz comparison of open sets for minimal free $\mathbb{Z}^d$-actions (Theorem 5.5).

Theorem 4.3. *Consider a minimal free dynamical system $(X, T, \mathbb{Z}^d)$. Let $N \in \mathbb{N}$ and $\varepsilon > 0$ be arbitrary. There exist two Rokhlin towers*

$$\mathcal{T}_0 := \{T^{-m}(\Omega_0) : m \in \{0, 1, ..., N_0 - 1\}^d\} \quad \text{and} \quad \mathcal{T}_1 := \{T^{-m}(\Omega_1) : m \in \{0, 1, ..., N_1 - 1\}^d\},$$

with $N_0, N_1 \geq N$ and $\Omega_0, \Omega_1 \subseteq X$ open, an open cover $\{U_1, U_2, ..., U_K\}$ of $X \setminus \bigcup_m T^{-m}(\Omega_0)$, and $h_1, h_2, ..., h_K \in \mathbb{Z}^d$ such that

(1) $T^{h_k}(U_k) \subseteq \bigcup_m T^{-m}(\Omega_1)$, $k = 1, 2, ..., K$;
(2) *the open sets*

$$T^{h_k}(U_k), \quad k = 1, 2, ..., K,$$

can be grouped as

$$\begin{cases} T^{h_1}(U_1), ..., T^{h_{s_1}}(U_{s_1}), \\ T^{h_{s_1+1}}(U_{s_1+1}), ..., T^{h_{s_2}}(U_{s_2}), \\ \cdots \\ T^{h_{s_{m-1}+1}}(U_{s_{m-1}+1}), ..., T^{h_{s_m}}(U_{s_m}), \end{cases}$$

for some $m \leq (\lfloor 2\sqrt{d} \rfloor + 1)^d$, *such that the open sets in each group are mutually disjoint;*

(3) *for each* $x \in \Omega_1$, *one has*

$$\frac{1}{N_1^d}\left|\left\{m \in \{0, 1, ..., N_1 - 1\}^d : T^m(x) \in \bigcup_{k=1}^{K} T^{n_k}(U_k)\right\}\right| < \varepsilon.$$

Proof. Applying Lemma 3.6 with $R_0 = 2N\sqrt{d}$, ε, and some $s \in (1, 2)$, together with some $N_1 > \max\{N(R_0, \varepsilon), N\}$ (in place of N) and $R_1 > \max\{R_0, 2N_1\sqrt{d}\}$, where $N(R_0, \varepsilon)$ is the constant of Lemma 3.4 with respect to ε and $2R_0 + 4 + \sqrt{d}/2$, there are two continuous equivariant $\mathbb{R}^d$-tilings $\mathcal{W}_{sH}$ and $\mathcal{W}_H$ for some (sufficiently large) $H > 0$, a finite open cover

$$U_1 \cup U_2 \cup \cdots \cup U_K \supseteq \beta_{R_0}(\mathcal{W}_{sH}),$$

and $n_1, n_2, ..., n_K \in \mathbb{Z}^d$ such that

(1) $T^{n_i}(U_i) \subseteq \iota_{R_1}(\mathcal{W}_H) \subseteq \iota_0(\mathcal{W}_H)$, $i = 1, 2, ..., K$;

(2) the open sets

$$T^{n_i}(U_i), \quad i = 1, 2, ..., K,$$

can be grouped as

$$\begin{cases} T^{n_1}(U_1), ..., T^{n_{s_1}}(U_{s_1}), \\ T^{n_{s_1+1}}(U_{s_1+1}), ..., T^{n_{s_2}}(U_{s_2}), \\ \cdots \\ T^{n_{s_{m-1}+1}}(U_{s_{m-1}+1}), ..., T^{n_{s_m}}(U_{s_m}), \end{cases}$$

with $m \leq (\lfloor 2\sqrt{d} \rfloor + 1)^d$, such that the open sets in each group are mutually disjoint;

(3) for each $x \in \iota_0(\mathcal{W}_H)$ and each $c \in \mathrm{int}(\mathcal{W}_H(x)_0) \cap \mathbb{Z}^d$ with $\mathrm{dist}(c, \partial\mathcal{W}_H) > N_1\sqrt{d}$, one has

$$\frac{1}{N_1^d}\left|\left\{n \in \{0, 1, ..., N_1 - 1\}^d : T^{c+n}(x) \in \bigcup_{i=1}^{K} T^{n_i}(U_i)\right\}\right| < \varepsilon.$$

Put

$$\Omega_0 = \{x \in X : \mathrm{dist}(0, \partial\mathcal{W}_{sH}(x)) > N\sqrt{d} \text{ and } \mathcal{W}_{sH}(x)_0 = W_{sH}(x, n),\ n = 0 \mod N\}.$$

Then

$$T^{-m}(\Omega_0), \quad m \in \{0, 1, ..., N_0 - 1\}^d$$

form a Rokhlin tower with $N_0 = N$, and by (4.3)

$$X \setminus \bigsqcup_{m \in \{0,1,...,N_0-1\}^d} T^{-m}(\Omega_0) \subseteq \{x \in X : \mathrm{dist}(0, \partial\mathcal{W}_{sH}(x)) \leq 2N\sqrt{d}\} = \beta_{2N\sqrt{d}}(\mathcal{W}_{sH}). \tag{4.5}$$

Thus, $U_1, U_2, ..., U_K$ form an open cover of $X \setminus \bigsqcup_{m \in \{0,1,...,N_0-1\}^d} T^{-m}(\Omega_0)$.

