ON THE CLASSIFICATION OF SIMPLE UNITAL C*-ALGEBRAS WITH FINITE DECOMPOSITION RANK

GEORGE A. ELLIOTT AND ZHUANG NIU

Dedicated to Richard V. Kadison on the occasion of his ninetieth birthday

Abstract. Let A be a unital simple separable C*-algebra satisfying the UCT. Assume that $\mathrm{dr}(A) < +\infty$, A is Jiang-Su stable, and $\mathrm{K}_0(A) \otimes \mathbb{Q} \cong \mathbb{Q}$. Then A is an ASH algebra (indeed, A is a rationally AH algebra).

1. Introduction

Let A be a simple separable nuclear unital C*-algebra. In [17], Matui and Sato showed that $A \otimes \mathrm{UHF}$ can be tracially approximated by finite dimensional C*-algebras (i.e., is TAF) if A is quasidiagonal with unique trace.

In this note, this result is enlarged upon as follows: the condition on the trace simplex is removed, at the cost of assuming the UCT, (still) finite nuclear dimension, and (still) that all traces are quasidiagonal—e.g., by assuming finite decomposition rank—see [2]—and (so far) of restricting the K_0-group to have torsion-free rank equal to one.

Theorem 1.1. *Let A be a simple unital separable C*-algebra satisfying the UCT. If $A \otimes Q$ has finite decomposition rank and $\mathrm{K}_0(A) \otimes \mathbb{Q} \cong \mathbb{Q}$, then $A \otimes Q \in \mathrm{TA}\mathcal{I}$. In particular, $A \otimes \mathcal{Z}$ is classifiable.*

This theorem can also be regarded as an abstract version (still in a special case) of the classification result of [9] and [7], where any simple unital locally approximately subhomogeneous C*-algebra is shown to be rationally tracially approximated by Elliott-Thomsen algebras (1-dimensional noncommutative CW complexes) ([7]) and hence to be classifiable ([9]).

2. The main result and the proof

In this note let us use Q to denote the UHF algebra with $\mathrm{K}_0(Q) \cong \mathbb{Q}$.

Definition 2.1 (N. Brown, [3])**.** Let A be a unital C*-algebra, and denote by $\mathrm{T}_{\mathrm{qd}}(A)$ the tracial states with the following property: For any $(\mathcal{F}, \varepsilon)$, there is a unital completely positive map $\phi : A \to Q$ such that

(1) $|\tau(a) - \mathrm{tr}(\phi(a))| < \varepsilon$, $a \in \mathcal{F}$, and
(2) $\|\phi(ab) - \phi(a)\phi(b)\| < \varepsilon$, $a, b \in \mathcal{F}$.

The research of the first author was supported by a Natural Sciences and Engineering Research Council of Canada (NSERC) Discovery Grant, and the research of the second author was supported by a Simons Foundation Collaboration Grant.

Remark 2.2. In the original definition of a quasidiagonal trace (Definition 3.3.1 of [3]), the UHF algebra Q was replaced by a matrix algebra. It is easy to see that these two approaches are equivalent.

Recall the tracial approximate uniqueness result of [6] and [13].

Theorem 2.3 (Theorem 4.15 of [6]; Theorem 5.3 of [13])**.** *Let A be a simple, unital, exact, separable C*-algebra satisfying the UCT. For any finite subset $\mathcal{F} \subseteq A$ and any $\varepsilon > 0$, there exist $n \in \mathbb{N}$ and a $\underline{\mathrm{K}}$-triple $(\mathcal{P}, \mathcal{G}, \delta)$ with the following property: For any admissible codomain B, and any three completely positive contractions $\phi, \psi, \xi : A \to B$ which are δ-multiplicative on $\mathcal{G}$, with ξ unital, ϕ and ψ nuclear, and $\phi_{\#}(p) = \psi_{\#}(p)$ in $\underline{\mathrm{K}}(B)$ for all $p \in \mathcal{P}$, and such that $\phi(1)$ and $\psi(1)$ are unitarily equivalent projections, there exists a unitary $u \in \mathcal{U}_{n+1}(B)$ such that*

$$\left\| u^* \begin{pmatrix} \phi(a) & \\ & n \cdot \xi(a) \end{pmatrix} u - \begin{pmatrix} \psi(a) & \\ & n \cdot \xi(a) \end{pmatrix} \right\| < \varepsilon, \quad a \in \mathcal{F}.$$

One may arrange that $u^*(\phi(1) \oplus n \cdot 1)u = \psi(1) \oplus n \cdot 1$.

