ON THE CLASSIFICATION OF SIMPLE UNITAL C*-ALGEBRAS WITH FINITE DECOMPOSITION RANK, II

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Abstract. Let A be a simple separable unital C*-algebra satisfying the UCT, and assume that A has finite decomposition rank. Let Q denote the UHF algebra with $\mathrm{K}_0(Q) = \mathbb{Q}$. Then $A \otimes Q$ can be tracially approximated by unital Elliott-Thomsen algebras, and therefore $A \otimes \mathcal{Z}$ is an ASH algebra (hence classifiable), where $\mathcal{Z}$ is the Jiang-Su algebra.

1. Introduction

Let us consider simple separable unital C*-algebras which have finite decomposition rank. We shall show that these C*-algebras can be rationally tracially approximated by unital Elliott-Thomsen algebras (assuming the UCT), and hence are ASH algebras by the classification result of [14] if they absorb the Jiang-Su algebra tensorially. More precisely, denoting by $\mathcal{C}_0$ the class of unital Elliott-Thomsen algebras with trivial K_1-group, one has

Theorem 1.1. *Let A be a separable unital simple C*-algebra satisfying the UCT. Assume that* $\mathrm{dr}(A \otimes Q) < +\infty$. *Then $A \otimes Q \in \mathrm{TA}\mathcal{C}_0$. In particular, $A \otimes \mathcal{Z}$ is classifiable.*

Let A be a simple unital separable locally ASH algebra. By Theorem A of [12], the decomposition rank of $A \otimes \mathcal{Z}$ is at most 2 (in fact the decomposition rank of $A \otimes Q$ is at most 1). Therefore, Theorem 1.1 gives another proof of the main result of [10]; that is,

Corollary 1.2 (Theorem 5.9 of [10])**.** *Let A be a simple unital separable locally ASH algebra. Then $A \otimes Q \in \mathrm{TA}\mathcal{C}_0$. In particular, $A \otimes \mathcal{Z}$ is classifiable.*

2. Main theorem and the proof

The idea in the proof of the main theorem is the same as that of [11]. Given an abstract C*-algebra A, a Q-stable Elliott-Thomsen algebra C, and a linear map $\gamma : \mathrm{Aff}(\mathrm{T}(A)) \to \mathrm{Aff}(\mathrm{T}(A))$, we shall lift the trace map approximately to an almost homomorphism from A to C. Since all traces of A are quasi-diagonal, the trace map can be approximately lifted to maps from A to the fibre of C (considered as a continuous field with fibre direct sums of Q).

However, in order to piece together these fibre maps to get a map from A to C, one requires that these maps to induce the same K_0-classes on certain projections of a matrix algebras of A (this condition is automatically satisfied if $\mathrm{K}_0(A) \otimes \mathbb{Q} \cong \mathbb{Q}$, as seen in [11]).

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If the spectrum of C is just several intervals, one then can perturb each fibre maps with the next lemma (Lemma 2.1) to smooth out the induced K_0-classes, and thus obtains the main theorem in the case that $\mathrm{K}_0(A\otimes Q)$ is a weakly unperforated Riesze group and the pairing map preserves extreme points.

For a general Elliott-Thomsen algebra C, recall that it is given by

$$C \cong \{(a, f) \in F_1 \oplus \mathrm{C}([0,1], F_2) : f(0) = \psi_0(a),\ f(1) = \psi_1(a)\},$$

where

$$F_1 := \underbrace{Q \oplus \cdots \oplus Q}_{p}, \quad F_2 := \underbrace{Q \oplus \cdots \oplus Q}_{l},$$

for some $p, l \in \mathbb{N} \cup \{0\}$ and $\psi_0, \psi_1 : F_1 \to F_2$ are unital homomorphisms. Before perturbing the maps to the fibre on the (open) interval, one first needs to perturb the maps to the fibre at infinity (from A to F_1) so that the induced K_0-classed is in $\ker([\psi_0] - [\psi_1])$. This perturbation indeed can be obtained by another application of Lemma 2.1.

Let us first consider the lemma.

Lemma 2.1. *Let A be a simple unital separable quasi-diagonal C*-algebra satisfying the UCT. Assume that $A \cong A \otimes Q$. Let a finite subset $\mathcal{G} \subseteq A$ and $\varepsilon_1, \varepsilon_2 > 0$ be given. Let $p_1, p_2, ..., p_s \in \mathrm{Proj}_\infty(A)$ be projections such that $[1]_0, [p_1]_0, [p_2]_0, ..., [p_s]_0 \in \mathrm{K}_0(A)$ are $\mathbb{Q}$-independent. Then there are a $\mathcal{G}$-ε_1-multiplicative c.p.c. (completely positive contractive) map $\sigma : A \to Q \otimes \mathcal{K}$ with $\sigma(1)$ a projection satisfying*

$$\mathrm{tr}(\sigma(1)) < \varepsilon_2,$$

and a $\delta > 0$, such that, for any $r_1, r_2, ..., r_s \in \mathbb{Q}$ with

$$|r_i| < \delta,\ i = 1, ..., s,$$

there is a $\mathcal{G}$-ε_1-multiplicative c.p.c. map $\mu : A \to Q \otimes \mathcal{K}$ such that

$$[\sigma(p_i)]_0 - [\mu(p_i)]_0 = r_i, \quad i = 1, ..., s,$$

and $\sigma(1) = \mu(1)$.

Proof. Let us agree that σ and μ are also understood to be required to be sufficiently multiplicative on $p_1, p_2, ..., p_s$ that the classes $[\sigma(p_i)]_0$ and $[\mu(p_i)]_0$ make sense. (Similarly for other c.p.c. approximately multiplicative maps, to be introduced below.) Since the classes $[1], [p_1], [p_2], ..., [p_s] \in \mathrm{K}_0(A)$ are $\mathbb{Q}$-independent, for each $i = 1, ..., s$, there is a homomorphism $\alpha_i : \mathrm{K}_0(A) \to \mathbb{Q} \cong \mathrm{K}_0(Q)$ such that

$$\alpha_i([p_i]) = 1, \quad \alpha_i([1]) = 0, \quad \alpha_i([p_j]) = 0, \quad j \neq i.$$

Noting that by the multicoefficient UCT ([8]), $\mathrm{KL}(A, Q) = \mathrm{Hom}(\mathrm{K}_0(A), \mathbb{Q})$, one may regard α_i as an element of $\mathrm{KL}(A, Q)$.

Since A is quasidiagonal, by Theorem 5.5 of [7], there are c.p.c. maps $\sigma_i, \mu_i : A \to Q \otimes \mathcal{K}$ such that σ_i and μ_i are $\mathcal{G}$-ε_1-multiplicative, $\sigma_i(1)$ and $\mu_i(1)$ are projections, and

$$[\sigma_i] - [\mu_i] = \alpha_i \quad \text{on } \{1, p_1, p_2, ..., p_s\}.$$

Note that since $\alpha_i([1]) = 0$, one has that $[\sigma_i(1)]_0 = [\mu_i(1)]_0$, and therefore by applying a unitary conjugacy, one may assume that $\sigma_i(1) = \mu_i(1) = P_i$ for a projection P_i. (If A is assumed to be nuclear, then the existence of σ_i and μ_i also follows from Corollary 5.1 of [2], Proposition 6.1.6 of [1], and Theorem 5.9 of [18].)

