ALL IRRATIONAL EXTENDED ROTATION ALGEBRAS ARE AF ALGEBRAS

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Abstract. Let $\theta \in [0,1]$ be any irrational number. It is shown that the extended rotation algebra $\mathcal{B}_\theta$ introduced in [8] is always an AF algebra.

1. Introduction

In [8], a natural embedding of the irrational rotation C*-algebra in a simple nuclear C*-algebra with trivial K_1-group was constructed, which gives rise to an isomorphism of ordered K_0-groups. For a dense G_δ of irrational numbers, the C*-algebra constructed (by adjoining natural spectral projections of the canonical unitary generators of the rotation algebra) was shown to be AF.

In the present paper, using the remarkable recent work of Matui and Sato ([18], [14], [13]), together with what might be called the Winter-Lin-Niu deformation technique ([20], [11], [12]), the new C*-algebra is shown to be AF for every irrational number.

By Remark 2.8 of [8], the flip of the rotation algebra extends to the larger algebra, and it is easily seen that the automorphism of order four known as the Fourier transform does also. It is an interesting question whether the whole of the natural $\mathrm{SL}(2,\mathbb{Z})$-action also extends. Also of interest is whether the (unique) extendibility of, say, the flip automorphism, determines the embedding up to an automorphism. (Note that, as pointed out in [9] and [6], all embeddings are approximately unitarily equivalent, but the question of when two differ by an automorphism—in particular, ours and the very concrete (if not absolutely unique) one of Pimsner and Voiculescu constructed in [16]—is clearly an important question—analogous to the basic question in subfactor theory.)

2. Irrational extended rotation algebras

Consider the C*-algebra $\mathrm{C}(\mathbb{T})$ as the canonical sub-C*-algebra of $\mathrm{L}^\infty(\mathbb{T})$, and denote by σ the automorphism of $\mathrm{L}^\infty(\mathbb{T})$ induced by translation by $e^{2\pi i\theta}$:

$$f(z) \mapsto f(e^{2\pi i\theta}z).$$

Note that $\mathrm{C}(\mathbb{T})$ is invariant under the action of σ.

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Note that $\mathrm{C}(\mathbb{T})$ is invariant under the action of σ.

Consider two collections of (closed, open, or half-open) subintervals $\{f_i\}_{i\in\Lambda_1}$ and $\{g_j\}_{i\in\Lambda_2}$ of $\mathbb{T}$, and still denote again by f_i and g_j the spectral projections of the canonical unitary $f(z)=z$ in $\mathrm{L}^\infty(\mathbb{T})$ corresponding to the subintervals f_i and g_j.

Consider the following two commutative C*-algebras:

$$\mathrm{C}(\Omega_u) := \mathrm{C}^*(\mathrm{C}(\mathbb{T}) \cup \{\sigma^{-k}(f_i);\ i\in\Lambda_1, k\in\mathbb{Z}\}) \subseteq \mathrm{L}^\infty(\mathbb{T})$$

and

$$\mathrm{C}(\Omega_v) := \mathrm{C}^*(\mathrm{C}(\mathbb{T}) \cup \{\sigma^{k}(g_j);\ j\in\Lambda_2, k\in\mathbb{Z}\}) \subseteq \mathrm{L}^\infty(\mathbb{T}),$$

where Ω_u and Ω_v denote the spectra of these algebras. Note that the rotation σ can be extended to an automorphism of $\mathrm{C}(\Omega_u)$ (or $\mathrm{C}(\Omega_v)$). Denote also by u and v the canonical generators of $\mathrm{C}(\mathbb{T})$ inside $\mathrm{C}(\Omega_u)$ and $\mathrm{C}(\Omega_v)$.

Definition 2.1. For an irrational number θ, and two collections of subintervals $\{f_i\}_{i\in\Lambda_1}$ and $\{g_j\}_{j\in\Lambda_2}$ of the unit circle $\mathbb{T}$, let us (as in [8]) refer to the universal C*-algebra generated by $\mathrm{C}(\Omega_u)$ and $\mathrm{C}(\Omega_v)$ with respect to the relations

(1) $uv = e^{2\pi i\theta}vu$,
(2) $u\sigma^k(g_j)u^* = \sigma^{k+1}(g_j)$ for any $j\in\Lambda_2$ and $k\in\mathbb{Z}$, and
(3) $v\sigma^{-k}(f_i)v^* = \sigma^{-k-1}(f_i)$ for any $i\in\Lambda_1$ and $k\in\mathbb{Z}$

as the (irrational) extended rotation algebra, and denote it by $\mathcal{B}_\theta$ $(=\mathcal{B}_\theta(\{f_i\},\{g_j\}))$.

Remark 2.2. If $\{f_i\}_{i\in\Lambda_1}$ and $\{g_j\}_{i\in\Lambda_2}$ are arbitrary intervals, then by 5.14 of [8], there is a short exact sequence

$$0 \longrightarrow \bigoplus \mathcal{K} \longrightarrow \mathcal{B}_\theta \longrightarrow \mathcal{B}'_\theta \longrightarrow 0\ ,$$

where $\mathcal{B}'_\theta$ is an extended rotation algebra which can be generated by half-open intervals with the same orientation, and $\mathcal{K}$ is the algebra of compact operators. In this paper, we will show that the C*-algebra $\mathcal{B}'_\theta$ is an AF algebra, and hence (by [3] and [5]) the C*-algebra $\mathcal{B}_\theta$ is AF as well.

If $\{f_i\}_{i\in\Lambda_1}$ and $\{g_j\}_{i\in\Lambda_2}$ are two collections of half-open subintervals of $\mathbb{T}$ with the same orientation, then there is another set of generators and relations for $\mathcal{B}_\theta$, as we shall now describe.

By Lemma 2.3 of [8], there is a θ-independent set of real numbers $\{a_k;\ k\in\Lambda_u\}$ for some countable index set Λ_u (finite or infinite) such that the C*-algebra $\mathrm{C}(\Omega_u)$ is generated by

$$\{\sigma^n(p_k), \sigma^n(e_k);\ n\in\mathbb{Z}, k\in\Lambda_u\},$$

where p_k is the spectral projection corresponding to $[a_k, a_k+\theta)$ or $(a_k, a_k+\theta]$, and e_k is the minimal projection corresponding to $\{a_k\}$ or zero. Since the half-open intervals f_i, $i\in\Lambda_1$, are chosen to have the same orientation, the projection e_k is always zero. Let us refer to the points $\{a_k;\ k\in\Lambda_u\}$ as the cutting points of the canonical unitary u. A similar argument also works for $\mathrm{C}(\Omega_v)$; let $\{b_l;\ l\in\Lambda_v\}$ denote the corresponding cutting points of v, where Λ_v is a countable index set.

