$\mathcal{Z}$-STABILITY OF TRANSFORMATION GROUP C*-ALGEBRAS

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Abstract. Let (X, Γ) be a free and minimal topological dynamical system, where X is a separable compact Hausdorff space and Γ is a countable infinite discrete amenable group. It is shown that if (X, Γ) has the Uniform Rokhlin Property (URP) and Cuntz comparison of open sets (COS), then $\mathrm{mdim}(X, \Gamma) = 0$ implies that $(\mathrm{C}(X) \rtimes \Gamma) \otimes \mathcal{Z} \cong \mathrm{C}(X) \rtimes \Gamma$, where mdim is the mean dimension of (X, Γ), $\mathcal{Z}$ is the Jiang-Su algebra, and $\mathrm{C}(X) \rtimes \Gamma$ is the transformation group C*-algebra of (X, Γ). In particular, in this case, $\mathrm{mdim}(X, \Gamma) = 0$ implies that the C*-algebra $\mathrm{C}(X) \rtimes \Gamma$ is classified by the Elliott invariant.

1. Introduction

Let Γ be a discrete amenable group, and let (Ω, μ) be a σ-finite standard measure space. Let $(\Omega, \mu) \curvearrowleft \Gamma$ be a free and ergodic action with absolutely continuous finite invariant measure. By the classification of injective von Neumann algebras, it is well known that the von Neumann II_1-factor $\mathrm{L}^\infty(\Omega, \mu) \rtimes \Gamma$ is isomorphic to the unique hyperfinite II_1-factor R. Thus, all such crossed products $\mathrm{L}^\infty(\Omega, \mu) \rtimes \Gamma$ are isomorphic.

In the topological setting, consider a compact separable Hausdorff space X, and consider a minimal and free action $X \curvearrowleft \Gamma$. Then the transformation group C*-algebra $\mathrm{C}(X) \rtimes \Gamma$ is simple separable unital nuclear and satisfies the UCT. Thus it is a very natural object for the Elliott's classification program of nuclear C*-algebras.

Many efforts have been devoted to the classifiability of $\mathrm{C}(X) \rtimes \Gamma$ (in term of the K-theoretical Elliott invariant); see, for instance, [35], [27] [26], [42], [38], [37], [45], etc. However, as shown by Giol and Kerr in [11], there exist minimal and free actions $X \curvearrowleft \mathbb{Z}$ such that the C*-algebras $A = \mathrm{C}(X) \rtimes \mathbb{Z}$ are not classified by the Elliott invariant, and these C*-algebras do not absorb the Jiang-Su algebra $\mathcal{Z}$ tensorially (i.e., $A \otimes \mathcal{Z} \not\cong A$).

The dynamical systems constructed in [11] have non-zero mean (topological) dimension; and in [9], it is shown that if a minimal and free $\mathbb{Z}$-action has zero mean dimension (this particularly includes all strictly ergodic systems and all minimal dynamical systems with finite topological entropy, see [28]), then the C*-algebra $\mathrm{C}(X) \rtimes \mathbb{Z}$ must be $\mathcal{Z}$-absorbing and is classifiable (see [10] and [7]).

In this paper, one considers an arbitrary discrete amenable group Γ, and studies the $\mathcal{Z}$-stability of $\mathrm{C}(X) \rtimes \Gamma$. Under the assumption that (X, Γ) has the Uniform Rokhlin Property (URP) and Cuntz comparison of Open Sets (COS), which are introduced in [33], one has that $\mathrm{mdim}(X, \Gamma) = 0$ implies that $(\mathrm{C}(X) \rtimes \Gamma) \otimes \mathcal{Z} \cong \mathrm{C}(X) \rtimes \Gamma$, where mdim is the mean

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dimension. In particular, this implies that $\mathrm{C}(X) \rtimes \Gamma$ is classified by its Elliott invariant (see, [14], [15], [8], [6], [39], and [3]).

Recall

Definition 1.1 (Definition 3.1 and Definition 4.1 of [33]). A topological dynamical system (X, Γ), where Γ is a discrete amenable group, is said to have Uniform Rokhlin Property (URP) if for any $\varepsilon > 0$ and any finite set $K \subseteq \Gamma$, there exist closed sets $B_1, B_2, ..., B_S \subseteq X$ and (K, ε)-invariant sets $\Gamma_1, \Gamma_2, ..., \Gamma_S \subseteq \Gamma$ such that

$$B_s\gamma, \quad \gamma \in \Gamma_s,\ s = 1, ..., S,$$

are mutually disjoint and

$$\mathrm{ocap}(X \setminus \bigsqcup_{s=1}^{S} \bigsqcup_{\gamma \in \Gamma_s} B_s\gamma) < \varepsilon,$$

where ocap denote the orbit capacity (see, for instance, Definition 5.1 of [29]).

The dynamical system (X, Γ) is said to have (λ, m)-Cuntz-comparison of open sets, where $\lambda \in (0, 1]$ and $m \in \mathbb{N}$, if for any open sets $E, F \subseteq X$ with

$$\mu(E) < \lambda\mu(F), \quad \mu \in \mathcal{M}_1(X, \Gamma),$$

where $\mathcal{M}_1(X, \Gamma)$ is the simplex of all invariant probability measures on X, then

$$\varphi_E \precsim \underbrace{\varphi_F \oplus \cdots \oplus \varphi_F}_{m} \quad \text{in } \mathrm{C}(X) \rtimes \Gamma,$$

where φ_E and φ_F are continuous functions supporting on E and F respectively.

The dynamical system (X, Γ) is said to have Cuntz comparison of Open Sets (COS) if it has (λ, m)-Cuntz-comparison on open sets for some λ and m.

Remark 1.2. The ideas of the (URP) have been used in [28], [17], and [18] to study zero mean dimension and small boundary property for $\mathbb{Z}$ or $\mathbb{Z}^k$-actions. One should also compare the (URP) to the almost finiteness of [21] which furthermore requires that the diameters of all level sets $B_s\gamma$, $\gamma \in \Gamma_s$, $s = 1, ..., S$, are arbitrarily small; the almost finiteness in measure is shown to be equivalent to the small boundary property ([21]).

For the (COS), one should compare it to the (topological) dynamical comparison, which was introduced by Winter and appears in [20] for general groups. For (Cuntz) comparison with multiplicities, see, for instance, [44]; dynamical comparison with multiplicities is also considered in [30]. The ideas of the dynamical comparison actually have a long history, see, for example, [12] and [13], and it is straightforward to verify that the dynamical comparison implies the (COS) (but whether the converse holds is unknown to the author). (A Cuntz subequivalence relation using only normalizers is considered in [25], and a version of (COS) using this strong version of Cuntz subequivalence is shown to imply the dynamical comparison.)

The properties of (URP) and (COS) have been verified for the following cases: any free minimal $\mathbb{Z}^d$-action has the (URP) and has $(\frac{1}{4}, (2\lfloor\sqrt{d}\rfloor + 1)^d + 1)$-Cuntz-comparison of open sets ([32]); any free and minimal Γ-action has the (URP) and has $(\frac{1}{4}, 1)$-Cuntz-comparison of open sets if Γ has subexponential growth and (X, Γ) is an extension of a Cantor system ([33]).

In [33], it is shown that if (X, Γ) has the (URP) and (COS), then the comparison radius of the C*-algebra $\mathrm{C}(X) \rtimes \Gamma$ is at most half of the mean dimension of (X, Γ). In particular, if $\mathrm{mdim}(X, \Gamma) = 0$, then the C*-algebra $\mathrm{C}(X) \rtimes \Gamma$ has the strict comparison of positive elements (see Definition 2.7), which, as a part of the Toms-Winter conjecture, should imply the $\mathcal{Z}$-stability (this has been verified in the case that the C*-algebra has finitely many extreme tracial states in [31], and then been generalized independently to the case that the set of extreme tracial states is finite dimensional in [36], [22], and [41], and then to the case that the algebra has Uniform Property Gamma in [2]) (in the forthcoming paper [24], it is shown that the (URP) and (COS) imply that the C*-algebra $\mathrm{C}(X) \rtimes \Gamma$, classifiable or not, always has stable rank one, and $\mathrm{C}(X) \rtimes \Gamma$ indeed satisfies the Toms-Winter conjecture).

Under the assumption that (X, Γ) has the small boundary property (SBP) (which implies zero mean dimension, see [29], and is shown in [28] and [18] to be equivalent to zero mean dimension in the case $\Gamma = \mathbb{Z}^d$), Kerr and Szabo show in [21] (Theorem 9.4) that the C*-algebra $\mathrm{C}(X) \rtimes \Gamma$ has the Uniform Property Gamma, and hence the strict comparison of positive elements implies $\mathcal{Z}$-stability for $\mathrm{C}(X) \rtimes \Gamma$.

In this note, one shows the following:

Theorem (Theorem 4.8)**.** *Let (X, Γ) be a free and minimal topological dynamical system with the (URP) and (COS). If (X, Γ) has mean dimension zero, then $(\mathrm{C}(X) \rtimes \Gamma) \otimes \mathcal{Z} \cong \mathrm{C}(X) \rtimes \Gamma$, where $\mathcal{Z}$ is the Jiang-Su algebra.*

In particular, let (X_1, Γ_1) and (X_2, Γ_2) be two free minimal dynamical systems with the (URP) and (COS), and zero mean dimension, then

$$\mathrm{C}(X_1) \rtimes \Gamma_1 \cong \mathrm{C}(X_2) \rtimes \Gamma_2$$

if and only if

$$\mathrm{Ell}(\mathrm{C}(X_1) \rtimes \Gamma_1) \cong \mathrm{Ell}(\mathrm{C}(X_2) \rtimes \Gamma_2),$$

where $\mathrm{Ell}(\cdot) = (\mathrm{K}_0(\cdot), \mathrm{K}_0^+(\cdot), [1], \mathrm{T}(\cdot), \rho, \mathrm{K}_1(\cdot))$ is the Elliott invariant. Moreover, these C-algebras are inductive limits of unital subhomogeneous C*-algebras.*

As a consequence, the following crossed-product C*-algebras are $\mathcal{Z}$-stable:

Corollary (Corollary 4.9)**.** *Let (X, Γ) be a free and minimal topological dynamical system with mean dimension zero. Assume that*

- *either $\Gamma = \mathbb{Z}^d$ for some $d \geq 1$, or*
- *(X, Γ) is an extension of a Cantor system and Γ has subexponetial growth.*

Then, the C-algebra $\mathrm{C}(X) \rtimes \Gamma$ is classified by the Elliott invariant and is an inductive limit of unital subhomogeneous C*-algebras.*

Two approaches are provided in this paper: The first approach is more self-contained and more C*-algebra oriented. It is to show that the C*-algebra $\mathrm{C}(X) \rtimes \Gamma$ is tracially $\mathcal{Z}$-stable; since $\mathrm{C}(X) \rtimes \Gamma$ is nuclear, it follows from [31] and [19] that $\mathrm{C}(X) \rtimes \Gamma$ actually is $\mathcal{Z}$-stable.

In the second approach (Section 5), one proves the following dynamical system statement:

$$\mathrm{mdim}0 + \mathrm{URP} \Rightarrow \mathrm{SBP},$$

which might be interesting by itself. If, in addition, the system is assumed to have the (COS), it follows from [33] that the C*-algebra $\mathrm{C}(X) \rtimes \Gamma$ has strict comparison of positive elements. Hence, with the SBP, the $\mathcal{Z}$-stability of $\mathrm{C}(X) \rtimes \Gamma$ also follows from the Theorem 9.4 and Corollary 9.5 of [21].

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2. Notation and preliminaries

2.1. Topological Dynamical Systems.

Definition 2.1. A topological dynamical system (X, Γ) consists of a separable compact Hausdorff space X, a discrete group Γ, and a homomorphism $\Gamma \to \mathrm{Homeo}(X)$, where $\mathrm{Homeo}(X)$ is the group of homeomorphisms of X, acting on X from the right. In this paper, we frequently omit the word topological, and just refer it as a dynamical system.

The dynamical system (X, Γ) is said to be free if $x\gamma = x$ implies $\gamma = e$, where $x \in X$ and $\gamma \in \Gamma$.

A closed set $Y \subseteq X$ is said to be invariant if

$$Y\gamma = Y, \quad \gamma \in \Gamma,$$

and the dynamical system (X, Γ) is said to be minimal if $\varnothing$ and X are the only invariant closed subsets.

Definition 2.2. A Borel measure μ on X is invariant if for any Borel set $E \subseteq X$, one has

$$\mu(E) = \mu(E\gamma), \quad \gamma \in \Gamma.$$

Denote by $\mathcal{M}_1(X, \Gamma)$ the set of all invariant Borel probability measures on X. It is a Choquet simplex under the weak* topology.