Consider the projection

$$P := \bigoplus_{i=1}^{s}(P_i \oplus P_i),$$

and the unital $\mathcal{G}$-ε_1-multiplicative unital c.p. map

$$\bigoplus_{i=1}^{s}(\sigma_i \oplus \mu_i) : A \to P(Q \otimes \mathcal{K})P.$$

Note that $P(Q\otimes\mathcal{K})P \cong Q$. Choose a projection $R \in Q \otimes \mathcal{K}$ with $0 < \mathrm{tr}(R) < \varepsilon_2$ and a rescaling

$$S : Q \otimes \mathcal{K} \to Q \otimes \mathcal{K},\ P \mapsto R.$$

Define

$$\sigma := S \circ (\bigoplus_{i=1}^{s}(\sigma_i \oplus \mu_i)) : A \to Q$$

and the strictly positive number

$$\delta := \frac{\mathrm{tr}(R)}{\mathrm{tr}(P)}.$$

(Here, tr denotes the tensor product of the traces on Q and $\mathcal{K}$, normalized in the usual way.) Let us show that σ and δ satisfy the condition of the lemma.

Let $r_1, r_2, ..., r_s \in \mathbb{Q}$ be given and satisfy

$$|r_i| < \delta, \quad i = 1, ..., s.$$

For each r_i, choose a projection $R_i \in Q \otimes \mathcal{K}$ with $\mathrm{tr}(R_i) = |r_i|$, and choose a rescaling

$$S_i : Q \otimes \mathcal{K} \to Q \otimes \mathcal{K},\ 1 \otimes e \mapsto R_i,$$

where e as before is a minimal non-zero projection of $\mathcal{K}$. Consider the maps

$$S_i \circ \sigma_i,\ S_i \circ \mu_i : A \to Q \otimes \mathcal{K}.$$

This pair then satisfies, for each $i = 1, ..., s$,

$$\begin{cases} [S_i \circ \sigma_i(p_i)] - [S_i \circ \mu_i(p_i)] = |r_i|\,, \\ [S_i \circ \sigma_i(1)] - [S_i \circ \mu_i(1)] = 0, \\ [S_i \circ \sigma_i(p_j)] - [S_i \circ \mu_i(p_j)] = 0, \qquad j \neq i. \end{cases}$$

Consider the direct sum maps

$$\tilde{\sigma} := (\bigoplus_{r_i>0} S_i \circ \sigma_i) \oplus (\bigoplus_{r_i<0} S_i \circ \mu_i)$$

and

$$\tilde{\mu} := (\bigoplus_{r_i>0} S_i \circ \mu_i) \oplus (\bigoplus_{r_i<0} S_i \circ \sigma_i).$$

Then

$$[\tilde{\sigma}(p_i)] - [\tilde{\mu}(p_i)] = r_i, \quad i = 1, ..., s.$$

Note that

$$\sigma = S \circ (\bigoplus_{i=1}^{s}(\sigma_i \oplus \mu_i)) = \bigoplus_{i=1}^{s}((S \circ \sigma_i) \oplus (S \circ \mu_i)). \tag{2.1}$$

For each S_i, since

$$\mathrm{tr}(S_i(P)) = \mathrm{tr}(P) \cdot \mathrm{tr}(S_i(1 \otimes e)) = \mathrm{tr}(P) \cdot \mathrm{tr}(R_i) = \mathrm{tr}(P)\,|r_i| < \mathrm{tr}(P)\delta = \mathrm{tr}(R) = \mathrm{tr}(S(P)),$$

there is a rescaling $T_i : Q \otimes \mathcal{K} \to Q \otimes \mathcal{K}$ such that

$$S = S_i \oplus T_i.$$

Therefore, by (2.1),

$$\begin{aligned}
\sigma &= \bigoplus_{i=1}^{s}(((S_i \oplus T_i) \circ \sigma_i) \oplus ((S_i \oplus T_i) \circ \mu_i)) \\
&= \bigoplus_{i=1}^{s}((S_i \circ \sigma_i) \oplus (T_i \circ \sigma_i)) \oplus \bigoplus_{i=1}^{s}((S_i \circ \mu_i) \oplus (T_i \circ \mu_i)) \\
&= (\bigoplus_{r_i > 0}((S_i \circ \sigma_i) \oplus (T_i \circ \sigma_i))) \oplus (\bigoplus_{r_i \leq 0}((S_i \circ \sigma_i) \oplus (T_i \circ \sigma_i))) \\
&\quad \oplus (\bigoplus_{r_i < 0}((S_i \circ \mu_i) \oplus (T_i \circ \mu_i))) \oplus (\bigoplus_{r_i \geq 0}((S_i \circ \mu_i) \oplus (T_i \circ \mu_i))) \\
&= \tilde{\sigma} \oplus \gamma,
\end{aligned}$$

where

$$\gamma = (\bigoplus_{r_i > 0}(T_i \circ \sigma_i)) \oplus (\bigoplus_{r_i \leq 0}((S_i \circ \sigma_i) \oplus (T_i \circ \sigma_i))) \oplus (\bigoplus_{r_i < 0}(T_i \circ \mu_i)) \oplus (\bigoplus_{r_i \geq 0}((S_i \circ \mu_i) \oplus (T_i \circ \mu_i))).$$

Consider the c.p.c. map

$$\mu := \tilde{\mu} \oplus \gamma.$$

One then has

$$[\sigma(p_i)] - [\mu(p_i)] = [\tilde{\sigma}(p_i)] - [\tilde{\mu}(p_i)] = r_i, \quad i = 1, ..., s,$$

as desired. □

Before we prove the main theorem, let us sketch how Lemma 2.1 is used to smooth out the K_0-classed induced by the maps at the fibre (then the rest of the argument is exact the same as that of Theorem 2.9 of [11]).

Let us start with two unital approximate homomorphisms $\phi_0, \phi_1 : A \to Q$ such that the K_0 difference $[\phi_1]_0 - [\phi_0]_1$ is small. Then, by Lemma 2.1, there are two approximate homomorphisms σ and μ with very small trace such that $[\sigma] - [\mu] = [\phi_1] - [\phi_0]$. Then replace ϕ_0 by the rescaling of $\phi_0 \oplus \sigma$, and replace ϕ_1 by the rescaling of $\phi_1 + \mu$. Then they have the same K_0 class. Since σ and μ have very small trace, this will not change the trace map very much.