Theorem 2.3 (Corollary 2.10 of [8]). *Assume that $\{f_i\}_{i\in\Lambda_1}$ and $\{g_j\}_{i\in\Lambda_2}$ are two collections of half-open subintervals of $\mathbb{T}$ with the same orientation. The C*-algebra $\mathcal{B}_\theta$ is the universal C*-algebra generated by positive elements $\{h_{u,k;\ k\in\Lambda_u}\}$ and $\{h_{v,l;\ l\in\Lambda_v}\}$ and unitaries u and v with respect to the relations*

(1) $uv = e^{2\pi i\theta} vu$,
(2) $\|h_{u,k}\| = \|h_{v,l}\| = 1$,
(3) $u = e^{2\pi i(h_{u,k}+a_k)}$, *and*
(4) $v = e^{2\pi i(h_{v,l}+b_l)}$,

where $\{a_k\}$ *and* $\{b_l\}$ *are as above.*

For the extended rotation algebra $\mathcal{B}_\theta$ in the case considered in Theorem 2.3 (which we shall usually consider now, unless otherwise specified—see Proposition 2.5 and Corollary 4.7), one has

Theorem 2.4 (Theorem 3.6 and 5.1 of [8])**.** *Assume that* $\{f_i\}_{i\in\Lambda_1}$ *and* $\{g_j\}_{i\in\Lambda_2}$ *are two collections of half-open subintervals of* $\mathbb{T}$ *with the same orientation. The C*-algebra* $\mathcal{B}_\theta$ *is simple and nuclear, and has a unique tracial state* τ*.*

Consider

$$B_u := \mathrm{C}^*\{f, u;\ f \in \mathrm{C}(\Omega_v)\} \quad \text{and} \quad B_v := \mathrm{C}^*\{f, v;\ f \in \mathrm{C}(\Omega_u)\}.$$

Then one has

$$B_u = \mathrm{C}(\Omega_v) \rtimes_\sigma \mathbb{Z} \quad \text{and} \quad B_v = \mathrm{C}(\Omega_u) \rtimes_\sigma \mathbb{Z}.$$

Both B_u and B_v contain the rotation algebra A_θ, and one also has that $\mathcal{B}_\theta = B_u *_{A_\theta} B_v$.

Proposition 2.5 (Proposition 3.4 and 5.3 of [8])**.** *There exist conditional expectations* $\mathbb{E}_u : \mathcal{B}_\theta \to \mathrm{C}(\Omega_u)$ *and* $\mathbb{E}_v : \mathcal{B}_\theta \to \mathrm{C}(\Omega_v)$*. Moreover, if* $\{f_i\}_{i\in\Lambda_1}$ *and* $\{g_j\}_{i\in\Lambda_2}$ *are half-open subintervals with the same orientation, then* $\mathbb{E}_u$ *and* $\mathbb{E}_v$ *are faithful.*

3. Strict comparison of positive elements

In this section, we shall show that any irrational extended rotation algebra has strict comparison for positive elements (Theorem 3.8). The technique we are going to use is that of the (one-sided) large sub-C*-algebra due to N. C. Phillips (based on the method of Putnam in [17]; see [15]). Since (as shown in [8]) irrational extended rotation algebras are simple and nuclear, and have a unique tracial state, by a result of Matui and Sato these C*-algebras are $\mathcal{Z}$-stable, i.e., $\mathcal{B}_\theta \otimes \mathcal{Z} \cong \mathcal{B}_\theta$.

Lemma 3.1. *Let* q *be a spectral projection of* v*, and let* $\varepsilon > 0$*. Then there is* $\delta > 0$ *such that if* e *is a spectral projection of* u *with support in an interval with length at most* δ*, one has*

$$\|eqe - \tau(q)e\| < \varepsilon,$$

where τ *is the canonical tracial state.*

Proof. The proof is almost the same as the proof of Lemma 5.7 of [8]. Let $f, g \in \mathrm{C}^*(v)$ be such that $f \geq q \geq g$ and

$$\tau(f - g) < \varepsilon/2.$$

Choose polynomials $F(v)$ and $G(v)$ such that

$$\|f - F\| < \varepsilon/2 \quad \text{and} \quad \|g - G\| < \varepsilon/2.$$

Denote by n the larger of the degrees of F and G. Since θ is irrational, there is $\delta > 0$ such that if e is a spectral projection of u with support in an interval with length at most δ, one has that $ev^ie = 0$ for all $1 \leq i \leq n$. In particular, this implies that $eFe = F(0)e$ and $eGe = G(0)e$, where $F(0)$ and $G(0)$ are the constant terms of F and G, respectively. Note that

$$|F(0) - \tau(q)| < \varepsilon/2 \quad \text{and} \quad |G(0) - \tau(q)| < \varepsilon/2,$$

and also note that

$$F(0)e = eFe \approx_{\varepsilon/2} efe \geq eqe \geq ege \approx_{\varepsilon/2} eGe = G(0)e.$$

Then

$$\tau(q)e + \varepsilon \geq eqe \geq \tau(q) - \varepsilon,$$

which is the conclusion of the lemma. □

Lemma 3.2. *Let $a \in \mathcal{B}_\theta$. For any $\varepsilon > 0$, there is δ such that if e is a spectral projection of u with support in an interval with length at most δ, then*

$$\|eae - \mathbb{E}_u(a)e\| < \varepsilon$$

(where $\mathbb{E}_u$ is the canonical conditional expectation from $\mathcal{B}_\theta$ to $\mathrm{C}(\Omega_u)$).

Proof. Choose $\sum_{j=1}^n c_j p_j q_j$ such that

$$\left\| a - \sum_{j=1}^n c_j p_j q_j \right\| < \varepsilon/3.$$

Then, by Lemma 3.1, there is $\delta > 0$ such that if e is a spectral projection of u with support in an interval with length at most δ, then

$$\left\| e(\sum_{j=1}^n c_j p_j q_j)e - (\sum_{j=1}^n c_j p_j \tau(q_j))e \right\| < \varepsilon/3.$$

Note that

$$\mathbb{E}_u(\sum_{j=1}^n c_j p_j q_j) = \sum_{j=1}^n c_j p_j \tau(q_j)$$

and, therefore,

$$\begin{aligned} \|eae - \mathbb{E}_u(a)e\| &\leq \left\| e(\sum_{j=1}^n c_j p_j q_j)e - \mathbb{E}_u(\sum_{j=1}^n c_j p_j q_j)e \right\| + 2\varepsilon/3 \\ &\leq \varepsilon, \end{aligned}$$

as desired. □

Lemma 3.3. *Assume that $\{f_i\}_{i \in \Lambda_1}$ and $\{g_j\}_{i \in \Lambda_2}$ are two collections of half-open subintervals of $\mathbb{T}$ each with the same orientation. Let b be a non-zero positive element of $\mathcal{B}_\theta$. Then there is a non-zero element $a \in \mathrm{C}(\Omega_u)$ such that $a \preceq b$. In particular, the C*-algebra $\mathcal{B}_\theta$ has the property (SP).*

Proof. By Proposition 2.5, the conditional expectation $\mathbb{E}_u$ is faithful. Choose ε with

$$0 < \varepsilon < \|\mathbb{E}_u(b)\|/2.$$

By Lemma 3.2, there is $e \in \mathrm{C}(\Omega_u)$ such that

$$\|ebe - \mathbb{E}_u(b)e\| < \varepsilon.$$

Moreover, again since $\mathbb{E}_u(b) \neq 0$, the spectral projection e can be chosen so that

$$|\|\mathbb{E}_u(b)e\| - \|\mathbb{E}_u(b)\|| < \frac{\varepsilon}{2}.$$

Put

$$a' := \mathbb{E}_u(b)e \in \mathrm{C}(\Omega_u)$$

and set $a = (a' - \varepsilon)_+$. Then

$$a \preceq ebe \preceq b.$$

Note that

$$\|a'\| = \|\mathbb{E}_u(b)e\| > \|\mathbb{E}_u(b)\| - \frac{\varepsilon}{2} > \frac{3\varepsilon}{2},$$

and hence $a \neq 0$, as desired. □

Lemma 3.4. *Let A be a C^*-algebra and let $a_1, a_2, ..., a_n \in A$. Then*

$$(a_1 + \cdots + a_n)^*(a_1 + \cdots + a_n) \leq na_1^*a_1 + \cdots + na_n^*a_n.$$