Remark 2.4. In the theorem above (and also Corollary 2.5 below), one assumes by convention that the finite subset $\mathcal{G}$ is sufficiently large and δ is sufficiently small that $[\phi(p)]$ is well defined for any $p \in \mathcal{P}$ if a map ϕ is δ-multiplicative on $\mathcal{G}$.

When $B = Q$, in fact one does not have to consider all the K-theory with coefficients. More precisely, one has

Corollary 2.5. *Let A be a simple, unital, exact, separable C*-algebra satisfying the UCT. For any finite subset $\mathcal{F} \subseteq A$ and any $\varepsilon > 0$, there exist $n \in \mathbb{N}$ and a $\underline{\mathrm{K}}$-triple $(\mathcal{P}, \mathcal{G}, \delta)$, with $\mathcal{P} \subseteq \mathrm{Proj}_\infty(A)$, with the following property: For any three completely positive contractions $\phi, \psi, \xi : A \to Q$ which are δ-multiplicative on $\mathcal{G}$, with $\phi(1) = \psi(1) = 1_Q - \xi(1)$ a projection, $[\phi(p)]_0 = [\psi(p)]_0$ in $\mathrm{K}_0(Q)$ for all $p \in \mathcal{P}$, and $\mathrm{tr}(\phi(1)) = \mathrm{tr}(\psi(1)) < 1/n$, there exists a unitary $u \in Q$ such that*

$$\|u^*(\phi(a) \oplus \xi(a))u - \psi(a) \oplus \xi(a)\| < \varepsilon, \quad a \in \mathcal{F}.$$

Proof. Applying Theorem 2.3 to $\mathcal{F}$ and $\varepsilon > 0$, one obtains $n_0 \in \mathbb{N}$ and a $\underline{\mathrm{K}}$-triple $(\widetilde{\mathcal{P}}, \mathcal{G}, \delta)$ with the property of Theorem 2.3. Set

$$n_0 + 1 = n \quad \text{and} \quad \widetilde{\mathcal{P}} \cap \mathrm{Proj}_\infty(A) = \mathcal{P}.$$

Let us show that n and $(\mathcal{P}, \mathcal{G}, \delta)$ have the desired property.

Let $\phi, \psi, \xi : A \to Q$ be completely positive contractions which are δ-multiplicative on $\mathcal{G}$, with $\phi(1) = \psi(1) = 1_Q - \xi(1)$ a projection, $[\phi(p)]_0 = [\psi(p)]_0$ in $\mathrm{K}_0(Q)$ for all $p \in \mathcal{P}$, and $\mathrm{tr}(\phi(1)) = \mathrm{tr}(\psi(1)) < 1/n$.

Decompose ξ approximately on $\mathcal{F} \subseteq A$ as a repeated direct sum

$$\underbrace{\xi' \oplus \cdots \oplus \xi'}_{n_0},$$

where $\xi' : A \to Q$ is again a completely positive contraction which is (necessarily, if the approximation is sufficiently good) δ-multiplicative on $\mathcal{G}$, and $(\xi' \oplus \cdots \oplus \xi')(1_A) = \xi(1_A)$. Since $\mathrm{tr}(\phi(1)) = \mathrm{tr}(\psi(1)) < 1/n$, one has that

$$\mathrm{tr}(\xi(1)) > (n-1)/n = n_0/n,$$

and so

$$\phi(1) = \psi(1) \preceq e,$$

where $e = \xi'(1)$. Then the maps $\phi \oplus \xi$ and $\psi \oplus \xi$ have the forms

$$\phi \oplus (n \cdot \xi'),\ \psi \oplus (n \cdot \xi') : A \to \mathrm{M}_{n_0+1}(eQe),$$

respectively. Note that eQe is stably isomorphic to Q, and therefore

$$\mathrm{K}_0(eQe, \mathbb{Z}/k\mathbb{Z}) = \{0\}, \quad k \in \mathbb{N} \setminus \{0\}, \quad \text{and} \quad \mathrm{K}_1(eQe, \mathbb{Z}/k\mathbb{Z}) = \{0\}, \quad k \in \mathbb{N} \cup \{0\}.$$

Together with the assumption $[\phi(p)]_0 = [\psi(p)]_0$ in $\mathrm{K}_0(Q)$ for all $p \in \mathcal{P}$, this implies

$$\phi_{\#}(p) = \psi_{\#}(p) \in \underline{\mathrm{K}}(eQe), \quad p \in \widetilde{\mathcal{P}}.$$