Of course, in this example, we do not need the full strength of Lemma 2.1. But, if the algebra C, in order to deal with all middle points, we need that the map σ to be independent of the K_0 difference so that the perturbation by σ of the map at one end point works for all middle points, and that is exactly the point of Lemma 2.1. In fact, this argument also works well for a general

Elliott-Thomsen algebra if the K-theory of the maps at infinity is matched (just compared the infinity points to any middle point, and apply the argument above).

So, for an general Elliott-Thomsen algebra, the problem is reduced to the points at infinity. What we do have are approximate homomorphism to the algebra at infinity (finite direct sum of Q), some projections at the abstract algebra A, and some projections at infinity, such that the trace of the image of each those projections of A is very close to the image of the corresponding projection at infinity. Our goal is to perturb the maps at infinity so that the image of those projections of A under the new maps satisfy the boundary condition. Of course the projections at infinity satisfy the condition (they come from projections in the point-line algebra), and these projections serve as goalposts. If we perturb the maps so that the induced K_0-class are exact the class of the projections at the infinity, then we reach the goal; but it is not clear if it is possible. So, we will move the goalposts slightly.

The way to move it is another application of Lemma 2.1. Similar as the interval algebras case, just compare the K_0-class induced by these maps and the K_0-class of the projections at infinity. Then Lemma 2.1 allows us to compensate the map to each point at infinity by a map μ (may depend on which point at the infinity, but it has very small trace) and compensate the projection at the corresponding point at the infinity by a map σ (which is independent of the points at the infinity), such that the new maps have the same K-class as the projection at infinity plus sigma—moving the goalposts. This movement is allowed, because we move the K-class of each projection at infinity by a multiple of σ, which always satisfy the boundary condition. Therefore, the K-class of the new map also satisfy the boundary condition.

Theorem 2.2. *Let A be a separable unital simple exact Q-stable C*-algebra satisfying the UCT. Assume that $\mathrm{T}(A) = \mathrm{T}_{\mathrm{qd}}(A)$. Then, for any finite subset $\mathcal{F} \subseteq A$ and any $\varepsilon > 0$, there are unital c.p. maps $\phi : A \to C$ and $\psi : C \to A$, where $C \in \mathcal{C}_0$, such that*

(1) *ϕ is $\mathcal{F}$-δ-multiplicative, ψ is an embedding, and*
(2) *$|\tau(\phi \circ \psi(a) - a)| < \varepsilon$, $a \in \mathcal{F}$, $\tau \in \mathrm{T}(A)$.*

Proof. One may assume that $\mathrm{T}(A) \neq \emptyset$. Applying Corollary 2.4 of [11] to A with respect to $(\mathcal{F} \cdot \mathcal{F}, \varepsilon/4)$, one obtains n and $(\mathcal{P}, \mathcal{G}, \delta)$. Write

$$\mathcal{P} = \{1, p_1, p_2, ..., p_s\}.$$

Without loss of generality, one may assume that $[1], [p_1], [p_2], ..., [p_s]$ are $\mathbb{Q}$-independent.

Applying Lemma 2.1 to the C*-algebra A with respect to $\mathcal{G}$, $\varepsilon_1 = \delta$, $\varepsilon_2 = \min\{\varepsilon/8, 1/4n\}$, and $\{p_1, p_2, ..., p_s\}$, one obtains a $\mathcal{G}$-δ-multiplicative c.p.c. map $\sigma : A \to Q$ and a constant $\delta_1 > 0$ which satisfy the condition of Lemma 2.1. Without loss of generality, one may assume that $\delta_1 < \varepsilon$.

Applying Lemma 2.1 again to A with respect to $\mathcal{G}$, $\varepsilon_1 = \delta$, $\varepsilon_2' = \min\{\varepsilon/4, \delta_1/16, 1/4n\}$, and $\{p_1, p_2, ..., p_s\}$, one obtains a $\mathcal{G}$-δ-multiplicative c.p.c. map $\sigma' : A \to Q$ and a constant $\delta_2 > 0$ which satisfies the condition of Lemma 2.1. Without loss of generality, one may assume that $\delta_2 < \varepsilon$.

By [9], since $A \cong A \otimes Q$, there is a unital inductive limit $C = \varinjlim(C_i, \iota_i)$ such that each C_i is isomorphic to a tensor product of a unital Elliott-Thomsen algebra and Q, with $\mathrm{K}_1(C_i) = \{0\}$, and there is an isomorphism

$$\Xi : ((\mathrm{K}_0(A), \mathrm{K}_0^+(A), [1_A]), \mathrm{Aff}(\mathrm{T}(A)), \rho_A) \cong ((\mathrm{K}_0(C), \mathrm{K}_0^+(C), [1_C]), \mathrm{Aff}(\mathrm{T}(C)), \rho_C).$$

Moreover, the maps ι_i may be chosen to be injective.

By Lemma 2.7 of [11], there is an approximate factorization, by means of unital positive maps,

$$\mathrm{Aff}(\mathrm{T}(A)) \xrightarrow{\varrho} \mathbb{R}^m \xrightarrow{\theta} \mathrm{Aff}(\mathrm{T}(A)),$$

such that

$$\|\theta(\varrho(\hat{f})) - \hat{f}\|_\infty < \min\{\varepsilon/16, \delta_1/16, \delta_2/16\}, \quad f \in \mathcal{F} \cup \mathcal{P}.$$

Therefore, by Lemma 2.8 of [11], after discarding finitely many terms of the sequence (C_i, ι_i), there is a unital positive linear map

$$\gamma : \mathrm{Aff}(\mathrm{T}(A)) \xrightarrow{\varrho} \mathbb{R}^m \longrightarrow \mathrm{Aff}(\mathrm{T}(C_1))$$

such that

$$\|(\iota_{1,\infty})_*(\gamma(\hat{f})) - \Xi(\hat{f})\|_\infty < \min\{\varepsilon/8, \delta_1/8, \delta_2/8\}, \quad f \in \mathcal{F} \cup \mathcal{P}. \tag{2.2}$$

Moreover, after discarding more terms, using continuity of K_0 as well as $\mathrm{Aff}\mathrm{T}$, there are $p'_1, ..., p'_k \in \mathrm{Proj}_\infty(C_1)$ such that

$$\|\rho_{C_1}(p'_i) - \gamma(\widehat{p_i})\|_\infty < \min\{\delta_1/4, \delta_2/4\}, \quad 1 \leq i \leq s. \tag{2.3}$$

Since C_1 is the tensor product of a unital Elliott-Thomsen algebra and Q, there are

$$F_1 := \underbrace{Q \oplus \cdots \oplus Q}_{p}, \quad F_2 := \underbrace{Q \oplus \cdots \oplus Q}_{l},$$

and unital homomorphisms $\psi_0, \psi_1 : F_1 \to F_2$ such that

$$C_1 \cong \{(a, f) \in F_1 \oplus \mathrm{C}([0,1], F_2) : f(0) = \psi_0(a),\ f(1) = \psi_1(a)\}.$$

Denote by $\pi : C_1 \to F_1$ the canonical homomorphism $\pi((a, f)) = a$. Let us also use $[0,1]_j$ to denote the interval in the spectrum of $\mathrm{C}([0,1], Q)$ corresponding to the jth copy of Q.