Definition 2.3. Let Γ be a (countable) discrete group. Let $K \subseteq \Gamma$ be a finite set and let $\delta > 0$. Then a finite set $F \subseteq \Gamma$ is said to be (K, ε)-invariant if

$$\frac{|FK\Delta F|}{|F|} < \varepsilon.$$

The group Γ is amenable if there is a sequence (Γ_n) of finite subsets of Γ such that for any (K, ε), the set Γ_n is (K, ε)-invariant if n is sufficiently large. The sequence (Γ_n) is called a Følner sequence.

The K-interior of a finite set $F \subseteq \Gamma$ is defined as

$$\mathrm{int}_K(F) = \{\gamma \in F : \gamma K \subseteq F\}.$$

Note that

$$|F \setminus \mathrm{int}_K(F)| \leq |K|\,|FK \setminus F| \leq |K|\,|FK\Delta F|\,,$$

and hence for any $\varepsilon > 0$, if F is $(K, \frac{\varepsilon}{|K|})$-invariant, then

$$\frac{|F \setminus \mathrm{int}_K(F)|}{|F|} < \varepsilon.$$

Definition 2.4 (see [29])**.** Consider a topological dynamical system (X, Γ), where Γ is amenable, and let $E \subseteq X$. The orbit capacity of E is defined by

$$\mathrm{ocap}(E) := \lim_{n\to\infty} \frac{1}{|\Gamma_n|} \sup_{x\in X} \sum_{\gamma\in\Gamma_n} \chi_E(x\gamma),$$

where (Γ_n) is a Følner sequence, and χ_E is the characteristic function of E. The limit always exists and is independent from the choice of the Følner sequence (Γ_n).

Definition 2.5 (see [16] and [29])**.** Let $\mathcal{U}$ be an open cover of X. Define

$$D(\mathcal{U}) = \min\{\mathrm{ord}(\mathcal{V}) : \mathcal{V} \text{ is an open cover of } X \text{ and } \mathcal{V} \preceq U\},$$

where

$$\mathrm{ord}(\mathcal{V}) = -1 + \sup_{x\in X} \sum_{V\in\mathcal{V}} \chi_V(x),$$

and $\mathcal{V} \preceq \mathcal{U}$ means that, for any $V \in \mathcal{V}$, there is $U \in \mathcal{U}$ with $V \subseteq U$.

Consider a topological dynamical system (X, Γ), where Γ is a discrete amenable group. The mean topological dimension is defined by

$$\mathrm{mdim}(X, \Gamma) := \sup_{\mathcal{U}} \lim_{n\to\infty} \frac{1}{|\Gamma_n|} D(\bigwedge_{\gamma\in\Gamma_n} \gamma^{-1}(\mathcal{U})),$$

where $\mathcal{U}$ runs over all finite open covers of X, (Γ_n) is a Følner sequence (the limit is independent from the choice of (Γ_n)), and $\alpha \wedge \beta$ denotes the open cover

$$\{U \cap V : U \in \alpha,\ V \in \beta\}$$

for any open covers α and β.

2.2. **Crossed product C*-algebras.** Consider a topological dynamical system (X, Γ). The (full) crossed product C*-algebra $A = \mathrm{C}(X) \rtimes \Gamma$ is defined to be the universal C*-algebra

$$\mathrm{C}^*\{f, u_\gamma;\ u_\gamma f u_\gamma^* = f(\cdot\gamma) = f \circ \gamma,\ u_{\gamma_1} u_{\gamma_2}^* = u_{\gamma_1\gamma_2^{-1}},\ u_e = 1,\ f \in \mathrm{C}(X),\ \gamma, \gamma_1, \gamma_2 \in \Gamma\}.$$

The C*-algebra A is nuclear (Corollary 7.18 of [43]) if Γ is amenable. If, moreover, (X, Γ) is minimal and topologically free, the C*-algebra A is simple (Theorem 5.16 of [5] and Théorème 5.15 of [47]), i.e., A has no non-trivial two-sided ideals. A is also called the transformation group C*-algebra of (X, Γ).

2.3. **Cuntz semigroups.**

Definition 2.6. Let A be a C*-algebra, and let $a, b \in A^+$. The element a is said to be Cuntz sub-equivalent to b, denoted by $a \precsim b$, if there are x_i, y_i, $i = 1, 2, ...$, such that

$$\lim_{i\to\infty} x_i b y_i = a,$$

and we say that a is Cuntz equivalent to b, denoted by $a \sim b$, if $a \precsim b$ and $b \precsim a$. Then the Cuntz semigroup of A, denoted by $\mathrm{W}(A)$, is defined as

$$(\mathrm{M}_\infty(A))^+/\sim$$

with the addition

$$[a] + [b] = \left[\begin{pmatrix} a & \\ & b \end{pmatrix}\right],$$

where $(\mathrm{M}_\infty(A))^+ := \bigcup_{n=1}^\infty \mathrm{M}_n^+(A)$ and $[\cdot]$ denotes the equivalence class.

Definition 2.7. Let A be a C*-algebra, let $\mathrm{T}(A)$ denote the set of all tracial states of A, equipped with the topology of pointwise convergence. Note that if A is unital, the set $\mathrm{T}(A)$ is a Choquet simplex.

Let a be a positive element of $\mathrm{M}_\infty(A)$ and $\tau \in \mathrm{T}(A)$; define

$$\mathrm{d}_\tau(a) = \lim_{n\to\infty} \tau(a^{\frac{1}{n}}),$$

where τ is extended naturally to $\mathrm{M}_\infty(A)$. The function

$$\mathrm{T}(A) \ni \tau \mapsto \mathrm{d}_\tau(a) \in \mathbb{R}^+$$

is the limit of an increasing sequence of strictly positive affine functions on $\mathrm{T}(A)$, so it is lower semicontinuous.

It is well known that if $a \precsim b$, then

$$\mathrm{d}_\tau(a) \leq \mathrm{d}_\tau(b), \quad \tau \in \mathrm{T}(A).$$

If the C*-algebra A satisfies the property that for any positive elements $a, b \in \mathrm{M}_\infty(A)$ with

$$\mathrm{d}_\tau(a) < \mathrm{d}_\tau(b), \quad \tau \in \mathrm{T}(A),$$

then $a \precsim b$, the C*-algebra A is said to have the strict comparison of positive elements.

Remark 2.8. Note that if $A = \mathrm{M}_n(\mathrm{C}_0(X))$, where X is a locally compact Hausdorff space, and τ be a trace of A. Then, for any positive element $a \in \mathrm{M}_\infty(A) \cong \mathrm{M}_\infty(\mathrm{C}_0(X))$ and any $\tau \in \mathrm{T}(A)$, one has

$$\tau(a) = \int_X \frac{1}{n}\mathrm{Tr}(a(x))d\mu_\tau \quad \text{and} \quad \mathrm{d}_\tau(a) = \int_X \frac{1}{n}\mathrm{rank}(a(x))d\mu_\tau,$$

where μ_τ is the Borel measure on X induced by τ.

Also recall

Definition 2.9 ([46])**.** Let A, B be C*-algebras, and let $\varphi : A \to B$ be a completely positive contractive linear map (c.p.c map). Then φ is said to be order zero if

$$a \perp b \Longrightarrow \varphi(a) \perp \varphi(b), \quad a, b \in A.$$

3. The Cuntz semigroup of $\mathrm{C}(X) \rtimes \Gamma$

In this section, let us show that $\mathrm{C}(X) \times \Gamma$ is tracially 0-divisible whenever (X, Γ) has the (URP), (COS), and mean dimension zero.

The following is a version of Theorem 3.4 of [40] for the C*-algebra $\mathrm{C}(X) \rtimes \Gamma$.

Proposition 3.1. *Let $A = \mathrm{C}(X) \rtimes \Gamma$, where (X, Γ) is free, minimal, has the (URP) and zero mean dimension. Then, for any continuous affine function $\alpha : \mathrm{T}(A) \to (0, \infty)$ and any $\varepsilon > 0$, there is a positive element $a \in \mathrm{M}_\infty(A)$ such that*

$$|\alpha(\tau) - \mathrm{d}_\tau(a)| < \varepsilon, \quad \forall \tau \in \mathrm{T}(A).$$

Proof. Without loss of generality, one may assume that $|\Gamma| = \infty$, as otherwise, the crossed product C*-algebra A is isomorphic to a matrix algebra, and the statement of the proposition clearly holds. One may also assume that

$$\varepsilon < \frac{1}{4}. \tag{3.1}$$

By Corollary 3.10 of [1], there is a positive element $a' \in A$ such that

$$\alpha(\tau) = \tau(a'), \quad \forall \tau \in \mathrm{T}(A).$$

Since the action is minimal, the algebra A is simple, and hence there is a $\delta \in (0, 1)$ such that

$$\tau(a') > \delta, \quad \forall \tau \in \mathrm{T}(A). \tag{3.2}$$

Also pick M such that

$$\tau(a') \leq \|a'\| < M, \quad \forall \tau \in \mathrm{T}(A). \tag{3.3}$$

Let $\varepsilon \in (0, \frac{1}{4})$ be arbitrary. Let $\mathcal{F} \subseteq A$ be an arbitrary finite set, and let

$$0 < \varepsilon' < \min\{\varepsilon, \frac{\delta}{4}\} \tag{3.4}$$

be arbitrary ($\mathcal{F}$ and ε' will be fixed in the next paragraph). Applying Theorem 3.9 of [33] (with $m = 1$) to $\{a'\} \cup \mathcal{F}$ (in place of $\{f_1, ..., f_n\}$), 1 (in place of h), and $\min\{\frac{\varepsilon'}{M+1}, \varepsilon'\delta\}$ (in place of δ), there exist $a'' \in A$, a finite set $\mathcal{F}' \subseteq A$, $h \in \mathrm{C}(X)^+$ (in place of p), and a sub-C*-algebra $C \subseteq A$ with $C \cong \bigoplus_{s=1}^{S} \mathrm{M}_{n_s}(\mathrm{C}_0(Z_s))$ and closed sets $[Z_s] \subseteq Z_s$ such that the following properties hold:

(1) for any $f \in \mathcal{F}$, there is $f' \in \mathcal{F}'$ such that $\|f - f'\| < \varepsilon'$, (Theorem 3.9 (1) of [33])
(2) $\|a' - a''\| < \varepsilon'$, $\|ha'' - a''h\| < \varepsilon'$, $\|hf' - f'h\| < \varepsilon'$, $\forall f' \in \mathcal{F}'$, (Theorem 3.9 (1)(2) of [33])
(3) $h \in C$, $ha''h \in C$, $hf'h \in C$, $\forall f' \in \mathcal{F}'$ (Theorem 3.9 (3) of [33]),
(4) $\|h\| \leq 1$, $\tau(1 - h) < \varepsilon'$, $\forall \tau \in \mathrm{T}_1(A)$, (Theorem 3.9 (5) of [33]),
(5) $\mu(X \setminus h^{-1}(1)) < \frac{\varepsilon'}{M+1}$, $\forall \mu \in \mathcal{M}_1(X, \Gamma)$, (Theorem 3.9 (5) of [33]),
(6) under the isomorphism $C \cong \bigoplus_{s=1}^{S} \mathrm{M}_{n_s}(\mathrm{C}_0(Z_s))$, the element h has the form

$$h = \bigoplus_{s=1}^{S} \mathrm{diag}\{h_{s,1}, ..., h_{s,n_s}\},$$

where $h_{s,i} : Z_s \to [0,1]$, and

$$\frac{1}{n_s}\left|\{1 \leq i \leq n_s : h_{s,i}(x) = 1\}\right| > 1 - \varepsilon', \quad x \in [Z_s],\ s = 1, ..., S,$$

(Theorem 3.9 (7) of [33]),

(7)
$$\frac{\dim([Z_s])}{n_s} < \varepsilon'\delta, \quad s = 1, 2, ..., S,$$

(Theorem 3.9 (4) of [33]),

(8) each n_s, $s = 1, ..., S$, is sufficiently large such that the interval $(2n_s\delta\varepsilon' + 1, 4n_s\varepsilon' - 1)$ contains at least one strictly positive integer (this follows from the assumption $|\Gamma| = \infty$ and the the proof of Theorem 3.9 [33], where $n_s = |\Gamma_s|$ for the Rokhlin tower (B_s, Γ_s), which can be arbitrarily large for the given $(\mathcal{F}, \varepsilon')$).