Put

$$\Omega_1 = \{x \in X : \mathrm{dist}(0, \partial\mathcal{W}_H(x)) > N_1\sqrt{d} \text{ and } \mathcal{W}_H(x)_0 = W_H(x, n),\ n = 0 \mod N_1\}.$$

Then

$$T^{-m}(\Omega_1), \quad m \in \{0, 1, ..., N_1 - 1\}^d,$$

form a Rokhlin tower, and by (4.3) (and the assumption that $R_1 > 2N_1\sqrt{d}$),

$$\bigsqcup_{m\in\{0,1,...,N_1-1\}^d} T^{-m}(\Omega_1) \supseteq \{x \in X : \mathrm{dist}(0, \partial\mathcal{W}_H) > 2N_1\sqrt{d}\} \supseteq \iota_{R_1}(\mathcal{W}_H). \tag{4.6}$$

Thus, $T^{-h_i}(U_i) \subseteq \bigsqcup_{m\in\{0,1,...,N_1-1\}^d} T^{-m}(\Omega_1)$.

If $x \in \Omega_1$ (hence $x \in \iota_0(\mathcal{W}_H)$ and $\mathrm{dist}(0, \partial\mathcal{W}_H) > N_1\sqrt{d}$), it then follows from (3) (with $c = 0$) that

$$\frac{1}{N_1^d}\left|\left\{m \in \{0, 1, ..., N_1 - 1\}^d : T^m(x) \in \bigcup_{k=1}^{K} T^{n_k}(U_k)\right\}\right| < \varepsilon,$$

as desired. □

5. Cuntz comparison of open sets, comparison radius, and the mean topological dimension

With the two-tower construction in the previous section, one is able to show that the C*-algebra $\mathrm{C}(X) \rtimes \mathbb{Z}^d$ has Cuntz-comparison on open sets (Theorem 5.5), and therefore the radius of comparison of $\mathrm{C}(X) \rtimes \mathbb{Z}^d$ is at most half of the mean dimension of $(X, T, \mathbb{Z}^d)$.

As a preparation, one has the following two very simple observations on the Cuntz semigroup of a C*-algebra.

Lemma 5.1. *Let A be a C*-algebra, and let $a_1, a_2, ..., a_m \in A$ be positive elements. Then*

$$[a_1] + [a_2] + \cdots + [a_m] \leq m[a_1 + a_2 + \cdots + a_m].$$

Proof. The lemma follows from the observation:

$$\begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_m \end{pmatrix} \leq \begin{pmatrix} a_1 + \cdots + a_m & & & \\ & a_1 + \cdots + a_m & & \\ & & \ddots & \\ & & & a_1 + \cdots + a_m \end{pmatrix}.$$

□

Lemma 5.2. *Let $U_1, U_2, ..., U_K \subseteq X$ be open sets which can be divided into M groups such that each group consists of mutually disjoint sets. Then*

$$[\varphi_{U_1}] + \cdots + [\varphi_{U_K}] \leq M[\varphi_{U_1\cup\cdots\cup U_K}] = M[\varphi_{U_1} + \cdots + \varphi_{U_K}]$$

Proof. Write $U_1, U_2, ..., U_K$ as

$$\{U_1, ..., U_{s_1}\}, \{U_{s_1+1}, ..., U_{s_2}\}, ..., \{U_{s_{m-1}+1}, ..., U_{s_M}\},$$

such that the open sets in each group are mutually disjoint. Then

$$[\varphi_{U_{s_i+1}}] + \cdots + [\varphi_{U_{s_{i+1}}}] = [\varphi_{U_{s_i+1}} + \cdots + \varphi_{U_{s_{i+1}}}] = [\varphi_{U_{s_i+1}\cup\cdots\cup U_{s_{i+1}}}], \quad i = 0, 1, ..., M-1,$$