Consider the corresponding multiplicity matrices

$$[\psi_0] = \begin{pmatrix} \lambda_{0,1,1} & \cdots & \lambda_{0,1,p} \\ \cdots & \cdots & \cdots \\ \lambda_{0,l,1} & \cdots & \lambda_{0,l,p} \end{pmatrix} \quad \text{and} \quad [\psi_1] = \begin{pmatrix} \lambda_{1,1,1} & \cdots & \lambda_{1,1,p} \\ \cdots & \cdots & \cdots \\ \lambda_{1,l,1} & \cdots & \lambda_{1,l,p} \end{pmatrix},$$

where $\lambda_{0,i,j}, \lambda_{1,i,j} \in \mathbb{Q} \cap [0,1]$. Note that since ψ_0 and ψ_1 are unital, the sum of each row of $[\psi_0]$ or $[\psi_1]$ is equal to 1.

Denote by $\gamma^* : \mathrm{T}(C_1) \to \mathrm{T}(A)$ the affine map induced by γ on tracial simplices. Since γ factors through $\mathbb{R}^m$ (so that γ^* factors through a finite dimensional simplex), there are $\tau_1, ..., \tau_m \in \mathrm{T}(A)$ and continuous functions $c_{j,1}, c_{j,2}, ..., c_{j,m} : [0,1]_j \to [0,1]$ such that

$$\gamma^*(\tau_t) = c_{j,1}(t)\tau_1 + c_{j,2}(t)\tau_2 + \cdots + c_{j,m}(t)\tau_m, \quad t \in [0,1]_j, \tag{2.4}$$

and

$$c_{j,1}(t) + c_{j,2}(t) + \cdots + c_{j,m}(t) = 1, \quad t \in [0,1]_j,$$

where $\tau_t \in \mathrm{T}(C_1)$ is determined by the Dirac measure concentrated at $t \in [0,1]_j$.

Denote by tr_e, $e = 1, ..., p$, the trace of C_1 induced by the eth copy of Q in F_1, and write

$$\gamma^*(\mathrm{tr}_e) = \alpha_{e,1}\tau_1 + \alpha_{e,2}\tau_2 + \cdots + \alpha_{e,m}\tau_m, \quad e = 1, ..., p, \tag{2.5}$$

for some $\alpha_{e,1}, ..., \alpha_{e,m} \in [0,1]$. Since

$$\tau_{0_j} = \lambda_{0,j,1} \cdot \mathrm{tr}_1 + \cdots + \lambda_{0,j,p} \cdot \mathrm{tr}_p \quad \text{and} \quad \tau_{1_j} = \lambda_{1,j,1} \cdot \mathrm{tr}_1 + \cdots + \lambda_{1,j,p} \cdot \mathrm{tr}_p, \quad j = 1, ..., l,$$

one has

$$\begin{aligned} & c_{j,1}(0_j)\tau_1 + c_{j,2}(0_j)\tau_2 + \cdots + c_{j,m}(0_j)\tau_m = \gamma^*(\tau_{0_j}) = \gamma^*(\lambda_{0,j,1} \cdot \mathrm{tr}_1 + \cdots + \lambda_{0,j,p} \cdot \mathrm{tr}_p) \\ = \ & (\sum_{e=1}^{p} \lambda_{0,j,e} \cdot \alpha_{e,1})\tau_1 + (\sum_{e=1}^{p} \lambda_{0,j,e} \cdot \alpha_{e,2})\tau_2 + \cdots + (\sum_{e=1}^{p} \lambda_{0,j,e} \cdot \alpha_{i,m})\tau_m. \end{aligned} \tag{2.6}$$

Therefore, putting

$$C(0) := \begin{pmatrix} c_{1,1}(0_1) & \cdots & c_{1,m}(0_1) \\ \cdots & \cdots & \cdots \\ c_{l,1}(0_l) & \cdots & c_{l,m}(0_l) \end{pmatrix} \quad \text{and} \quad \Theta := \begin{pmatrix} \alpha_{1,1} & \cdots & \alpha_{1,m} \\ \cdots & \cdots & \cdots \\ \alpha_{p,1} & \cdots & \alpha_{p,m} \end{pmatrix},$$

one has

$$C(0) = [\psi_0] \cdot \Theta. \tag{2.7}$$

The same argument shows that

$$C(1) = [\psi_1] \cdot \Theta, \tag{2.8}$$

where $C(1)$ is defined in a way similar to $C(0)$, with 0_j replaced by 1_j, $j = 1, ..., l$.

Since $\tau_1, \tau_2, ..., \tau_m \in \mathrm{T}_{\mathrm{qd}}(A)$, there are c.p.c. maps $\phi_k : A \to Q$, $k = 1, ..., m$, such that each ϕ_k is $\mathcal{G}$-δ-multiplicative, and

$$|\mathrm{tr}(\phi_k(f)) - \tau_k(f)| < \min\{\delta_1/16m, \delta_2/16m\}, \quad f \in \mathcal{F} \cup \mathcal{P}. \tag{2.9}$$

For each $t \in [0,1]_j$, there is a open neighbourhood U such that for any $s \in U$, one has

$$|c_{j,k}(s) - c_{j,k}(t)| < 1/8mn.$$

Since $[0,1]_j$ is compact, there is a partition $0 = t_0 < t_1 < \cdots < t_{h-1} < t_h = 1$ such that

$$|c_{j,k}(s) - c_{j,k}(t_i)| < 1/8mn, \quad s \in [t_{i-1}, t_i]. \tag{2.10}$$

Moreover, we may assume that this partition is fine enough that

$$\left|\gamma(\hat{f})(\tau_t) - \gamma(\hat{f})(\tau_{t_i})\right| < \varepsilon/8, \quad f \in \mathcal{F} \cup \mathcal{P}, \ t \in [t_{i-1}, t_i], \tag{2.11}$$

and

$$\frac{\pi}{h-2} < \frac{\varepsilon}{4}. \tag{2.12}$$

For each $\alpha_{e,k}$, $e = 1, ..., p$, $k = 1, ..., m$, pick a rational number $\tilde{r}_{e,k}$ such that

$$\tilde{r}_{e,1} + \tilde{r}_{e,2} + \cdots + \tilde{r}_{e,m} = 1,$$

and

$$|\tilde{r}_{e,k} - \alpha_{e,k}| < \min\{\delta_1/32m, \delta_2/32m, 1/16mn\}, \quad k = 1, ..., m. \tag{2.13}$$

For each $i = 1, 2, ..., h-1$, pick rational numbers $r_{i,1,j}, r_{i,2,j}, ..., r_{i,m,j} \in [0,1]$ such that

$$r_{i,1,j} + \cdots + r_{i,m,j} = 1,$$

and

$$|r_{i,k,j} - c_{j,k}(t_i)| < \min\{\delta_1/16m, \delta_2/16m, 1/8mn\}, \quad k = 1, ..., m. \tag{2.14}$$

(The cases $i = 0$ and $i = h$ will be considered later, based on $\{\tilde{r}_{e,k} : \ e = 1, ..., p, \ k = 1, ..., m\}$.)