Put

$$a_1' = h^{\frac{1}{2}} a'' h^{\frac{1}{2}}.$$

First, note that, with ε' sufficiently small, by Properties (4) and (2), for any $\tau \in \mathrm{T}(A)$,

$$\tau(a_1') = \tau(h^{\frac{1}{2}} a'' h^{\frac{1}{2}}) \approx_\varepsilon \tau((1-h)^{\frac{1}{2}} a'' (1-h)^{\frac{1}{2}} + h^{\frac{1}{2}} a'' h^{\frac{1}{2}}) \approx_\varepsilon \tau(a'') \approx_\varepsilon \tau(a'). \tag{3.5}$$

One asserts that with $\mathcal{F}$ sufficiently large and ε' sufficiently small further, one has

$$M > \tau(\pi(a_1')) > \delta, \quad \forall \tau \in \mathrm{T}(\pi(C)), \tag{3.6}$$

where π is the standard quotient map from $C \cong \bigoplus_{s=1}^S \mathrm{M}_{n_s}(\mathrm{C}_0(Z_s))$ to $\bigoplus_{s=1}^S \mathrm{M}_{n_s}(\mathrm{C}_0([Z_s]))$. Then, fix the pair $(\mathcal{F}, \varepsilon')$.

Indeed, suppose the contrary, there then exist a sequence of finite subsets $\mathcal{F}_i' \subseteq A$, $i = 1, 2, ...$, with dense union and a sequence of positive numbers ε_i, $i = 1, 2, ...$, decreasing to 0, sub-C*-algebras $C_i \subseteq A$, $i = 1, 2, ...$, elements $a_i'' \in A$, $i = 1, 2, ...$, and positive elements $h_i \in C_i$, $i = 1, 2, ...$, such that

- $\|a' - a_i''\| < \varepsilon_i$,
- $\left\|h_i^{\frac{1}{4}} f' - f' h_i^{\frac{1}{4}}\right\| < \varepsilon_i$, $\forall f' \in \mathcal{F}_i'$,
- $h_i a_i'' h_i \in C_i$, $h_i \in C_i$, and $h_i f' h_i \in C_i$, $\forall f' \in \mathcal{F}_i'$, so that

$$h_i^{\frac{1}{2}} a_i'' h_i^{\frac{1}{2}} \in C_i, \quad h_i^{\frac{1}{4}} a_i'' h_i^{\frac{1}{4}} \in C_i, \quad h_i^{\frac{1}{2}} f' h_i^{\frac{1}{2}} \in C_i, \quad \text{and} \quad h_i^{\frac{1}{4}} f' h_i^{\frac{1}{4}} \in C_i, \quad \forall f' \in \mathcal{F}_i',$$

- there exists $\tau_i \in \mathrm{T}(\pi(C_i))$ such that

$$\tau_i(\pi_i(h_i^{\frac{1}{2}} a_i'' h_i^{\frac{1}{2}})) \leq \delta \quad \text{or} \quad \tau_i(\pi_i(h_i^{\frac{1}{2}} a_i'' h_i^{\frac{1}{2}})) \geq M, \tag{3.7}$$

  where π_i is the standard quotient map from $C_i \cong \bigoplus_{s=1}^S \mathrm{M}_{n_s}(\mathrm{C}_0(Z_s))$ to $\bigoplus_{s=1}^S \mathrm{M}_{n_s}(\mathrm{C}_0([Z_s]))$,
- $\tau(\pi(h_i)) > 1 - \varepsilon_i$, $\forall \tau \in \mathrm{T}(\pi(C_i))$ (this follows from Property (6) and Remark 2.8).

Consider the linear functional

$$\rho_i : A \ni a \mapsto \tau_i(\pi_i(h_i^{\frac{1}{2}} a h_i^{\frac{1}{2}})) \in \mathbb{C},$$

and note that

$$\|\rho_i\| = \rho_i(1_A) = \tau_i(\pi(h_i)) > 1 - \varepsilon_i.$$

Also note that for, any $a, b \in \mathcal{F}_i'$,

$$\begin{aligned}\rho_i(ab) &= \tau_i(\pi_i(h_i^{\frac{1}{2}}abh_i^{\frac{1}{2}})) \approx_{2\varepsilon_i} \tau_i(\pi_i(h_i^{\frac{1}{4}}ah_i^{\frac{1}{4}}h_i^{\frac{1}{4}}bh_i^{\frac{1}{4}})) = \tau_i(\pi_i(h_i^{\frac{1}{4}}bh_i^{\frac{1}{4}}h_i^{\frac{1}{4}}ah_i^{\frac{1}{4}}))\\ &\approx_{2\varepsilon_i} \tau_i(\pi_i(h_i^{\frac{1}{2}}bah_i^{\frac{1}{2}})) = \rho_i(ba).\end{aligned}$$

Thus, any accumulation point of $\{\rho_i\}$, say ρ_∞, is actually a tracial state. However, by (3.7), there exists an accumulation point, still denoted by ρ_∞, such that

$$\rho_\infty(a') = \lim_{i\to\infty} \tau_i(\pi_i(h^{\frac{1}{2}}a_i''h_i^{\frac{1}{2}})) \leq \delta \quad \text{or} \quad \rho_\infty(a') = \lim_{i\to\infty} \tau_i(\pi_i(h^{\frac{1}{2}}a_i''h_i^{\frac{1}{2}})) \geq M,$$

which contradicts to (3.2) or (3.3). This proves the assertion.

Denote by Z the (abstract) disjoint union of Z_s, $s = 1, ..., S$, and denote by $[Z]$ the (abstract) disjoint union of $[Z_s]$, $s = 1, ..., S$. Consider $\pi(a_1') \in \pi(C)$, and consider the continuous function

$$[Z] \ni x \mapsto \mathrm{Tr}(\pi(a_1')(x)) \in (0, +\infty).$$

For each $s = 1, 2, ..., S$, by Property (8), one picks an integer

$$\Delta_s \in (2n_s\delta\varepsilon' + 1, 4n_s\varepsilon' - 1). \tag{3.8}$$

Define

$$f : [Z] \ni x \mapsto \lceil \mathrm{Tr}(\pi(a_1')(x)) \rceil + \Delta_s, \quad \text{if } x \in [Z_s],$$

and

$$g : [Z] \ni x \mapsto \lfloor \mathrm{Tr}(\pi(a_1')(x)) \rfloor - \Delta_s, \quad \text{if } x \in [Z_s],$$

where $\lfloor t \rfloor = \max\{k \in \mathbb{Z} : k \leq t\}$ and $\lceil t \rceil = \min\{k \in \mathbb{Z} : k \geq t\}$. Note that by (3.1), (3.4), (3.6), and (3.8), for any $x \in [Z_s]$, $s = 1, ..., S$, one has

$$\begin{aligned}\lfloor \mathrm{Tr}(\pi(a_1')(x)) \rfloor - \Delta_s &\geq \lfloor \mathrm{Tr}(\pi(a_1')(x)) \rfloor - (4n_s\varepsilon' - 1)\\ &\geq \mathrm{Tr}(\pi(a_1')(x)) - 1 - (4n_s\varepsilon' - 1)\\ &= n_s\mathrm{tr}(\pi(a_1')(x)) - 4n_s\varepsilon'\\ &> n_s\delta - 4n_s\varepsilon' = n_s(\delta - 4\varepsilon') > 0.\end{aligned}$$

That is, the function g is a positive. Also note that for any $x \in [Z_s]$, $s = 1, ..., S$, by (3.1), (3.4), (3.6), and (3.8) again,

$$\begin{aligned}f(x) &\leq \max\{\lceil \mathrm{Tr}(\pi(a_1')(y)) \rceil + \Delta_s : y \in [Z_s]\}\\ &\leq \max\{\mathrm{Tr}(\pi(a_1')(y)) + 4n_s\varepsilon' : y \in [Z_s]\}\\ &= n_s \max\{\mathrm{tr}(\pi(a_1')(y)) + 4\varepsilon' : y \in [Z_s]\}\\ &\leq n_s(M + 1).\end{aligned}$$

Therefore f and g satisfy

(a) g is positive upper semicontinuous and f is lower semicontinuous,
(b) $0 < g(x) < \mathrm{Tr}(\pi(a_1')(x)) < f(x) \leq n_s(M + 1)$, $\forall x \in [Z_s]$, and
(c) $4\mathrm{dim}([Z_s]) < 4\varepsilon'\delta n_s < 2\Delta_s - 2 < f(x) - g(x) \leq 2\Delta_s + 2 < 8\varepsilon' n_s$, $\forall x \in [Z_s]$ (by Property (7) and (3.8)).

It then follows from Proposition 2.9 of [40] that there is a positive element $a''' \in \mathrm{M}_\infty(\pi(C))$ such that

$$g(x) < \mathrm{rank}(a'''(x)) < f(x), \quad \forall x \in [Z_s].$$

Extend a''' to an element of $\mathrm{M}_\infty(C) \subseteq \mathrm{M}_\infty(A)$ and denote it by a. One then has that for any $x \in [Z]$, with $n(x) := n_s$ if $x \in [Z_s]$,

$$\begin{aligned} &\left|\frac{1}{n(x)}\mathrm{rank}(a(x)) - \mathrm{tr}(a_1'(x))\right| \\ = &\left|\frac{1}{n(x)}\mathrm{rank}(a'''(x)) - \mathrm{tr}(a_1'(x))\right| \\ \leq &\left|\frac{1}{n(x)}\mathrm{Tr}(a_1'(x)) - \mathrm{tr}(a_1'(x))\right| + \frac{1}{n(x)}(f(x) - g(x)) \\ < & \ 8\varepsilon' < 8\varepsilon. \end{aligned} \tag{3.9}$$

Note that the element a can be chosen so that for any $x \in Z_s \setminus [Z_s]$, $s = 1, ..., S$,

$$\mathrm{rank}(a(x)) \ \leq \ \max\{f(x) : x \in [Z_s]\} \leq n_s(M+1).$$

Now, let $\tau \in \mathrm{T}(A)$ be arbitray, and let μ_τ denote the Borel measure on Z induced by the restriction of τ to C. Note that $1 - \varepsilon < \|\mu_\tau\| \leq 1$ (since $\tau(h) \geq 1 - \varepsilon' > 1 - \varepsilon$, by Property (4)), and also note that, by Property (5),

$$\mu_\tau(Z \setminus [Z]) \leq \mathrm{d}_\tau(\tilde{c} - h) \leq \mathrm{d}_\tau(1_A - h) < \mu(X \setminus h^{-1}(1)) < \frac{\varepsilon'}{M+1} < \frac{\varepsilon}{M+1},$$

where $\tilde{c} \geq h$ is some strict positive element of $C \subseteq A$, and μ is the invariant measure on X corresponding to τ (μ is not μ_τ). Therefore,

$$\int_{Z\setminus[Z]} \frac{1}{n(x)}\mathrm{rank}(a(x))d\mu_\tau \leq \int_{Z\setminus[Z]} (M+1)d\mu_\tau < \varepsilon,$$

where $n(x) = n_s$ if $x \in Z_s$, and (by (3.3) and Property (2))

$$\int_{Z\setminus[Z]} \mathrm{tr}(a_1'(x))d\mu_\tau \leq \int_{Z\setminus[Z]} \|a_1'\| \, d\mu_\tau \leq \int_{Z\setminus[Z]} \|a''\| \, d\mu_\tau \leq \int_{Z\setminus[Z]} (M+\varepsilon')d\mu_\tau < \varepsilon.$$

In particular

$$\left|\int_{Z\setminus[Z]} \frac{1}{n(x)}\mathrm{rank}(a(x))d\mu_\tau - \int_{Z\setminus[Z]} \mathrm{tr}(a_1'(x))d\mu_\tau\right| < 2\varepsilon.$$

Together with (3.5) and (3.9), one has

$$\begin{aligned}
\mathrm{d}_\tau(a) &= \int_Z \frac{1}{n(x)}\mathrm{rank}(a(x))d\mu_\tau \\
&= \int_{[Z]} \frac{1}{n(x)}\mathrm{rank}(a(x))d\mu_\tau + \int_{Z\setminus[Z]} \frac{1}{n(x)}\mathrm{rank}(a(x))d\mu_\tau \\
&\approx_{2\varepsilon} \int_{[Z]} \frac{1}{n(x)}\mathrm{rank}(a(x))d\mu_\tau + \int_{Z\setminus[Z]} \mathrm{tr}(a_1'(x))d\mu_\tau \\
&\approx_{8\varepsilon} \int_{[Z]} \mathrm{tr}(a_1'(x))d\mu_\tau + \int_{Z\setminus[Z]} \mathrm{tr}(a_1'(x))d\mu_\tau \\
&= \int_Z \mathrm{tr}(a_1'(x))d\mu_\tau = \tau(a_1') \\
&\approx_{3\varepsilon} \tau(a').
\end{aligned}$$

Since ε is arbitrary, this proves the desired conclusion. □

Corollary 3.2. *Let (X, Γ) be a free and minimal dynamical system with the (URP) and (COS). If (X, Γ) has mean dimension zero, then, for any positive contraction $a \in \mathrm{M}_\infty(A)$, any $k \in \mathbb{N}$, and any $\varepsilon > 0$, there is an order zero map*

$$\phi : \mathrm{M}_k(\mathbb{C}) \to \mathrm{Her}(a),$$

where $\mathrm{Her}(a)$ *is the hereditary sub-C*-algebra generated by a, such that*

$$\tau(\phi(1_k)) > \tau(a) - \varepsilon, \quad \forall \tau \in \mathrm{T}(A).$$

That is, A is tracially 0-divisible in the sense of Definition 3.5(ii) of [44].