Thus, it follows from Theorem 2.3 that there is a unitary $w \in \mathrm{M}_{n_0+1}(eQe)$ such that

$$\left\| w^* \begin{pmatrix} \phi(a) & \\ & n_0 \cdot \xi'(a) \end{pmatrix} w - \begin{pmatrix} \psi(a) & \\ & n_0 \cdot \xi'(a) \end{pmatrix} \right\| < \varepsilon, \quad a \in \mathcal{F},$$

and

$$w^*(\phi(1) \oplus n_0 \cdot e)w = \psi(1) \oplus n_0 \cdot e. \tag{2.1}$$

Note that

$$\phi(1) \oplus (n_0 \cdot e) = \psi(1) \oplus (n_0 \cdot e) = 1_Q.$$

By (2.1), a straightforward calculation shows that

$$u := 1_Q w 1_Q$$

is a unitary of Q. Clearly, if the approximation of ξ by $\xi' \oplus \cdots \oplus \xi'$ on $\mathcal{F}$ is sufficiently good, then, in Q,

$$\|u^*(\phi(a) \oplus \xi(a))u - \psi(a) \oplus \xi(a)\| < \varepsilon, \quad a \in \mathcal{F},$$

as desired. □

Definition 2.6. Recall that an abelian group G is said to be of (torsion free) rank one if $G \otimes \mathbb{Q} \cong \mathbb{Q}$.

Lemma 2.7. *Let Δ be a compact metrizable Choquet simplex. Then, for any finite subset $\mathcal{F} \subseteq \mathrm{Aff}(\Delta)$ and any $\varepsilon > 0$, there exist $m \in \mathbb{N}$ and unital (pointwise) positive linear maps ϱ and θ,*

$$\mathrm{Aff}(\Delta) \xrightarrow{\varrho} \mathbb{R}^m \xrightarrow{\theta} \mathrm{Aff}(\Delta),$$

where the unit of $\mathbb{R}^m$ is $(1, ..., 1)$, such that

$$\|\theta(\varrho(f)) - f\|_\infty < \varepsilon, \quad f \in \mathcal{F}.$$

Proof. By Theorem 5.2 of [12] and its corollary, there is an increasing sequence of finite-dimensional subspaces of $\mathrm{Aff}(\Delta)$ with dense union, containing the canonical order unit $1 \in \mathrm{Aff}(\Delta)$, and such that each map $\mathbb{R}^{m_k} \to \mathbb{R}^{m_{k+1}}$ and $\mathbb{R}^{m_k} \hookrightarrow \mathrm{Aff}(\Delta)$ is positive, with respect to the canonical (pointwise) order relations:

$$\mathbb{R}^{m_1} \hookrightarrow \mathbb{R}^{m_2} \hookrightarrow \cdots \hookrightarrow \mathrm{Aff}(\Delta).$$

(The authors are indebted to David Handelman for reminding us of [12].)

Without loss of generality, one may assume that $\mathcal{F} \subseteq \mathbb{R}^{m_1}$, and hence one only has to extend the identity map of $\mathbb{R}^{m_1}$ to a positive unital map $\varrho : \mathrm{Aff}(\Delta) \to \mathbb{R}^{m_1}$.

Write $\mathbb{R}^{m_1} = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \cdots \oplus \mathbb{R}e_{m_1}$, and consider the unital positive functionals

$$\rho_i : \mathbb{R}^{m_1} \ni (x_1, x_2, ..., x_{m_1}) \mapsto x_i \in \mathbb{R}, \quad i = 1, ..., m_1.$$

By the Riesz Extension Theorem ([18]), each ρ_i can be extended to a unital positive linear functional $\tilde{\rho}_i : \mathrm{Aff}(\Delta) \to \mathbb{R}$. Then the map

$$\varrho : \mathrm{Aff}(\Delta) \ni f \mapsto (\tilde{\rho}_1(f), \tilde{\rho}_2(f), ..., \tilde{\rho}_{m_1}(f)) \in \mathbb{R}^{m_1}$$

has the desired property. □

Lemma 2.8. *Let $C = \varinjlim(C_n, \iota_n)$ be a unital inductive system of C*-algebras such that C is simple. Let $(\mathbb{R}^m, \|\cdot\|_\infty, u)$ be a finite-dimensional ordered Banach space with order unit u, and let $\gamma : \mathbb{R}^m \to \mathrm{Aff}(\mathrm{T}(C))$ be a unital positive linear map. Then, for any finite set $\mathcal{F} \subseteq \mathbb{R}^m$ and any $\varepsilon > 0$, there are n and a unital positive linear map $\gamma_n : \mathbb{R}^m \to \mathrm{Aff}(\mathrm{T}(C_n))$ such that*

$$\|\gamma(a) - \iota_{n,\infty} \circ \gamma_n(a)\| < \varepsilon, \quad a \in \mathcal{F}.$$