Consider the maps

$$\hat{\varphi}_e := \tilde{r}_{e,1}\phi_1 \oplus \cdots \oplus \tilde{r}_{e,m}\phi_m : A \to Q, \quad e = 1, ..., p,$$

and

$$\tilde{\varphi}_{i,j} := r_{i,1,j}\phi_1 \oplus \cdots \oplus r_{i,m,j}\phi_m : A \to Q, \quad i = 1, ..., h-1, \ j = 1, ..., l.$$

Note that it follows from (2.4), (2.9), and (2.14) that

$$|\mathrm{tr}(\tilde{\varphi}_{i,j}(f)) - \gamma^*(\tau_{t_i})(f)| < \min\{\delta_1/4, \delta_2/4\} < \varepsilon/4, \quad f \in \mathcal{F} \cup \mathcal{P}, \ i = 1, ..., h-1, \tag{2.15}$$

and it follows from (2.5), (2.9), and (2.13) that

$$|\mathrm{tr}(\hat{\varphi}_e(f)) - \gamma^*(\mathrm{tr}_e)(f)| < \min\{\delta_1/4, \delta_2/4\} < \varepsilon/4, \quad f \in \mathcal{F} \cup \mathcal{P}, \ e = 1, ..., p. \tag{2.16}$$

Set

$$\hat{\varphi} := \hat{\varphi}_1 \oplus \cdots \oplus \hat{\varphi}_p,$$

and consider the quantities

$$d_e(p_i) := \mathrm{tr}([\hat{\varphi}_e(p_i)]) - \mathrm{tr}([\pi_e(p_i')]) \in \mathbb{Q}, \quad e = 1, ..., p,$$

where π_e is the canonical quotient map from C_1 to the eth copy of Q in F_1. One has that for any $1 \leq i \leq s$,

$$\begin{aligned} |d_e(p_i)| &< |\gamma^*(\mathrm{tr}_e)(p_i) - \mathrm{tr}([\pi_e(p_i')])| + \delta_2/4 \quad \text{(by (2.16))} \\ &< |\mathrm{tr}(\pi_e(p_i')) - \mathrm{tr}([\pi_e(p_i')])| + \delta_2/4 + \delta_2/4 \quad \text{(by (2.3))} \\ &< \delta_2. \end{aligned}$$

Therefore, by Lemma 2.1, there are $\mathcal{G}$-δ-multiplicative c.p.c. maps $\mu_1, ..., \mu_p : A \to Q$ such that

$$\sigma'(1_A) = \mu_e(1_A) = q, \quad e = 1, ..., p,$$

for a projection $q \in Q \otimes \mathcal{K}$ with

$$\mathrm{tr}(q) < \varepsilon_2' \tag{2.17}$$

and

$$[\sigma'(p_i)] - [\mu_e(p_i)] = d_e(p_i) = [\hat{\varphi}_e(p_i)] - [\pi_e(p_i)], \quad e = 1, ..., p. \tag{2.18}$$

Define

$$\mu := \mu_1 \oplus \cdots \oplus \mu_p : A \to F_1$$

and consider the direct sum map

$$\mu \oplus \hat{\varphi} : A \to (1 + \mu(1_A))F_1 \otimes \mathcal{K}(1 + \mu(1_A)).$$

By (2.18), one has

$$[\mu \oplus \hat{\varphi}](p_i) = [\pi](p_i) + ([\sigma'](p_i), ..., [\sigma'](p_i)) \in \ker([\psi_0] - [\psi_1]) \subseteq \mathbb{Q}^p. \tag{2.19}$$

(Note that $([\sigma'](p_i), ..., [\sigma'](p_i))$ is always in $\ker([\psi_0] - [\psi_1])$.)

Rescaling each $\mu_e \oplus \hat{\varphi}_e$, $e = 1, ..., p$, into a unital map

$$\widehat{\varphi}_e := \frac{1}{1 + \mathrm{tr}(q)}(\mu_e \oplus \hat{\varphi}_e),$$

consider the direct sum

$$\widehat{\varphi} = \widehat{\varphi}_1 \oplus \cdots \oplus \widehat{\varphi}_p : A \to F_1.$$

Since $\mu_1(1_A) = \mu_2(1_A) = \cdots = \mu_p(1_A) = q$, by (2.19), one still has

$$[\widehat{\phi}(p_i)] \in \ker([\psi_0] - [\psi_1]), \quad i = 1, ..., s. \tag{2.20}$$

For each $1 \leq j \leq l$, define

$$\tilde{\varphi}_{0,j} = \lambda_{0,j,1}\widehat{\varphi}_1 \oplus \cdots \oplus \lambda_{0,j,p}\widehat{\varphi}_p,$$

and

$$\tilde{\varphi}_{h,j} = \lambda_{1,j,1}\widehat{\varphi}_1 \oplus \cdots \oplus \lambda_{1,j,p}\widehat{\varphi}_p.$$

By (2.20), one has that

$$[\tilde{\varphi}_{0,j}(p)] = [\tilde{\varphi}_{h,j}(p)], \quad p \in \mathcal{P}. \tag{2.21}$$

The maps $\tilde{\varphi}_{0,j}$ and $\tilde{\varphi}_{h,j}$ have the following decompositions:

$$\tilde{\varphi}_{0,j} = \frac{1}{1 + \mathrm{tr}(q)}(\mu_{0,j} \oplus (\sum_{e=1}^{p} \lambda_{0,j,e}\tilde{r}_{e,1})\phi_1 \oplus \cdots \oplus (\sum_{e=1}^{p} \lambda_{0,j,e}\tilde{r}_{e,m})\phi_m)$$

and

$$\tilde{\varphi}_{h,j} = \frac{1}{1 + \mathrm{tr}(q)}(\mu_{1,j} \oplus (\sum_{e=1}^{p} \lambda_{1,j,e}\tilde{r}_{e,1})\phi_1 \oplus \cdots \oplus (\sum_{e=1}^{p} \lambda_{1,j,e}\tilde{r}_{e,m})\phi_m),$$

respectively, where

$$\mu_{0,j} = \lambda_{0,j,1}\mu_1 \oplus \cdots \oplus \lambda_{0,j,p}\mu_p$$

and

$$\mu_{1,j} = \lambda_{1,j,1}\mu_1 \oplus \cdots \oplus \lambda_{1,j,p}\mu_p.$$

Put

$$r_{0,k,j} := \frac{1}{1 + \mathrm{tr}(q)} \sum_{e=1}^{p} \lambda_{0,j,e}\tilde{r}_{e,k} \quad \text{and} \quad r_{1,k,j} := \frac{1}{1 + \mathrm{tr}(q)} \sum_{e=1}^{p} \lambda_{1,j,e}\tilde{r}_{e,k}.$$