Proof. Since (X, Γ) has mean dimension zero, by Theorem 4.8 of [33], the C*-algebra A has strict comparison of positive elements.

Let $a \in \mathrm{M}_\infty(A)$ be a positive contraction, and consider the lower semicontinuous affine function

$$\mathrm{T}(A) \ni \tau \mapsto \frac{1}{k}\mathrm{d}_\tau(a) \in (0, \infty).$$

Then, by Proposition 3.1 and the proof of Theorem 5.3 of [1] that there is a positive element $x \in A \otimes \mathcal{K}$ such that

$$\mathrm{d}_\tau(x) = \frac{1}{k}\mathrm{d}_\tau(a), \quad \forall \tau \in \mathrm{T}(A).$$

Indeed, pick a sequence (α_n) of strictly positive continuous affine maps on $\mathrm{T}(A)$ such that

(1) $\alpha_n(\tau) < \alpha_{n+1}(\tau)$, $n = 1, 2, ...$, $\tau \in \mathrm{T}(A)$, and
(2) $\lim_{n\to\infty} \alpha_n(\tau) = \frac{1}{k}\mathrm{d}_\tau(a)$, $\tau \in \mathrm{T}(A)$.

Since $\mathrm{T}(A)$ is compact, for each n, there is ε_n such that

$$\alpha_{n+1}(\tau) - \alpha_n(\tau) > \varepsilon_n, \quad \forall \tau \in \mathrm{T}(A).$$

Then, for each α_n, $n = 1, 2, ...$, by Proposition 3.1, there is $a_n \in \mathrm{M}_\infty(A)$ such that

$$|\alpha_n(\tau) - \mathrm{d}_\tau(a_n)| < \frac{1}{2}\min\{\varepsilon_{n-1}, \varepsilon_n\},$$

where $\varepsilon_0 = 1$. Then

(1) $\mathrm{d}_\tau(a_n) < \mathrm{d}_\tau(a_{n+1})$, $n = 1, 2, ...$, $\tau \in \mathrm{T}(A)$, and
(2) $\lim_{n\to\infty} \mathrm{d}_\tau(a_n) = \frac{1}{k}\mathrm{d}_\tau(a)$, $\tau \in \mathrm{T}(A)$.

Since A has strict comparison of positive elements, one has $a_n \precsim a_{n+1}$, $n = 1, 2, ...$

By [4] (Theorem 1(i) and Appendix 6), supremum of every increasing sequence of $\mathrm{W}(A\otimes\mathcal{K})$ exists, and hence there is a positive element $x \in A \otimes \mathcal{K}$ such that

$$\mathrm{d}_\tau(x) = \frac{1}{k}\mathrm{d}_\tau(a), \quad \forall \tau \in \mathrm{T}(A).$$

For each pair of positive numbers $\delta_1 < \delta_2$, define the continuous function

$$f_{\delta_1,\delta_2}(t) = \begin{cases} 0, & t \leq \delta_1, \\ \frac{t-\delta_1}{\delta_2-\delta_1}, & \delta_1 < t < \delta_2, \\ 1, & t \geq \delta_2. \end{cases}$$

Also consider the continuous function

$$f_\varepsilon(t) := \max\{t - \varepsilon, 0\}, \quad t \in \mathbb{R}.$$

Then, since A is simple, with a sufficiently small $\delta > 0$ (see, Remark 2.7 of [44]), one has

$$\tau(f_{2\delta,3\delta}(x)) > \frac{1}{k}\tau(f_\varepsilon(a)) > \frac{1}{k}(\tau(a) - \varepsilon), \quad \forall \tau \in \mathrm{T}(A)$$

and use the simplicity again, there is $\delta' > 0$ such that

$$\tau(f_{\delta/2,\delta}(x)) < \mathrm{d}_\tau(x) - \delta' = \frac{1}{k}(\mathrm{d}_\tau(a)) - \delta', \quad \forall \tau \in \mathrm{T}(A).$$

Thus, with a perturbation of x, there is a positive element $x' \in \mathrm{M}_\infty(A)$ such that,

$$\tau(f_{2\delta,3\delta}(x')) > \frac{1}{k}(\tau(a) - \varepsilon), \quad \forall \tau \in \mathrm{T}(A) \tag{3.10}$$

and

$$\tau(f_{\delta/2,\delta}(x')) < \frac{1}{k}\mathrm{d}_\tau(a) - \delta', \quad \forall \tau \in \mathrm{T}(A).$$

Note that

$$k\mathrm{d}_\tau(f_{\delta,2\delta}(x')) < k\tau(f_{\delta/2,\delta}(x')) < \mathrm{d}_\tau(a), \quad \forall \tau \in \mathrm{T}(A).$$

Since A has strict comparison, one has $k[f_{\delta/2,\delta}(x')] < [a]$ in $\mathrm{W}(A)$. By Proposition 2.12 of [44], there is an order zero map $\phi : \mathrm{M}_k(\mathbb{C}) \to \mathrm{Her}(a)$ such that

$$\phi(e_{1,1}) \approx f_{2\delta,3\delta}(x'),$$

where $a \approx b$ denotes the relation $a = vv^*$, $b = v^*v$ for some v. In particular, by (3.10),

$$\tau(\phi(1_k)) = k\tau(\phi(e_{1,1})) = k\tau(f_{2\delta,3\delta}(x')) > \tau(a) - \varepsilon, \quad \forall \tau \in \mathrm{T}(A),$$

as desired. □

Remark 3.3. Note that a straightforward argument shows that there is m such that for any $k \in \mathbb{N}$, there is $x \in \mathrm{W}(A)$ such that

$$kx \leq [1_A] \leq m(k+1)x,$$

whenever (X, Γ) has the (URP) and (COS), even without mean dimension zero. Then, as a natural question, is the C*-algebra $A = \mathrm{C}(X) \rtimes \Gamma$ always tracially m-divisible for some $m \in \mathbb{N}$ if (X, Γ) has the (URP) and (COS), but without any assumptions on mean dimension?

4. Approximate central order zero maps from $\mathrm{M}_k(\mathbb{C})$ to $\mathrm{C}(X) \rtimes \Gamma$ and the $\mathcal{Z}$-stability of $\mathrm{C}(X) \rtimes \Gamma$

One considers the $\mathcal{Z}$-stability of $\mathrm{C}(X) \rtimes \Gamma$ in this section. First, one has the following lemma which essentially is Theorem 3.9 of [33], stating that the C*-algebra $A = \mathrm{C}(X) \rtimes \Gamma$ can be (weakly) tracially approximated by homogeneous C*-algebras, but with an extra conclusion that there is an element h in the homogeneous sub-C*-algebra, which is approximately central in A, large in trace, and is orthogonal to the elements with smaller trace in the decomposition obtained from the tracial approximation.

Lemma 4.1. *Let (X, Γ) be a free topological dynamical system with the (URP). Then, for any finite set $\{f_1, f_2, ..., f_n\} \subseteq \mathrm{C}(X) \rtimes \Gamma$ and any $\varepsilon > 0$, there exist a C*-algebra $C \subseteq \mathrm{C}(X) \rtimes \Gamma$ with $C \cong \bigoplus_{s=1}^{S} \mathrm{M}_{k_s}(\mathrm{C}_0(U_s))$ for some $k_s \in \mathbb{N}$ and locally compact Hausdorff spaces U_s, $s = 1, ..., S$, a positive contraction $h \in \mathrm{C}(X) \cap C$, and $f_1^{(0)}, f_1^{(1)}, f_2^{(0)}, f_2^{(1)}, ..., f_n^{(0)}, f_n^{(1)} \in \mathrm{C}(X) \rtimes \Gamma$ such that*

(1) $\left\| f_i - (f_i^{(0)} + f_i^{(1)}) \right\| < \varepsilon$, $1 \leq i \leq n$,
(2) $f_i^{(1)} \in C$, $1 \leq i \leq n$,
(3) $\left\| f_i^{(0)} h \right\| = 0$, $1 \leq i \leq n$,
(4) $\left\| [f_i^{(1)}, h] \right\| < \varepsilon$, $1 \leq i \leq n$, *and*
(5) $\tau(1 - h^2) < \varepsilon$, $\forall \tau \in \mathrm{T}(\mathrm{C}(X) \rtimes \Gamma)$.

Proof. The proof is similar to that of Theorem 3.9 of [33], but without dealing with mean dimension.

Denote by A the crossed product C*-algebra $\mathrm{C}(X) \rtimes \Gamma$. Without loss of generality, one may assume

$$f_i = \sum_{\gamma \in \mathcal{N}} f_{i,\gamma} u_\gamma$$

for some finite set $\mathcal{N} \subseteq \Gamma$ with $e \in \mathcal{N} = \mathcal{N}^{-1}$, and some $f_{i,\gamma} \in \mathrm{C}(X)$. Denote by

$$M = \max\{1, \|f_{i,\gamma}\| : i = 1, ..., n, \gamma \in \mathcal{N}\}.$$

For the given $\varepsilon > 0$, choose $\varepsilon_1 \in (0, \varepsilon)$ such that if a positive element $a \in A$ with $\|a\| \leq 1$ satisfies

$$\|af_i - f_i a\| < \varepsilon_1, \quad 1 \leq i \leq n,$$

then

$$\left\| a^{\frac{1}{2}} f_i - f_i a^{\frac{1}{2}} \right\| < \frac{\varepsilon}{2}, \quad 1 \leq i \leq n.$$

Pick a natural number

$$L > \frac{M\,|\mathcal{N}|}{\varepsilon_1},$$

and pick a sufficiently large finite set $K \subseteq \Gamma$ and a sufficiently small positive number δ so that if a finite set $\Gamma_0 \subseteq \Gamma$ is (K, δ)-invariant, then

$$\frac{|\Gamma_0 \setminus \mathrm{int}_{\mathcal{N}^{L+1}}(\Gamma_0)|}{|\Gamma_0|} < \frac{\varepsilon}{2}. \tag{4.1}$$

Since (X, Γ) has the (URP), there exist closed sets $B_1, B_2, ..., B_S \subseteq X$ and (K, δ)-invariant sets $\Gamma_1, \Gamma_2, ..., \Gamma_S \subseteq \Gamma$ such that

$$B_s\gamma, \quad \gamma \in \Gamma_s,\ s = 1, ..., S,$$

are mutually disjoint and

$$\mathrm{ocap}(X \setminus \bigsqcup_{s=1}^{S} \bigsqcup_{\gamma\in\Gamma_s} B_s\gamma) < \frac{\varepsilon}{2}.$$

Pick two open sets $U_s, V_s \subseteq X$, $s = 1, 2, ..., S$, satisfying

$$U_s \supseteq V_s \supseteq B_s, \quad U_s \supseteq \overline{V_s},$$

and

$$U_s\gamma, \quad \gamma \in \Gamma_s,\ s = 1, ..., S,$$

are mutually disjoint.