Proof. Denote by e_i, $i = 1, ..., m$, the standard basis of $\mathbb{R}^m$, and write

$$u = c_1 e_1 + \cdots + c_m e_m,$$

where $c_1, ..., c_m > 0$. Since C is simple, each affine function $\gamma(e_i)$ is strictly positive on $\mathrm{T}(C)$. Since $\mathrm{T}(C)$ is compact, there is δ_i such that

$$\gamma(e_i)(\tau) > \delta_i, \quad \tau \in \mathrm{T}(C),\ 1 \leq i \leq m. \tag{2.2}$$

Without loss of generality, one may assume that $\mathcal{F} = \{e_1, e_2, ..., e_m\}$.

Pick C_n and $e'_1, e'_2, ..., e'_{m-1} \in \mathrm{Aff}(\mathrm{T}(C_n))$ such that

$$\|\iota_{n,\infty}(e'_i) - \gamma(e_i)\|_\infty < \min\{\varepsilon, \frac{\delta_i}{2}, \frac{c_m\delta_m}{2(c_1 + \cdots + c_{m-1})}, \frac{c_m\varepsilon}{2(c_1 + \cdots + c_{m-1})}\}, \quad 1 \leq i \leq m-1.$$

In particular, by (2.2),

$$\iota_{n,\infty}(e'_i)(\tau) \geq \delta_i/2, \quad \tau \in \mathrm{T}(C),\ 1 \leq i \leq m-1.$$

Setting

$$e'_m := \frac{1}{c_m}(1 - c_1 e'_1 - \cdots - c_{m-1}e'_{m-1}) \in \mathrm{Aff}(\mathrm{T}(C_n)),$$

one has

$$\begin{aligned} \|\iota_{n,\infty}(e'_m) - \gamma(e_m)\|_\infty &= \|\frac{1}{c_m}(1 - c_1\iota_{n,\infty}(e'_1) - \cdots - c_{m-1}\iota_{n,\infty}(e'_{m-1})) - \\ &\quad \frac{1}{c_m}(1 - c_1\gamma(e_1) - \cdots - c_{m-1}\gamma(e_{m-1}))\|_\infty \\ &\leq \min\{\delta_m/2, \varepsilon\}. \end{aligned}$$

In particular, by (2.2),

$$\iota_{n,\infty}(e'_m)(\tau) \geq \delta_m/2, \quad \tau \in \mathrm{T}(C).$$

Then, considering instead the images of $e'_1, e'_2, ..., e'_m$ in a building block further out (replacing n by the later index), one may assume that

$$e'_i(\tau) > \delta_i/4, \quad \tau \in \mathrm{T}(C_n),\ 1 \leq i \leq m.$$

In particular, all the affine functions $e'_i \in \mathrm{Aff}(\mathrm{T}(C_n))$ are positive. Define $\gamma_n : \mathbb{R}^m \to \mathrm{Aff}(\mathrm{T}(C_{n_0}))$ by

$$\gamma_n(e_i) = e'_i, \quad 1 \leq i \leq m.$$

It is clear that γ_n satisfies the condition of the lemma. □

Theorem 2.9. *Let A be a separable simple unital exact C*-algebra satisfying the UCT. Assume that $\mathrm{T}(A) = \mathrm{T}_{\mathrm{qd}}(A)$ and that $\mathrm{K}_0(A)$ is of rank one. Then, for any finite set $\mathcal{F} \subseteq A \otimes Q$ and any $\varepsilon > 0$, there are unital completely positive linear maps $\phi : A \otimes Q \to I$ and $\psi : I \to A \otimes Q$, where I is an interval algebra, such that*

(1) *ϕ is $\mathcal{F}$-δ-multiplicative, ψ is an embedding, and*
(2) *$|\tau(\psi \circ \phi(a) - a)| < \varepsilon$, $a \in \mathcal{F}$, $\tau \in \mathrm{T}(A \otimes Q)$.*

Proof. If $\mathrm{T}(A) = \emptyset$, then the conclusion holds trivially (with $I = \{0\}$). Otherwise, assuming, as we may, that $A \cong A \otimes Q$, we have $\mathrm{K}_0(A) \cong \mathbb{Q}$ (as order-unit groups).