Then

$$\tilde{\varphi}_{0,j} = \frac{\mu_{0,j}}{1 + \mathrm{tr}(q)} \oplus r_{0,1,j}\phi_1 \oplus \cdots \oplus r_{0,m,j}\phi_m$$

and

$$\tilde{\varphi}_{h,j} = \frac{\mu_{1,j}}{1 + \mathrm{tr}(q)} \oplus r_{1,1,j}\phi_1 \oplus \cdots \oplus r_{1,1,j}\phi_m.$$

Note that by (2.13), (2.7), and (2.17), one has

$$|r_{0,k,j} - c_{j,k}(0_j)| < \left|\sum_{e=1}^{p} \lambda_{0,j,e}\tilde{r}_{e,k} - c_{j,k}(0_j)\right| + c_{j,k}(0_j)\frac{\mathrm{tr}(q)}{1+\mathrm{tr}(q)} < \min\{\delta_1/32m, 1/16mn\} + c_{j,k}(0_j)\varepsilon_2', \tag{2.22}$$

and, for the same reason (by (2.13), (2.8), and (2.17)), one has

$$|r_{h,k,j} - c_{j,k}(1_j)| < \min\{\delta_1/32m, 1/16mn\} + c_{j,k}(1_j)\varepsilon_2'. \tag{2.23}$$

Since $\mathrm{tr}(q) < \varepsilon_2'$, by (2.13) and (2.6), one has

$$\begin{aligned} &\left|\mathrm{tr}(\tilde{\varphi}_{0,j}(f)) - \gamma^*(\tau_{0_j})(f)\right| \\ <& \left|\mathrm{tr}(\sum_{k=1}^{m}(\sum_{e=1}^{p}\lambda_{0,j,e}\tilde{r}_{e,k})\phi_k(f)) - \gamma^*(\tau_{t_i})(f)\right| + \varepsilon_2' \\ <& \left|\mathrm{tr}(\sum_{k=1}^{m}(\sum_{e=1}^{p}\lambda_{0,j,e}\alpha_{e,k})\phi_k(f)) - \gamma^*(\tau_{t_i})(f)\right| + \varepsilon_2' + \delta_1/16 \\ =& \varepsilon_2' + \delta_1/16 < \delta_1/4 < \varepsilon/4. \end{aligned} \tag{2.24}$$

Then, for each $p \in \mathcal{P}$,

$$\begin{aligned} |\mathrm{tr}([\tilde{\varphi}_{i,j}(p)]) - \mathrm{tr}([\tilde{\varphi}_{0,j}(p)])| &< \left|\gamma^*(\tau_{t_i})(p) - \gamma^*(\tau_{0_j})(p)\right| + \delta_1/2 \quad \text{(by (2.15), (2.24))} \\ &= \left|\gamma(\hat{p})(\tau_{t_i}) - \gamma(\hat{p})(\tau_{0_j})\right| + \delta_1/2 \\ &< \left|\tau_{t_i}(p') - \tau_{0_j}(p')\right| + \delta_1/2 + \delta_1/2 \quad \text{(by (2.3))} \\ &= \delta_1. \end{aligned}$$

Define

$$d_{i,j}(p) := \mathrm{tr}([\tilde{\varphi}_{i,j}(p)]) - \mathrm{tr}([\tilde{\varphi}_{0,j}(p)]), \quad p \in \mathcal{P}.$$

Then, by Lemma 2.1, there is a $\mathcal{G}$-δ-multiplicative c.p.c. map $\mu_{i,j} : A \to Q$ such that

$$[\sigma(p)] - [\mu_{i,j}(p)] = d_{i,j}(p), \quad p \in \mathcal{P}, \tag{2.25}$$

and

$$\sigma(1_A) = \mu_{i,j}(1_A) = \theta, \quad 1 \le i \le h-1, 1 \le j \le l,$$

for a projection $\theta \in Q \otimes \mathcal{K}$ satisfying

$$\mathrm{tr}(\theta) < \varepsilon_2 = \min\{\varepsilon/8, 1/4n\}. \tag{2.26}$$

Consider the various direct sum maps

$$\bar{\varphi}_e := \sigma \oplus \widehat{\varphi}_e : A \to (1+\theta)(Q \otimes \mathcal{K})(1+\theta),$$

$$\bar{\varphi}_\infty := \bar{\varphi}_1 \oplus \cdots \oplus \bar{\varphi}_p,$$

$$\bar{\varphi}_{0,j} := \lambda_{0,j,1}\bar{\varphi}_1 \oplus \cdots \oplus \lambda_{0,j,p}\bar{\varphi}_p = \sigma \oplus \tilde{\varphi}_{0,j} : A \to (1+\theta)(Q \otimes \mathcal{K})(1+\theta),$$

$$\bar{\varphi}_{h,j} := \lambda_{h,j,1}\bar{\varphi}_1 \oplus \cdots \oplus \lambda_{h,j,p}\bar{\varphi}_p = \sigma \oplus \tilde{\varphi}_{h,j} : A \to (1+\theta)(Q \otimes \mathcal{K})(1+\theta),$$

and

$$\bar{\varphi}_{i,j} := \mu_{i,j} \oplus \tilde{\varphi}_{i,j} : A \to (1+\theta)(Q \otimes \mathcal{K})(1+\theta).$$

Then, by (2.25), one has that for any $p \in \mathcal{P}$ and $1 \leq i \leq h-1$,

$$\mathrm{tr}([\bar{\varphi}_{i,j}(p)]) - \mathrm{tr}([\bar{\varphi}_{0,j}(p)]) = ([\sigma(p)] - [\mu_{i,j}(p)]) - ([\tilde{\varphi}_{i,j}(p)] - [\tilde{\varphi}_{0,j}(p)]) = 0.$$

It also follows from (2.21) that

$$\mathrm{tr}[\bar{\varphi}_{h,j}(p)] - \mathrm{tr}[\bar{\varphi}_{0,j}(p)] = \mathrm{tr}[\tilde{\varphi}_{h,j}(p)] - \mathrm{tr}[\tilde{\varphi}_{0,j}(p)] = 0, \quad p \in \mathcal{P}.$$

Therefore,

$$\mathrm{tr}([\bar{\varphi}_{i_1,j}(p)]) = \mathrm{tr}([\bar{\varphi}_{i_2,j}(p)]), \quad p \in \mathcal{P}, 0 \leq i_1, i_2 \leq h.$$