Consider the sub-C*-algebra

$$C := \mathrm{C}^*\{u_\gamma^* f : f \in \mathrm{C}_0(U_s), \gamma \in \Gamma_s, s = 1, 2, ..., S\} \subseteq \mathrm{C}(X) \rtimes \Gamma, \tag{4.2}$$

which, by Lemma 3.12 of [33], is isomorphic to

$$\bigoplus_{s=1}^{S} \mathrm{M}_{|\Gamma_s|}(\mathrm{C}_0(U_s)).$$

For each $s = 1, 2, ..., S$, pick continuous functions $\chi_{U_s}, \chi_{V_s} : X \to [0, 1]$ such that

$$\chi_{U_s}|_{V_s} = 1, \quad \chi_{V_s}|_{B_s} = 1, \quad \chi_{U_s}|_{X\setminus U_s} = 0, \quad \text{and} \quad \chi_{V_s}|_{X\setminus V_s} = 0. \tag{4.3}$$

Note that $\chi_{U_s}, \chi_{V_s} \in C$, and

$$\chi_{U_s} f,\ \chi_{V_s} f \in C, \quad f \in \mathrm{C}(X). \tag{4.4}$$

For each Γ_s, $s = 1, 2, ..., S$, define the subsets

$$\begin{cases} \Gamma_{s,L+1} &= \mathrm{int}_{\mathcal{N}^{L+1}}(\Gamma_s), \\ \Gamma_{s,L} &= \mathrm{int}_{\mathcal{N}^{L}}(\Gamma_s) \setminus \mathrm{int}_{\mathcal{N}^{L+1}}(\Gamma_s), \\ \Gamma_{s,L-1} &= \mathrm{int}_{\mathcal{N}^{L-1}}(\Gamma_s) \setminus \mathrm{int}_{\mathcal{N}^{L}}(\Gamma_s), \\ \vdots & \vdots\ \vdots \\ \Gamma_{s,0} &= \Gamma_s \setminus \mathrm{int}_{\mathcal{N}}(\Gamma_s). \end{cases}$$

Then, for any $\gamma \in \mathcal{N}$, one has

$$\Gamma_{s,l}\gamma \subseteq \Gamma_{s,l-1} \cup \Gamma_{s,l} \cup \Gamma_{s,l+1}, \quad 1 \leq l \leq L. \tag{4.5}$$

Indeed, pick an arbitrary $\gamma' \in \Gamma_{s,l}$. By the construction, one has

$$\gamma'\mathcal{N}^l \subseteq \Gamma_s \quad \text{but} \quad \gamma'\mathcal{N}^{l+1} \nsubseteq \Gamma_s. \tag{4.6}$$

Therefore

$$\gamma'\gamma\mathcal{N}^{l-1} \subseteq \gamma'\mathcal{N}^{l} \subseteq \Gamma_s$$

and hence $\gamma'\gamma \in \mathrm{int}_{\mathcal{N}^{l-1}}\Gamma_s$ (since $e \in \mathcal{N}^{l-1}$).

Thus, to show (4.5), one only has to show that $\gamma'\gamma \notin \mathrm{int}_{\mathcal{N}^{l+2}}\Gamma_s$. Suppose $\gamma'\gamma\mathcal{N}^{l+2} \subseteq \Gamma_s$. Since $\mathcal{N}$ is symmetric, one has $\gamma^{-1} \in \mathcal{N}$; hence $\mathcal{N}^{l+1} \subseteq \gamma\mathcal{N}^{l+2}$ and

$$\gamma'\mathcal{N}^{l+1} \subseteq \gamma'\gamma\mathcal{N}^{l+2} \subseteq \Gamma_s,$$

which contradicts (4.6).

Also note that

$$\Gamma_{s,L+1}\gamma \subseteq \Gamma_{s,L+1} \cup \Gamma_{s,L}. \tag{4.7}$$

For each $\gamma \in \Gamma_s$, define

$$\ell(\gamma) = l, \quad \text{if } \gamma \in \Gamma_{s,l}.$$

By (4.5) and (4.7), the function ℓ satisfies

$$|\ell(\gamma'\gamma) - \ell(\gamma)| \leq 1, \quad \gamma' \in \mathcal{N},\ \gamma \in \Gamma_{s,1} \cup \cdots \cup \Gamma_{s,L+1}. \tag{4.8}$$

Define

$$h_U = \sum_{s=1}^{S}\sum_{l=1}^{L+1}\sum_{\gamma\in\Gamma_{s,l}} \frac{l-1}{L}(\chi_{U_s}\circ\gamma^{-1}) = \sum_{s=1}^{S}\sum_{l=1}^{L+1}\sum_{\gamma\in\Gamma_{s,l}} \frac{l-1}{L}u_\gamma^*\chi_{U_s}u_\gamma \in \mathrm{C}(X)\cap C,$$

and

$$h_V = \sum_{s=1}^{S}\sum_{l=1}^{L+1}\sum_{\gamma\in\Gamma_{s,l}} \frac{l-1}{L}(\chi_{V_s}\circ\gamma^{-1}) = \sum_{s=1}^{S}\sum_{l=1}^{L+1}\sum_{\gamma\in\Gamma_{s,l}} \frac{l-1}{L}u_\gamma^*\chi_{V_s}u_\gamma \in \mathrm{C}(X)\cap C.$$

Note that, by (4.3),

$$\begin{aligned} h_Uh_V &= (\sum_{s=1}^{S}\sum_{l=1}^{L+1}\sum_{\gamma\in\Gamma_{s,l}} \frac{l-1}{L}u_\gamma^*\chi_{U_s}u_\gamma)(\sum_{s=1}^{S}\sum_{l=1}^{L+1}\sum_{\gamma\in\Gamma_{s,l}} \frac{l-1}{L}u_\gamma^*\chi_{V_s}u_\gamma) \\ &= \sum_{s=1}^{S}\sum_{l=1}^{L+1}\sum_{\gamma\in\Gamma_{s,l}} \frac{l-1}{L}u_\gamma^*\chi_{U_s}\chi_{V_s}u_\gamma \\ &= \sum_{s=1}^{S}\sum_{l=1}^{L+1}\sum_{\gamma\in\Gamma_{s,l}} \frac{l-1}{L}u_\gamma^*\chi_{V_s}u_\gamma = h_V, \end{aligned}$$

and hence

$$(1-h_U)h_V = 0. \tag{4.9}$$

By (4.3) (and (4.1)),

$$\begin{aligned}\operatorname{ocap}(X\setminus h_V^{-1}(1)) &\leq \max\{\frac{|\Gamma_s\setminus \operatorname{int}_{\mathcal{N}^{L+1}}(\Gamma_s)|}{|\Gamma_s|}: s=1,...,S\}\\ &\quad +\operatorname{ocap}(X\setminus \bigsqcup_{s=1}^{S}\bigsqcup_{\gamma\in\Gamma_s} B_s\gamma)\leq \frac{\varepsilon}{2}+\frac{\varepsilon}{2}\\ &< \varepsilon,\end{aligned}$$

and therefore

$$\tau(1-h_V^2)<\varepsilon,\quad \tau\in \mathrm{T}(A).$$

Note that, by the construction of C (see (4.2)),

$$\chi_{U_s}^{\frac{1}{2}}u_\gamma\in C,\quad \gamma\in\Gamma_s.$$

Hence, for each $\gamma'\in\mathcal{N}$, since $\gamma\gamma'\in\Gamma_s$, $\gamma\in\Gamma_{s,l}$, $l=1,2,...,L+1$, one has

$$h_Uu_{\gamma'}=\sum_{s=1}^{S}\sum_{l=1}^{L+1}\sum_{\gamma\in\Gamma_{s,l}}\frac{l-1}{L}u_\gamma^*\chi_{U_s}u_{\gamma\gamma'}=\sum_{s=1}^{S}\sum_{l=1}^{L+1}\sum_{\gamma\in\Gamma_{s,l}}\frac{l-1}{L}(u_\gamma^*\chi_{U_s}^{\frac{1}{2}})(\chi_{U_s}^{\frac{1}{2}}u_{\gamma\gamma'})\in C,$$

and therefore,

$$h_Uu_\gamma h_U\in C,\quad \gamma\in\mathcal{N}.$$

For any $f\in \mathrm{C}(X)$, by (4.4), one has

$$h_Uf=\sum_{s=1}^{S}\sum_{l=1}^{L+1}\sum_{\gamma\in\Gamma_{s,l}}\frac{l-1}{L}u_\gamma^*\chi_{U_s}u_\gamma f=\sum_{s=1}^{S}\sum_{l=1}^{L+1}\sum_{\gamma\in\Gamma_{s,l}}\frac{l-1}{L}u_\gamma^*\chi_{U_s}(u_\gamma fu_\gamma^*)u_\gamma\in C,$$

and therefore

$$h_Uf_ih_U\in C,\quad 1\leq i\leq n.$$

Note that, for each $\gamma'\in\mathcal{N}$, by (4.8),

$$\begin{aligned}&\left\|u_{\gamma'}^*h_Uu_{\gamma'}-h_U\right\|\\ =&\left\|\sum_{s=1}^{S}\sum_{l=1}^{L+1}\sum_{\gamma\in\Gamma_{s,l}}\frac{l-1}{L}\chi_{U_s}\circ(\gamma'\gamma)^{-1}-\sum_{s=1}^{S}\sum_{l=1}^{L+1}\sum_{\gamma\in\Gamma_{s,l}}\frac{l-1}{L}\chi_{U_s}\circ\gamma^{-1}\right\|\\ =&\max\{\left|\frac{\ell(\gamma'\gamma)-1}{L}-\frac{\ell(\gamma)-1}{L}\right|:\gamma\in\Gamma_s\setminus\Gamma_{s,0},\ s=1,2,...,S\}\\ <&\frac{1}{L}<\frac{\varepsilon_1}{M\left|\mathcal{N}\right|},\end{aligned}$$

and hence

$$\|h_Uf_i-f_ih_U\|<\varepsilon_1,\quad i=1,2,...,n. \tag{4.10}$$

The same argument also shows that

$$\|h_Vf_i-f_ih_V\|<\varepsilon_1<\varepsilon,\quad i=1,2,...,n. \tag{4.11}$$

It follows from (4.10) and the choice of ε_1 that

$$\left\|h_U^{\frac{1}{2}}f_i - f_i h_U^{\frac{1}{2}}\right\| < \frac{\varepsilon}{2} \quad \text{and} \quad \left\|(1-h_U)^{\frac{1}{2}}f_i - f_i(1-h_U)^{\frac{1}{2}}\right\| < \frac{\varepsilon}{2}, \quad i = 1, 2, ..., n,$$

and hence

$$\left\|f_i - ((1-h_U)^{\frac{1}{2}}f_i(1-h_U)^{\frac{1}{2}} + h_U^{\frac{1}{2}}f_i h_U^{\frac{1}{2}})\right\| < \varepsilon, \quad 1 \leq i \leq n.$$

Put

$$f_i^{(0)} = (1-h_U)^{\frac{1}{2}}f_i(1-h_U)^{\frac{1}{2}} \quad \text{and} \quad f_i^{(1)} = h_U^{\frac{1}{2}}f_i h_U^{\frac{1}{2}}.$$

By (4.9),

$$f_i^{(0)}h_V = 0, \quad i = 1, ..., n.$$

One also has, by (4.11),

$$f_i^{(1)}h_V = h_U^{\frac{1}{2}}f_i h_U^{\frac{1}{2}}h_V = h_U^{\frac{1}{2}}f_i h_V h_U^{\frac{1}{2}} \approx_\varepsilon h_U^{\frac{1}{2}}h_V f_i h_U^{\frac{1}{2}} = h_V h_U^{\frac{1}{2}}f_i h_U^{\frac{1}{2}} = h_V f_i^{(1)}.$$

Thus

$$\left\|f_i^{(1)}h_V - h_V f_i^{(1)}\right\| < \varepsilon, \quad i = 1, ..., n.$$

Then the element $h := h_V$ satisfies the lemma. □

Definition 4.2 ([19])**.** A unital C*-algebra A is said to be tracially $\mathcal{Z}$-stable if for any finite set $\mathcal{F} \subseteq A$, any $\varepsilon > 0$, and any non-zero positive element $a \in A$, there is a c.p.c. order zero map $\varphi : \mathrm{M}_2(\mathbb{C}) \to A$ such that

(1) $\|[\varphi(x), f]\| < \varepsilon$, $\forall x \in \mathrm{M}_2(\mathbb{C})$, $\|x\| \leq 1$, $f \in \mathcal{F}$,
(2) $1_A - \varphi(1_2) \precsim a$.

Based on [31], for nuclear C*-algebras, the tracial $\mathcal{Z}$-stability is shown to be equivalent to the $\mathcal{Z}$-stability in [19]:

Theorem 4.3. *Let A be a simple separable unital nuclear C*-algebra. Then $A \cong A \otimes \mathcal{Z}$ if and only if A is tracially $\mathcal{Z}$-stable, where $\mathcal{Z}$ is the Jiang-Su algebra.*

Remark 4.4. In general, there are non-nuclear C*-algebras which are tracially $\mathcal{Z}$-stable but not $\mathcal{Z}$-stable (see [34]).

The following two lemmas are simple observations.