Apply Corollary 2.5 to A with respect to $(\mathcal{F}\cdot\mathcal{F}, \varepsilon/4)$ to obtain n and $(\mathcal{P}, \mathcal{G}, \delta)$. Since $\mathrm{K}_0(A) = \mathbb{Q}$ (unique unital identification), we may suppose that $\mathcal{P} = \{1_A\}$.

By Theorem 3.9 of [20], there is a simple unital inductive limit $C = \varinjlim(C_i, \iota_i)$ such that $\mathrm{K}_0(C) = \mathbb{Q}$ (unital identification), $C_i = \mathrm{M}_{k_i}(\mathrm{C}([0,1]))$, the maps ι_i are injective, and there is an isomorphism

$$\Xi : \mathrm{Aff}(\mathrm{T}(A)) \cong \mathrm{Aff}(\mathrm{T}(C)).$$

By Lemma 2.7, there is an approximate factorization, by means of unital positive maps,

$$\mathrm{Aff}(\mathrm{T}(A)) \xrightarrow{\varrho} \mathbb{R}^m \xrightarrow{\theta} \mathrm{Aff}(\mathrm{T}(A)),$$

such that

$$\|\theta(\varrho(\hat{f})) - \hat{f}\|_\infty < \varepsilon/16, \quad f \in \mathcal{F}.$$

Therefore, by Lemma 2.8, after discarding finitely many terms of the sequence (C_i, ι_i), there is a unital positive linear map

$$\gamma : \mathrm{Aff}(\mathrm{T}(A)) \xrightarrow{\varrho} \mathbb{R}^m \longrightarrow \mathrm{Aff}(\mathrm{T}(C_1))$$

such that

$$\|(\iota_{1,\infty})_*(\gamma(\hat{f})) - \Xi(\hat{f})\|_\infty < \varepsilon/8, \quad f \in \mathcal{F}. \tag{2.3}$$

Denote by $\gamma^* : \mathrm{T}(C_1) \to \mathrm{T}(A)$ the affine map induced by γ on tracial simplices. Since γ factors through $\mathbb{R}^m$ (so that γ^* factors through a finite dimensional simplex), there are $\tau_1, ..., \tau_m \in \mathrm{T}(A)$ and continuous functions $c_1, c_2, ..., c_m : [0,1] \to [0,1]$ such that

$$\gamma^*(\tau_t) = c_1(t)\tau_1 + c_2(t)\tau_2 + \cdots + c_m(t)\tau_m, \quad t \in [0,1], \tag{2.4}$$

and

$$c_1(t) + c_2(t) + \cdots + c_m(t) = 1, \quad t \in [0,1],$$

where $\tau_t \in \mathrm{T}(C_1)$ is determined by the Dirac measure concentrated at $t \in [0,1]$.

Since $\tau_1, \tau_2, ..., \tau_m \in \mathrm{T}_{\mathrm{qd}}(A)$, there are unital completely positive linear maps $\phi_k : A \to Q$, $k = 1, 2, ..., m$, such that each ϕ_k is $\mathcal{G}$-δ-multiplicative, and

$$|\mathrm{tr}(\phi_k(f)) - \tau_k(f)| < \varepsilon/16m, \quad f \in \mathcal{F}. \tag{2.5}$$

For each $t \in [0,1]$, there is a open neighbourhood U such that for any $s \in U$, one has

$$|c_k(s) - c_k(t)| < 1/4mn.$$

(Recall that n is the constant from Corollary 2.5, as in the second paragraph of the proof.) Since $[0,1]$ is compact, there is a partition $0 = t_0 < t_1 < \cdots < t_{l-1} < t_l = 1$ such that

$$|c_k(s) - c_k(t_j)| < 1/4mn, \quad s \in [t_{j-1}, t_j]. \tag{2.6}$$

Moreover, we may assume that this partition is fine enough that

$$|\gamma(\hat{f})(\tau_t) - \gamma(\hat{f})(\tau_{t_j})| < \varepsilon/8, \quad f \in \mathcal{F},\ t \in [t_{j-1}, t_j]. \tag{2.7}$$