In particular,

$$(a_1 + \cdots + a_n)^*(a_1 + \cdots + a_n) \preceq a_1^*a_1 \oplus \cdots \oplus a_n^*a_n.$$

Proof. For any $a, b \in A$, since $(a-b)^*(a-b) \geq 0$, one has

$$a^*b + b^*a \leq a^*a + b^*b. \tag{3.1}$$

Then

$$\begin{aligned}(a_1 + \cdots + a_n)^*(a_1 + \cdots + a_n) &= \sum_{i=1}^n a_i^*a_i + \sum_{i<j}^n (a_i^*a_j + a_j^*a_i)\\ &\leq \sum_{i=1}^n a_i^*a_i + \sum_{i<j}^n (a_i^*a_i + a_j^*a_j)\\ &= n\sum_{i=1}^n a_i^*a_i,\end{aligned}$$

as desired. □

Lemma 3.5. *Assume that $\{f_i\}_{i\in\Lambda_1}$ and $\{g_j\}_{i\in\Lambda_2}$ are two collections of half-open subintervals of $\mathbb{T}$ with the same orientation. For any $a \in \mathcal{B}_\theta$, any $\varepsilon > 0$, and any $b \in \mathcal{B}_\theta \setminus \{0\}$, there are $c \in A_\theta$ and $g \in \mathcal{B}_\theta$ such that*

1. $\|a - (c + g)\| < \varepsilon$,
2. $g^*g \preceq b$ *in* $\mathcal{B}_\theta$.

Proof. By Lemma 3.3, one may assume that $b \in \mathrm{C}(\Omega_u)$, and hence one may assume that b is a projection in A_θ.

By Theorem 1 of [9], one may choose $\sum_{i=1}^{n} c_i p_i q_i$ such that

$$\left\| a - \sum_{i=1}^{n} c_i p_i q_i \right\| < \varepsilon/2,$$

where p_i and q_i are spectral projections in $\mathrm{C}(\Omega_u)$ and $\mathrm{C}(\Omega_v)$, respectively. Then choose a projection $e \in A_\theta$ with $e < b$ and $4n[e] < [b]$.

Since (by [17]) the C*-algebra $\mathrm{C}(\Omega_u) \rtimes \mathbb{Z}$ is an A$\mathbb{T}$ algebra, it has strict comparison of positive elements. Then, for each spectral projection p_i, we may choose f_i^-, f_i and f_i^+ such that

$$p_i = f_i^- + f_i + f_i^+,$$

with $f_i \in \mathrm{C}^*(u)$ and

$$f_i^-, f_i^+ \preceq e, \quad i = 1, ..., n.$$

Similarly, one also chooses g_i^-, g_i and g_i^+ such that

$$q_i = g_i^- + g_i + g_i^+,$$

with $g_i \in \mathrm{C}^*(v)$ and

$$g_i^-, g_i^+ \preceq e, \quad i = 1, ..., n.$$

Then,

$$\begin{aligned} \sum_{i=1}^{n} c_i p_i q_i &= \sum_{i=1}^{n} c_i (f_i^- + f_i + f_i^+)(g_i^- + g_i + g_i^+) \\ &= \sum_{i=1}^{n} c_i f_i g_i + \sum_{i=1}^{n} c_i f_i^- q_i + \sum_{i=1}^{n} c_i f_i^+ q_i + \sum_{i=1}^{n} c_i f_i q g_i^- + \sum_{i=1}^{n} c_i f_i q g_i^+. \end{aligned}$$

Put

$$c = \sum_{i=1}^{n} c_i f_i g_i$$

and

$$g = \sum_{i=1}^{n} c_i f_i^- q_i + \sum_{i=1}^{n} c_i f_i^+ q_i + \sum_{i=1}^{n} c_i f_i q g_i^- + \sum_{i=1}^{n} c_i f_i q g_i^+.$$

By Lemma 3.4, one has

$$g^* g \preceq \bigoplus_{i=1}^{n} f_i^- \oplus \bigoplus_{i=1}^{n} f_i^+ \oplus \bigoplus_{i=1}^{n} g_i^- \oplus \bigoplus_{i=1}^{n} g_i^+ \tag{3.2}$$

$$\preceq \bigoplus^{4n} e \preceq b, \tag{3.3}$$

as desired. □

We thank N. C. Phillips for communicating to us the following two lemmas (Lemma 1.9 and Lemma 1.11 of [15]).

Lemma 3.6 (1.9 of [15])**.** *Let A be a C^*-algebra, let $a, b \in A$ be positive, and let $\alpha, \beta \geq 0$. Then*

$$(a + b - (\alpha + \beta))_+ \preceq (a - \alpha)_+ + (b - \beta)_+ \preceq (a - \alpha)_+ \oplus (b - \beta)_+.$$

Lemma 3.7 (1.11 of [15])**.** *Let A be a C^*-algebra, and let $a, b \in A$ be such that $0 \leq a \leq b$. Let $\varepsilon > 0$. Then $(a - \varepsilon)_+ \preceq (b - \varepsilon)_+$.*

Theorem 3.8. *Assume that $\{f_i\}_{i \in \Lambda_1}$ and $\{g_j\}_{i \in \Lambda_2}$ are two collections of half-open subintervals of $\mathbb{T}$ with the same orientation. The irrational extended rotation algebra $\mathcal{B}_\theta$ has strict comparison of positive elements.*

Proof. Let a, b be positive elements of $\mathcal{B}_\theta$ (or of a matrix algebra over $\mathcal{B}_\theta$) such that $d_\tau(a) < d_\tau(b) - \delta$ for some $\delta > 0$, where τ is the canonical trace.

Suppose that 0 is not an isolated point of $\operatorname{sp}(b)$.

Choose $\delta_1 > 0$ such that

$$d_\tau((b - \eta)_+) > d_\tau(b) - \delta/4, \quad \text{for all } \eta \in (0, \delta_1). \tag{3.4}$$

Fix $\varepsilon > 0$ with $\varepsilon < \delta_1/9$. Since 0 is not an isolated point of $\operatorname{sp}(b)$, we may also assume that $h_{(0,\varepsilon/2)}(b) \neq 0$ and $h_{(\varepsilon/2,\varepsilon)}(b) \neq 0$ for continuous positive functions $h_{(0,\varepsilon/2)}$ and $h_{(\varepsilon/2,\varepsilon)}$ with supports in $(0, \varepsilon/2)$ and $(\varepsilon/2, \varepsilon)$, respectively.