Lemma 4.5. *Let A be a unital C*-algebra, and let τ be a tracial state of A. Assume $a, b \in A$ are positive elements with norm at most* 1 *and*

$$\tau(1-a) < \varepsilon \quad \text{and} \quad \tau(1-b) < \varepsilon,$$

then

$$\tau(ab) > 1 - 2\varepsilon.$$

Proof. It follows from the assumption that

$$1 - \varepsilon < \tau(a) \quad \text{and} \quad -\varepsilon < \tau(b-1).$$

Also note that

$$0 \leq \tau((1-a)^{\frac{1}{2}}(1-b)(1-a)^{\frac{1}{2}}) = \tau((1-a)(1-b)) = \tau(1-a-b+ab),$$

and so

$$\tau(a+b-1) \leq \tau(ab).$$

Then

$$1-2\varepsilon = (1-\varepsilon)-\varepsilon < \tau(a)+\tau(b-1) = \tau(a+b-1) \leq \tau(ab),$$

as desired. □

Lemma 4.6. *Let A be a C^*-algebra, and let $\varphi : \mathrm{M}_k(\mathbb{C}) \to A$ be a c.p.c. order zero map with*

$$\tau(1_A - \varphi(1_k)) < \varepsilon, \quad \forall \tau \in \mathrm{T}(A),$$

for some $\varepsilon > 0$. Then there is a c.p.c. order zero map $\varphi' : \mathrm{M}_k(\mathbb{C}) \to A$ such that

$$\|\varphi' - \varphi\| < \sqrt{\varepsilon}$$

and

$$\mathrm{d}_\tau(1_A - \varphi'(1_k)) < \sqrt{\varepsilon}, \quad \forall \tau \in \mathrm{T}(A).$$

Proof. Since φ has order zero, it follows from Theorem 1.2 of [46] that there is

$$h \in \mathcal{M}(\mathrm{C}^*(\varphi(\mathrm{M}_k))) \cap (\mathrm{C}^*(\varphi(\mathrm{M}_k)))'$$

and a unital homomorphism

$$\tilde{\varphi} : \mathrm{M}_k(\mathbb{C}) \to \mathcal{M}(\mathrm{C}^*(\varphi(\mathrm{M}_k))) \cap (h)'$$

such that

$$\varphi(a) = \tilde{\varphi}(a)h, \quad \forall a \in \mathrm{M}_k(\mathbb{C}).$$

Note that $h = \varphi(1_k)$.

Let $\tau \in \mathrm{T}(A)$ be arbitrary, and denote by μ_τ the probability measure induced by τ on $\mathrm{sp}(h) \subseteq [0,1]$. Since $\tau(1_A - h) < \varepsilon$, one has

$$\begin{aligned} 1-\varepsilon &< \int_{[0,1]} t\mathrm{d}\mu_\tau = \int_{[0,1-\sqrt{\varepsilon}]} t\mathrm{d}\mu_\tau + \int_{(1-\sqrt{\varepsilon},1]} t\mathrm{d}\mu_\tau \\ &\leq (1-\sqrt{\varepsilon})\mu_\tau([0,1-\sqrt{\varepsilon}]) + (1-\mu_\tau([0,1-\sqrt{\varepsilon}])), \end{aligned}$$

and hence

$$\mu_\tau([0,1-\sqrt{\varepsilon}]) < \sqrt{\varepsilon}.$$

Set $f(t) = \min\{\frac{t}{1-\sqrt{\varepsilon}}, 1\}$. Consider $f(h)$ and the c.p.c. order zero map

$$\varphi' := \tilde{\varphi}(a)f(h), \quad \forall a \in \mathrm{M}_k(\mathbb{C}).$$

Note that $\|h - f(h)\| < \sqrt{\varepsilon}$; one has that

$$\|\varphi - \varphi'\| < \sqrt{\varepsilon}.$$

On the other hand, for any $\tau \in \mathrm{T}(A)$, one has

$$\mathrm{d}_\tau(1-\varphi'(1_k)) = \mathrm{d}_\tau(1-f(h)) = \mu_\tau([0,1-\sqrt{\varepsilon})) < \sqrt{\varepsilon},$$

as desired. □

Proposition 4.7. *Let (X, Γ) be a free and minimal topological dynamical system with the (URP), and assume $\mathrm{C}(X) \rtimes \Gamma$ is tracially m-almost divisible for some $m \in \mathbb{N}$ (see Definition 3.5(ii) of [44]). For any finite set $\{f_1, f_2, ..., f_n\} \subseteq \mathrm{C}(X) \rtimes \Gamma$, any $\varepsilon > 0$, and any $k \in \mathbb{N}$, then there is a c.p.c. order zero map $\phi : \mathrm{M}_k(\mathbb{C}) \to \mathrm{C}(X) \rtimes \Gamma$ such that*

(1) $\|[\phi(a), f_i]\| < \varepsilon$, $\forall a \in \mathrm{M}_k(\mathbb{C})$ *with* $\|a\| = 1$ *and* $1 \leq i \leq n$, *and*
(2) $\mathrm{d}_\tau(1_A - \phi(1_k)) < \varepsilon$, $\forall \tau \in \mathrm{T}(A)$.

Proof. Denote by $A = \mathrm{C}(X) \rtimes \Gamma$. By Lemma 4.6, it is enough to show that for any given $\varepsilon > 0$ and any finite set $\{f_1, f_2, ..., f_n\} \subseteq A$, there is a c.p.c. order-zero map $\phi : \mathrm{M}_k(\mathbb{C}) \to A$ such that

(1) $\|[\phi(a), f_i]\| < \varepsilon$, $\forall a \in \mathrm{M}_k(\mathbb{C})$ with $\|a\| = 1$ and $1 \leq i \leq n$, and
(2) $\tau(1_A - \phi(1_k)) < \varepsilon$, $\forall \tau \in \mathrm{T}(A)$.

Since order zero maps from $\mathrm{M}_k(\mathbb{C})$ are weakly stable (see Proposition 2.5 of [23]), one is able to pick $\delta > 0$ sufficiently small such that if a c.p.c. map $\rho : \mathrm{M}_k(\mathbb{C}) \to A$ satisfies

$$a \perp b \Rightarrow \|\rho(a)\rho(b)\| < \delta, \quad \forall a, b \in \mathrm{M}_k(\mathbb{C}),\ \|a\| = \|b\| = 1,$$

there is a c.p.c order zero map $\theta : \mathrm{M}_k(\mathbb{C}) \to A$ such that

$$\|\rho(a) - \theta(a)\| < \frac{\varepsilon}{4}, \quad \forall a \in \mathrm{M}_k(\mathbb{C}),\ \|a\| = 1.$$

By Lemma 4.1, there are $f_1^0, f_1^{(1)}, f_2^0, f_2^{(1)}, ..., f_n^0, f_n^{(1)} \in A$, a C*-algebra $C \subseteq A$ with $C \cong \bigoplus_{s=1}^S \mathrm{M}_{k_s}(\mathrm{C}_0(U_s))$ for some locally compact Hausdorff spaces U_s, $s = 1, ..., S$, a positive contraction $h \in A$ such that

$$\left\| f_i - (f_i^0 + f_i^{(1)}) \right\| < \frac{\varepsilon}{8}, \quad 1 \leq i \leq n, \tag{4.12}$$

$$h \in C \quad \text{and} \quad f_i^{(1)} \in C, \quad 1 \leq i \leq n, \tag{4.13}$$

$$\left\| f_i^{(0)} h^{\frac{1}{2}} \right\| < \frac{\varepsilon}{16}, \quad 1 \leq i \leq n, \tag{4.14}$$

$$\left\| [f_i^{(1)}, h^{\frac{1}{2}}] \right\| < \frac{\varepsilon}{24}, \quad 1 \leq i \leq n, \tag{4.15}$$

and

$$\tau(1 - h) < \frac{\varepsilon}{4}, \quad \forall \tau \in \mathrm{T}(A). \tag{4.16}$$

Consider the unitization $\tilde{C} = C + \mathbb{C}1_A$, and note that

$$\tilde{C} \cong \{f \in \mathrm{C}(\{\infty\} \cup \bigsqcup_{s=1}^S U_s, \bigoplus_{s=1}^S \mathrm{M}_{k_s}(\mathbb{C})) : f(\infty) \in \mathbb{C}1\}.$$

Since the compact Hausdorff space $\{\infty\} \cup \bigsqcup_{s=1}^S U_s$ is an inverse limit of finite dimensional CW-complexes, with a small perturbation of $f_1^{(1)}, f_2^{(1)}, ..., f_n^{(1)}$, and h, one may assume that $\tilde{C}$ (and C) has finite nuclear dimension and Equations (4.12)–(4.16) still hold (this is the reason that $f_i^{(0)} h^{\frac{1}{2}}$ in (4.14) is not required to be 0, such as in Lemma 4.1).

Since A is assumed to be tracially m-divisible, applying Lemma 5.11 of [44] to $\tilde{B}$ and using (4.13), one obtains a c.p.c. order zero map $\varphi : \mathrm{M}_k(\mathbb{C}) \to A$ such that

$$\left\| [\varphi(a), f_i^{(1)}] \right\| < \frac{\varepsilon}{24}, \quad \forall a \in \mathrm{M}_k(\mathbb{C}),\ \|a\| = 1,\ 1 \leq i \leq n, \tag{4.17}$$

$$\| [\varphi(a), h] \| < \delta, \quad \forall a \in \mathrm{M}_k(\mathbb{C}),\ \|a\| = 1, \tag{4.18}$$

and

$$\tau(1_A - \varphi(1_k)) < \frac{\varepsilon}{4}, \quad \forall \tau \in \mathrm{T}(A). \tag{4.19}$$

Consider the c.p.c. map

$$\mathrm{M}_k(\mathbb{C}) \ni a \mapsto h^{\frac{1}{2}} \varphi(a) h^{\frac{1}{2}} \in A.$$

Then, for any elements $a, b \in \mathrm{M}_k(\mathbb{C})$ with $a \perp b$ and $\|a\| = \|b\| = 1$, one has (by (4.18))

$$(h^{\frac{1}{2}} \varphi(a) h^{\frac{1}{2}})(h^{\frac{1}{2}} \varphi(b) h^{\frac{1}{2}}) = h^{\frac{1}{2}} \varphi(a) h \varphi(b) h^{\frac{1}{2}} \approx_\delta h^{\frac{3}{2}} \varphi(a) \varphi(b) h^{\frac{1}{2}} = 0,$$

and hence, by the choice of δ, there exists a c.p.c order zero map $\phi : \mathrm{M}_k(\mathbb{C}) \to A$ such that

$$\left\| \phi(a) - h^{\frac{1}{2}} \varphi(a) h^{\frac{1}{2}} \right\| < \frac{\varepsilon}{4}, \quad \forall a \in \mathrm{M}_k(\mathbb{C}),\ \|a\| = 1. \tag{4.20}$$

Then, for any $a \in \mathrm{M}_k(\mathbb{C})$ with $\|a\| = 1$ and any $1 \leq i \leq n$, one has

$$\begin{aligned}
\| [\phi(a), f_i] \| &< \left\| [h^{\frac{1}{2}} \varphi(a) h^{\frac{1}{2}}, f_i] \right\| + \frac{\varepsilon}{2} \qquad \text{(by (4.20))} \\
&< \left\| [h^{\frac{1}{2}} \varphi(a) h^{\frac{1}{2}}, f_i^{(0)} + f_i^{(1)}] \right\| + \frac{3\varepsilon}{4} \qquad \text{(by (4.12))} \\
&= \left\| [h^{\frac{1}{2}} \varphi(a) h^{\frac{1}{2}}, f_i^{(0)}] \right\| + \left\| [h^{\frac{1}{2}} \varphi(a) h^{\frac{1}{2}}, f_i^{(1)}] \right\| + \frac{3\varepsilon}{4} \\
&< \frac{\varepsilon}{8} + \frac{\varepsilon}{8} + \frac{3\varepsilon}{4} = \varepsilon \qquad \text{(by (4.14), (4.15) and (4.17))}.
\end{aligned}$$

Moreover, applying Lemma 4.5 with (4.16) and (4.19), together with (4.20), one has

$$\tau(\phi(1_k)) \approx_{\frac{\varepsilon}{4}} \tau(h^{\frac{1}{2}} \varphi(1_k) h^{\frac{1}{2}}) = \tau(h \varphi(1_k)) > 1 - \frac{\varepsilon}{2}, \quad \forall \tau \in \mathrm{T}(A),$$

as desired. □

Theorem 4.8. *Let (X, Γ) be a free and minimal topological dynamical system with the (URP) and (COS). If (X, Γ) has mean dimension zero, then $(\mathrm{C}(X) \rtimes \Gamma) \otimes \mathcal{Z} \cong \mathrm{C}(X) \rtimes \Gamma$.*

In particular, let (X_1, Γ_1) and (X_2, Γ_2) be two free minimal topological dynamical systems with the (URP) and (COS), and zero mean dimension, then

$$\mathrm{C}(X_1) \rtimes \Gamma_1 \cong \mathrm{C}(X_2) \rtimes \Gamma_2$$

if, and only if,

$$\mathrm{Ell}(\mathrm{C}(X_1) \rtimes \Gamma_1) \cong \mathrm{Ell}(\mathrm{C}(X_2) \rtimes \Gamma_2),$$

where $\mathrm{Ell}(\cdot) = (\mathrm{K}_0(\cdot), \mathrm{K}_0^+(\cdot), [1], \mathrm{T}(\cdot), \rho, \mathrm{K}_1(\cdot))$ is the Elliott invariant. Moreover, these C^*-algebras are inductive limits of unital subhomogeneous C^*-algebras.