and together with the lemma above, one has

$$\begin{aligned} [\varphi_{U_1}] + \cdots + [\varphi_{U_K}] &= [\varphi_{U_1} + \cdots + \varphi_{U_{s_1}}] + \cdots + [\varphi_{U_{s_{m-1}+1}} + \cdots + \varphi_{U_{s_M}}] \\ &= [\varphi_{U_1\cup\cdots\cup U_{s_1}}] + \cdots + [\varphi_{U_{s_{m-1}+1}\cup\cdots\cup U_{s_M}}] \\ &\leq M[\varphi_{U_1\cup\cdots\cup U_K}], \end{aligned}$$

as desired. □

Definition 5.3. Consider a topological dynamical system (X, Γ), where X is a compact metrizable space and Γ is a discrete group acting on X from the right, and consider a Rokhlin tower

$$\mathcal{T} = \{\Omega\gamma,\ \gamma \in \Gamma_0\},$$

where $\Omega \subseteq X$ is open and $\Gamma_0 \subseteq \Gamma$ is a finite set containing the unit e of the discrete group Γ. Define the C*-algebra

$$\mathrm{C}^*(\mathcal{T}) := \mathrm{C}^*\{u_\gamma \mathrm{C}_0(\Omega),\ \gamma \in \Gamma_0\} \subseteq \mathrm{C}(X) \rtimes \Gamma.$$

By Lemma 3.11 of [24], it is canonically isomorphic to $\mathrm{M}_{|\Gamma_0|}(\mathrm{C}_0(\Omega))$, and

$$\mathrm{C}_0(\bigcup_{\gamma \in \Gamma_0} \Omega\gamma) \ni \phi \mapsto \mathrm{diag}\{\phi|_{\Omega\gamma_1}, \phi|_{\Omega\gamma_2}, ..., \phi|_{\Omega\gamma_{|\Gamma_0|}}\} \in \mathrm{M}_{|\Gamma_0|}(\mathrm{C}_0(\Omega))\}$$

under this isomorphism.

The following comparison result essentially is a special case of Theorem 7.8 of [24].

Lemma 5.4 (Theorem 7.8 of [24])**.** *Let Z be a locally compact metrizable space, and consider $\mathrm{M}_n(\mathrm{C}_0(Z))$. Let $a, b \in \mathrm{M}_n(\mathrm{C}_0(Z))$ be two positive diagonal elements, i.e.,*

$$a(t) = \mathrm{diag}\{a_1(t), a_2(t), ..., a_n(t)\} \quad \text{and} \quad b(t) = \mathrm{diag}\{b_1(t), b_2(t), ..., b_n(t)\}$$

for some positive continuous functions $a_1, ..., a_n, b_1, ..., b_n : Z \to \mathbb{R}$. If

$$\mathrm{rank}(a(t)) \leq \frac{1}{4}\mathrm{rank}(b(t)), \quad t \in Z$$

and

$$4 < \mathrm{rank}(b(t)), \quad t \in Z,$$

then $a \precsim b$ in $\mathrm{M}_n(\mathrm{C}_0(Z))$.

Proof. It is enough to show that $(a - \varepsilon)_+ \precsim b$ for arbitrary $\varepsilon > 0$. For a given $\varepsilon > 0$, there is a compact subset $D \subseteq Z$ such that $(a-\varepsilon)_+$ is supported inside D. Denote by $\pi : \mathrm{M}_n(\mathrm{C}_0(Z)) \to \mathrm{M}_n(\mathrm{C}(D))$ the restriction map. One then has

$$\mathrm{rank}(\pi((a-\varepsilon)_+)(t)) \leq \frac{1}{4}\mathrm{rank}(\pi(b)(t)), \quad t \in D$$

and

$$\frac{1}{n} < \frac{1}{4n}\mathrm{rank}(b(t)), \quad t \in D.$$

By Theorem 7.8 of [24], one has that $\pi((a-\varepsilon)_+) \precsim \pi(b)$ in $\mathrm{M}_n(\mathrm{C}(D))$, that is, there is a sequence $(v_k) \subseteq \mathrm{M}_n(\mathrm{C}(D))$ such that $v_k(\pi(b))v_k^* \to \pi((a-\varepsilon)_+)$ as $k \to \infty$. Extend each v_k to a function in $\mathrm{M}_n(\mathrm{C}_0(Z))$, and still denote it by v_k. It is clear that the new sequence (v_k) satisfies $v_k b v_k^* \to (a-\varepsilon)_+$ as $k \to \infty$, and hence $(a-\varepsilon)_+ \precsim b$, as desired. □

Theorem 5.5. *Let $(X, T, \mathbb{Z}^d)$ be a minimal free dynamical system, and let $E, F \subseteq X$ be open sets such that*

$$\mu(E) \leq \frac{1}{4}\nu(F), \quad \mu \in \mathcal{M}_1(X, T, \mathbb{Z}^d).$$