By Lemma 3.5, there are $b_0 \in A_\theta$ and $b_1 \in \mathcal{B}_\theta$ such that $\left\|(b - \varepsilon)^{1/2} - (b_0 + b_1)\right\|$ is sufficiently small that

$$\|(b - \varepsilon)_+ - (b_0 + b_1)^*(b_0 + b_1)\| < \varepsilon \tag{3.5}$$

and also

$$b_1^* b_1 \preceq h_{(0,\varepsilon/2)}(b). \tag{3.6}$$

Moreover, we may assume that

$$\|(b - 8\varepsilon)_+ - ((b_0 + b_1)^*(b_0 + b_1) - 7\varepsilon)_+\| < \varepsilon. \tag{3.7}$$

Then, by Lemma 3.4,

$$b_0^* b_0 = (b_0 + b_1 - b_1)^*(b_0 + b_1 - b_1) \leq 2(b_0 + b_1)^*(b_0 + b_1) + 2b_1^* b_1, \tag{3.8}$$

and then by Lemma 3.7 and Lemma 3.6,

$$\begin{aligned} (b_0^* b_0 - 3\varepsilon)_+ &\preceq (2(b_0 + b_1)^*(b_0 + b_1) + 2b_1^* b_1 - 3\varepsilon)_+ && (3.9)\\ &\preceq (2(b_0 + b_1)^*(b_0 + b_1) - 2\varepsilon)_+ + (2b_1^* b_1 - \varepsilon)_+ && (3.10)\\ &\preceq (b - \varepsilon)_+ + h_{(0,\varepsilon/2)}(b) \quad \text{(by (3.5) and (3.6))}. && (3.11) \end{aligned}$$

In particular,

$$d_\tau((b_0^* b_0 - 3\varepsilon)_+) \leq d_\tau(b).$$

On the other hand,

$$\begin{aligned}
(b-9\varepsilon)_+ &= ((b-8\varepsilon)_+-\varepsilon)_+ && (3.12)\\
&\preceq ((b_0+b_1)^*(b_0+b_1)-7\varepsilon)_+ \quad \text{(by (3.7))} && (3.13)\\
&\preceq (2b_0^*b_0+2b_1^*b_1-7\varepsilon)_+ && (3.14)\\
&\preceq (2b_0^*b_0-6\varepsilon)_+ + (2b_1b_1^*-\varepsilon)_+ && (3.15)\\
&\preceq (b_0^*b_0-3\varepsilon)_+\oplus b_1^*b_1. && (3.16)
\end{aligned}$$

In particular,

$$d_\tau((b-9\varepsilon)_+)\le d_\tau((b_0^*b_0-3\varepsilon)_+)+d_\tau(b_1^*b_1),$$

and hence

$$d_\tau((b_0^*b_0-3\varepsilon)_+)\ge d_\tau((b-9\varepsilon)_+)-d_\tau(b_1^*b_1)>d_\tau(b)-\delta/4-\delta/4=d_\tau(b)-\delta/2. \tag{3.17}$$

Applying Lemma 3.5 to $a^{1/2}$, we define $a_0\in A_\theta$ and $a_1\in\mathcal{B}_\theta$ such that

$$\|a-(a_0+a_1)^*(a_0+a_1)\|<\varepsilon, \tag{3.18}$$

$$a_1^*a_1\preceq h_{(\varepsilon/2,\varepsilon)}(b), \tag{3.19}$$

and

$$\|(a-7\varepsilon)_+-((a_0+a_1)^*(a_0+a_1)-7\varepsilon)_+\|<\varepsilon. \tag{3.20}$$

Then, the same argument as for (3.12) shows that

$$(a-8\varepsilon)_+\preceq(a_0^*a_0-3\varepsilon)_+ + a_1^*a_1, \tag{3.21}$$

and since

$$a_0^*a_0\le 2(a_0+a_1)^*(a_0+a_1)+2a_1^*a_1,$$

the same argument as for (3.9) shows that

$$\begin{aligned}
(a_0^*a_0-3\varepsilon)_+ &\preceq ((a_0+a_1)^*(a_0+a_1)-\varepsilon)+(a_1^*a_1-\varepsilon/2)_+ && (3.22)\\
&\preceq a\oplus a_1^*a_1. && (3.23)
\end{aligned}$$

Therefore, by (3.17),

$$d_\tau((a_0^*a_0-3\varepsilon)_+)\le d_\tau(a)+d_\tau(a_1^*a_1)<d_\tau(a)+\delta/2<d_\tau(b)-\delta/2<d_\tau((b_0^*b_0-3\varepsilon)_+).$$

Note that $(a_0^*a_0-3\varepsilon)_+\in A_\theta$ and $(b_0^*b_0-3\varepsilon)_+\in A_\theta$. Since (by [7] or [2]) A_θ has strict comparison on positive elements, one has

$$(a_0^*a_0-3\varepsilon)_+\preceq(b_0^*b_0-3\varepsilon)_+,$$

and hence

$$\begin{aligned}
(a-8\varepsilon)_+ &\preceq (a_0^*a_0-3\varepsilon)_+ + a_1^*a_1 \quad \text{(by (3.21))} && (3.24)\\
&\preceq (b_0^*b_0-3\varepsilon)_+ + a_1^*a_1 && (3.25)\\
&\preceq (b-\varepsilon)_+ + h_{(0,\varepsilon/2)}(b)+h_{(\varepsilon/2,\varepsilon)}(b) \quad \text{(by (3.9))} && (3.26)\\
&\preceq b. && (3.27)
\end{aligned}$$

Since ε is arbitrary, and the left side converges to a as ε converges to zero, by inspection of the definition of Cuntz comparison, one has

$$a \preceq b.$$

Suppose now that 0 is an isolated point of $\mathrm{sp}(b)$. Then the range projection of b in the bidual of $\mathcal{B}_\theta$ belongs to $\mathcal{B}_\theta$, and is Cuntz equivalent to b, and so we may assume that b is a projection. Since $\mathcal{B}_\theta$ has property (SP), if $b \neq 0$ (as we may suppose), there is a non-zero projection $p < b$ such that $\tau(p) < \tau(b) - d_\tau(a)$. Pick a positive element $c \in p\mathcal{B}_\theta p$ with $\mathrm{sp}(c) = [0, 1]$, and consider the positive element

$$b' := (b - p) + c.$$

Then

$$d_\tau(b') \geq \tau(b - p) > d_\tau(a),$$

$\mathrm{sp}(b') = [0, 1]$, and $b' < b$. By the first part of the proof, one has that

$$a \preceq b'.$$

Since $b' < b$, we again have $a \preceq b$. □

Corollary 3.9. *Assume that $\{f_i\}_{i\in\Lambda_1}$ and $\{g_j\}_{i\in\Lambda_2}$ are two collections of half-open subintervals of $\mathbb{T}$ with the same orientation. Then the irrational extended rotation algebra $\mathcal{B}_\theta$ is $\mathcal{Z}$-stable.*

Proof. By Theorem 5.1 and Theorem 3.6 of [8], $\mathcal{B}_\theta$ is simple and has a unique tracial state. By Corollary 7.5 of [8], $\mathcal{B}_\theta$ is nuclear. Hence, by Theorem 1.1 of [13], $\mathcal{B}_\theta$ is $\mathcal{Z}$-stable. □

4. Quasidiagonality and the UCT

In this section, let us show that any $\mathcal{B}_\theta$ is quasidiagonal and satisfies the UCT. Then, by a result of Matui and Sato in [14] and a recent classification theorem, it will follow that the C*-algebra $\mathcal{B}_\theta$ is an AF algebra.