Note that one also has

$$\pi_j \circ \psi_0 \circ \bar{\varphi}_\infty = \lambda_{0,j,1}\bar{\varphi}_1 \oplus \cdots \oplus \lambda_{0,j,p}\bar{\varphi}_p = \bar{\varphi}_{0,j} \tag{2.27}$$

and

$$\pi_j \circ \psi_1 \circ \bar{\varphi}_\infty = \lambda_{1,j,1}\bar{\varphi}_1 \oplus \cdots \oplus \lambda_{1,j,p}\bar{\varphi}_p = \bar{\varphi}_{h,j}. \tag{2.28}$$

Renormalize each $\bar{\varphi}_{i,j}$, $i = 0, ..., l$ into a unital homomorphism and denote it still by $\bar{\varphi}_{i,j}$. Equations (2.27) and (2.28) still hold, and one still has

$$\mathrm{tr}([\bar{\varphi}_{i_1,j}(p)]) = \mathrm{tr}([\bar{\varphi}_{i_2,j}(p)]), \quad p \in \mathcal{P}, 0 \leq i_1, i_2 \leq h, \tag{2.29}$$

so that by (2.15) and (2.26) and the construction of $\bar{\varphi}_{i,j}$, one has

$$|\mathrm{tr}(\bar{\varphi}_{i,j}(f) - \gamma^*(\tau_{t_i})(f))| \quad \leq \quad |\mathrm{tr}(\tilde{\varphi}_{i,j}(f) - \gamma^*(\tau_{t_i})(f))| + 2\mathrm{tr}(\theta) < \varepsilon/2. \tag{2.30}$$

Note that one has the decomposition

$$\bar{\varphi}_{i,j} = \frac{1}{1+\mathrm{tr}(\theta)}\nu_{i,j} \oplus \frac{r_{i,1,j}}{1+\mathrm{tr}(\theta)}\phi_1 \oplus \cdots \oplus \frac{r_{i,m,j}}{1+\mathrm{tr}(\theta)}\phi_m : A \to Q,$$

where

$$\nu_{0,j} := \sigma \oplus \frac{\mu_{0,j}}{1+\mathrm{tr}(q)}, \quad \nu_{h,j} := \sigma \oplus \frac{\mu_{1,j}}{1+\mathrm{tr}(q)},$$

and

$$\nu_{i,j} := \mu_{i,j}$$

otherwise.

For each $i = 0, ..., l-1$, compare the two maps

$$\bar{\varphi}_{i,j} := \frac{1}{1+\mathrm{tr}(\theta)}\nu_{i,j} \oplus \frac{r_{i,1,j}}{1+\mathrm{tr}(\theta)}\phi_1 \oplus \cdots \oplus \frac{r_{i,m,j}}{1+\mathrm{tr}(\theta)}\phi_m : A \to Q,$$

and

$$\bar{\varphi}_{i+1,j} := \frac{1}{1+\mathrm{tr}(\theta)}\nu_{i+1,j} \oplus \frac{r_{i+1,1,j}}{1+\mathrm{tr}(\theta)}\phi_1 \oplus \cdots \oplus \frac{r_{i+1,m,j}}{1+\mathrm{tr}(\theta)}\phi_m : A \to Q,$$

and define

$$\psi_{i,j} = 0 \oplus \frac{\min\{r_{i,1,j}, r_{i+1,1,j}\}}{1+\mathrm{tr}(\theta)}\phi_1 \oplus \cdots \oplus \frac{\min\{r_{i,m,j}, r_{i+1,m,j}\}}{1+\mathrm{tr}(\theta)}\phi_m : A \to Q.$$

By (2.26), (2.10), (2.22), (2.23), and (2.14), one has

$$\begin{aligned} & |\mathrm{tr}(1-\psi_{i,j}(1))| \\ < \;& \mathrm{tr}(q)+\mathrm{tr}(\theta)+|r_{i,1,j}-r_{i+1,1,j}|+\cdots+|r_{i,m,j}-r_{i+1,m,j}| \\ < \;& \varepsilon_2+2\varepsilon_2'+\frac{1}{4n}<\frac{1}{n}. \end{aligned} \tag{2.31}$$

On the other hand, by (2.29), for any $p\in\mathcal{P}$, one has

$$[(\bar\varphi_{i,j}\ominus\psi_j)(p)]=[\bar\varphi_{i,j}(p)]-[(\psi_{i,j}(p))]=[\bar\varphi_{i+1,j}(p)]-[(\psi_{i,j}(p))]=[(\phi_{j+1}\ominus\psi_j)(p)]. \tag{2.32}$$

By (2.31) and (2.32), the hypothesis of Corollary 2.5 of [11] is satisfied. Therefore, by the conclusion of Corollary 2.5 of [11], there is a unitary $u_{i+1,j}\in Q$ such that

$$\left\|\bar\varphi_{i,j}(f)-u_{i+1,j}^*\bar\varphi_{i+1,j}(f)u_{i+1,j}\right\|<\varepsilon/4,\quad f\in\mathcal{F}\cdot\mathcal{F},\ 0\leq j\leq h-1.$$

Define $v_{0,j}=1$, and set

$$u_{i,j}u_{i-1,j}\cdots u_{1,j}=v_{i,j},\quad i=1,...,h.$$

Then, for any $0\leq i\leq h-1$ and any $f\in\mathcal{F}\cdot\mathcal{F}$, one has

$$\begin{aligned} & \|\mathrm{Ad}(v_{i,j})\circ\bar\varphi_{i,j}(f)-\mathrm{Ad}(v_{i+1,j})\circ\bar\varphi_{i+1,j}(f)\| \\ = \;& \|(u_{i,j}\cdots u_{1,j})^*\bar\varphi_{i,j}(f)(u_{i,j}\cdots u_{1,j})-(u_{i+1,j}\cdots u_{1,j})^*\bar\varphi_{j+1}(f)(u_{i+1,j}\cdots u_{1,j})\| \\ = \;& \left\|\bar\varphi_{i,j}(f)-u_{i+1,j}^*\bar\varphi_{i+1,j}(f)u_{i+1,j}\right\|<\varepsilon/4. \end{aligned}$$

Replacing each homomorphism $\bar\varphi_{i,j}$ by $\mathrm{Ad}(v_{i,j})\circ\bar\varphi_{i,j}$ for $i=1,...,h-1$, and still denoting it by $\bar\varphi_{i,j}$, one has

$$\|\bar\varphi_{i,j}(f)-\bar\varphi_{i+1,j}(f)\|<\varepsilon/4,\quad f\in\mathcal{F}\cdot\mathcal{F},\ 0\leq i\leq h-2.$$

Note that

$$\|(u_{h,j}v_{h-1,j})\bar\varphi_{h-1,j}(f)(v_{h-1,j}^*u_{h,j}^*)-\bar\varphi_{h,j}(f)\|<\varepsilon/4,\quad f\in\mathcal{F}\cdot\mathcal{F}.$$