For each $j = 0, 1, ..., l$, pick rational numbers $r_{j,1}, r_{j,2}, ..., r_{j,m} \in [0,1]$ such that

$$r_{j,1} + \cdots + r_{j,m} = 1$$

and

$$|r_{j,k} - c_k(t_j)| < \min\{\varepsilon/16m, 1/4mn\}, \quad k = 1, ..., m. \tag{2.8}$$

Write $r_{j,k} = q_{j,k}/p$ where $q_{j,k}, p \in \mathbb{N}$, and then define

$$\varphi_j := (\underbrace{\phi_1 \oplus \cdots \oplus \phi_1}_{q_{j,1}}) \oplus \cdots \oplus (\underbrace{\phi_m \oplus \cdots \oplus \phi_m}_{q_{j,m}}) : A \to Q.$$

Note that it follows from (2.4), (2.5), and (2.8) that

$$\left|\mathrm{tr}(\varphi_j(f)) - \gamma^*(\tau_{t_j})(f)\right| < \varepsilon/4, \quad f \in \mathcal{F}. \tag{2.9}$$

By (2.8), (2.6), one has that

$$\frac{|q_{j,k} - q_{j+1,k}|}{p} < \frac{1}{mn}, \quad k = 1, ..., m,\ j = 0, ..., l-1. \tag{2.10}$$

For each $j = 0, ..., l-1$, compare the direct sum maps

$$\varphi_j = (\underbrace{\phi_1 \oplus \cdots \oplus \phi_1}_{q_{j,1}}) \oplus \cdots \oplus (\underbrace{\phi_m \oplus \cdots \oplus \phi_m}_{q_{j,m}})$$

and

$$\varphi_{j+1} = (\underbrace{\phi_1 \oplus \cdots \oplus \phi_1}_{q_{j+1,1}}) \oplus \cdots \oplus (\underbrace{\phi_m \oplus \cdots \oplus \phi_m}_{q_{j+1,m}}),$$

and consider the common direct summand of these two maps,

$$\psi_j := (\underbrace{\phi_1 \oplus \cdots \oplus \phi_1}_{\min\{q_{j,1}, q_{j+1,1}\}}) \oplus \cdots \oplus (\underbrace{\phi_m \oplus \cdots \oplus \phi_m}_{\min\{q_{j,m}, q_{j+1,m}\}}).$$

By (2.10), one has

$$|\mathrm{tr}(1 - \psi_j(1))| = \frac{1}{p}\sum_{k=1}^{m} |q_{j,k} - q_{j+1,k}| < \frac{1}{n}.$$

On the other hand, since φ_j and φ_{j+1} are unital, one has

$$[(\varphi_j \ominus \psi_j)(1_A)]_0 = 1 - \mathrm{tr}(\psi_j(1_A)) = [(\varphi_{j+1} \ominus \psi_j)(1_A)]_0.$$

Recall that $\mathcal{P} = \{1_A\}$. By the conclusion of Corollary 2.5 there is a unitary u_{j+1} such that

$$\left\|\varphi_j(f) - u_{j+1}^*\varphi_{j+1}(f)u_{j+1}\right\| < \varepsilon/4, \quad f \in \mathcal{F}\cdot\mathcal{F},\ 0 \leq j \leq l-1.$$

Define $v_0 = 1$, and set

$$u_j u_{j-1} \cdots u_1 = v_j, \quad j = 1, ..., l.$$

Then, for any $0 \leq j \leq l-1$ and any $f \in \mathcal{F}\cdot\mathcal{F}$, one has

$$\begin{aligned} & \|\mathrm{Ad}(v_j)\circ\varphi_i(f) - \mathrm{Ad}(v_{j+1})\circ\varphi_{j+1}(f)\| \\ = & \ \|(u_j \cdots u_1)^*\varphi_i(f)(u_j \cdots u_1) - (u_{j+1} \cdots u_1)^*\varphi_{j+1}(f)(u_{j+1} \cdots u_1)\| \\ = & \ \left\|\varphi_j(f) - u_{j+1}^*\varphi_{j+1}(f)u_{j+1}\right\| < \varepsilon/4. \end{aligned}$$

Replacing each homomorphism φ_j by $\mathrm{Ad}(v_j)\circ\varphi_j$ for $j = 1, ..., l$, and still denoting it by φ_j, one has

$$\|\varphi_j(f) - \varphi_{j+1}(f)\| < \varepsilon/4, \quad f \in \mathcal{F}\cdot\mathcal{F},\ 0 \leq j \leq l-1. \tag{2.11}$$