Theorem 4.1. *Assume that $\{f_i\}_{i\in\Lambda_1}$ and $\{g_j\}_{i\in\Lambda_2}$ are two collections of half-open subintervals of $\mathbb{T}$ with the same orientation. Then, for any irrational θ, the extended rotation algebra $\mathcal{B}_\theta$ is quasidiagonal.*

Proof. Since $\mathcal{B}_\theta$ is nuclear (7.5 of [8]), it is enough to show that $\mathcal{B}_\theta$ can be (unitally) embedded into $\frac{\prod_{\lambda=1}^{\infty} \mathrm{M}_{n_\lambda}(\mathbb{C})}{\bigoplus_{\lambda=1}^{\infty} \mathrm{M}_{n_\lambda}(\mathbb{C})}$ for suitable natural numbers n_λ.

Let m_λ, n_λ be natural numbers such that $m_\lambda/n_\lambda \to \theta$ as $\lambda \to \infty$. Set $\omega_\lambda = e^{2\pi i m_\lambda/n_\lambda}$,

$$u_\lambda := \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \ddots & \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathrm{M}_{n_\lambda}(\mathbb{C}),$$

and

$$v_\lambda := \mathrm{diag}\{\omega_\lambda, \omega_\lambda^2, \cdots, \omega_\lambda^{n_\lambda}\} \in \mathrm{M}_{n_\lambda}(\mathbb{C}).$$

For each $k \in \Lambda_u$, pick

$$h_{u,k,\lambda} = \frac{1}{2\pi i}\log(u_\lambda e^{-ia_k})$$

with $0 \leq h_{u,k,\lambda} \leq 1$, and for each $l \in \Lambda_v$, pick

$$h_{v,l,\lambda} = \frac{1}{2\pi i}\log(v_\lambda e^{-ib_l})$$

with $0 \leq h_{v,l,\lambda} \leq 1$.

Consider the elements

$$u := \widetilde{(u_\lambda)},\ v := \widetilde{(v_\lambda)},\ h_{u,k} := \widetilde{(h_{u,k,\lambda})},\ \text{and}\ h_{v,l} := \widetilde{(h_{v,l,\lambda})}$$

in $\frac{\prod_{\lambda=1}^{\infty}\mathrm{M}_{n_\lambda}(\mathbb{C})}{\bigoplus_{\lambda=1}^{\infty}\mathrm{M}_{n_\lambda}(\mathbb{C})}$. We shall show that that the C*-algebra generated by these elements is isomorphic to $\mathcal{B}_\theta$; the theorem follows.

By Theorem 2.3, it is enough to show that u, v, $h_{u,k}$, $h_{v,l}$ satisfy the relations

(1) $uv = e^{2\pi i\theta}vu$,
(2) $\|h_{u,k}\| = \|h_{v,l}\| = 1$,
(3) $u = e^{2\pi i(h_{u,k}+a_k)}$, and
(4) $v = e^{2\pi i(h_{v,l}+b_l)}$.

One only has to verify Condition (1) since the other conditions are satisfied straightforwardly. A calculation shows that

$$u_\lambda v_\lambda u_\lambda^* v_\lambda^* = e^{2\pi i m_\lambda/n_\lambda},$$

and hence

$$\lim_{\lambda\to\infty} u_\lambda v_\lambda u_\lambda^* v_\lambda^* = \lim_{\lambda\to\infty} e^{2\pi i m_\lambda/n_\lambda} = e^{2\pi i\theta},$$

which implies

$$uv = e^{2\pi i\theta}vu$$

in $\frac{\prod_{\lambda=1}^{\infty}\mathrm{M}_{n_\lambda}(\mathbb{C})}{\bigoplus_{\lambda=1}^{\infty}\mathrm{M}_{n_\lambda}(\mathbb{C})}$. Therefore, the elements u, v, $h_{u,k}$, $h_{v,l}$ generate a copy of $\mathcal{B}_\theta$ in $\frac{\prod_{\lambda=1}^{\infty}\mathrm{M}_{n_\lambda}(\mathbb{C})}{\bigoplus_{\lambda=1}^{\infty}\mathrm{M}_{n_\lambda}(\mathbb{C})}$, as desired. $\square$

Let us now show that the C*-algebra $\mathcal{B}_\theta$ satisfies the UCT. It will be convenient to show at the same time, for use in the final classification, that $\mathrm{K}_1(\mathcal{B}_\theta) = \{0\}$, and that $\mathrm{K}_0(\mathcal{B}_\theta)$ is torsion free.

Note that $\mathcal{B}_\theta = B_u *_{A_\theta} B_v$. (In the case that there is only one cutting point for each of u and v, it follows directly from the Cuntz-Germain-Thomsen exact sequence that $\mathrm{K}_0(\mathcal{B}_\theta) = \mathbb{Z} + \theta\mathbb{Z}$ and $\mathrm{K}_1(\mathcal{B}_\theta) = \{0\}$.) Denote by i_u and i_v the embeddings of A_θ into B_u and B_v, respectively, and denote by j_u and j_v the embeddings of B_u and B_v into $\mathcal{B}_\theta$, respectively.

Before looking at $\mathrm{K}_*(\mathcal{B}_\theta)$, let us consider the C*-algebras B_u and B_v, and rewrite them as certain amalgamated free products.

Recall that $B_u = \mathrm{C}(\Omega_v) \rtimes_\sigma \mathbb{Z}$, and suppose that there are only finitely many cutting points $\{b_l;\ l \in \Lambda_v\}$ on the unitary v. Put $0 = b_1 < b_2 < \cdots < b_{|\Lambda_v|} < 1$.

For each $l \in \Lambda_v$, denote by I_l the closed interval $[b_l, b_{l+1}]$ (assume $b_{|\Lambda_v|+1} = 1$), and consider the C*-algebra $\bigoplus_{l\in\lambda_v} \mathrm{C}(I_l)$. For each $l \in \Lambda_v$, define a function $h_l : [0,1] \to [0,1]$ by

$$h_l : t \mapsto \begin{cases} t - b_l, & \text{if } t \in \dot{\bigcup}_{s\geq l} I_s, \\ t + (1 - b_l), & \text{otherwise.} \end{cases}$$

Then $\{1, h_l;\ l \in \Lambda_v\}$ is a set of generators for the C*-algebra $\bigoplus_{l\in\lambda_v} \mathrm{C}(I_l)$. Regard the unitary v as the function

$$t \mapsto e^{2\pi i t}, \quad t \in \dot{\bigcup}_l I_l,$$

in $\bigoplus_{l\in\Lambda_v} \mathrm{C}(I_l)$. Then a direct verification shows that

$$v = e^{2\pi i (h_l + b_l)}, \quad l \in \Lambda_v.$$

On the other hand, in the concrete C*-algebra B_u, the commutative C*-algebra $\mathrm{C}(\Omega_v)$ contains a copy of $\bigoplus_{l\in\Lambda_v} \mathrm{C}(I_l)$. Let $\bar{h}_l$ denote the generator in $\mathrm{C}(\Omega_v)$ corresponding to the element h_l. Then, there is a homomorphism ϕ from the amalgamated free product $A_\theta *_{\mathrm{C}(\mathbb{T})} (\bigoplus_{l\in\Lambda_v} \mathrm{C}(I_l))$ to B_u induced by

$$\phi(u) = u,\ \phi(h_l) = \bar{h}_l, \quad l \in \Lambda_v. \tag{4.1}$$

Since the image contains $\{u^{-n}\bar{h}_l u^n;\ n \in \mathbb{Z}, l \in \Lambda_v\}$, it contains all the elements of $\mathrm{C}(\Omega_v)$, and therefore ϕ is surjective. In the following, let us show that the map ϕ is also injective.