Then,

$$[\varphi_E] \leq ((2\lfloor\sqrt{d}\rfloor + 1)^d + 1)[\varphi_F]$$

in the Cuntz semigroup of $\mathrm{C}(X) \rtimes \mathbb{Z}^d$*. In other words, the C*-algebra* $\mathrm{C}(X) \rtimes \mathbb{Z}^d$ *has* $(\frac{1}{4}, (2\lfloor\sqrt{d}\rfloor + 1)^d + 1)$*-Cuntz-comparison on open sets (see Definition 1.1).*

Proof. Let E and F be open sets satisfying the condition of the theorem. Let $\varepsilon > 0$ be arbitrary. In order to prove the statement of the theorem, it is enough to show that

$$(\varphi_E - \varepsilon)_+ \precsim \underbrace{\varphi_F \oplus \cdots \oplus \varphi_F}_{(2\lfloor\sqrt{d}\rfloor+1)^d+1}.$$

For the given ε, pick a compact set $E' \subseteq E$ such that

$$(\varphi_E - \varepsilon)_+(x) = 0, \quad x \notin E'. \tag{5.1}$$

By the assumption of the theorem, one has that

$$\mu(E') < \frac{1}{4}\mu(F), \quad \mu \in \mathcal{M}_1(X, T, \mathbb{Z}^n), \tag{5.2}$$

and then there is $N \in \mathbb{N}$ such that for any $M > N$ and any $x \in X$,

$$\frac{1}{M^d}\{m \in \{0, 1, ..., M-1\}^d : T^{-m}(x) \in E'\} < \frac{1}{4}\frac{1}{M^d}\{m \in \{0, 1, ..., M-1\}^d : T^{-m}(x) \in F\}. \tag{5.3}$$

Otherwise, there are sequences $N_k \in \mathbb{N}$, $x_k \in X$, $k = 1, 2, ...$, such that $N_k \to \infty$ as $k \to \infty$, and for any k,

$$\frac{1}{N_k^d}\{m \in \{0, 1, ..., N_k-1\}^d : T^{-m}(x_k) \in E'\} \geq \frac{1}{4}\frac{1}{N_k^d}\{m \in \{0, 1, ..., N_k-1\}^d : T^{-m}(x_k) \in F\}.$$

That is

$$4\delta_{N_k,x_k}(E') \geq \delta_{N_k,x_k}(F), \quad k = 1, 2, ..., \tag{5.4}$$

where $\delta_{N_k,x_k} = \frac{1}{N_k^d}\sum_{m\in\{0,1,...,N_k-1\}^d}\delta_{T^{-m}(x_k)}$ and δ_y is the Diract measure concentrated at y. Let δ_∞ be a limit point of $\{\delta_{N_k,x_k}, k = 1, 2, ...\}$ and it is clear that $\delta_\infty \in \mathcal{M}_1(X, T, \mathbb{Z}^d)$. Passing to a subsequence of k, one has

$$\begin{aligned}
\delta_\infty(F) &\leq \liminf_{k\to\infty}\delta_{N_k,x_k}(F) && (F \text{ is open}) \\
&\leq 4\liminf_{k\to\infty}\delta_{N_k,x_k}(E') && (\text{by } (5.4)) \\
&\leq 4\limsup_{k\to\infty}\delta_{N_k,x_k}(E') \\
&\leq 4\delta_\infty(E'), && (E' \text{ is closed})
\end{aligned}$$

which contradicts to (5.2).

With (5.1) and (5.3), one has that for any $M > N$ and any $x \in X$,

$$\begin{aligned}
&\frac{1}{M^d}\{m \in \{0,1,...,M-1\}^d : (\varphi_E - \varepsilon)_+(T^{-m}(x)) > 0\} \\
\leq\ & \frac{1}{M^d}\{m \in \{0,1,...,M-1\}^d : T^{-m}(x) \in E'\} \\
<\ & \frac{1}{4}\frac{1}{M^d}\{m \in \{0,1,...,M-1\}^d : T^{-m}(x) \in F\} \\
=\ & \frac{1}{4}\frac{1}{M^d}\{m \in \{0,1,...,M-1\}^d : \varphi_F(T^{-m}(x)) > 0\}.
\end{aligned} \tag{5.5}$$

Also note that since $(X, T, \mathbb{Z}^d)$ is minimal, there is $\delta > 0$ such that for any $M > N$,

$$\frac{1}{4M^d}\left|\{m \in \{0,1,...,M-1\}^d : \varphi_F(T^{-m}(x)) > 0\}\right| > \delta, \quad x \in X \tag{5.6}$$

Let

$$\mathcal{T}_0 = \{T^{-m}(\Omega_0), \quad m \in \{0,1,...,N_0-1\}^d\}$$

and

$$\mathcal{T}_1 = \{T^{-m}(\Omega_1), \quad m \in \{0,1,...,N_1-1\}^d\}$$

be the two towers obtained from Theorem 4.3 with respect to $\max\{N, \sqrt[d]{\frac{1}{\delta}}\}$ and δ. Denote by $U_1, U_2, ..., U_K$ and $n_1, n_2, ..., n_K \in \mathbb{Z}^d$ be the open sets and group elements, respectively, obtained from Theorem 4.3.