Define a unital completely positive linear map $\phi : A \to C_1$ by

$$\phi(f)(t) := \frac{t_{j+1} - t}{t_{j+1} - t_j}\varphi_j(f) + \frac{t - t_j}{t_{j+1} - t_j}\varphi_{j+1}(f), \quad \text{if } t \in [t_j, t_{j+1}].$$

Then, by (2.11), the map ϕ is $\mathcal{F}$-ε-multiplicative. By (2.9) and (2.7), one has

$$\|\phi_*(\hat{f}) - \gamma(\hat{f})\|_\infty < \varepsilon/2, \quad f \in \mathcal{F}. \tag{2.12}$$

Note that A and C have cancellation for projections, and also $\mathrm{K}_0^+(A) = \mathrm{K}_0^+(C) = \mathbb{Q}^+$ (unital identification) and $\mathrm{Aff}(\mathrm{T}(A)) \cong \mathrm{Aff}(\mathrm{T}(C))$. By Theorem 4.4 and Corollary 6.8 of [8] (see also Theorem 2.6 of [5] and Theorem 5.5 of [4], expressed in terms of W instead of Cu), it follows that the Cuntz semigroup of A and the Cuntz semigroup of C are isomorphic. Applied to the canonical unital map $\mathrm{Cu}(C_1) \to \mathrm{Cu}(C) \cong \mathrm{Cu}(A)$, Theorem 1 of [19] implies that there is a unital homomorphism $\psi : C_1 \to A$ giving rise to this map, and in particular such that

$$\psi_* = \Xi^{-1} \circ (\iota_{1,\infty})_* \quad \text{on } \mathrm{Aff}(\mathrm{T}(C_1)). \tag{2.13}$$

Since the ideal of $\mathrm{Cu}(C_1)$ killed by the map $\mathrm{Cu}(C_1) \to \mathrm{Cu}(C) \cong \mathrm{Cu}(A)$ is zero, as the map $C_1 \to C$ is an embedding, it follows that the map $C_1 \to A$ is also an embedding. By (2.12), (2.13), and (2.3), one then has

$$\|\phi_* \circ \psi_*(\hat{f}) - \hat{f}\|_\infty < \varepsilon, \quad f \in \mathcal{F},$$

as desired. □

Theorem 2.10. *Let A be a separable simple unital C*-algebra satisfying the UCT. Assume that $A \otimes Q$ has finite nuclear dimension, $\mathrm{T}(A) = \mathrm{T}_{\mathrm{qd}}(A)$, and $\mathrm{K}_0(A) \otimes \mathbb{Q} = \mathbb{Q}$ (identification of order-unit groups). Then $A \otimes Q \in \mathrm{TA}\mathcal{I}$.*

Proof. This follows from Theorem 2.9 above and Theorem 2.2 of [21] directly. □

Proof of Theorem 1.1. By Proposition 8.5 of [2], as $A \otimes Q$ has finite decomposition rank, $\mathrm{T}(A \otimes Q) = \mathrm{T}_{\mathrm{qd}}(A \otimes Q)$. Furthermore, by [11], $A \otimes Q$ is stably finite and nuclear and so by [1] and [10], $\mathrm{T}(A) \neq \emptyset$. Then $\mathrm{K}_0(A \otimes Q) = \mathbb{Q}$ (as order-unit groups), and the statement follows from Theorem 2.10. (The classifiability of $A \otimes \mathcal{Z}$ holds by [22], [15], [16], and [14].) □

References

[1] B. Blackadar and M. Rørdam. Extending states on preordered semigroups and existence of the quasitrace on C*-algebras. *J. Algebra*, 152(1):240–247, 1992.

[2] J. Bosa, N. P. Brown, Y. Sato, A. Tikuisis, S. White, and W. Winter. Covering dimension of C*-algebras and 2-coloured classification. 06 2015. URL: `http://arxiv.org/abs/1506.03974`, `arXiv:1506.03974`.

[3] N. P. Brown. Invariant means and finite representation theory of C*-algebras. *Mem. Amer. Math. Soc.*, 184(865):viii+105, 2006. URL: `http://dx.doi.org/10.1090/memo/0865`, `doi:10.1090/memo/0865`.

[4] N. P. Brown, F. Perera, and A. S. Toms. The Cuntz semigroup, the Elliott conjecture, and dimension functions on C^*-algebras. *J. Reine Angew. Math.*, 621:191–211, 2008. URL: `http://dx.doi.org/10.1515/CRELLE.2008.062`, `doi:10.1515/CRELLE.2008.062`.