Since Q is AF, the exponential length of Q is at most π, and hence there are unitaries

$$1=z_1,z_2,...,z_{h-2},z_{h-1}=u_{h,j}v_{h-1,j}$$

such that (by (2.12))

$$\|z_i-z_{i-1}\|<\pi/(h-2)<\varepsilon/4.$$

Replacing each homomorphism $\bar\varphi_{i,j}$ by $\mathrm{Ad}(z_i^*)\circ\bar\varphi_{i,j}$ for $i=1,...,h-1$, and still denoting it by $\bar\varphi_{i,j}$, one then has

$$\|\bar\varphi_{i,j}(f)-\bar\varphi_{i+1,j}(f)\|<\varepsilon/2,\quad f\in\mathcal{F}\cdot\mathcal{F},\ 0\leq i\leq h-1.$$

Define $\phi_c: A\to\mathrm{C}([0,1],F_2)$ by

$$\phi_c(f)(t):=\frac{t_{i+1}-t}{t_{i+1}-t_i}\bar\varphi_{i,j}(f)+\frac{t-t_i}{t_{i+1}-t_i}\bar\varphi_{i+1,j}(f),\quad \text{if } t\in[t_i,t_{i+1}]_j,$$

and define $\phi: A\to C_1$ by

$$\phi(f):=(\bar\varphi_\infty(f),\phi_c(f)).$$

The map ϕ is well defined. Indeed, by (2.27) and (2.28), for any $f \in A$,

$$\phi_c(f)(0_j) = \bar{\varphi}_{0,j}(f) = \lambda_{0,j,1}\bar{\varphi}_1(f) \oplus \cdots \oplus \lambda_{0,j,p}\bar{\varphi}_p(f) = \pi_j \circ \psi_0(\bar{\varphi}_\infty(f))$$

and

$$\phi_c(f)(1_j) = \bar{\varphi}_{h,j}(f) = \lambda_{1,j,1}\bar{\varphi}_1(f) \oplus \cdots \oplus \lambda_{1,j,p}\bar{\varphi}_p(f) = \pi_j \circ \psi_1(\bar{\varphi}_\infty(f)).$$

The map ϕ is $\mathcal{F}$-ε-multiplicative. By (2.30) and (2.11), one has

$$\|\phi_*(\hat{f}) - \gamma(\hat{f})\|_\infty < \varepsilon/2, \quad f \in \mathcal{F}. \tag{2.33}$$

Note that A and C have cancellation for projections, and, also, $\mathrm{K}_0^+(A) \cong \mathrm{K}_0^+(C)$ (unital identification) and $\mathrm{Aff}(\mathrm{T}(A)) \cong \mathrm{Aff}(\mathrm{T}(C))$ (in a way compatible with the K_0-pairing). By Theorem 4.4 and Corollary 6.8 of [13] (see also Theorem 2.6 of [6] and Theorem 5.5 of [5], expressed in terms of W instead of Cu), it follows that the Cuntz semigroup of A and the Cuntz semigroup of C are isomorphic. Applied to the canonical unital map $\mathrm{Cu}(C_1) \to \mathrm{Cu}(C) \cong \mathrm{Cu}(A)$, Theorem 1 of [20] (applicable as $\mathrm{K}_1(C_1) = \{0\}$, A has stable rank one, and C_1 is unital and has stable rank one—by Theorem 5 (i) of [20], the functor $\mathrm{Cu}^\sim$ classifies homomorphisms from C_1 if and only if Cu classifies homomorphisms from C_1, and by Theorem 1 of [20], the functor $\mathrm{Cu}^\sim$ classifies homomorphisms from C_1) implies that there is a unital homomorphism $\psi : C_1 \to A$ giving rise to this map, and in particular such that

$$\psi_* = \Xi^{-1} \circ (\iota_{1,\infty})_* \quad \text{on } \mathrm{Aff}(\mathrm{T}(C_1)). \tag{2.34}$$

Since the ideal of $\mathrm{Cu}(C_1)$ killed by the map $\mathrm{Cu}(C_1) \to \mathrm{Cu}(C) \cong \mathrm{Cu}(A)$ is zero, as the map $C_1 \to C$ is an embedding, it follows that the map $C_1 \to A$ is also an embedding. By (2.33), (2.34), and (2.2), one then has

$$\|\psi_* \circ \phi_*(\hat{f}) - \hat{f}\|_\infty < \varepsilon, \quad f \in \mathcal{F},$$

as desired. □

Theorem 2.3. *Let A be a separable unital simple C*-algebra satisfying the UCT. Assume that $\mathrm{T}(A) = \mathrm{T}_{\mathrm{qd}}(A)$, and that $A \otimes Q$ has finite nuclear dimension. Then $A \otimes \mathcal{Z}$ is classifiable. Moreover, if $\mathrm{T}(A) \neq \emptyset$, one has that $A \otimes Q \in \mathrm{TA}\mathcal{C}_0$.*

Proof. If $\mathrm{T}(A) = \emptyset$, then $A \otimes \mathcal{Z}$ is purely infinite, and hence is classifiable by the classification theorem of Kirchberg and Phillips ([16], [19])

If $\mathrm{T}(A) \neq \emptyset$, then it follows from Theorem 2.2 above and Theorem 2.2 of [21] that $A \otimes Q \in \mathrm{TA}\mathcal{C}_0$. Then, by [14], $A \otimes \mathcal{Z}$ is classifiable. □

Proof of Theorem 1.1. By Proposition 8.5 of [4], as $A \otimes Q$ has finite decomposition rank, $\mathrm{T}(A \otimes Q) = \mathrm{T}_{\mathrm{qd}}(A \otimes Q)$. Furthermore, by [17], $A \otimes Q$ is stably finite and nuclear and so by [3] and [15], $\mathrm{T}(A) \neq \emptyset$. Then the statement follows from Theorem 2.3. □

Remark 2.4. By the classification result, it follows that the class of Theorem 1.1 (finite decomposition rank) coincides with the class of Theorem 2.3 (assuming $\mathrm{T}(A) \neq \emptyset$). Namely, as pointed out in the proof of Theorem 1.1, the class of Theorem 1.1 is contained in the class of Theorem 2.3. Conversely, an algebra in the class of Theorem 2.3 is, as shown in [14], is isomorphic to an

ASH algebra with decomposition rank at most three (in fact, it has decomposition rank at most two by Theorem A of [12]), and so belongs to the class of Theorem 1.1.

Together with Theorem 7.5 of [4], one has

Corollary 2.5. *Let A be a simple separable unital $\mathcal{Z}$-stable nuclear C*-algebra such that* $\mathrm{T}(A)$ *is a Bauer simplex. If* $\mathrm{T}(A) = \mathrm{T}_{\mathrm{qd}}(A)$ *and if A satisfies the UCT, A is classifiable.*

Proof. One may assume that $\mathrm{T}(A) \neq \emptyset$. It then follows from Theorem 7.5 of [4] that $\mathrm{dr}(A) \leq 1$, and hence A is classifiable by Theorem 1.1. □

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