Pick $\chi_0 \in \mathrm{C}(X)^+$ such that

$$\begin{cases} \chi_0(x) = 1, & x \notin \bigcup_{k=1}^K U_k, \\ \chi_0(x) > 0, & x \in \bigsqcup_{m\in\{0,1,...,N_0-1\}^d} T^{-m}(\Omega_0), \\ \chi_0(x) = 0, & x \notin \bigsqcup_{m\in\{0,1,...,N_0-1\}^d} T^{-m}(\Omega_0). \end{cases} \tag{5.7}$$

Note that then $(1-\chi_0)$ is supported in $U_1 \cup U_2 \cup \cdots \cup U_K$. Consider

$$(\varphi_E - \varepsilon)_+ = (\varphi_E - \varepsilon)_+(1-\chi_0) + (\varphi_E - \varepsilon)_+\chi_0.$$

Then, for any $x \in \Omega_0$, it follows from (5.5) and (5.7) that

$$\begin{aligned}
&\left|\{m \in \{0,1,...,N_0-1\}^d : ((\varphi_E - \varepsilon)_+\chi_0)(T^{-m}(x)) > 0\}\right| \\
=\ & \left|\{m \in \{0,1,...,N_0-1\}^d : (\varphi_E - \varepsilon)_+(T^{-m}(x)) > 0\}\right| \\
<\ & \frac{1}{4}\left|\{m \in \{0,1,...,N_0-1\}^d : \varphi_F(T^{-m}(x)) > 0\}\right| \\
=\ & \frac{1}{4}\left|\{m \in \{0,1,...,N_0-1\}^d : (\varphi_F\chi_0)(T^{-m}(x)) > 0\}\right|.
\end{aligned}$$

Therefore, under the isomorphism $\mathrm{C}^*(\mathcal{T}_0) \cong \mathrm{M}_{N_0^d}(\mathrm{C}_0(\Omega_0))$, one has

$$\mathrm{rank}(((\varphi_E - \varepsilon)_+\chi_0)(x)) \leq \frac{1}{4}\mathrm{rank}((\varphi_F\chi_0)(x)), \quad x \in \Omega_0.$$

Moreover, it follows from (5.6) and the fact that $N_0 > \sqrt[d]{\frac{1}{\delta}}$ that for any $x \in \Omega_0$,

$$\frac{1}{4N_0^d}\mathrm{rank}((\varphi_F\chi_0)(x)) = \frac{1}{4N_0^d}\left|\{m \in \{0,1,...,N_0-1\}^d : \varphi_F(T^{-m}(x)) > 0\}\right| > \delta > \frac{1}{N_0^d}.$$

Thus, by Lemma 5.4, one has that

$$(\varphi_E-\varepsilon)_+\chi_0 \precsim \varphi_F\chi_0 \precsim \varphi_F. \tag{5.8}$$

Consider $(\varphi_E-\varepsilon)_+(1-\chi_0)$. Since $(1-\chi_0)$ is supported in $U_1\cup U_2\cup\cdots\cup U_K$, one has that

$$(\varphi_E-\varepsilon)_+(1-\chi_0)\precsim(1-\chi_0)\precsim\varphi_{U_1\cup\cdots\cup U_K}\sim\varphi_{U_1}+\cdots+\varphi_{U_K}\precsim\varphi_{U_1}\oplus\cdots\oplus\varphi_{U_K}.$$

On the other hand, by Lemma 5.2,

$$\varphi_{T^{n_1}(U_1)}\oplus\cdots\oplus\varphi_{T^{n_K}(U_K)}\precsim\bigoplus_{(2\lfloor\sqrt{d}\rfloor+1)^d}(\varphi_{T^{n_1}(U_1)}+\cdots+\varphi_{T^{n_K}(U_K)}).$$

Note that $\varphi_{U_i}\sim\varphi_{T^{n_i}(U_i)}$, $i=1,2,...,K$, and one has

$$(\varphi_E-\varepsilon)_+(1-\chi_0)\precsim\bigoplus_{(2\lfloor\sqrt{d}\rfloor+1)^d}(\varphi_{T^{n_1}(U_1)\cup\cdots\cup T^{n_K}(U_K)}). \tag{5.9}$$

By Theorem 4.3,

$$\frac{1}{N_1^d}\left|\left\{m\in\{0,1,...,N_1-1\}^d : T^{-m}(x)\in\bigcup_{k=1}^{K}T^{n_k}(U_k)\right\}\right|<\delta,\quad x\in\Omega_1. \tag{5.10}$$