Proof. It follows from Corollary 3.2 that $\mathrm{C}(X) \rtimes \Gamma$ is tracially 0-divisible. It follows from Theorem 4.8 of [33] that $\mathrm{C}(X) \rtimes \Gamma$ has strict comparison of positive elements. Together with Proposition 4.7 and the simplicity of $\mathrm{C}(X) \rtimes \Gamma$, one has that $\mathrm{C}(X) \rtimes \Gamma$ is tracially $\mathcal{Z}$-stable. Since $\mathrm{C}(X) \rtimes \Gamma$ is nuclear, it is $\mathcal{Z}$-stable, as desired. □

Corollary 4.9. *Let (X, Γ) be a free and minimal topological dynamical system with mean dimension zero. Assume that*

- *either $\Gamma = \mathbb{Z}^d$ for some $d \geq 1$, or*
- *(X, Γ) is an extension of a Cantor system and Γ has subexponetial growth.*

Then, the C-algebra $\mathrm{C}(X) \rtimes \Gamma$ is classified by the Elliott invariant and is an inductive limit of unital subhomogeneous C*-algebras.*

Proof. It follows from [33] and [32] that the dynamical systems being considered have the (URP) and (COS). The statement then follows from Theorem 4.8. □

5. An alternative approach: $\mathrm{mdim}0 + \mathrm{URP} \Rightarrow \mathrm{SBP}$

In this section, one considers the zero mean dimension together with the (URP), and shows that these two conditions actually implies that the dynamical system has the small boundary property (SBP). Together with [21] and [33], this gives another proof of Theorem 4.8. One should note that the ideas of the (URP) have been used in [28], [17], and [18] to show that zero mean dimension implies small boundary property for $\mathbb{Z}$ or $\mathbb{Z}^k$-actions, and the proof of the following theorem actually depends on [18].

Theorem 5.1. *Let (X, Γ) be a free topological dynamical system with the (URP). If*

$$\mathrm{mdim}(X, \Gamma) = 0,$$

then (X, Γ) has the (SBP).

Proof. It follows from Lemma 5.5 and Corollary 5.4 of [18] that, in order to show that (X, Γ) has the (SBP), it is enough to show that for any continuous function $f : X \to \mathbb{R}$ and any $\varepsilon > 0$, there is a continuous function $g : X \to \mathbb{R}$ such that

(1) $\|f - g\| < \varepsilon$, and
(2) $\mathrm{ocap}(\{x \in X : g(x) = 0\}) < \varepsilon$.

Let $f : X \to \mathbb{R}$ and $\varepsilon > 0$ be given. Pick $\mathcal{U}$ to be a finite open cover of X such that

$$|f(x) - f(y)| < \frac{\varepsilon}{3}, \quad \forall x, y \in U,\ \forall U \in \mathcal{U}. \tag{5.1}$$

Since $\mathrm{mdim}(X, \Gamma) = 0$, there is (K, ε'), where $K \subseteq \Gamma$ is a finite set and $\varepsilon' > 0$, such that if $\Gamma_0 \subseteq \Gamma$ is (K, ε')-invariant, there is an open cover $\mathcal{V}$ such that

(1) $\mathcal{V}$ refines $\bigwedge_{\gamma \in \Gamma_0} \mathcal{U}\gamma$, and
(2) $\mathrm{ord}(\mathcal{V}) < \frac{\varepsilon}{3} |\Gamma_0|$.

Since (X, Γ) has the (URP), there are closed sets $B_1, B_2, ..., B_S$ and (K, ε')-invariant sets $\Gamma_1, \Gamma_2, ..., \Gamma_S \subseteq \Gamma$ such that

$$B_s\gamma, \quad \forall \gamma \in \Gamma_s,\ 1 \leq s \leq S$$

are mutually disjoint and

$$\operatorname{ocap}(X \setminus \bigsqcup_{s=1}^{S} \bigsqcup_{\gamma\in\Gamma_s} B_s\gamma) < \frac{\varepsilon}{3}. \tag{5.2}$$

Pick a small neighborhood U_s of each B_s, $s = 1, 2, ..., S$, such that

$$U_s\gamma, \quad \forall\gamma \in \Gamma_s,\ 1 \leq s \leq S,$$

are still mutually disjoint.

For each $s = 1, 2, ..., S$, since Γ_s is (K, ε')-invariant, there is an open cover $\mathcal{V}$ of X such that

(1) $\mathcal{V}$ refines $\bigwedge_{\gamma\in\Gamma_0} \mathcal{U}\gamma$, and
(2) $\operatorname{ord}(\mathcal{V}) < \frac{\varepsilon}{3} |\Gamma_s|$.

Then, consider the collection of open sets

$$\mathcal{V}_s := \{V \cap U_s : V \in \mathcal{V}\}.$$

Note that $\mathcal{V}_s$ covers B_s and for any $V \in \mathcal{V}_s$ and any $\gamma \in \Gamma_s$, there is $U \in \mathcal{U}$ such that

$$V\gamma \subseteq U.$$

For each $\mathcal{V}_s$, $s = 1, 2, ..., S$, pick continuous functions

$$\phi_V^{(s)} : X \to [0, 1], \quad V \in \mathcal{V}_s$$

such that

$$(\phi_V^{(s)})^{-1}((0, 1]) \subseteq V, \quad \forall V \in \mathcal{V}_s,$$
$$\sum_{V\in\mathcal{V}_s} \phi_V^{(s)}(x) \leq 1, \quad \forall x \in X, \quad \text{and}$$
$$\sum_{V\in\mathcal{V}_s} \phi_V^{(s)}(x) = 1, \quad \forall x \in B_s.$$

Also define

$$W_s = \{x \in X : \sum_{V\in\mathcal{V}_s} \phi_V^{(s)}(x) > 0\} \subseteq U_s.$$

For each $\mathcal{V}_s$, $s = 1, 2, ..., S$, also consider the simplicial complex Δ_s spanned by $[V]$, $V \in \mathcal{V}_s$, with

$$[V_0], [V_1], ..., [V_d]$$

span a simplex if and only if

$$V_0 \cap V_1 \cap \cdots \cap V_d \neq \varnothing.$$

Note that

$$\dim(\Delta_s) = \operatorname{ord}(\mathcal{V}_s) \leq \operatorname{ord}(\mathcal{V}) \leq \frac{\varepsilon}{3} |\Gamma_s| . \tag{5.3}$$

Define the map

$$\eta_s : X \ni x \mapsto \sum_{V\in\mathcal{V}_s} \phi_V^{(s)}(x)[V] \in \mathrm{C}\Delta_s, \tag{5.4}$$

where $\mathrm{C}\Delta_s$ is the cone over Δ_s, i.e., $\mathrm{C}\Delta_s$ consists of

$$t_0[V_0]+\cdots+t_d[V_d],\quad \sum_{i=0}^{d}t_i\leq 1,\ t_i\in[0,1],\ i=0,1,...,d,$$

whenever $V_0, V_1, ..., V_d\in\mathcal{V}_s$ satisfy

$$V_0\cap\cdots\cap V_d\neq\varnothing.$$

Assume all of the cones $\mathrm{C}\Delta_s$, $s=1,2,...,S$, share the same zero vertex, which is denoted by $\mathbf{0}$. Note that

$$\eta_s(B_s)\subseteq\Delta_s.$$

For each $V\in\mathcal{V}_s$, pick a point $x_V^*\in V$, and define

$$\tilde{f}=f\cdot(1-\sum_{s=1}^{S}\sum_{\gamma\in\Gamma_s}\sum_{V\in\mathcal{V}_s}(\phi_V^{(s)}\circ\gamma^{-1}))+\sum_{s=1}^{S}\sum_{\gamma\in\Gamma_s}\sum_{V\in\mathcal{V}_s}f(x_V^*\gamma)(\phi_V^{(s)}\circ\gamma^{-1})\in\mathrm{C}(X).$$

Then, using (5.1) in the last step, one has that, for any $x\in X$,

$$\begin{aligned}
&\left|f(x)-\tilde{f}(x)\right|\\
=\ &\left|f(x)-(f(x)(1-\sum_{s=1}^{S}\sum_{\gamma\in\Gamma_s}\sum_{V\in\mathcal{V}_s}\phi_V^{(s)}(x\gamma^{-1}))+\sum_{s=1}^{S}\sum_{\gamma\in\Gamma_s}\sum_{V\in\mathcal{V}_s}f(x_V^*\gamma)\phi_V^{(s)}(x\gamma^{-1}))\right|\\
=\ &\left|f(x)(1-\sum_{s=1}^{S}\sum_{\gamma\in\Gamma_s}\sum_{V\in\mathcal{V}_s}\phi_V^{(s)}(x\gamma^{-1})+\sum_{s=1}^{S}\sum_{\gamma\in\Gamma_s}\sum_{V\in\mathcal{V}_s}\phi_V^{(s)}(x\gamma^{-1}))-\right.\\
&\left.(f(x)(1-\sum_{s=1}^{S}\sum_{\gamma\in\Gamma_s}\sum_{V\in\mathcal{V}_s}\phi_V^{(s)}(x\gamma^{-1}))+\sum_{s=1}^{S}\sum_{\gamma\in\Gamma_s}\sum_{V\in\mathcal{V}_s}f(x_V^*\gamma)\phi_V^{(s)}(x\gamma^{-1}))\right|\\
=\ &\left|\sum_{s=1}^{S}\sum_{\gamma\in\Gamma_s}\sum_{V\in\mathcal{V}_s}(f(x)-f(x_V^*\gamma))\phi_V^{(s)}(x\gamma^{-1}))\right|\\
\leq\ &\sum_{s=1}^{S}\sum_{\gamma\in\Gamma_s}\sum_{V\in\mathcal{V}_s}\left|f(x)-f(x_V^*\gamma)\right|\phi_V^{(s)}(x\gamma^{-1}))\\
<\ &\frac{\varepsilon}{3}.
\end{aligned}$$

That is,

$$\left\|f-\tilde{f}\right\|<\frac{\varepsilon}{3}. \tag{5.5}$$

Define the linear function $F_s\colon\mathrm{C}\Delta_s\to\mathbb{R}^{|\Gamma_s|}$ by

$$F_s([V])=\bigoplus_{\gamma\in\Gamma_s}f(x_V^*\gamma)\in\mathbb{R}^{|\Gamma_s|};$$

that is

$$F_s(t_0[V_0]+t_1[V_1]+\cdots+t_d[V_d])=\bigoplus_{\gamma\in\Gamma_s}(\sum_{i=0}^{d}t_if(x_{V_i}^*\gamma))\in\mathbb{R}^{|\Gamma_s|},$$

whenever $V_0 \cap V_1 \cap \cdots \cap V_d \neq \varnothing$. (In particular, $F_s(\mathbf{0}) = 0$.)

Then, together with (5.4), one has

$$\tilde{f} = f(1 - \sum_{s=1}^{S} \sum_{\gamma \in \Gamma_s} \sum_{V \in \mathcal{V}_s} (\phi_V^{(s)} \circ \gamma^{-1})) + \sum_{s=1}^{S} \sum_{\gamma \in \Gamma_s} \pi_{s,\gamma} \circ F_s \circ \eta_s \circ \gamma^{-1},$$

where $\pi_{s,\gamma}$ is the projection of $\mathbb{R}^{|\Gamma_s|}$ to the γ-coordinate.