Lemma 4.2. *Under the assumption that $|\Lambda_v| < \infty$, the map ϕ defined in (4.1) is injective. In particular, the C*-algebra B_u is isomorphic to $A_\theta *_{\mathrm{C}(\mathbb{T})} (\bigoplus_{l\in\Lambda_v} \mathrm{C}(I_l))$.*

Proof. The argument is similar to that of Theorem 2.9 of [8]. Set

$$B'_u = A_\theta *_{\mathrm{C}(\mathbb{T})} (\bigoplus_{l\in\Lambda_v} \mathrm{C}(I_l)).$$

Choose a faithful representation π of B'_u on some Hilbert space $\mathcal{H}$, and let us still use the same notation for the images of the elements of B'_u as for the elements themselves.

Since $v = e^{2\pi i(h_l+b_l)}$, one has

$$h_l = \frac{1}{2\pi i} \log(e^{-2\pi i b_l} v) + e_l, \tag{4.2}$$

where e_l is a subprojection of the spectral projection $E_v(\{e^{2\pi i b_l}\})$.

Consider the positive elements $g_1 := f_1(h_l)$ and $g_2 := f_2(h_l)$, where

$$f_1(x) = \begin{cases} 0 & \text{if } 0 \leq x \leq 1/2, \\ \text{linear} & \text{if } 1/2 \leq x \leq 1-\theta, \\ 1 & \text{otherwise,} \end{cases}$$

and

$$f_2(x) = \begin{cases} 1 & \text{if } 0 \leq x \leq \theta, \\ \text{linear} & \text{if } \theta \leq x \leq 1/2, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$g_1 = f_1(\frac{1}{2\pi i}\log(v)) + e_l,$$

$$g_2 = f_2(\frac{1}{2\pi i}\log(v)) + (E_v(\{e^{2\pi i b_l}\}) - e_l),$$

and hence

$$(4.3)\qquad g_1(ug_2u^*) = (f_1(\frac{1}{2\pi i}\log(v)) + e_l)(uf_2(\frac{1}{2\pi i}\log(v))u^* + u(E_v(\{e^{2\pi i b_l}\}) - e_l)u^*)$$

$$(4.4)\qquad = f_1(\frac{1}{2\pi i}\log(v)) \cdot uf_2(\frac{1}{2\pi i}\log(v))u^* + e_l + u(E_v(\{e^{2\pi i b_l}\}) - e_l)u^*$$

$$(4.5)\qquad = E_v((b_l - \theta, b_l)) + e_l + u(E_v(\{e^{2\pi i b_l}\}) - e_l)u^*.$$

Therefore, the element $g_1ug_2u^*$ is a projection. Let us define

$$d_{l,n} := u^{-n-1}(g_1(vg_2u^*))u^{n+1}.$$

Then the elements

$$\{v, d_{l,n};\ n \in \mathbb{Z}, k \in \Lambda_v\}$$

satisfy the relation $\mathcal{R}'$ of [8], and by Lemma 2.6 of [8], the C*-algebra generated by $\{v, d_{l,n};\ n \in \mathbb{Z}, k \in \Lambda_v\}$ is isomorphic to $\mathrm{C}(\Omega_v)$ under the map

$$v \mapsto z,\ d_{l,n} \mapsto \sigma^{-n}(\chi_{[b_l, b_l+\theta)}).$$

Therefore, there is a homomorphism

$$\psi : B_u \cong \mathrm{C}(\Omega_v) \rtimes \mathbb{Z} \to B'_u$$

with

$$\psi(u) = u \quad \text{and} \quad \psi(\sigma^{-n}(\chi_{[b_l,b_l+\theta)})) = d_{l,n}.$$

In particular, by (4.2) and (4.3), one has that

$$\psi(\bar{h}_l) = h_l, \quad l \in \Lambda_v,$$

and hence $\psi \circ \phi = \mathrm{id}_{B'_u}$, which implies that the map ϕ is injective. □

Lemma 4.3. *Consider the C*-algebra B_u (or B_v). Let $a \in K_0(B_u)$ with $na \in (i_u)_0(K_0(A_\theta))$ for some non-zero $n \in \mathbb{N}$. Then $a \in (i_u)_0(K_0(A_\theta))$.*

Proof. Assume that $|\Lambda_v| < \infty$. By Lemma 4.2 and Theorem 6.4 of [19], a straightforward calculation shows that the sequence

$$0 \longrightarrow \mathrm{K}_0(\mathrm{C}(\mathbb{T})) \xrightarrow{\iota} \mathrm{K}_0(A_\theta) \oplus (\bigoplus \mathrm{K}_0(\mathrm{C}(I_l))) \xrightarrow{(i_u)_0 - (\eta)_0} \mathrm{K}_0(B_u) \longrightarrow 0$$

is exact, where

$$\iota(1) = (0,1) \oplus (1, ..., 1),$$

and η is the embedding of $\bigoplus_l \mathrm{C}(I_l)$ into B_u.

Let $(a, b) \oplus (c_1, ..., c_{|\Lambda_v|-1}) \in \mathrm{K}_0(A_\theta) \oplus (\bigoplus \mathrm{K}_0(\mathrm{C}(I_l)))$ be a representative of a. One then has that

$$((na, nb) \oplus (nc_1, ..., nc_{|\Lambda_v|-1})) - ((a', b') \oplus (0, ..., 0)) = (0, m) \oplus (m, ..., m)$$

for some $a', b', m \in \mathbb{Z}$. In particular, this implies that m is divisible by n, and

$$c_1 = \cdots = c_{|\Lambda_v|-1} = m/n.$$

Then the element

$$(a, b) \oplus (c_1, ..., c_{|\Lambda_v|-1}) - (0, m/n) \oplus (m/n, ..., m/n) = (a, b - m/n) \oplus (0, ..., 0)$$

is still a representative of a, and it is in the image of $\mathrm{K}_0(A_\theta)$, as desired.

If $|\Lambda_v| = |\{b_1, b_2, ..., b_i, ...\}| = \infty$, denote by $\Omega_{v,n}$ for each $n = 1, 2, ...$ the commutative C*-algebra generated by the spectral projections $\{\chi_{[b_l+k\theta, b_l+(k+1)\theta)};\ i = 1, ..., n, k \in \mathbb{Z}\}$, and consider the C*-algebra crossed product

$$B_{u,n} := \mathrm{C}(\Omega_{v,n}) \rtimes_\sigma \mathbb{Z}.$$

Then, as a sub-C*-algebra of B_u, each $B_{u,n}$ contains A_θ, and $B_u = \overline{\bigcup_{n=1}^{\infty} B_{u,n}}$. The conclusion follows from the preceding case, that there are only finitely many cutting points. ☐

Lemma 4.4. *With the setting as above,*

1. *$K_1(\mathcal{B}_\theta) = \{0\}$, and $K_0(\mathcal{B}_\theta)$ is torsion free,*
2. *The map $(i_u, -i_v) : A_\theta \to B_u \oplus B_v$ induces an injective map on the K-groups.*

Proof. Since $B_u = \mathrm{C}(\Omega_v) \rtimes \mathbb{Z}$ and the action of σ on Ω_v has no nontrivial clopen subset, a direct calculation using the Pimsner-Voiculescu six-term exact sequence shows that $\mathrm{K}_1(B_u)$ is isomorphic to $\mathbb{Z}$, and is generated by the canonical unitary u. The same argument also works for B_v. In particular, this implies that $(i_u, -i_v)$ (or (i_u, i_v)) induces an isomorphism between $\mathrm{K}_1(A_\theta)$ and $\mathrm{K}_1(B_u) \oplus \mathrm{K}_1(B_v)$.