Let $\chi_1: X\to[0,1]$ be a continuous function such that

$$\begin{cases}\chi_1(x)>0, & x\in\bigsqcup_{m\in\{0,1,...,N_1-1\}^d}T^{-m}(\Omega_1),\\ \chi_1(x)=0, & x\notin\bigsqcup_{m\in\{0,1,...,N_1-1\}^d}T^{-m}(\Omega_1).\end{cases}$$

Then

$$\frac{1}{4N_1^d}\mathrm{rank}((\varphi_F\chi_1)(x))=\frac{1}{4N_1^d}\left|\{m\in\{0,1,...,N_1-1\}^d:\varphi_F(T^{-m}(x))>0\}\right|>\delta>\frac{1}{N_1^d},\quad x\in\Omega_1,$$

and hence, for any $x\in\Omega_1$, with (5.10), one has

$$\begin{aligned}\mathrm{rank}(\varphi_{T^{n_1}(U_1)\cup\cdots\cup T^{n_K}(U_K)}(x)) &= \left|\left\{m\in\{0,1,...,N_1-1\}^d : T^{-m}(x)\in\bigcup_{k=1}^{K}T^{n_k}(U_k)\right\}\right|\\ &< N_1^d\delta<\frac{1}{4}\mathrm{rank}((\varphi_F\chi_1)(x)).\end{aligned}$$

By Lemma 5.4,

$$\varphi_{T^{n_1}(U_1)\cup\cdots\cup T^{n_K}(U_K)}\precsim\varphi_F\chi_1\precsim\varphi_F,$$

and together with (5.9) and (5.8),

$$\begin{aligned}(\varphi_E-\varepsilon)_+ &\precsim (\varphi_E-\varepsilon)_+(1-\chi_0)\oplus(\varphi_E-\varepsilon)_+\chi_0\\ &\precsim (\bigoplus_{(2\lfloor\sqrt{d}\rfloor+1)^d}(\varphi_{T^{n_1}(U_1)\cup\cdots\cup T^{n_K}(U_K)}))\oplus\varphi_F\\ &\precsim (\bigoplus_{(2\lfloor\sqrt{d}\rfloor+1)^d}\varphi_F)\oplus\varphi_F,\end{aligned}$$

as desired. □

Theorem 5.6. *Let $(X, T, \mathbb{Z}^d)$ be a minimal free dynamical system. Then*

$$\mathrm{rc}(\mathrm{C}(X) \rtimes \mathbb{Z}^d) \leq \frac{1}{2}\mathrm{mdim}(X, T, \mathbb{Z}^d).$$

Proof. By Theorem 5.5, the C*-algebra $\mathrm{C}(X) \rtimes \mathbb{Z}^d$ has the (COS). By Theorem 4.2, the dynamical system $(X, T, \mathbb{Z}^d)$ has the (URP). Then the statement follows directly from Theorem 4.8 of [24]. □

The following corollary generalizes Corollary 4.9 of [4] (where $d = 1$) and generalizes the classifiability result of [29] (where $\dim(X) < \infty$).

Theorem 5.7. *Let $(X, T, \mathbb{Z}^d)$ be a minimal free dynamical system with mean dimension zero, then $\mathrm{C}(X) \rtimes \mathbb{Z}^d$ is classified by its Elliott invariant. In particular, if $\dim(X) < \infty$, or $(X, T, \mathbb{Z}^d)$ has at most countably many ergodic measures, or $(X, T, \mathbb{Z}^d)$ has finite topological entropy, then $\mathrm{C}(X) \rtimes \mathbb{Z}^d$ is classified by its Elliott invariant.*

Proof. By Theorem 4.2 and Theorem 5.5, the dynamical system $(X, \mathbb{Z}^d)$ has the (URP) and (COS). Then the statement follows from Theorem 4.8 of [25] □

Remark 5.8. In [17], it is also shown that the (URP) and (COS) implies that the C*-algebra $\mathrm{C}(X) \rtimes \Gamma$, classifiable or not, always has (topological) stable rank one; and the C*-algebra $\mathrm{C}(X) \rtimes \Gamma$ satisfies the Toms-Winter conjecture (i.e., it is classifiable if, and only if it has the strict comparison of positive elements). Therefore, as corollaries of Theorem 5.6, the C*-algebra $\mathrm{C}(X) \rtimes \mathbb{Z}^d$, classifiable or not, always has stable rank one, and the C*-algebra $\mathrm{C}(X) \rtimes \mathbb{Z}^d$ satisfies the Toms-Winter conjecture.

The following is a generalization of Corollary 5.7 of [4].