Consider the restriction $F_s|_{\Delta_s}$, and apply Lemma 5.7 of [18] to Δ_s; there is a linear map $\tilde{F}_s : \Delta_s \to \mathbb{R}^{|\Gamma_s|}$ such that

$$\left\| F_s(x) - \tilde{F}_s(x) \right\|_\infty < \frac{\varepsilon}{3}, \quad x \in \Delta_s, \tag{5.6}$$

and

$$\left| \{\gamma \in \Gamma_s : \pi_{s,\gamma}(\tilde{F}_s(x)) = 0\} \right| \leq \mathrm{dim}\Delta_s, \quad x \in \Delta_s. \tag{5.7}$$

Extend $\tilde{F}_s$ to $\mathrm{C}\Delta_s$ linearly by assigning $\tilde{F}_s(\mathbf{0}) = 0$, and still denote it by $\tilde{F}_s$. Since the map F_s is linear on $\mathrm{C}\Delta_s$ and $F_s(\mathbf{0}) = 0$, by (5.6) and (5.7), one has

$$\left\| F_s(x) - \tilde{F}_s(x) \right\|_\infty < \frac{\varepsilon}{3}, \quad x \in \mathrm{C}\Delta_s, \tag{5.8}$$

and

$$\left| \{\gamma \in \Gamma_s : \pi_{s,\gamma}(\tilde{F}_s(x)) = 0\} \right| \leq \mathrm{dim}\Delta_s, \quad x \in \mathrm{C}\Delta_s \setminus \{\mathbf{0}\}. \tag{5.9}$$

Put

$$g = f(1 - \sum_{s=1}^{S} \sum_{\gamma \in \Gamma_s} \sum_{V \in \mathcal{V}_s} (\phi_V^{(s)} \circ \gamma^{-1})) + \sum_{s=1}^{S} \sum_{\gamma \in \Gamma_s} \pi_{s,\gamma} \circ \tilde{F}_s \circ \eta_s \circ \gamma^{-1}, \tag{5.10}$$

and then, for any $x \in X$,

$$\begin{aligned} & \left| \tilde{f}(x) - g(x) \right| \\ = & \left| \sum_{s=1}^{S} \sum_{\gamma \in \Gamma_s} \pi_{s,\gamma} \circ F_s \circ \eta_s(x\gamma^{-1}) - \sum_{s=1}^{S} \sum_{\gamma \in \Gamma_s} \pi_{s,\gamma} \circ \tilde{F}_s \circ \eta_s(x\gamma^{-1}) \right| \\ = & \left| \sum_{s=1}^{S} \sum_{\gamma \in \Gamma_s} (\pi_{s,\gamma} \circ F_s \circ \eta_s(x\gamma^{-1}) - \pi_{s,\gamma} \circ \tilde{F}_s \circ \eta_s(x\gamma^{-1})) \right|. \end{aligned}$$

If $x \notin \bigsqcup_{s=1}^{S} \bigsqcup_{\gamma \in \Gamma_s} W_s\gamma$, then

$$\eta_s(x\gamma^{-1}) = \mathbf{0}, \quad \gamma \in \Gamma_s, \ s = 1, ..., S.$$

Hence

$$\pi_{s,\gamma} \circ F_s \circ \eta_s(x\gamma^{-1}) = \pi_{s,\gamma} \circ \tilde{F}_s \circ \eta_s(x\gamma^{-1}) = 0, \quad \gamma \in \Gamma_s, \ s = 1, ..., S,$$

and

$$\tilde{f}(x) = g(x). \tag{5.11}$$

If $x \in \bigsqcup_{s=1}^{S} \bigsqcup_{\gamma\in\Gamma_s} W_s\gamma \subseteq \bigsqcup_{s=1}^{S} \bigsqcup_{\gamma\in\Gamma_s} U_s\gamma$, then there exist $s_0 \in \{1, ..., S\}$ and $\gamma_0 \in \Gamma_{s_0}$ such that

$$x \text{ is only in } U_{s_0}\gamma_0.$$

Then

$$\eta_s(x\gamma^{-1}) = \mathbf{0}, \quad \gamma \in \Gamma_s,\ s \neq s_0,$$

and

$$\eta_{s_0}(x\gamma^{-1}) = \mathbf{0}, \quad \gamma \neq \gamma_0.$$

Hence, by (5.8),

$$\begin{aligned} & \left|\sum_{s=1}^{S}\sum_{\gamma\in\Gamma_s}(\pi_{s,\gamma}\circ F_s\circ\eta_s(x\gamma^{-1}) - \pi_{s,\gamma}\circ\tilde{F}_s\circ\eta_s(x\gamma^{-1}))\right| \\ = & \left|\pi_{s_0,\gamma_0}\circ F_{s_0}\circ\eta_{s_0}(x\gamma_0^{-1}) - \pi_{s_0,\gamma_0}\circ\tilde{F}_{s_0}\circ\eta_{s_0}(x\gamma_0^{-1})\right| \\ < & \frac{\varepsilon}{3}, \end{aligned}$$

and

$$\left|\tilde{f}(x) - g(x)\right| < \frac{\varepsilon}{3}.$$

Together with (5.11), one has

$$\left\|\tilde{f} - g\right\| < \frac{\varepsilon}{3};$$

and together with (5.5), one has

$$\|f - g\| < \frac{2\varepsilon}{3} < \varepsilon.$$

Let us estimate

$$\mathrm{ocap}(\{x \in X : g(x) = 0\}).$$

First, note that for an arbitrary $x \in B_s$, where $s \in \{1, 2, ..., S\}$, one has that $\eta_s(x) \neq \mathbf{0}$, and hence by (5.9),

$$|\{\gamma \in \Gamma_s : g(x\gamma) = 0\}| = \left|\{\gamma \in \Gamma_s : \pi_{s,\gamma}(\tilde{F}_s(\eta_s(x))) = 0\}\right| \leq \dim\Delta_s. \tag{5.12}$$

Let $\Gamma_0 \subseteq \Gamma$ be a finite set which is sufficiently invariant such that

$$\frac{\left|\mathrm{int}_{\bigcup_{s=1}^{S}(\Gamma_s^2)^{-1}}\Gamma_0\right|}{|\Gamma_0|} > 1 - \frac{\varepsilon}{3}, \tag{5.13}$$

and (by (5.2)),

$$\frac{1}{|\Gamma_0|}\left|\{\gamma \in \Gamma_0 : x\gamma \in X \setminus \bigsqcup_{s=1}^{S}\bigsqcup_{\gamma\in\Gamma_s} B_s\gamma\}\right| < \frac{\varepsilon}{3}, \quad x \in X. \tag{5.14}$$

Let $x \in X$ be arbitrary, and consider the orbit $x\Gamma$. The partition

$$X = (X \setminus \bigsqcup_{c=1}^{S}\bigsqcup_{\gamma\in\Gamma_s} B_s\gamma) \sqcup \bigsqcup_{c=1}^{S}\bigsqcup_{\gamma\in\Gamma_s} B_s\gamma$$

induces a partition of $x\Gamma$; since the action is free, this induces a partition of Γ:

$$\Gamma = \Lambda \sqcup \bigsqcup_{i=1}^{\infty} c_i \Gamma_{s(i)},$$

where

$$\Lambda = \{\gamma \in \Gamma : x\gamma \in X \setminus \bigsqcup_{s=1}^{S} \bigsqcup_{\gamma \in \Gamma_s} B_s \gamma\},$$

$$s(i) \in \{1, 2, ..., S\} \quad \text{and} \quad c_i \in \Gamma, \quad i = 1, 2, ...,$$

satisfy

$$xc_i \in B_{s(i)}. \tag{5.15}$$

Restrict this partition to Γ_0, one has

$$\Gamma_0 = (\Gamma_0 \cap \Lambda) \cup \bigsqcup_{c_i \Gamma_{s(i)} \nsubseteq \Gamma_0} (\Gamma_0 \cap (c_i \Gamma_{s(i)})) \cup \bigsqcup_{c_i \Gamma_{s(i)} \subseteq \Gamma_0} c_i \Gamma_{s(i)}. \tag{5.16}$$

A straightforward calculation shows that if $\gamma \in \Gamma_0 \cap (c_i \Gamma_{s(i)})$ and $c_i \Gamma_{s(i)} \nsubseteq \Gamma_0$, then $\gamma \notin \mathrm{int}_{(\Gamma_{s(i)}^2)^{-1}} \Gamma_0$. Therefore

$$\bigsqcup_{c_i \Gamma_{s(i)} \nsubseteq \Gamma_0} (\Gamma_0 \cap c_i \Gamma_{s(i)}) \subseteq \Gamma_0 \setminus \mathrm{int}_{\bigcup_{s=1}^{S} (\Gamma_{s(i)}^2)^{-1}} (\Gamma_0) =: \partial_{\bigcup_{s=1}^{S} (\Gamma_{s(i)}^2)^{-1}} \Gamma_0,$$

and, by (5.16), (5.15), (5.12), (5.14), (5.13), and (5.3),

$$\begin{aligned}
&\frac{1}{\Gamma_0} \left|\{\gamma \in \Gamma_0 : g(x\gamma) = 0\}\right| \\
= \ & \frac{|\Gamma_0 \cap \Lambda|}{|\Gamma_0|} + \frac{\left|\bigsqcup_{c_i \Gamma_{s(i)} \nsubseteq \Gamma_0} (\Gamma_0 \cap c_i \Gamma_{s(i)})\right|}{|\Gamma_0|} + \frac{1}{|\Gamma_0|} \sum_{c_i \Gamma_{s(i)} \subseteq \Gamma_0} \left|\{\gamma \in c_i \Gamma_{s(i)} : g(x\gamma) = 0\}\right| \\
= \ & \frac{|\Gamma_0 \cap \Lambda|}{|\Gamma_0|} + \frac{\left|\bigsqcup_{c_i \Gamma_{s(i)} \nsubseteq \Gamma_0} (\Gamma_0 \cap c_i \Gamma_{s(i)})\right|}{|\Gamma_0|} + \frac{1}{|\Gamma_0|} \sum_{c_i \Gamma_{s(i)} \subseteq \Gamma_0} \left|\{\gamma \in \Gamma_{s(i)} : g((xc_i)\gamma) = 0\}\right| \\
\leq \ & \frac{|\Gamma_0 \cap \Lambda|}{|\Gamma_0|} + \frac{\left|\partial_{\bigcup_{s=1}^{S} (\Gamma_{s(i)}^2)^{-1}} \Gamma_0\right|}{|\Gamma_0|} + \frac{1}{|\Gamma_0|} \sum_{c_i \Gamma_{s(i)} \subseteq \Gamma_0} \mathrm{dim} \Delta_{s(i)} \\
\leq \ & \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\sum_{c_i \Gamma_{s(i)} \subseteq \Gamma_0} \mathrm{dim} \Delta_{s(i)}}{\sum_{c_i \Gamma_{s(i)} \subseteq \Gamma_0} \left|\Gamma_{s(i)}\right|} \\
< \ & \frac{2\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Since x is arbitrary, this implies

$$\mathrm{ocap}(\{x \in X : g(x) = 0\}) < \varepsilon,$$

as desired. □

Remark 5.2. Note that if $\Gamma = \mathbb{Z}^d$, it follows from Theorem 1.10.1 and Theorem 1.10.3 of [17] that

$$\mathrm{TRP} + \mathrm{mdim0} \Leftrightarrow \mathrm{SBP},$$

where TRP stands for the Topological Rokhlin Property in the sense of 1.9 of [17] ($\mathrm{edim}(X, \mathbb{Z}^d) \leq l$ densely for some $l \in \mathbb{N}$ is actually not needed in Theorem 1.10.3). It is easy to see that URP implies TRP. Therefore, in this case, the statement of Theorem 5.1 is covered by Theorem 1.10.3 of [17]. It was also proved later in [18] (Corollary 5.4) that

$$\mathrm{mdim0} \Leftrightarrow \mathrm{SBP}$$

for any $\mathbb{Z}^d$-actions with marker property.

With the Uniform Property Gamma and [2], Kerr and Szabo has the following:

Theorem 5.3 (Corollary 9.5 of [21])**.** *Assume that* (X, Γ) *has the (SBP). Then,* $\mathrm{C}(X) \rtimes \Gamma$ *has the strict comparison if and only if it is* $\mathcal{Z}$*-stable.*

Thus, together with Theorem 5.1 and Theorem 4.8 of [33], one has the following:

Alternative proof of Theorem 4.8. Since (X, Γ) is assumed to have the (URP) and mdim0, by Theorem 5.1, it has the (SBP). Therefore, by Theorem 5.3, in order to prove the theorem, it is enough to show that $\mathrm{C}(X) \rtimes \Gamma$ has the strict comparison of positive elements. But since (X, Γ) has the (COS) and mdim0, the strict comparison property follows from Theorem 4.8 of [33]. ☐

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