For the injectivity on K_0-groups, by applying the standard trace on B_u (or B_v), one has that the map ι_u (or ι_v) also induces an embedding of $\mathrm{K}_0(A_\theta)$ into $\mathrm{K}_0(B_u)$ (or $\mathrm{K}_0(B_v)$). In particular, the map $(i_u, -i_v)$ (or (i_u, i_v)) induces an injective map from $\mathrm{K}_1(A_\theta)$ to $\mathrm{K}_1(B_u) \oplus \mathrm{K}_1(B_v)$.

By Theorem 6.4 of [19], one has the exact sequence

$$\begin{array}{ccccc} \mathrm{K}_0(A_\theta) & \xrightarrow{(i_{u0}, i_{v0})} & \mathrm{K}_0(B_u) \oplus \mathrm{K}_0(B_v) & \xrightarrow{j_{u0} - j_{v0}} & \mathrm{K}_0(\mathcal{B}_\theta) \\ \uparrow & & & & \downarrow \\ \mathrm{K}_1(\mathcal{B}_\theta) & \xleftarrow[j_{u1} - j_{v1}]{} & \mathrm{K}_1(B_u) \oplus \mathrm{K}_1(B_v) & \xleftarrow[(i_{u1}, i_{v1})]{} & \mathrm{K}_1(A_\theta). \end{array}$$

Since $(i_{u1}, i_{v1}) : \mathrm{K}_1(A_\theta) \to \mathrm{K}_1(B_u) \oplus \mathrm{K}_1(B_v)$ is an isomorphism, one has that $\mathrm{K}_1(\mathcal{B}_\theta)$ embeds into $\mathrm{K}_0(A_\theta)$ with image the kernel of the map $(i_{u0}, i_{v0}) : \mathrm{K}_0(A_\theta) \to \mathrm{K}_0(B_u) \oplus \mathrm{K}_0(B_v)$. But the map (i_{u0}, i_{v0}) is injective, as shown above. Therefore, $\mathrm{K}_1(\mathcal{B}_\theta) = \{0\}$.

Let us show that $\mathrm{K}_0(\mathcal{B}_\theta)$ is torsion free. As shown above, one has that

$$\mathrm{K}_0(\mathcal{B}_\theta) = (\mathrm{K}_0(B_u) \oplus \mathrm{K}_0(B_v))/(i_{u0}, i_{v0})(\mathrm{K}_0(A_\theta)).$$

Let $(a, b) \in \mathrm{K}_0(B_u) \oplus \mathrm{K}_0(B_v)$ with $n\overline{(a, b)} = 0$ for some nonzero $n \in \mathbb{Z}$; that is

$$n(a, b) = ((i_u)_0(c), (i_v)_0(c))$$

for some $c \in \mathrm{K}_0(A_\theta)$, and hence

$$na = (i_u)_0(c) \quad \text{and} \quad nb = (i_v)_0(c).$$

By Lemma 4.3, one has that

$$a \in (i_u)_0(\mathrm{K}_0(A_\theta)) \quad \text{and} \quad b \in (i_v)_0(\mathrm{K}_0(A_\theta)).$$

Denote by $a', b' \in \mathrm{K}_0(A_\theta)$ the preimages of a, b, respectively. Since the maps $(i_u)_0$ and $(i_v)_0$ are injective, one has

$$na' = c = nb'.$$

Since $\mathrm{K}_0(A_\theta)$ is torsion free, one has $a' = b'$, and therefore

$$(a, b) = ((i_u)_0(a'), (i_v)_0(b')) \in ((i_u)_0, (i_v)_0)(\mathrm{K}_0(A_\theta)),$$

which implies

$$\overline{(a, b)} = 0 \in \mathrm{K}_0(\mathcal{B}_\theta).$$

This shows that the group $\mathrm{K}_0(\mathcal{B}_\theta)$ is torsion free. □

Let A, B be C*-algebras. In what follows, let

$$\gamma(A, B) : \mathrm{KK}(A, B) \to \mathrm{Hom}(\mathrm{K}_*(A), \mathrm{K}_*(B))$$

denote the canonical homomorphism. Let us also use the same notation for the analogous homomorphism with domain $\mathrm{E}(A, B)$.

Proposition 4.5 (23.8.1 of [1])**.** *Let A be a separable C*-algebra. Suppose that for every separable C*-algebra B with divisible K-groups, $\gamma(A, B)$ is an isomorphism. Then for every separable C*-algebra B, the exact sequence of the UCT holds for A and B.*

Theorem 4.6. *For any irrational θ, the extended rotation algebra $\mathcal{B}_\theta$ satisfies the UCT.*

Proof. Since $\mathcal{B}_\theta$ is nuclear, the group $\mathrm{E}(\mathcal{B}_\theta, D)$ and $\mathrm{KK}(\mathcal{B}_\theta, D)$ are canonically isomorphic for any separable D. Therefore by Proposition 4.5, it is enough to show that $\gamma(\mathcal{B}_\theta, D)$ is an isomorphism between $\mathrm{E}(\mathcal{B}_\theta, D)$ and $\mathrm{Hom}(\mathrm{K}_*(\mathcal{B}_\theta), \mathrm{K}_*(D))$ for any separable D with divisible K-groups.

By Theorem 6.3 of [19], one has the exact sequence

(4.6)
$$\begin{array}{ccccc} \mathrm{E}(A_\theta, D) & \xleftarrow{i_u^* - i_v^*} & \mathrm{E}(B_u, D) \oplus \mathrm{E}(B_v, D) & \xleftarrow{(j_u^*, j_v^*)} & \mathrm{E}(\mathcal{B}_\theta, D) \\ \downarrow & & & & \uparrow \\ \mathrm{E}(\mathcal{B}_\theta, SD) & \xrightarrow[(j_u^*, j_v^*)]{} & \mathrm{E}(B_u, SD) \oplus \mathrm{E}(B_v, SD) & \xrightarrow[i_u^* - i_v^*]{} & \mathrm{E}(A_\theta, SD). \end{array}$$

Applying the functor γ to the lower-right corner of (4.6), one has the commutative diagram

$$\begin{array}{ccc} \mathrm{E}(B_u, SD) \oplus \mathrm{E}(B_v, SD) & \xrightarrow{i_u^* - i_v^*} & \mathrm{E}(A_\theta, SD) \\ \Big\downarrow {\scriptstyle \gamma(B_u, SD) \oplus \gamma(B_v, SD)} & & \Big\downarrow {\scriptstyle \gamma(A_\theta, SD)} \\ \mathrm{Hom}(\mathrm{K}_*(B_u), \mathrm{K}_*(SD)) \oplus \mathrm{Hom}(\mathrm{K}_*(B_v), \mathrm{K}_*(SD)) & \xrightarrow[i_u^* - i_v^*]{} & \mathrm{Hom}(\mathrm{K}_*(A_\theta), \mathrm{K}_*(SD)). \end{array}$$

By Lemma 4.4 (2), the map $(i_u, -i_v): A_\theta \to B_u \oplus B_v$ induces an embedding of K-groups. Then, since $\mathrm{K}_*(D)$ is divisible, the map $i_u^* - i_v^*$ in the bottom row is a surjective homomorphism. Since the C*-algebras B_u, B_v, and A_θ are nuclear and satisfy the UCT, the vertical maps induced by the functor γ are isomorphisms, and therefore the map

$$i_u^* - i_v^* : \mathrm{E}(B_u, SD) \oplus \mathrm{E}(B_v, SD) \to \mathrm{E}(A_\theta, SD)$$

must be surjective.