Corollary 5.9. *Let $(X_1, T_1, \mathbb{Z}^{d_1})$ and $(X_2, T_2, \mathbb{Z}^{d_2})$ be minimal free dynamical systems where $d_1, d_2 \in \mathbb{N}$. Then the tensor product C*-algebra $(\mathrm{C}(X_1) \rtimes \mathbb{Z}^{d_1}) \otimes (\mathrm{C}(X_2) \rtimes \mathbb{Z}^{d_2})$ is classified by its Elliott invariant.*

Proof. Note that

$$(\mathrm{C}(X_1) \rtimes \mathbb{Z}^{d_1}) \otimes (\mathrm{C}(X_2) \rtimes \mathbb{Z}^{d_2}) \cong \mathrm{C}(X_1 \times X_2) \rtimes (\mathbb{Z}^{d_1} \times \mathbb{Z}^{d_2}),$$

where $\mathbb{Z}^{d_1} \times \mathbb{Z}^{d_2}$ acting on $X_1 \times X_2$ by

$$(T_1 \times T_2)^{(n_1, n_2)}((x_1, x_2)) = (T_1^{n_1}(x_1), T_2^{n_2}(x_2)), \quad n_1 \in \mathbb{Z}^{d_1},\ n_2 \in \mathbb{Z}^{d_2}.$$

By the argument of Remark 5.8 of [4], one has

$$\mathrm{mdim}(X_1 \times X_2, T_1 \times T_2, \mathbb{Z}^{d_1} \times \mathbb{Z}^{d_2}) = 0,$$

and the statement then follows from Theorem 5.7. □

References

[1] G. A. Elliott, G. Gong, H. Lin, and Z. Niu. On the classification of simple amenable C*-algebras with finite decomposition rank, II. 07 2015. URL: http://arxiv.org/abs/1507.03437, arXiv:1507.03437.

[2] G. A. Elliott and Z. Niu. On the radius of comparison of a commutative C*-algebra. *Canad. Math. Bull.*, 56(4):737–744, 2013. doi:10.4153/CMB-2012-012-9.

[3] G. A. Elliott and Z. Niu. On the classification of simple amenable C*-algebras with finite decomposition rank. In R. S. Doran and E. Park, editors, *"Operator Algebras and their Applications: A Tribute to Richard V. Kadison", Contemporary Mathematics*, volume 671, pages 117–125. Amer. Math. Soc., 2016. arXiv:http://dx.dot.org/10.1090/conm/671/13506.

[4] G. A. Elliott and Z. Niu. The C*-algebra of a minimal homeomorphism of zero mean dimension. *Duke Math. J.*, 166(18):3569–3594, 2017. doi:10.1215/00127094-2017-0033.

[5] Bernardo Freitas Paulo Da Costa. Deux exemples sur la dimension moyenne d'un espace de courbes de Brody. *Ann. Inst. Fourier (Grenoble)*, 63(6):2223–2237, 2013. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=3237445.

[6] J. Giol and D. Kerr. Subshifts and perforation. *J. Reine Angew. Math.*, 639:107–119, 2010. URL: http://dx.doi.org/10.1515/CRELLE.2010.012, doi:10.1515/CRELLE.2010.012.

[7] G. Gong, H. Lin, and Z. Niu. Classification of finite simple amenable $\mathcal{Z}$-stable C*-algebras, I. C*-algebras with generalized tracial rank one. *C. R. Math. Acad. Sci. Soc. R. Can.*, 42(3):63–450, 2020.

[8] G. Gong, H. Lin, and Z. Niu. Classification of finite simple amenable $\mathcal{Z}$-stable C*-algebras, II. C*-algebras with rational generalized tracial rank one. *C. R. Math. Acad. Sci. Soc. R. Can.*, 42(4):451–539, 2020.

[9] M. Gromov. Topological invariants of dynamical systems and spaces of holomorphic maps. I. *Math. Phys. Anal. Geom.*, 2(4):323–415, 1999. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=1742309, doi:10.1023/A:1009841100168.

[10] Y. Gutman. Embedding $\mathbb{Z}^k$-actions in cubical shifts and $\mathbb{Z}^k$-symbolic extensions. *Ergodic Theory Dynam. Systems*, 31(2):383–403, 2011. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=2776381, doi:10.1017/S0143385709001096.

[11] Y. Gutman, E. Lindenstrauss, and M. Tsukamoto. Mean dimension of $\mathbb{Z}^k$-actions. *Geom. Funct. Anal.*, 26(3):778–817, 2016. URL: http://dx.doi.org/10.1007/s00039-016-0372-9, doi:10.1007/s00039-016-0372-9.

[12] Yonatan Gutman. Embedding topological dynamical systems with periodic points in cubical shifts. *Ergodic Theory Dynam. Systems*, 37(2):512–538, 2017. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=3614036, doi:10.1017/etds.2015.40.

[13] Yonatan Gutman and Masaki Tsukamoto. Mean dimension and a sharp embedding theorem: extensions of aperiodic subshifts. *Ergodic Theory Dynam. Systems*, 34(6):1888–1896, 2014. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=3272776, doi:10.1017/etds.2013.30.