[5] N. P. Brown and A. S. Toms. Three applications of the Cuntz semigroup. *Int. Math. Res. Not. IMRN*, (19):Art. ID rnm068, 14, 2007.

[6] M. Dădărlat and S. Eilers. On the classification of nuclear C*-algebras. *Proc. London Math. Soc. (3)*, 85(1):168–210, 2002. URL: `http://dx.doi.org/10.1112/S0024611502013679`, `doi:10.1112/S0024611502013679`.

[7] G. A. Elliott, G. Gong, H. Lin, and Z. Niu. The classification of simple separable unital locally ASH-algebras. 06 2015. URL: `http://arxiv.org/abs/1506.02308`, `arXiv:1506.02308`.

[8] G. A. Elliott, L. Robert, and L. Santiago. The cone of lower semicontinuous traces on a C*-algebra. *Amer. J. Math*, 133(4):969–1005, 2011.

[9] G. Gong, H. Lin, and Z. Niu. Classification of finite simple amenable $\mathcal{Z}$-stable C*-algebras. 01 2015. URL: `http://arxiv.org/abs/1501.00135`, `arXiv:1501.00135`.

[10] U. Haagerup. Quasitraces on exact C*-algebras are traces. *C. R. Math. Acad. Sci. Soc. R. Can.*, 36(2-3):67–92, 2014.

[11] E. Kirchberg and W. Winter. Covering dimension and quasidiagonality. *Internat. J. Math.*, 15(1):63–85, 2004. URL: `http://dx.doi.org/10.1142/S0129167X04002119`, `doi:10.1142/S0129167X04002119`.

[12] A. J. Lazar and J. Lindenstrauss. Banach spaces whose duals are L_1 spaces and their representing matrices. *Acta Math.*, 126:165–193, 1971.

[13] H. Lin. Stable approximate unitary equivalence of homomorphisms. *J. Operator Theory*, 47(2):343–378, 2002.

[14] H. Lin. Asymptotic unitary equivalence and classification of simple amenable C*-algebras. *Invent. Math.*, 183(2):385–450, 2011. URL: http://dx.doi.org/10.1007/s00222-010-0280-9, doi:10.1007/s00222-010-0280-9.

[15] H. Lin. Localizing the Elliott conjecture at strongly self-absorbing C*-algebras, II. *J. Reine Angew. Math.*, 692:233–243, 2014. doi:DOI10.1515/crelle-2012-0182.

[16] H. Lin and Z. Niu. Lifting KK-elements, asymptotic unitary equivalence and classification of simple C*-algebras. *Adv. Math.*, 219(5):1729–1769, 2008. URL: http://dx.doi.org/10.1016/j.aim.2008.07.011, doi:10.1016/j.aim.2008.07.011.

[17] H. Matui and Y. Sato. Decomposition rank of UHF-absorbing C*-algebras. *Duke Math. J.*, 163(14):2687–2708, 2014. URL: http://dx.doi.org/10.1215/00127094-2826908, doi:10.1215/00127094-2826908.

[18] M. Riesz. Sur le problème des moments. iii. *Ark. F. Mat. Astr. O. Fys*, 17(16):1–52, 1923.

[19] L. Robert. Classification of inductive limits of 1-dimensional NCCW complexes. *Adv. Math.*, 231(5):2802–2836, 2012. URL: http://dx.doi.org/10.1016/j.aim.2012.07.010, doi:10.1016/j.aim.2012.07.010.

[20] K. Thomsen. Inductive limits of interval algebras: the tracial state space. *Amer. J. Math.*, 116(3):605–620, 1994. URL: http://dx.doi.org/10.2307/2374993, doi:10.2307/2374993.

[21] W. Winter. Classifying crossed product C*-algebras. 08 2013. URL: http://arxiv.org/abs/1308.5084, arXiv:1308.5084.

[22] W. Winter. Localizing the Elliott conjecture at strongly self-absorbing C*-algebras. *J. Reine Angew. Math.*, 692:193–231, 2014. doi:DOI10.1515/crelle-2012-0082.

Department of Mathematics, University of Toronto, Toronto, Ontario, Canada M5S 2E4
E-mail address: elliott@math.toronto.edu

Department of Mathematics, University of Wyoming, Laramie, WY 82071, USA.
E-mail address: zniu@uwyo.edu