By exactness of the sequence (4.6), one then has that the map $\mathrm{E}(A_\theta, SD) \to \mathrm{E}(\mathcal{B}_\theta, D)$ is zero, and therefore the map

$$(j_u^*, j_v^*) : \mathrm{E}(\mathcal{B}_\theta, D) \to \mathrm{E}(B_u, D) \oplus \mathrm{E}(B_v, D)$$

is injective.

Let us consider the map $\gamma(\mathcal{B}_\theta, D)$ and show that it is an isomorphism. Applying the functor γ to the top part of (4.6), one has the commutative diagram

(4.7)
$$\begin{array}{ccccc} \mathrm{Hom}(\mathrm{K}_*(A_\theta), \mathrm{K}_*(D)) & \xleftarrow{i_u^*-i_v^*} & \bigoplus_{\bullet=u,v} \mathrm{Hom}(\mathrm{K}_*(B_\bullet), \mathrm{K}_*(D)) & \xleftarrow{(j_u^*, j_v^*)} & \mathrm{Hom}(\mathrm{K}_*(\mathcal{B}_\theta), \mathrm{K}_*(D)) \\ \uparrow{\scriptstyle \gamma(A_\theta, D)} & & \uparrow{\scriptstyle \gamma(B_u,D)\oplus\gamma(B_v,D)} & & \uparrow{\scriptstyle \gamma(\mathcal{B}_\theta, D)} \\ \mathrm{E}(A_\theta, D) & \xleftarrow{i_u^*-i_v^*} & \bigoplus_{\bullet=u,v} \mathrm{E}(B_\bullet, D) & \xleftarrow{(j_u^*, j_v^*)} & \mathrm{E}(\mathcal{B}_\theta, D). \end{array}$$

Since the map (j_u^*, j_v^*) in the bottom row is injective, and as before, the first two vertical maps (actually, we only need the middle one here) are isomorphisms, the map $\gamma(\mathcal{B}_\theta, D)$ must be injective.

Let us show that $\gamma(\mathcal{B}_\theta, D)$ is also surjective. Note that the sequence

$$0 \longrightarrow \mathrm{K}_0(A_\theta) \xrightarrow{(i_u^*, -i_v^*)} \mathrm{K}_0(B_u) \oplus \mathrm{K}_0(B_v) \xrightarrow{j_u^*+j_v^*} \mathrm{K}_0(\mathcal{B}_\theta) \longrightarrow 0$$

is exact, and $\mathrm{K}_1(\mathcal{B}_\theta) = \{0\}$. Then a direct calculation shows that the top sequence of (4.7) is exact in the middle, and the map (j_u^*, j_v^*) (in the top row) is injective. Since the C*-algebras A_θ, B_u, and B_v satisfy the UCT, the maps $\gamma(A_\theta, D)$, $\gamma(B_u, D)$, and $\gamma(B_v, D)$ are isomorphisms. Let $a \in \mathrm{Hom}(\mathrm{K}_0(\mathcal{B}_\theta), \mathrm{K}_0(D))$, and denote the image of a in $\mathrm{E}(B_u, D) \oplus \mathrm{E}(B_v, D)$ by a'. Then, by the exactness of the top sequence, the element a' must be sent to 0 in $\mathrm{E}(A_\theta, D)$, whence, by the exactness of the lower sequence, there is an element $a'' \in E(\mathcal{B}_\theta, D)$ which is sent to a'. Since (j_u^*, j_v^*) is injective also at the level of Hom (in the top row), the element a'' must be sent to a under the map $\gamma(\mathcal{B}_\theta, D)$. This shows that the map $\gamma(\mathcal{B}_\theta, D)$ is surjective, as desired. □

Corollary 4.7. *For arbitrary collections of sub-intervals $\{f_i\}_{i\in\Lambda_1}$ and $\{g_j\}_{j\in\Lambda_2}$, the irrational extended rotation algebra $\mathcal{B}_\theta = \mathcal{B}_\theta(\{f_i\}, \{g_j\})$ is an AF algebra.*

Proof. Suppose that $\{f_i\}_{i\in\Lambda_1}$ and $\{g_j\}_{j\in\Lambda_2}$ are two collections of half-open intervals with the same orientation; then $\mathcal{B}_\theta$ is simple, unital, nuclear, and has a unique tracial state. It is quasidiagonal by Theorem 4.1. Hence by Theorem 6.1 of [14], $\mathcal{B}_\theta \otimes \mathcal{Q}$ is TAF for the universal UHF algebra $\mathcal{Q}$. In other words, $\mathcal{B}_\theta$ is rationally TAF.

By Theorem 4.6, $\mathcal{B}_\theta$ satisfies the UCT; and by Corollary 3.9, it is $\mathcal{Z}$-stable. Therefore, it is covered by the classification theorem of [20], [11], and [12]. By Lemma 4.4 (1), the group $\mathrm{K}_1(\mathcal{B}_\theta)$ is zero and $\mathrm{K}_0(\mathcal{B}_\theta)$ is torsion free. Also, as $\mathcal{B}_\theta$ is $\mathcal{Z}$-stable, by [10], the ordered group $\mathrm{K}_0(\mathcal{B}_\theta)$ is unperforated. Since $\mathcal{B}_\theta$ has a unique tracial state, the ordered group $\mathrm{K}_0(\mathcal{B}_\theta)$ has a unique state. (It is the same to show that $\mathrm{K}_0(\mathcal{B}_\theta \otimes \mathcal{Q})$ has a unique state, but this holds as $\mathcal{B}_\theta \otimes \mathcal{Q}$ has a unique tracial state and is TAF—see above.) Furthermore, the image of $\mathrm{K}_0(\mathcal{B}_\theta)$ is dense in $\mathbb{R}$ (it contains the subgroup $\mathbb{Z} + \mathbb{Z}\theta$). Therefore, $\mathrm{K}_0(\mathcal{B}_\theta)$ is a Riesz group.

It follows by [4] that there is an AF algebra with the same invariant, which is also covered by this classification theorem, and so the C*-algebra $\mathcal{B}_\theta$ is isomorphic to that AF algebra.

For the general case, that $\{f_i\}_{i\in\Lambda_1}$ and $\{g_j\}_{j\in\Lambda_2}$ are two collections of arbitrary intervals, by 5.14 of [8], there is a short exact sequence

$$0 \longrightarrow \bigoplus \mathcal{K} \longrightarrow \mathcal{B}_\theta \longrightarrow \mathcal{B}'_\theta \longrightarrow 0\ ,$$

where $\mathcal{B}'_\theta$ is an extended rotation algebra which can be generated by half-open intervals with the same orientation, and $\mathcal{K}$ is the algebra of compact operators. Since the previous argument shows that $\mathcal{B}'_\theta$ is an AF algebra, the C*-algebra $\mathcal{B}_\theta$ is an extension of AF algebras, and therefore (by [3] and [5]) it is an AF algebra as well. □

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