[14] Yonatan Gutman and Masaki Tsukamoto. Embedding minimal dynamical systems into Hilbert cubes. *Invent. Math.*, 221(1):113–166, 2020. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=4105086, doi:10.1007/s00222-019-00942-w.

[15] U. Haagerup. Quasitraces on exact C*-algebras are traces. *C. R. Math. Acad. Sci. Soc. R. Can.*, 36(2-3):67–92, 2014.

[16] D. Kerr and G. Szabo. Almost finiteness and the small boundary property. 07 2018. URL: https://arxiv.org/pdf/1807.04326, arXiv:1807.04326.

[17] C. Li and Z. Niu. Stable rank of $\mathrm{C}(X) \rtimes \Gamma$. *arXiv: 2008.03361*, 2020.

[18] Hanfeng Li. Sofic mean dimension. *Adv. Math.*, 244:570–604, 2013. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=3077882, doi:10.1016/j.aim.2013.05.005.

[19] Hanfeng Li and Bingbing Liang. Mean dimension, mean rank, and von Neumann–Lück rank. *J. Reine Angew. Math.*, 739:207–240, 2018. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=3808261, doi:10.1515/crelle-2015-0046.

[20] E. Lindenstrauss. Mean dimension, small entropy factors and an embedding theorem. *Inst. Hautes Études Sci. Publ. Math.*, (89):227–262 (2000), 1999. URL: http://www.numdam.org/item?id=PMIHES_1999__89__227_0.

[21] E. Lindenstrauss and B. Weiss. Mean topological dimension. *Israel J. Math.*, 115:1–24, 2000. URL: http://dx.doi.org/10.1007/BF02810577, doi:10.1007/BF02810577.

[22] Elon Lindenstrauss and Masaki Tsukamoto. From rate distortion theory to metric mean dimension: variational principle. *IEEE Trans. Inform. Theory*, 64(5):3590–3609, 2018. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=3798396, doi:10.1109/TIT.2018.2806219.

[23] Shinichiroh Matsuo and Masaki Tsukamoto. Brody curves and mean dimension. *J. Amer. Math. Soc.*, 28(1):159–182, 2015. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=3264765, doi:10.1090/S0894-0347-2014-00798-0.

[24] Z. Niu. Comparison radius and mean topological dimension: Rokhlin property, comparison of open sets, and subhomogeneous C*-algebras. *J. Analyse Math., accepted*, 2020.

[25] Z. Niu. $\mathcal{Z}$-stability of $\mathrm{C}(X) \rtimes \Gamma$. *Trans. Amer. Math. Soc*, accepted.

[26] N. C. Phillips. The C*-algebra of a minimal homeomorphism with finite mean dimension has finite radius of comparison. 05 2016. URL: https://arxiv.org/abs/1605.07976, arXiv:1605.07976.

[27] M. Rørdam. On the structure of simple C*-algebras tensored with a UHF-algebra. II. *J. Funct. Anal.*, 107(2):255–269, 1992. URL: http://dx.doi.org/10.1016/0022-1236(92)90106-S, doi:10.1016/0022-1236(92)90106-S.

[28] R. Schneider. *Convex bodies: the Brunn-Minkowski theory*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1993. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=1216521, doi:10.1017/CBO9780511526282.

[29] G. Szabó. The Rokhlin dimension of topological $\mathbb{Z}^m$-actions. *Proc. Lond. Math. Soc. (3)*, 110(3):673–694, 2015. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=3342101, doi:10.1112/plms/pdu065.

[30] G. Szabo, J. Wu, and J. Zacharias. Rokhlin dimension for actions of residually finite groups. 08 2014. URL: http://arxiv.org/abs/1408.6096, arXiv:1408.6096.

[31] A Tikuisis, S. White, and W. Winter. Quasidiagonality of nuclear C*-algebras. *Ann. of Math. (2), to appear*, 09. URL: http://arxiv.org/abs/1509.08318, arXiv:1509.08318.

[32] A. S. Toms. Flat dimension growth for C*-algebras. *J. Funct. Anal.*, 238(2):678–708, 2006.

[33] A. S. Toms and W. Winter. Minimal dynamics and K-theoretic rigidity: Elliott's conjecture. *Geom. Funct. Anal.*, 23(1):467–481, 2013. URL: http://dx.doi.org/10.1007/s00039-012-0208-1, doi:10.1007/s00039-012-0208-1.

[34] Masaki Tsukamoto. Deformation of Brody curves and mean dimension. *Ergodic Theory Dynam. Systems*, 29(5):1641–1657, 2009. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=2545021, doi:10.1017/S014338570800076X.

[35] Masaki Tsukamoto. Mean dimension of the dynamical system of Brody curves. *Invent. Math.*, 211(3):935–968, 2018. URL: https://mathscinet.ams.org/mathscinet-getitem?mr=3763403, doi:10.1007/s00222-017-0758-9.

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